
Robust Outlier Arm Identification

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Abstract

We study the problem of Robust Outlier Arm Identification (ROAI), where the goal is to identify arms whose expected rewards deviate substantially from the majority, by adaptively sampling from their reward distributions. We compute the outlier threshold using the median and median absolute deviation of the expected rewards. This is a robust choice for the threshold compared to using the mean and standard deviation, since it can identify outlier arms even in the presence of extreme outlier values. Our setting is different from existing pure exploration problems where the threshold is pre-specified as a given value or rank. This is useful in applications where the goal is to identify the set of promising items but the cardinality of this set is unknown, such as finding promising drugs for a new disease or identifying items favored by a population. We propose two δ -PAC algorithms for ROAI, which includes the first UCB-style algorithm for outlier detection, and derive upper bounds on their sample complexity. We also prove a matching, up to logarithmic factors, worst case lower bound for the problem, indicating that our upper bounds are generally unimprovable. Experimental results show that our algorithms are both robust and about 5x sample efficient compared to state-of-the-art.

1. Introduction

Multi-armed bandits are commonly used to identify the optimal items (arms) among multiple candidates through adaptive queries (pure exploration setting (Jamieson & Nowak, 2014)). Every item is associated with an unknown probability distribution, and when a bandit algorithm selects (pulls) an item, it observes a value (reward) sampled from this distribution. Depending on its objective and the history of observed values, the bandit algorithm has to decide which

item to sample at every time t , so as to identify the optimal items using as few samples as possible. Pure exploration bandit algorithms have been proposed for various objectives, such as identifying arms with the largest rewards (Jamieson et al., 2014; Jamieson & Nowak, 2014; Chen et al., 2016), identifying arms above a given threshold (Locatelli et al., 2016; Mukherjee et al., 2017; Xu et al., 2019) or clustering arms (Katariya et al., 2018; 2019).

In this paper, we study bandit algorithms for identifying outlier arms. Outlier arms are defined as those with expected rewards that are outliers relative to the overall set of expected rewards (e.g., arms with expected rewards that are several deviations above the mean/median of the overall set of expected rewards). The outlier detection problem has wide applications in scientific discovery (Grün et al., 2015; Chaudhary et al., 2015), fraud detection (Porwal & Mukund, 2018), medicine (Schiff et al., 2017), and public health (Hauskrecht et al., 2013). In contrast to passive outlier detection algorithms which identify outlier items using a pre-sampled dataset, bandit algorithms actively query items with the goal of identifying outliers using as few samples as possible. This is important because it can lead to early detection of fraud for example. Outlier arms subsume *good* arms with expected rewards substantially above the average, and most applications mentioned in good arm identification (Kano et al., 2019) apply to our setting.

As observed in Zhuang et al. (2017), bandit outlier detection cannot be reduced to best arm(s) identification in bandits because of the inherent double exploration dilemma - the threshold is unknown and any algorithm must balance exploring individual arms and exploring the outlier threshold. Zhuang et al. (2017) define the outlier threshold $\bar{\theta}$ using the k -sigma rule applied to the mean $\bar{\mu}$ and standard deviation $\bar{\sigma}$ of the expected rewards i.e., $\bar{\theta} = \bar{\mu} + k \cdot \bar{\sigma}$. However this threshold can fail to identify the correct outlier arms because the mean and standard deviation are themselves sensitive to outlier values (non-robust estimators). It can also miss outliers when the number of arms is small. In this paper, we define the threshold using the k -sigma rule applied to the *median* and the *median absolute deviation*, which are robust estimators with the highest possible breakdown point 0.5. This is the recommended practice in literature (Hampel, 1974; Huber, 2004; Swallow & Kianifard, 1996), and emphasized by Leys et al. (2013) in their aptly titled paper:

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“Detecting outliers: Do not use standard deviation around the mean, use absolute deviation around the median”. Similarly, Chung et al. (2008) conduct extensive experiments to compare the two methods and show that the median-based threshold identifies outliers that were missed by the mean-based threshold. We show through our theoretical and empirical results that this robust threshold not only identifies outliers more accurately, but it also requires fewer samples to do so than the mean-based threshold.

1.1. Contributions and Paper Organization

We make the following contributions. In Section 2 we formally define the Robust Outlier Arm Identification (ROAI) problem with justifications from Huber’s ϵ -contamination model. In Section 3, we propose two algorithms for the ROAI problem, which includes the first UCB-style algorithm for outlier detection. We theoretically prove the correctness of our algorithms and derive their sample complexity upper bounds in Section 4. A matching, up to logarithmic factors, worst case lower bound is provided in Section 5, indicating our upper bounds are generally tight. We further generalize our algorithms to settings with known contamination upper bound in Section 6. Experiments conducted in Section 7 show that our algorithms are *both* more robust and more sample efficient than previous state-of-the-art. We conclude our paper in Section 8 with open problems. All proofs are deferred to the Appendix due to lack of space.

1.2. Related Work

The pure exploration problem in the multi-armed bandit setting has a long history, starting from the work of (Bechhofer, 1958; Paulson et al., 1964). The aim of pure exploration is to identify an arm or arms with certain properties. For example, the best-arm identification problem involves correctly deciding which arm has the largest expected reward. The instance-dependent sample complexity bound on the best arm identification problem was analyzed/improved by (Even-Dar et al., 2002; 2006; Gabillon et al., 2012; Karnin et al., 2013; Jamieson et al., 2014; Jamieson & Nowak, 2014; Chen et al., 2016). The problem was also generalized to the setting of identifying the top- m arm (Kalyanakrishnan et al., 2012; Chen et al., 2017a;b); the thresholding bandit (Locatelli et al., 2016; Mukherjee et al., 2017; Xu et al., 2019) which identifies all arms with expected reward above a given threshold θ ; and the good arm identification problem (Kano et al., 2019; Katz-Samuels & Jamieson, 2019), where for a given ϵ “good arms” have expected reward within ϵ of the largest. Lower bounds developed in the pure exploration setting (Mannor & Tsitsiklis, 2004; Chen & Li, 2015; Kaufmann et al., 2016; Garivier & Kaufmann, 2016; Simchowitz et al., 2017) shed light on the optimality of existing algorithms.

In all of the above settings, the subset of arms of interest is determined by a user-defined parameter, e.g., m , θ , and ϵ . Outlier arm identification cannot be cast in these settings, since the cut-off cannot be a prespecified threshold or rank. The cut-off depends on the overall distribution of expected rewards, which is unknown in advance. In other words, outlier arm identification has an *instance-dependent identification target*. Bandit problems with instance-dependent identification targets have attracted some attention recently. One of the work (Katariya et al., 2019) studies the problem of identifying the largest gap in the ordering of the expected rewards, which provides a natural separation of the arms into two groups or clusters. Another line of work (Zhuang et al., 2017) focuses on identifying outlier arms with an outlier threshold adaptive to the bandit instance. Specifically, they use the threshold $\bar{\theta} = \bar{\mu} + k \cdot \bar{\sigma}$, with $\bar{\mu}$ and $\bar{\sigma}$ being the mean and standard deviation of distribution of expected rewards, respectively. The parameter k is usually chosen as 2 or 3 according to the famous three-sigma rule.

Our work focuses on robust and sample-efficient approaches to the outlier arm identification problem. We model our setting through Huber’s ϵ -contamination model (Huber et al., 1964) and apply robust estimators with the highest possible breakdown point (Donoho & Huber, 1983; Rousseeuw & Hubert, 2011), i.e., median and median absolute deviation (MAD), in building the outlier threshold. Robust statistics were previously incorporated in the bandit setting (Altschuler et al., 2019), but they mainly deal with traditional settings, i.e., best arm identification, with each reward distribution being contaminated rather than identifying instance-adaptive subsets. Although our work could also be generalized to the setting with contaminated reward distribution by incorporating their techniques, we do not pursue this direction here.

2. Problem Setting and Notations

We consider the standard multi-armed bandit setting where there are n arms and the reward of each arm follows a 1-subgaussian distribution with mean y_i . The goal of the agent is to identify outlier arms whose expected rewards *substantially* deviate from the *majority*, in the fixed confidence and pure exploration setting. Without loss of generality, we assume $y_i \geq y_{i+1}$ and $n = 2m - 1$, so that the median arm is unambiguous.¹ We also only consider identifying outliers with high rewards; identifying outliers with low rewards is analogous. Let $y_{(m)} = \text{median}\{y_i\}$ denote the expected reward of the *median* arm, and let $\text{AD}_i = |y_i - y_{(m)}|$ represent the absolute deviation of arm i from the median. Let $\text{AD}_{(m)} = \text{median}\{|y_i - y_{(m)}|\}$ denote the Median Absolute Deviation (MAD) of expected reward. Note that $y_{(m)}$

¹If $n = 2m$, we choose the median as m without loss of generality.

and $\text{AD}_{(m)}$ serve as the first two robust moments of the means of the underlying bandit instance $\{y_i\}_{i=1}^n$. We define outlier arms to be arms whose mean is greater than the threshold θ given by

$$\theta = y_{(m)} + k \cdot \text{AD}_{(m)}, \quad (1)$$

where $k \geq 1$ is a user-specified parameter. The goal of the agent is to *identify outlier arms using as few samples as possible*. Specifically, we are interested in designing adaptive algorithms that return the subset of outlier arms $S_o = \{i \in [n] : y_i > \theta\}$ (we assume $y_i \neq \theta, \forall i \in [n]$). We call this setting *Robust Outlier Arm Identification* (ROAI). For a given error probability $\delta \in (0, 1)$, we say an algorithm is δ -PAC if it correctly identifies S_o with probability at least $1 - \delta$ using a finite number of samples.

Our choice of the threshold is justified under Huber's ϵ -contamination model, where with probability $1 - \epsilon$ the mean y_i is drawn from an unknown *meta* distribution P with mean μ and standard deviation σ , and with probability ϵ the mean y_i is drawn from a contamination distribution. Note that sample median and MAD enjoy the highest possible breakdown point 0.5 (Donoho & Huber, 1983; Rousseeuw & Hubert, 2011). Hence, our threshold in Eq. (1) (up to scaling of $\text{AD}_{(m)}$) is a more robust estimator of the true threshold as compared to existing thresholds constructed using the sample mean and sample standard deviation (which have a breakdown point of 0) (Zhuang et al., 2017). Furthermore, for many common meta distributions including the normal and uniform distribution, Altschuler et al. (2019) prove tight non-asymptotic concentration results for the median and MAD constructed from contaminated samples.

Given our assumption of $y_i \geq y_{i+1}$, let the outlier set be $S_o = \{1, \dots, n_1\}$ where n_1 is *unknown*. For a given set $\{z_i\}_{i=1}^n$, we use $z_{(k)}$ to denote the k -th largest value in $\{z_i\}$; particularly, we use $z_{(m)} := \text{median}\{z_i\}$.

3. Algorithms

We formally introduce our algorithms in the section. We first provide a subroutine for constructing confidence intervals (CIs) of various quantities including the outlier threshold in Section 3.1; and then introduce our elimination- and LUCB-style algorithms in Section 3.2.

For any arm $i \in [n]$ and time t , we use $S_{i,t}$ and $N_{i,t}$ to denote the sum of rewards and number of pulls; and use $\hat{y}_{i,t} = S_{i,t}/N_{i,t}$ to denote the empirical mean reward. For any quantity $q \in \{y_i, y_{(m)}, \text{AD}_i, \text{AD}_{(m)}, \theta\}$, we use $L_{q,t}, U_{q,t}, \mathcal{I}_{q,t}$ to denote the lower bound, upper bound, and the CI respectively of q at time t .

3.1. Construction of Confidence Intervals (CIs)

The CI of individual arms i can easily be constructed using Hoeffding's inequality as $[L_{y_i,t}, U_{y_i,t}] = [\hat{y}_{i,t} - \beta_{N_{i,t}}, \hat{y}_{i,t} + \beta_{N_{i,t}}]$, where $\beta_s = \sqrt{\log(4ns^2/\delta)}/2s$.

The construction of CIs for the median ($\mathcal{I}_{y_{(m)},t}$), MAD ($\mathcal{I}_{\text{AD}_{(m)},t}$), and the outlier threshold ($\mathcal{I}_{\theta,t}$), which are needed for ascertaining whether an arm is an outlier, is explained in Algorithm 1. On line 1, the CI $\mathcal{I}_{y_{(m)},t}$ is constructed using the CIs of all arms. This is necessary because the identity of the median arm may be unknown. If the median arm can be unambiguously determined, this CI reduces to the CI of the median-th arm. The CI $\mathcal{I}_{\text{AD}_{(m)},t}$ is similarly constructed from $\mathcal{I}_{\text{AD}_i,t}$. We set $\widehat{\text{AD}}_{i,t}$ and $\hat{\theta}_t$ as the midpoint of their corresponding confidence intervals.

Algorithm 1 Construction of Confidence Intervals

Input: CIs of individual arms $\{\mathcal{I}_{y_i,t}\}_{i=1}^n$

Output: CIs $\mathcal{I}_{y_{(m)},t}, \mathcal{I}_{\text{AD}_i,t}, \mathcal{I}_{\text{AD}_{(m)},t}, \mathcal{I}_{\theta,t}$

- 1: $L_{y_{(m)},t} = \text{median}\{L_{y_i,t}\}$
 $U_{y_{(m)},t} = \text{median}\{U_{y_i,t}\}$
 $\mathcal{I}_{y_{(m)},t} = [L_{y_{(m)},t}, U_{y_{(m)},t}]$
 - 2: **for** $i = 1, \dots, n$ **do**
 - 3: $L_{\text{AD}_i,t} = \max\{L_{y_i,t} - U_{y_{(m)},t}, L_{y_{(m)},t} - U_{y_i,t}\}$
 $U_{\text{AD}_i,t} = \max\{U_{y_i,t} - L_{y_{(m)},t}, U_{y_{(m)},t} - L_{y_i,t}\}$
 $\mathcal{I}_{\text{AD}_i,t} \in [L_{\text{AD}_i,t}, U_{\text{AD}_i,t}]$
 $\widehat{\text{AD}}_{i,t} = (U_{\text{AD}_i,t} + L_{\text{AD}_i,t}) / 2$
 - 4: **end for**
 - 5: $L_{\text{AD}_{(m)},t} = \text{median}\{L_{\text{AD}_i,t}\}$
 $U_{\text{AD}_{(m)},t} = \text{median}\{U_{\text{AD}_i,t}\}$
 $\mathcal{I}_{\text{AD}_{(m)},t} = [L_{\text{AD}_{(m)},t}, U_{\text{AD}_{(m)},t}]$
 - 6: $L_{\theta,t} = L_{y_{(m)},t} + k \cdot L_{\text{AD}_{(m)},t}$
 $U_{\theta,t} = U_{y_{(m)},t} + k \cdot U_{\text{AD}_{(m)},t}$
 $\mathcal{I}_{\theta,t} = [L_{\theta,t}, U_{\theta,t}]$ and $\hat{\theta}_t = (U_{\theta,t} + L_{\theta,t}) / 2$
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3.2. Algorithms

We introduce our elimination-style (Even-Dar et al., 2006) algorithm `ROAIElim` and LUCB-style (Kalyanakrishnan et al., 2012) algorithm `ROAILUCB` in this section. Any pure exploration bandit algorithm is specified through its sampling, stopping, and recommendation rule (Kaufmann et al., 2016). The stopping and recommendation rules are the same for both algorithms. We stop at the first time t such that $\{i \in [n] : \mathcal{I}_{y_i,t} \cap \mathcal{I}_{\theta,t} \neq \emptyset\} = \emptyset$, and upon stopping we output the empirical subset of outlier arms $\hat{S}_{o,t} = \{i \in [n] : \hat{y}_{i,t} > \hat{\theta}_t\}$. We present our two algorithms next.

ROAIElim: The pseudocode of ROAIElim is given in Algorithm 2. At round t , ROAIElim constructs three active sets for the median, the MAD, and the threshold. Each of these active sets contains arms whose CIs overlap with the respective CI. Since the threshold is constructed from the median and the MAD, any of these arms can contribute towards shrinking the CI of the threshold, and hence ROAIElim samples all arms in the union of these active sets.

Algorithm 2 ROAIElim

Input: Error tolerance ϵ , probability of failure δ , and outlier detection parameter k

Output: Subset of outlier arms $\hat{S}_{o,t}$

- 1: Initialize $A_{E,1} = A_{E,1}^{\text{median}} = A_{E,1}^{\text{MAD}} = A_{E,1}^\theta = [n]$
- 2: **for** $t = 1, 2, \dots$ **do**
- 3: Sample arms in $A_{E,t}$ and update $\{\mathcal{I}_{i,t}\}_{i \in A_{E,t}}$
- 4: Update CIs using Algorithm 1
- 5: Set

$$A_{E,t+1}^{\text{median}} = \{i \in [n] : \mathcal{I}_{y_i,t} \cap \mathcal{I}_{y_{(m)},t} \neq \emptyset\} \cap A_{E,t}^{\text{median}}$$

$$A_{E,t+1}^{\text{MAD}} = \{i \in [n] : \mathcal{I}_{\text{AD}_i,t} \cap \mathcal{I}_{\text{AD}_{(m)},t} \neq \emptyset\} \cap A_{E,t}^{\text{MAD}}$$

$$A_{E,t+1}^\theta = \{i \in [n] : \mathcal{I}_{y_i,t} \cap \mathcal{I}_{\theta,t} \neq \emptyset\} \cap A_{E,t}^\theta$$

$$A_{E,t+1} = A_{E,t+1}^{\text{median}} \cup A_{E,t+1}^{\text{MAD}} \cup A_{E,t+1}^\theta$$

- 6: If $A_{E,t+1}^\theta = \emptyset$, stop and return $\hat{S}_{o,t}$
- 7: **end for**

ROAILUCB: The pseudocode of ROAILUCB is presented in Algorithm 3. We use the notation $J_{\kappa_i,t}$ to denote κ_i arms with the largest empirical means $\{\hat{y}_{i,t}\}$, and $J_{\kappa_i,t}^{\text{AD}}$ to denote the κ_i arms with the largest empirical absolute deviations $\{\widehat{\text{AD}}_{i,t}\}$. Since we are mainly interested in shrinking confidence intervals around the median quantity, we set $\kappa_1 = m - 1$ and $\kappa_2 = m$.

Motivated by the LUCB algorithm (Kalyanakrishnan et al., 2012), ROAILUCB finds the 4 arms at the median boundary, 4 arms at the MAD boundary, and 2 arms at the threshold boundary, and samples arms in the union of these sets. Unlike ROAIElim, ROAILUCB samples at most 10 arms in each round.

4. Analysis

In Section 4.1, we discuss correctness and sample complexity results of our algorithms. We compare the robustness and sample complexity of our algorithms with previous work in Section 4.2. The proofs can be found in the Appendix.

Algorithm 3 ROAILUCB

Input: Error tolerance ϵ , probability of failure δ , and outlier detection parameter k

Output: Subset of outlier arms $\hat{S}_{o,t}$

- 1: Initialize $A_{L,1} = [n]$
- 2: **for** $t = 1, 2, \dots$ **do**
- 3: Sample arms in $A_{L,t}$ and update $\{\mathcal{I}_{y_i,t}\}_{i \in A_{L,t}}$
- 4: Update CIs using Algorithm 1
- 5: Set

$$A_{L,t+1}^{\text{median}} = \arg \min_{i \in J_{m-1,t}} \{L_{y_i,t}\} \cup \arg \min_{i \in J_{m,t}} \{L_{y_i,t}\}$$

$$\cup \arg \max_{i \notin J_{m-1,t}} \{U_{y_i,t}\} \cup \arg \max_{i \notin J_{m,t}} \{U_{y_i,t}\}$$

$$A_{L,t+1}^{\text{MAD}} = \arg \min_{i \in J_{m-1,t}^{\text{AD}}} \{L_{\text{AD}_i,t}\} \cup \arg \min_{i \in J_{m,t}^{\text{AD}}} \{L_{\text{AD}_i,t}\}$$

$$\cup \arg \max_{i \notin J_{m-1,t}^{\text{AD}}} \{U_{\text{AD}_i,t}\} \cup \arg \max_{i \notin J_{m,t}^{\text{AD}}} \{U_{\text{AD}_i,t}\}$$

$$A_{L,t+1}^\theta = \arg \min_{i \in \hat{S}_{o,t}} \{L_{y_i,t}\} \cup \arg \max_{i \in \hat{S}_n,t} \{U_{y_i,t}\}$$

$$\cap \{i \in [n] : \mathcal{I}_{y_i,t} \cap \mathcal{I}_{\theta,t} \neq \emptyset\}$$

$$A_{L,t+1} = A_{L,t+1}^{\text{median}} \cup A_{L,t+1}^{\text{MAD}} \cup A_{L,t+1}^\theta$$

- 6: If $A_{L,t+1}^\theta = \emptyset$, stop and return $\hat{S}_{o,t}$
- 7: **end for**

4.1. Correctness and Sample Complexity

Lemma 1 shows the correctness of CIs in Algorithm 1. We use it to prove the correctness of our algorithms in Theorem 1.

Lemma 1. *Suppose*

$$\mathbb{P}(\forall t \in \mathbb{N}, \forall i \in [n], y_i \in \mathcal{I}_{y_i,t}) \geq 1 - \delta.$$

Then the CIs returned by Algorithm 1 are valid with probability $1 - \delta$, i.e., for $q \in \{y_{(m)}, \{\text{AD}_i\}_{i=1}^n, \text{AD}_{(m)}, \theta\}$,

$$\mathbb{P}(\forall t \in \mathbb{N}, q \in \mathcal{I}_{q,t}) \geq 1 - \delta.$$

Theorem 1 (Correctness). ROAIElim and ROAILUCB are δ -PAC.

In order to state our sample complexity bounds, we first introduce some new notations. Define

$$\Delta_i^\theta = |\theta - y_i|, \quad \Delta_*^\theta = \min_{i \in [n]} \{\Delta_i^\theta\},$$

$$\Delta_i^{\text{median}} = |y_{(m)} - y_i|, \quad \Delta_i^{\text{MAD}} = |\text{AD}_{(m)} - \text{AD}_i|,$$

$$\Delta_i^* = \max\{\Delta_*^\theta, \min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}\}. \quad (2)$$

Theorem 2 (Sample Complexity). *With probability at least $1 - \delta$, the sample complexity of ROAIElim and ROAILUCB is upper bounded by*

$$Ck^2 \sum_{i=1}^n \frac{\log(nk/\delta\Delta_i^*)}{(\Delta_i^*)^2}, \quad (3)$$

where C is a universal constant.

The sample complexity is inversely proportional to Δ_i^* defined in Eq. (2). In order to interpret the sample complexity, we consider two cases. If there exists arms whose means are close to the threshold θ , i.e., Δ_*^θ is small, then in order to classify these arms correctly, we need to estimate θ and consequently the median and the MAD accurately. Hence the complexity of sampling an arm depends on its gaps from $y_{(m)}$, $\text{AD}_{(m)}$, θ . Conversely, if all the arm means are widely separated from the threshold, i.e., Δ_*^θ is large and there is a clear distinction between normal and outlier arms, then we do not need to estimate θ accurately, and the sample complexity is $O(n/(\Delta_*^\theta)^2)$.

We highlight that the proof of Theorem 2 is non-trivial and cannot be reduced to existing techniques. The existing works (Kalyanakrishnan et al., 2012; Katariya et al., 2018) deal with scenarios where the positions of the separating boundaries depend only on the arm means, and furthermore they are user-specified. This holds true only for the median in our case, it does not hold for the AD, MAD, and the threshold because their values do not depend on a single arm. The CIs of these estimators have varying degree of uncertainty and we quantify these in our Lemmas. The technical contributions may be of independent interest and we refer the reader to our proofs in the Appendix.

4.2. Comparison to Previous Work

We compare our setting and analysis to algorithms by Zhuang et al. (2017), which is the only work study outlier detection in the bandit setting.

To deal with the *unknown* μ and σ , (Zhuang et al., 2017) use the sample mean $\bar{\mu} = \sum_{i=1}^n y_i/n$ and sample standard deviation $\bar{\sigma} = \sqrt{\sum_{i=1}^n (y_i - \bar{\mu})^2/n}$ to approximate μ and σ , respectively, and define the outlier threshold to be $\bar{\theta} = \bar{\mu} + k \cdot \bar{\sigma}$. As discussed in Section 2, these estimators have a breakdown point 0 and are very sensitive to outliers; a single extreme outlier arm can ruin their threshold.

Algorithms developed in (Zhuang et al., 2017) also require the reward distribution of the arms to be strictly bounded; our analysis is general and works for any sub-gaussian distributions.

Finally, although a direct comparison of sample complexities is not possible due to different definitions of outlier thresholds, we empirically see that our algorithms require

fewer samples to achieve the same error rate.

5. Lower bound

In this section, we study lower bound on the expected number of samples needed to identify outlier arms by any δ -PAC algorithm, where the outlier threshold is defined by Eq. (1).

Our lower bound is instance-dependent. Recall that our upper bound scales like $\tilde{O}(\sum_{i \in [n]} 1/(\Delta_i^*)^2)$ where Δ_i^* is given by Eq. (2). The problem is easy when Δ_i^* is large, and the upper bound could potentially be large when Δ_i^θ is small. In this section we argue that this is unavoidable. We show that if Δ_i^θ is small enough, there exists a lower bound that matches the upper bound up to logarithmic factors. This indicates that our sample complexity upper bounds are generally *unimprovable*.

We apply the change of measure technique (Kaufmann et al., 2016), which give a lower bound in terms of the KL-divergence. To connect the KL-divergence to the Euclidean distance in our upper bound, we assume that the reward distribution of each arm is $\mathcal{N}(y_i, 1)$.² We use D_{y_i} to denote the distribution $\mathcal{N}(y_i, 1)$ as it is fully characterized by its mean y_i .

For a bandit instance $D_y = (D_{y_1}, \dots, D_{y_n})$, assume without loss of generality that $y_i \geq y_{i+1}$ and that each arm is unambiguously identifiable as an outlier or normal arm, i.e., $y_i \neq \theta, \forall i \in [n]$. We use $\mathbb{E}_y(\cdot)$ to represent the expectation with respect to the bandit instance D_y and randomness in the algorithm. We develop lower bounds for the following subset of bandit instances.

Definition 1. *Let $\mathcal{M}_{n,\rho} = \{D_y = (D_{y_1}, \dots, D_{y_n}) : y_i \neq \theta\}$ be a subset of bandit instances with θ defined in Eq. (1) and $k \geq 2$, and satisfying the following two conditions.*

1. *There exists a unique median $y_{(m)}$ and a unique MAD $\text{AD}_{(m)}$, with*

$$\eta := 1/2 \cdot \min_{i \in \{m, m-1\}} \{y_{(i)} - y_{(i+1)}, \text{AD}_{(i)} - \text{AD}_{(i+1)}\}.$$

2. *There exists a constant $\rho < \eta$ such that at least two arms l_1 and l_2 such that $\rho/2 < \theta - y_{l_i} < \rho$, and at least two arms u_1 and u_2 such that $\rho/2 < y_{u_i} - \theta < \rho$; furthermore, there exists no arm with mean in $[\theta - \rho/2, \theta + \rho/2]$.*

It is easy to see that $\mathcal{M}_{n,\rho} \neq \emptyset$ for reasonably large n . The conditions in Definition 1 are essentially to make sure that slightly changing the median $y_{(m)}$ or the MAD $\text{AD}_{(m)}$ will incur a change in the set of outlier arms. Then, for

²The lower bound could be generalized to other distributions, as discussed in (Kaufmann et al., 2016).

any δ -PAC algorithm to correctly identify the subset of outlier arms, it is necessary to accurately identify the outlier threshold, which eventually leads to a matching sample complexity lower bound. We state our lower bound for the subset of bandit instances $\mathcal{M}_{n,\rho}$ next.

Theorem 3. *Suppose bandit instance $D_y \in \mathcal{M}_{n,\rho}$. Then for $\delta \leq 0.15$, any δ -PAC outlier arm identification algorithm \mathcal{A} with outlier threshold constructed as in Eq. (1) and an almost surely finite stopping time τ , we have that*

$$\mathbb{E}_y[\tau] \geq \sum_{i \in [n]} \frac{1}{5(\Delta_i^*)^2} \log\left(\frac{1}{2.4\delta}\right).$$

In general for bandit instances outside $\mathcal{M}_{n,\rho}$ but with non-empty subset of outlier arms, the outlier identification problem is at least as hard as the top- n_1 arm identification problem where n_1 is the number of outlier arms given by an oracle. Thus, any lower bound for top- n_1 arm identification, e.g., Theorem 4 in (Kaufmann et al., 2016), applies as a general lower bound for the outlier arm identification problem.

6. Heuristic to Reduce Sample Complexity

The sample complexity of our algorithms is inversely proportional to $(\Delta_i^*)^2$ (see Eq. (2)), which could be as small as $(\min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\})^2$ if Δ_i^* is small. As n increases, there can be many arms with small Δ_i^{median} or Δ_i^{MAD} and the sample complexity can be high as a result. In general, we cannot circumvent this cost if the outlier threshold is constructed as in Eq. (1).

However, it might not be necessary to always construct outlier threshold using all n arms, and one heuristic approach is to construct threshold only from a subset of arms. Suppose we know, from an oracle, an upper bound $c < 0.5$ on the fraction of arms drawn from the contaminated distribution, we could then randomly draw a subset $\Omega \subseteq [n]$ of arms with cardinality $|\Omega| \geq 2\lceil nc \rceil + 1$. The cardinality requirement makes sure the fraction of contamination within the subset Ω is smaller than 0.5 so that the median and MAD are not arbitrarily destroyed by outliers; but of course the threshold constructed crucially depends on the selection of Ω . Although the outlier set computed from this modified threshold could differ from the outlier set computed from $[n]$, we could potentially enjoy a smaller sample complexity. We next state an upper bound on the sample complexity in this setting.³ Empirical examinations of the performance are summarized in Section 7.2.

Corollary 1. *Suppose we run Algorithm 3 with $y_{(m)}$, $\text{AD}_{(m)}$ and θ constructed using arms in $\Omega \subseteq [n]$. Then, with probability at least $1 - \delta$, the sample complexity is*

³See Appendix F for details of the algorithm.

upper bounded by

$$Ck^2 \sum_{i \in \Omega} \frac{\log(nk/(\delta\Delta_i^*))}{(\Delta_i^*)^2} + C \sum_{i \notin \Omega} \frac{\log(n/(\delta\Delta_i^\theta))}{(\Delta_i^\theta)^2},$$

where $\Delta_i^* = \max\{\Delta_i^\theta, \min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}\}$ and C is a universal constant.

7. Experiments

We conduct three experiments. In Section 7.1, we verify the tightness of our sample complexity upper bounds in Section 4.1. In Section 7.2, we compare our setting to the non-robust version proposed by Zhuang et al. (2017) and empirically confirm the robustness of our thresholds as discussed in Section 4.2. Finally, in Section 7.3, we compare the anytime performance of our algorithms with baselines on a synthetic and a real-world dataset. For ease of comparison, we make the fraction of contamination deterministic rather than random as in the original Huber’s contamination model. All our results are averaged over 500 runs. Error bar in Fig. 2, Fig. 4 and Fig. 5 are rescaled by $2/\sqrt{500}$. Our code is publicly available (Zhu et al., 2020).

7.1. Sample Complexity

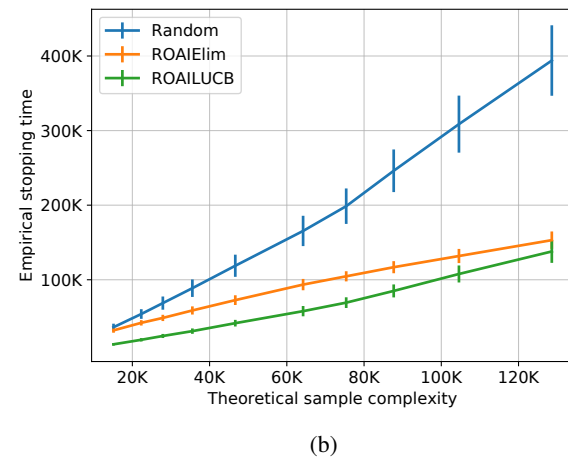
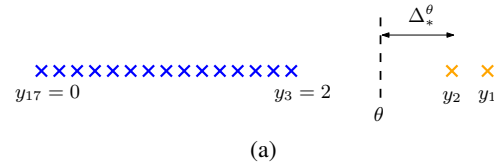


Figure 1. (a) Configuration of the arm means, we vary Δ_*^θ to change hardness (b) Theoretical upper bound vs empirical stopping time, the linear relationship shows that our upper bounds are correct up to constants.

In order to test that the hardness predicted by our upper

bound scales correctly, we first plot the empirical stopping time of each algorithm against the theoretical sample complexity (Theorem 2 with $C = 10$). We choose the arm configuration in Fig. 1(a) containing 15 normal arms (in blue) with fixed means equally distributed from 0 to 2, an outlier threshold $\theta \approx 2.837$, and 2 outlier arms (in orange) above the outlier threshold. The distance between the outlier arms is fixed at 0.2. We decrease Δ_*^θ from 0.6 to 0.2, and this changes the theoretical sample complexity. Note that the threshold does not change. The reward of each arm is normally distributed with standard deviation 0.5. In Fig. 1(b), we plot the empirical stopping time of our algorithms against the theoretical sample complexity, and we see a linear relationship between the two, which suggests that our sample complexity in Theorem 2 is correct up to constants. Fig. 1(b) also shows that our adaptive algorithms always outperform random sampling, and the gains increase with the hardness of the problem.

7.2. Setting Comparison

In this section, we compare the robustness of our outlier threshold and the sample complexity upper bound of our algorithms to the threshold and algorithms considered by Zhuang et al. (2017). We introduce the nomenclature of the algorithms next. Round Robin (RR) and Weighted Round Robin (WRR) are algorithms proposed by Zhuang et al. (2017) which use a non-robust outlier threshold. We denote by ROAI- $\lambda n\epsilon$ the algorithm suggested in Section 6 that constructs the outlier threshold from a subset Ω of arms with $|\Omega| = \max\{\lambda[n\epsilon] + 1, 15\}$. For each run of this experiment, we generate the means of normal arms from $\mathcal{N}(0.3, 0.075^2)$ (clipped to the three-sigma range), and the means of outlier arms from $\text{Unif}(x, 1)$. We draw 105 arms in total. We multiply MAD with $1/(\Phi^{-1}(3/4)) \approx 1.4826$ to make it consistent for the true scale of normal distribution (Leys et al., 2013).

We first study robustness. In Fig. 2, we generate outlier arms from $\text{Unif}(0.7, 1)$ and vary the fraction ϵ of contaminated arms from 0 to 0.2, and compare the robustness of the proposed outlier threshold from different algorithms. We measure the robustness as deviation of the proposed threshold from the true threshold. The true threshold is chosen according to the three-sigma rule. It is clear that our outlier thresholds are much more robust to contamination.

We next compare the upper bounds on the sample complexity of different algorithms. We generate 10 outlier arms from $\text{Unif}(x, 1)$ with x varying from 0.6 to 0.9. In Fig. 3, we plot the median sample complexity upper bounds of each algorithm in log scale, ignoring universal constants. We notice that under these contamination settings, our sample complexity upper bounds are orders of magnitude smaller than the ones proposed in Zhuang et al. (2017). From Fig. 2

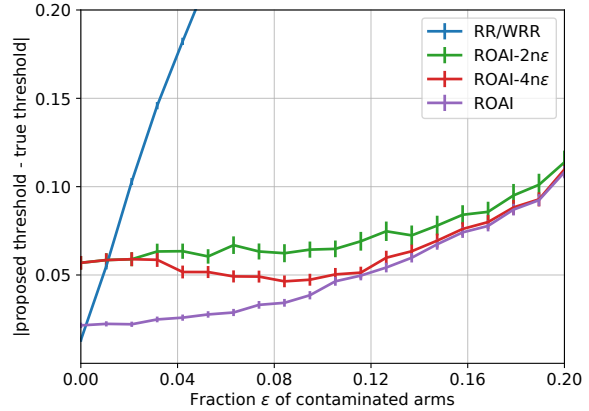


Figure 2. Deviation of the proposed outlier threshold from the true threshold as a function of the contamination level ϵ . This shows that our threshold is robust to contamination.

and Fig. 3, we also see the trade-off between robustness and sample complexity for our generalized algorithms suggested in Section 6.

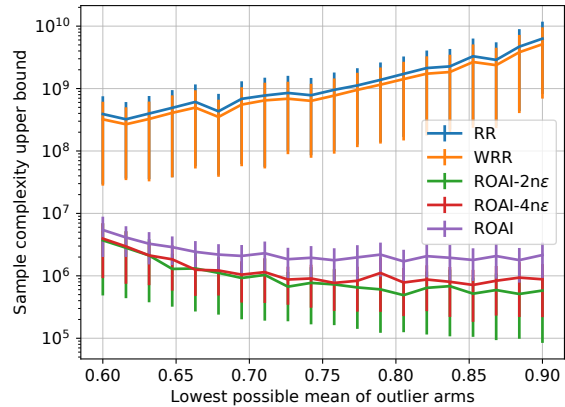


Figure 3. Sample complexity upper bounds as a function of the lowest possible mean of outlier arms, our upper bounds are smaller.

7.3. Anytime Performance

In this section, we examine the anytime empirical error rate of ROAIUCB, ROAIElim, random sampling and RR/WRR (Zhuang et al., 2017). Similar to Section 7.2, we generate 100 normal arm means from $\mathcal{N}(0.3, 0.075^2)$ and 5 outlier means from $\text{Unif}(0.8, 1)$. We draw rewards of each arm from a Bernoulli distribution with respect to its mean. We use Bernoulli distributions here as algorithms in Zhuang et al. (2017) only apply to arms with a strictly bounded

distribution. In order to simulate a run, we randomly draw means according to these two distributions and then draw rewards from these arms with fixed means till the end of the run. Under this setting, both our threshold (median-MAD) and the threshold in Zhuang et al. (2017) (mean-standard deviation) will lie in $[0.525, 0.8]$ with high probability. We filter out instances where the outlier sets (with respect to both thresholds) do not match the ground truth. The averaged minimum gap $\min\{|y_i - \theta|\}$ is 0.062 according to our threshold, and 0.063 according to theirs. In Fig. 4, we plot the fraction of times any algorithm fails to identify the correct set of outlier arms. We notice that ROAILUCB requires about 5x fewer samples than RR/WRR for the same error rate. Notice that RR is essentially random sampling with their threshold, and hence we use our threshold in the algorithm labeled Random. The empirical performance of RR/WRR is worse than Random.

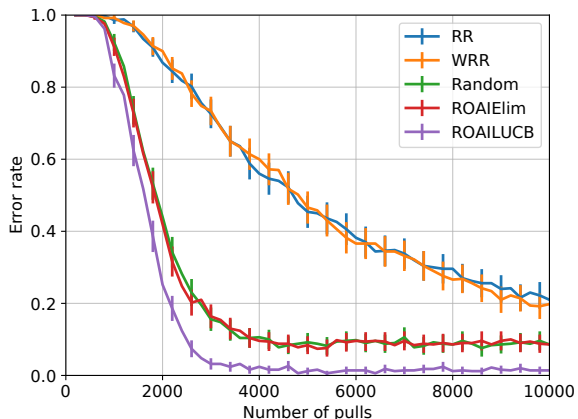


Figure 4. Fraction of times the outlier set is misidentified on synthetic data.

We also compare the performance of all algorithms on the real-world Wine Quality dataset (Sathe & Aggarwal, 2016), which is widely used to compare outlier detection algorithms. This dataset contains 129 wines, each having 13 features. 10 of these wines are labeled as outliers in the dataset. To obtain a 1d representation of each wine, we projected data points on the first principal component and then rescaled them to $[0, 1]$. We deleted 6 values closest to the threshold in this 1d representation so that the outlier set is the same according to both definitions. The 123 means thus obtained are plotted in Fig. 5(a) with the top-5 outliers in orange. We simulate each arm as a Bernoulli distribution. As in the previous experiment, ROAILUCB greatly outperforms other algorithms, and RR/WRR is worse than random sampling.

The fact that ROAIElim performs similar to random sampling in terms of the anytime error rate is not new (Jamieson

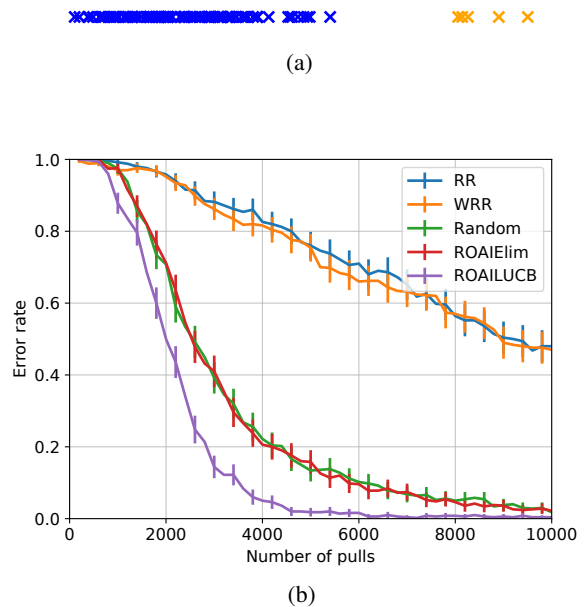


Figure 5. (a) 1d means obtained from the Wine Quality dataset (b) Fraction of times the outlier set is misidentified on this dataset.

et al., 2014), elimination-style algorithms are very conservative initially. Fig. 1(b) does show that ROAIElim outperforms random sampling in terms of the empirical stopping time.

8. Conclusion

This paper studies robust outlier arm identification problem, a pure exploration problem with instance-adaptive identification target in the multi-armed bandit setting. We propose two algorithms ROAIElim and ROAILUCB, and theoretically derive their correctness and sample complexity upper bounds. We also provide a matching, up to log factors, worst case lower bound, indicating our upper bounds are generally tight. We conduct experiments to show our algorithms are both robust and about 5x sample efficient compared to state-of-the-art.

We leave open several questions. First, the sample complexity of our algorithms is large when Δ_*^θ is small. We propose a heuristic to partially address this issue if an upper bound on the contamination ϵ is known in Section 6. Another potential approach is to add an error tolerance to allow arms close the threshold being misclassified, but that adds another user-specific parameter. We also leave open the problem of obtaining a tight instance dependent lower bound. Our current lower bound, even though instance-dependent, works only in the worst case, and we reduce the problem to top- n_1 arm identification in the general case.

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A. Correctness Analysis: Proof of Lemma 1 and Theorem 1

We first define two events W' and W as followings:

$$W' = \bigcap_{t \in \mathbb{N}} \bigcap_{i \in [n]} \{y_i \in \mathcal{I}_{y_i, t}\}$$

and

$$W = \bigcap_{t \in \mathbb{N}} \left\{ \bigcap_{i \in [n]} \{y_i \in \mathcal{I}_{y_i, t}\} \quad \bigcap_{q \in \{y_{(m)}, \{\text{AD}_i\}_{i=1}^n, \text{AD}_{(m)}, \theta\}} \{q \in \mathcal{I}_{q, t}\} \right\}.$$

A byproduct of the proof of Lemma 1 is $W' \implies W$.

Lemma 1. *Suppose*

$$\mathbb{P}(\forall t \in \mathbb{N}, \forall i \in [n], y_i \in \mathcal{I}_{y_i, t}) \geq 1 - \delta.$$

Then the CIs returned by Algorithm 1 are valid with probability $1 - \delta$, i.e., for $q \in \{y_{(m)}, \{\text{AD}_i\}_{i=1}^n, \text{AD}_{(m)}, \theta\}$,

$$\mathbb{P}(\forall t \in \mathbb{N}, q \in \mathcal{I}_{q, t}) \geq 1 - \delta.$$

Proof. Under event W' , which is assumed to hold with probability at least $1 - \delta$, we show that the construction of confidence intervals through Algorithm 1 are valid and tight.⁴

We first show $y_{(m)} \in \mathcal{I}_{y_{(m)}, t}$ through contradiction: Suppose $y_{(m)} < L_{y_{(m)}, t}$, we then know there exists at least m arms with means greater than $y_{(m)}$, which contradicts the fact that there could have at most $m - 1$ arms with means greater than $y_{(m)}$; similarly, we know that $y_{(m)} > U_{y_{(m)}, t}$ cannot hold. We thus know $y_{(m)} \in \mathcal{I}_{y_{(m)}, t}$. The confidence interval of $y_{(m)}$ is tight in the sense that we could have $y_{(m)} = L_{y_{(m)}, t}$ or $y_{(m)} = U_{y_{(m)}, t}$: It is possible that all means of arms are on the right or left end points of their corresponding confidence intervals.

We next show $\text{AD}_i \in [L_{\text{AD}_i, t}, U_{\text{AD}_i, t}]$. We first argue $\text{AD}_i \geq L_{\text{AD}_i, t}$: When $\mathcal{I}_{y_i, t} \cap \mathcal{I}_{y_{(m)}, t} \neq \emptyset$, we have both $L_{y_i, t} - U_{y_{(m)}, t} \leq 0$ and $L_{y_{(m)}, t} - U_{y_i, t} \leq 0$; since $\text{AD}_i \geq 0$ by definition, we have $\text{AD}_i \geq L_{\text{AD}_i, t} = 0$. When $\mathcal{I}_{y_i, t} \cap \mathcal{I}_{y_{(m)}, t} = \emptyset$, we have one of $L_{y_i, t} - U_{y_{(m)}, t}$ and $L_{y_{(m)}, t} - U_{y_i, t}$ represents the smallest distance between two points within their corresponding confidence intervals and the other one represents the negative of the largest distance between two points within their corresponding confidence intervals; since AD_i cannot be smaller than the smallest distance between two confidence intervals, we have $\text{AD}_i \geq L_{\text{AD}_i, t}$. To prove $\text{AD}_i \leq U_{\text{AD}_i, t}$: We notice that no matter two confidence intervals overlap or not, we have $\max\{U_{y_i, t} - L_{y_{(m)}, t}, U_{y_{(m)}, t} - L_{y_i, t}\}$ represents the largest distance between two points within their corresponding confidence intervals; thus, $\text{AD}_i \leq U_{\text{AD}_i, t}$. The confidence interval $\mathcal{I}_{\text{AD}_i, t}$ is tight in the sense that we could have $\text{AD}_i = U_{\text{AD}_i, t}$ or $\text{AD}_i = L_{\text{AD}_i, t}$.

The proof of $\text{AD}_{(m)} \in \mathcal{I}_{\text{AD}_{(m)}, t}$ is similar to the proof of $y_{(m)} \in \mathcal{I}_{y_{(m)}, t}$; and the proof of $\theta \in \mathcal{I}_{\theta, t}$ follows from the fact that $\theta = y_{(m)} + k \cdot \text{AD}_{(m)}$ and the construction of $\mathcal{I}_{y_{(m)}, t}$ and $\mathcal{I}_{\text{AD}_{(m)}, t}$.

We notice that above analyses show $W' \implies W$. Thus, $\mathbb{P}(\neg W) \leq \mathbb{P}(\neg W') \leq \delta$. \square

Theorem 1 (Correctness). *ROAIElim and ROAILUCB are δ -PAC.*

Proof. We first notice that, under event W , when $A_{E, t}^\theta = \emptyset$ or $A_{L, t}^\theta = \emptyset$, we have the confidence interval of all arms being separated from the confidence interval of outlier threshold; and thus we could correctly output the subset of outlier arms.

We next show that $\mathbb{P}(W) \geq 1 - \delta$: Since we have $W' \implies W$, which leads to $\mathbb{P}(\neg W) \leq \mathbb{P}(\neg W')$; we thus only need to show $\mathbb{P}(\neg W') < \delta$ in the following. Based on Hoeffding's inequality and the construction of confidence bound for

⁴We set $L_{\text{AD}_i, t} = \max\{L_{y_i, t} - U_{y_{(m)}, t}, L_{y_{(m)}, t} - U_{y_i, t}\}$ in Algorithm 1 in order to provide better estimations of $\widehat{\text{AD}}_{i, t}$ in experiments, especially when t is small. Here, we actually prove with a slightly tighter version $L_{\text{AD}_i, t} = \max\{0, L_{y_i, t} - U_{y_{(m)}, t}, L_{y_{(m)}, t} - U_{y_i, t}\}$, as absolute deviation is always non-negative. Our analysis works for the situations $L_{\text{AD}_i, t} = \max\{L_{y_i, t} - U_{y_{(m)}, t}, L_{y_{(m)}, t} - U_{y_i, t}\}$ as well: (1) for ROAIElim, one could simply change Definition 2, and the subsequent Lemma 3 and Lemma 8 still hold; (2) for ROAILUCB, Lemma 10 still hold.

individual arms, we actually allocate $\delta/(2nN_{i,t}^2)$ failure probability to arm i at the $N_{i,t}$ -th pull. The fact that

$$\sum_{i \in [n]} \sum_{N_{i,t}=1}^{\infty} \frac{\delta}{2nN_{i,t}^2} = \frac{\pi^2 \delta}{12} < \delta$$

directly leads to the desired result $\mathbb{P}(\neg W') < \delta$.

We prove that both algorithms stop in finite time in our sample complexity analysis. \square

B. Sample Complexity Analysis: Restate of Theorem 2

Recall

$$\begin{aligned} \Delta_i^\theta &= |\theta - y_i|, & \Delta_*^\theta &= \min_{i \in [n]} \{\Delta_i^\theta\}, \\ \Delta_i^{\text{median}} &= |y_{(m)} - y_i|, & \Delta_i^{\text{MAD}} &= |\text{AD}_{(m)} - \text{AD}_i|, \\ \Delta_i^* &= \max\{\Delta_*^\theta, \min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}\}. \end{aligned}$$

We restate Theorem 2 as below. The proofs of `ROAIElim` and `ROAILUCB` can be found in Appendix C and Appendix D, respectively. The factor k appears in the upper bound in Theorem 2 as $\mathcal{I}_{\text{AD}_{(m)},t}$ is enlarged by a factor of k when constructing $\mathcal{I}_{\theta,t}$.

Theorem 2 (Sample Complexity). *With probability at least $1 - \delta$, the sample complexity of `ROAIElim` and `ROAILUCB` is upper bounded by*

$$Ck^2 \sum_{i=1}^n \frac{\log(nk/\delta\Delta_i^*)}{(\Delta_i^*)^2}, \quad (3)$$

where C is a universal constant.

We also provide the following remark, which will be referred frequently in our formal sample complexity analysis.

Remark 1. *There exists a universal constant C such that for any $\lambda > 0$, when $s > C \frac{\log(n/\delta\lambda)}{\lambda^2}$, we have $\beta_s < \lambda$ with*

$$\beta_s = \sqrt{\frac{\log(4ns^2/\delta)}{2s}}.$$

C. Sample Complexity Analysis of `ROAIElim`

We conduct the sample complexity analysis of `ROAIElim` on top of intersecting confidence intervals introduced in Appendix C.1.

We start by providing some supporting lemmas; we then characterize the confidence interval of the median $y_{(m)}$, the MAD $\text{AD}_{(m)}$ and the outlier threshold θ before getting into the sample complexity analysis. We will upper bound sample complexity under the good event W , which happens with probability of at least $1 - \delta$.

C.1. Intersecting Confidence Interval

Start from now on and up to the proof of sample complexity of `ROAIElim` (Theorem 4), we will conduct our analysis with respect to the following intersecting confidence intervals:

$$\mathcal{I}'_{y_i,t} = \bigcap_{t' \leq t} \mathcal{I}_{y_i,t'} = \left[\max_{t' \leq t} L_{y_i,t'}, \min_{t' \leq t} U_{y_i,t'} \right] =: [L'_{y_i,t}, U'_{y_i,t}] \quad (4)$$

for $\forall t \in \mathbb{N}, i \in [n]$. It's easy to see that $\mathcal{I}'_{y_i,t}$ is a valid confidence interval and $\mathcal{I}'_{y_i,t} \subseteq \mathcal{I}_{y_i,t} = [L_{y_i,t}, U_{y_i,t}]$, a property will be used frequently. A formal analysis for the correctness of $\mathcal{I}'_{y_i,t}$ could be obtained from Lemma 9 in (Katariya et al., 2019). We apply $\mathcal{I}'_{y_i,t}$ in Algorithm 1 and Algorithm 2.

We analyze confidence intervals of the median, the median absolute deviation and the threshold, with the help of this tighter confidence interval. For convenience, we will keep using notation $\mathcal{I}_{y_i,t} = [L_{y_i,t}, U_{y_i,t}]$ to actually represent $\mathcal{I}'_{y_i,t} = [L'_{y_i,t}, U'_{y_i,t}]$; and $\mathcal{I}_{y^{(m)},t}, \mathcal{I}_{AD_i,t}, \mathcal{I}_{AD^{(m)},t}, \mathcal{I}_{\theta,t}$ to represent the generated confidence intervals through Algorithm 1 with the intersecting confidence intervals as input. There will be no ambiguity of this slightly abuse of notation as original confidence intervals are not used anymore up to the end of Appendix C.

C.2. Supporting Definition and Lemmas

Lemma 2. $\forall i \in [n]$ and $\forall t \in \mathbb{N}$, we have

$$[L_{y_i,t}, U_{y_i,t}] \subseteq [y_i - 2\beta_{N_{i,t}}, y_i + 2\beta_{N_{i,t}}].$$

Proof. We only need to prove $L_{y_i,t} \geq y_i - 2\beta_{N_{i,t}}$ and the other side is symmetric. Combining $L_{y_i,t} \geq \hat{y}_{i,t} - \beta_{N_{i,t}}$ with the fact that $\hat{y}_{i,t} \geq y_i - \beta_{N_{i,t}}$ leads to the desired result.⁵ \square

Definition 2. We define a $\text{FindAD}([l_1, u_1], [l_2, u_2])$ operator with two confidence intervals as input and output a single confidence interval of absolute deviation as following

$$\text{FindAD}([l_1, u_1], [l_2, u_2]) = [\max\{0, l_1 - u_2, l_2 - u_1\}, \max\{u_1 - l_2, u_2 - l_1\}]. \quad (5)$$

Note that FindAD is symmetric with respect to its inputs.

Lemma 3. Suppose $[l_1, u_1] \subseteq [l'_1, u'_1]$, then we have $\text{FindAD}([l_1, u_1], [l_2, u_2]) \subseteq \text{FindAD}([l'_1, u'_1], [l_2, u_2])$.

Proof. This is almost self-evident by combining Eq. (5) with the fact that $u'_1 \geq u_1$ and $l'_1 \leq l_1$. \square

Lemma 4. Given two set $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$. If $a_i \geq b_i$ for each $i \in [n]$, then we have $a_{(j)} \geq b_{(j)}$ for any $j \in [n]$.

Proof. We prove the result through contradiction. Suppose $a_{(j)} < b_{(j)}$, then there exists a subset $S \subseteq [n]$ with $|S| \geq n - j + 1$ such that $\forall i \in S, a_i < b_{(j)}$; this results in at most $j - 1$ items among $\{a_i\}_{i=1}^n$ are greater than or equal to $b_{(j)}$. On the other side, since $a_i \geq b_i$ and $b_{(j)}$ is the j -th largest item among $\{b_i\}_{i=1}^n$, there should have at least j items among $\{a_i\}_{i=1}^n$ being greater or equal to $b_{(j)}$, which leads to a contradiction. \square

Lemma 5. Let $\mathcal{I}_{y_i,t} = [L_{y_i,t}, U_{y_i,t}]$ represents the intersecting confidence intervals of arms i . Suppose $\mathcal{I}_{y^{(m)},t}, \mathcal{I}_{AD_i,t}, \mathcal{I}_{AD^{(m)},t}$ and $\mathcal{I}_{\theta,t}$ are generated from Algorithm 1 with input $\{\mathcal{I}_{y_i,t}\}_{i=1}^n$, then for any $t' \leq t$, we have

$$\begin{aligned} \mathcal{I}_{y^{(m)},t} &\subseteq \mathcal{I}_{y^{(m)},t'}, & \mathcal{I}_{AD_i,t} &\subseteq \mathcal{I}_{AD_i,t'}, \\ \mathcal{I}_{AD^{(m)},t} &\subseteq \mathcal{I}_{AD^{(m)},t'}, & \mathcal{I}_{\theta,t} &\subseteq \mathcal{I}_{\theta,t'}. \end{aligned}$$

Proof. We first notice $[L_{y_i,t}, U_{y_i,t}] \subseteq [L_{y_i,t'}, U_{y_i,t'}]$ according to the construction of intersecting confidence intervals in Eq. (4).

To prove $\mathcal{I}_{y^{(m)},t} \subseteq \mathcal{I}_{y^{(m)},t'}$, we show $L_{y^{(m)},t} \geq L_{y^{(m)},t'}$ here and the other side is similar: Since $L_{y^{(m)},t} = \text{median}\{L_{y_i,t}\}$ and $L_{y^{(m)},t'} = \text{median}\{L_{y_i,t'}\}$ and $L_{y_i,t} \geq L_{y_i,t'}$, invoking Lemma 4 with $j = m$ lead to the desired result.

To prove $\mathcal{I}_{AD_i,t} \subseteq \mathcal{I}_{AD_i,t'}$, we notice that $[L_{y_i,t}, U_{y_i,t}] \subseteq [L_{y_i,t'}, U_{y_i,t'}]$ and $[L_{y^{(m)},t}, U_{y^{(m)},t}] \subseteq [L_{y^{(m)},t'}, U_{y^{(m)},t'}]$. Thus, invoking Lemma 3 twice we have $\mathcal{I}_{AD_i,t} = \text{FindAD}([L_{y_i,t}, U_{y_i,t}], [L_{y^{(m)},t}, U_{y^{(m)},t}]) \subseteq \text{FindAD}([L_{y_i,t'}, U_{y_i,t'}], [L_{y^{(m)},t'}, U_{y^{(m)},t'}]) = \mathcal{I}_{AD_i,t'}$.

The proof of $\mathcal{I}_{AD^{(m)},t} \subseteq \mathcal{I}_{AD^{(m)},t'}$ is similar to the proof of $\mathcal{I}_{y^{(m)},t} \subseteq \mathcal{I}_{y^{(m)},t'}$ after noticing $\mathcal{I}_{AD_i,t} \subseteq \mathcal{I}_{AD_i,t'}$; the proof of $\mathcal{I}_{\theta,t} \subseteq \mathcal{I}_{\theta,t'}$ is a direct consequence of $\mathcal{I}_{y^{(m)},t} \subseteq \mathcal{I}_{y^{(m)},t'}$, $\mathcal{I}_{AD^{(m)},t} \subseteq \mathcal{I}_{AD^{(m)},t'}$ and the construction of $\mathcal{I}_{\theta,t}$ described in Algorithm 1. \square

⁵Recall we use $L_{y_i,t}$ to present the lower bound of intersecting confidence intervals; and that's why we have ' $L_{y_i,t} \geq \hat{y}_{i,t} - \beta_{N_{i,t}}$ ' rather than ' $L_{y_i,t} = \hat{y}_{i,t} - \beta_{N_{i,t}}$ '.

C.3. Confidence Interval of θ

Lemma 6. In ROAIElim, at time t , we have

$$\mathcal{I}_{y_{(m)},t} \subseteq [y_{(m)} - 2\beta_t, y_{(m)} + 2\beta_t].$$

Proof. Recall $\mathcal{I}_{y_{(m)},t} = [L_{y_{(m)},t}, U_{y_{(m)},t}]$. We only prove here that $y_{(m)} - 2\beta_t \leq L_{y_{(m)},t}$ here, and the other side could be proved similarly.

Let

$$S_{top,t}^{\text{median}} = \{i \in [n] : L_{y_i,t} > U_{y_{(m)},t}\}, \quad S_{bottom,t}^{\text{median}} = \{i \in [n] : U_{y_i,t} < L_{y_{(m)},t}\}.$$

Due to the application of intersecting confidence intervals, one could see that $S_{top,t}^{\text{median}}$ represents an *identified*, whether at time t or a previous time step $t' < t$, subset of arms with means greater than $y_{(m)}$, and thus $|S_{top,t}^{\text{median}}| \leq m - 1$; similarly, $S_{bottom,t}^{\text{median}}$ represents the *identified* subset of arms with means smaller than $y_{(m)}$, and $|S_{bottom,t}^{\text{median}}| \leq m - 1$. Since $A_{E,t}^{\text{median}}$ in Algorithm 2 essentially represents the subset of arms un-distinguished from the median $y_{(m)}$ up to time t , we have

$$A_{E,t}^{\text{median}} = [n] \setminus (S_{top,t}^{\text{median}} \cup S_{bottom,t}^{\text{median}}).$$

Suppose $|S_{top,t}^{\text{median}}| = k_{top,t} \leq m - 1$. We first notice that $S_{top,t}^{\text{median}}$ contains $k_{top,t}$ arms with lower bounds greater than $L_{y_{(m)},t}$: for any $i_a \in S_{top,t}^{\text{median}}$, there exists $t' \leq t$ such that $L_{y_{i_a},t} \geq L_{y_{i_a},t'} > U_{y_{(m)},t'} \geq L_{y_{(m)},t}$, where the first inequality comes from Lemma 5 and the last inequality comes from the fact that: if $U_{y_{(m)},t'} < L_{y_{(m)},t}$, we have $\mathcal{I}_{y_{(m)},t'} \cap \mathcal{I}_{y_{(m)},t} = \emptyset$ contradicting with event W .

Suppose $|S_{bottom,t}^{\text{median}}| = k_{bottom,t} \leq m - 1$. We could similarly notice that $S_{bottom,t}^{\text{median}}$ contains $k_{bottom,t}$ arms with lower bounds smaller than $L_{y_{(m)},t}$: for any $i_b \in S_{bottom,t}^{\text{median}}$, there exist $t' \leq t$ such that $L_{y_{i_b},t} \leq U_{y_{i_b},t'} < L_{y_{(m)},t'} \leq L_{y_{(m)},t}$, where the first inequality comes from the fact that $\mathcal{I}_{y_{i_b},t} \cap \mathcal{I}_{y_{i_b},t'} \neq \emptyset$ and the last inequality comes from Lemma 5.

Thus, to identify $L_{y_{(m)},t}$, we only need to identify the $(m - k_{top,t})$ -th largest lower bound among arms in $A_{E,t}^{\text{median}}$. For $i \in A_{E,t}^{\text{median}}$, we have

$$y_i - 2\beta_t = y_i - 2\beta_{N_{i,t}} \leq L_{y_i,t}$$

according to Lemma 2 and the fact that arms in $A_{E,t}^{\text{median}}$ are pulled t times as $A_{E,t}^{\text{median}} \subseteq A_{E,t}$.

Invoking Lemma 4 with $j = m - k_{top,t}$ concludes the proof as $y_{(m)} - 2\beta_t$ is the $(m - k_{top,t})$ -th largest quantity among $\{y_i - 2\beta_t\}$ for $i \in A_{E,t}^{\text{median}}$. \square

Lemma 7. In ROAIElim, at time t , for $\forall i \in A_{E,t}^{\text{MAD}}$, we have

$$\mathcal{I}_{AD_i,t} \subseteq [AD_i - 4\beta_t, AD_i + 4\beta_t].$$

Proof. We are first going to quantify the *extended* confidence interval of AD_i , namely, $[\tilde{L}_{AD_i,t}, \tilde{U}_{AD_i,t}] \supseteq [L_{AD_i,t}, U_{AD_i,t}]$.

According to Lemma 6 and Lemma 2, we have $[y_{(m)} - 2\beta_t, y_{(m)} + 2\beta_t] \supseteq \mathcal{I}_{y_{(m)},t}$ and $[y_i - 2\beta_{N_{i,t}}, y_i + 2\beta_{N_{i,t}}] \supseteq \mathcal{I}_{y_i,t}$. Feeding $[y_{(m)} - 2\beta_t, y_{(m)} + 2\beta_t]$ and $[y_i - 2\beta_{N_{i,t}}, y_i + 2\beta_{N_{i,t}}]$ into the FindAD operator and apply Lemma 3 twice leads to

$$[\tilde{L}_{AD_i,t}, \tilde{U}_{AD_i,t}] := [AD_i - 2\beta_t - 2\beta_{N_{i,t}}, AD_i + 2\beta_t + 2\beta_{N_{i,t}}] \supseteq [L_{AD_i,t}, U_{AD_i,t}].$$

We conclude the proof with $\beta_{N_{i,t}} = \beta_t$ for $\forall i \in A_{E,t}^{\text{MAD}}$. \square

Lemma 8. In ROAIElim, at time t , we have

$$\mathcal{I}_{AD_{(m)},t} \subseteq [AD_{(m)} - 4\beta_t, AD_{(m)} + 4\beta_t].$$

Proof. The proof of this Lemma largely follows from the proof of Lemma 6. As before, we will give a proof for $AD_{(m)} - 4\beta_t \leq L_{AD_{(m)},t}$, and the other side could be proved similarly.

Let

$$S_{top,t}^{\text{MAD}} = \{i \in [n] : L_{AD_i,t} > U_{AD_{(m)},t}\}, \quad S_{bottom,t}^{\text{MAD}} = \{i \in [n] : U_{AD_i,t} < L_{AD_{(m)},t}\}.$$

According to Lemma 5, one could see that $S_{top,t}^{\text{MAD}}$ represents an *identified*, whether at time t or a previous time step $t' < t$, subset of arms with absolute deviations greater than $\text{AD}_{(m)}$, and thus $|S_{top,t}^{\text{MAD}}| \leq m - 1$; similarly, $S_{bottom,t}^{\text{MAD}}$ represents the *identified* subset of arms with absolute deviations smaller than $\text{AD}_{(m)}$, and $|S_{bottom,t}^{\text{MAD}}| \leq m - 1$. Since $A_{E,t}^{\text{MAD}}$ in Algorithm 2 essentially represents the subset of arms un-distinguished from the median absolute deviation $\text{AD}_{(m)}$ up to time t , we have

$$A_{E,t}^{\text{MAD}} = [n] \setminus (S_{top,t}^{\text{MAD}} \cup S_{bottom,t}^{\text{MAD}}).$$

Suppose $|S_{top,t}^{\text{MAD}}| = k_{top,t} \leq m - 1$. We first notice that $S_{top,t}^{\text{MAD}}$ contains $k_{top,t}$ arms with lower bounds on absolute deviation greater than $L_{\text{AD}_{(m)},t}$: for any $i_a \in S_{top,t}^{\text{MAD}}$, there exists $t' \leq t$ such that $L_{\text{AD}_{i_a},t} \geq L_{\text{AD}_{i_a},t'} > U_{\text{AD}_{(m)},t'} \geq L_{\text{AD}_{(m)},t}$, where the first inequality comes from Lemma 5 and the last inequality come from the fact that: if $U_{\text{AD}_{(m)},t'} < L_{\text{AD}_{(m)},t}$, we have $\mathcal{I}_{\text{AD}_{(m)},t'} \cap \mathcal{I}_{\text{AD}_{(m)},t} = \emptyset$ contradicting with event W .

Suppose $|S_{bottom,t}^{\text{MAD}}| = k_{bottom,t} \leq m - 1$. We could similarly notice that $S_{bottom,t}^{\text{MAD}}$ contains $k_{bottom,t}$ arms with lower bounds smaller than $L_{\text{AD}_{(m)},t}$: for any $i_b \in S_{bottom,t}^{\text{MAD}}$, there exist $t' \leq t$ such that $L_{\text{AD}_{i_b},t} \leq U_{\text{AD}_{i_b},t'} < L_{\text{AD}_{(m)},t'} \leq L_{\text{AD}_{(m)},t}$, where the first inequality comes from the fact that $\mathcal{I}_{\text{AD}_{i_b},t} \cap \mathcal{I}_{\text{AD}_{i_b},t'} \neq \emptyset$ and the last inequality comes from Lemma 5.

Thus, to identify $L_{\text{AD}_{(m)},t}$, we only need to identify the $(m - k_{top,t})$ -th largest lower bound among arms in $A_{E,t}^{\text{MAD}}$. For $i \in A_{E,t}^{\text{MAD}}$, according to Lemma 7, we have

$$\text{AD}_i - 4\beta_t = \text{AD}_i - 2\beta_t - 2\beta_{N_{i,t}} \leq L_{\text{AD}_{i,t}}.$$

Invoking Lemma 4 with $j = m - k_t$ and with respect to $i \in A_{E,t}^{\text{MAD}}$ leads to the desired result as $\text{AD}_{(m)} - 4\beta_t$ is the $(m - k_{top,t})$ -th largest quantity among $\{\text{AD}_i - 4\beta_t\}$ for $i \in A_{E,t}^{\text{MAD}}$. \square

Lemma 9. *In ROAIElim, at time t , we have*

$$\mathcal{I}_{\theta,t} \subseteq [\theta - (2 + 4k)\beta_t, \theta + (2 + 4k)\beta_t].$$

Proof. Combining Lemma 6 and 8 with Eq. (1) immediately gives the desired result. \square

C.4. Sample complexity analysis

The sample complexity of ROAIElim could be characterized in the following theorem.

Theorem 4. *With probability of at least $1 - \delta$, the sample complexity of ROAIElim is upper bounded by*

$$O\left(\sum_{i=1}^n \frac{\log\left(\frac{n/\delta \tilde{\Delta}_i^*}{(\tilde{\Delta}_i^*)^2}\right)}{(\tilde{\Delta}_i^*)^2}\right),$$

where

$$\tilde{\Delta}_i^* = \max\{\Delta_{i,t}^\theta/(1+k), \min\{\Delta_{i,t}^\theta/(1+k), \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}\}.$$

Proof. ROAIElim stops sampling an arm i if either the algorithm stops, or if arm i is eliminated from the active set $A_{E,t}$. We analyze the sample complexity of both these events. The number of samples of arm i is then the minimum of these two sample complexities.

ROAIElim stops when $A_{E,t}^\theta = \emptyset$. When $A_{E,t}^\theta \neq \emptyset$, for all $i \in A_{E,t}^\theta$, we have $\beta_{N_{i,t}} = \beta_t$. Notice that, based on Lemma 2 and 9, when $(2 + 4k)\beta_t + 2\beta_t < \Delta_i^\theta$, $i \notin A_{E,t}^\theta$. Remark 1 immediately shows that there exists a constant C such that when $N_{i,t} \geq C \frac{(1+k)^2 \log(n(1+k)/\delta \Delta_i^\theta)}{(\Delta_i^\theta)^2}$, arm i is guaranteed to be expelled from $A_{E,t}^\theta$. As another consequence, no arm will be pulled more than $C \frac{(1+k)^2 \log(n(1+k)/\delta \Delta_i^\theta)}{(\Delta_i^\theta)^2}$ times as that's when the algorithm stops, i.e., $A_{E,t}^\theta = \emptyset$.

We next calculate the number of samples before $i \notin A_{E,t}$. Note that $A_{E,t} = A_{E,t}^{\text{median}} \cup A_{E,t}^{\text{MAD}} \cup A_{E,t}^\theta$, thus we only need to further consider when arm i is out of set $A_{E,t}^{\text{median}}$ and set $A_{E,t}^{\text{MAD}}$. Based on Lemma 2 and 6, we know that when $2\beta_t + 2\beta_t < \Delta_i^{\text{median}}$, we have $i \notin A_{E,t}^{\text{median}}$. Remark 1 shows that there exists a constant C such that when

$N_{i,t} \geq C \frac{\log(n/\delta \Delta_i^{\text{median}})}{(\Delta_i^{\text{median}})^2}$, we have $2\beta_t + 2\tilde{\beta}_t < \Delta_i^{\text{median}}$, i.e., $i \notin A_{E,t}^{\text{median}}$. In a similar manner, invoking Lemma 7 and Lemma 8, we have $i \notin A_{E,t}^{\text{MAD}}$ when $N_{i,t} \geq C \frac{\log(n/\delta \Delta_i^{\text{MAD}})}{(\Delta_i^{\text{MAD}})^2}$.

To summarize, the total number of pulls on arm $i \in [n]$ could be upper bounded by

$$O\left(\sum_{i=1}^n \frac{\log(n/\delta \tilde{\Delta}_i^*)}{(\tilde{\Delta}_i^*)^2}\right).$$

□

D. Sample Complexity Analysis of ROAILUCB

We first prove some supporting Lemmas in Appendix D.1, D.2, D.3 and D.4, and then move to the proof of sample complexity in Appendix D.5. As before, we will prove sample complexity upper bound under the good event W , which happens with probability of at least $1 - \delta$. During the analysis, for any quantity indexed by two arguments q and t (time step), if q itself already indicates the time step, we will simply drop the second argument t . For example, we will simplify $L_{y_{i_0,t},t}$ as $L_{y_{i_0,t}}$.

D.1. Supporting Lemma and Notation

Recall that we assume the number of arms $n = 2m - 1$ is odd, $y_i \geq y_{i+1}$ and the set of outlier arms is $S_o = \{1, \dots, n_1\}$. In the identifiable case, i.e., $y_i \neq \theta$, we then know $\theta \in (y_{n_1}, y_{n_1+1})$.

Lemma 10. *For any $i \in [n]$ and $t \in \mathbb{N}$, we have $\text{len}(\mathcal{I}_{AD_{i,t}}) \leq \text{len}(\mathcal{I}_{y_i,t}) + \text{len}(\mathcal{I}_{y_{(m)},t})$.*

Proof. Suppose $\mathcal{I}_{y_i,t} = [L_{y_i,t}, U_{y_i,t}] = [\hat{y}_{i,t} - \beta_{N_{i,t}}, \hat{y}_{i,t} + \beta_{N_{i,t}}]$, and $\mathcal{I}_{y_{(m)},t} = [L_{y_{(m)},t}, U_{y_{(m)},t}] = [\tilde{y}_t - \tilde{\beta}_t, \tilde{y}_t + \tilde{\beta}_t]$ with \tilde{y}_t defined as the midpoint of $[L_{y_{(m)},t}, U_{y_{(m)},t}]$ and $\tilde{\beta}_t = \text{len}(\mathcal{I}_{y_{(m)},t})/2$. Let $\mathcal{I}_{AD_{i,t}} = [L_{AD_{i,t}}, U_{AD_{i,t}}]$, by construction of $\mathcal{I}_{AD_{i,t}}$ in Algorithm 1,⁶ we have

$$U_{AD_{i,t}} = \max\{U_{y_i,t} - L_{y_{(m)},t}, U_{y_{(m)},t} - L_{y_i,t}\} = |\hat{y}_{i,t} - \tilde{y}_t| + \beta_{N_{i,t}} + \tilde{\beta}_t.$$

On the other side, we have

$$L_{AD_{i,t}} = \max\{0, L_{y_i,t} - U_{y_{(m)},t}, L_{y_{(m)},t} - U_{y_i,t}\} = \max\{0, |\hat{y}_{i,t} - \tilde{y}_t| - \beta_{N_{i,t}} - \tilde{\beta}_t\} \geq |\hat{y}_{i,t} - \tilde{y}_t| - \beta_{N_{i,t}} - \tilde{\beta}_t,$$

where the second inequality comes from the fact that $\max\{L_{y_i,t} - U_{y_{(m)},t}, L_{y_{(m)},t} - U_{y_i,t}\} = |\hat{y}_{i,t} - \tilde{y}_t| - \beta_{N_{i,t}} - \tilde{\beta}_t$.

Thus, we have $\text{len}(\mathcal{I}_{AD_{i,t}}) = U_{AD_{i,t}} - L_{AD_{i,t}} \leq 2\beta_{N_{i,t}} + 2\tilde{\beta}_t = \text{len}(\mathcal{I}_{y_i,t}) + \text{len}(\mathcal{I}_{y_{(m)},t})$. □

We next define the following set of constants, which we shall refer to frequently in our analysis.

$$\begin{cases} c_1^\theta = \frac{y_{n_1} + \theta}{2}, \\ c_2^\theta = \frac{y_{n_1+1} + \theta}{2}, \\ c_1^{\text{median}} = \frac{y_{(m-1)} + y_{(m)}}{2}, \\ c_2^{\text{median}} = \frac{y_{(m)} + y_{(m+1)}}{2}, \\ c_1^{\text{MAD}} = \frac{AD_{(m-1)} + AD_{(m)}}{2}, \\ c_2^{\text{MAD}} = \frac{AD_{(m)} + AD_{(m+1)}}{2}. \end{cases} \quad (6)$$

For $i \in \{1, 2\}$, we could potentially have $c_i^{\text{median}} = y_{(m)}$ or $c_i^{\text{MAD}} = AD_{(m)}$ when there exists multiple medians among $\{y_i\}$ or $\{AD_i\}$; we have $c_i^\theta \neq \theta$ due to assumption on identifiability. Note that these constants are only used in analysis, and Algorithm 3 proceeds without knowing constants defined in Eq. (6).

⁶We consider the version with slight modification $L_{AD_{i,t}} = \max\{0, L_{y_i,t} - U_{y_{(m)},t}, L_{y_{(m)},t} - U_{y_i,t}\}$; the reasoning is mentioned in the proof of Lemma 1.

D.2. Analysis for Outlier Identification

In this section, we define $\text{NEEDY}_{i,t}^\theta$, which denotes the event arm i /threshold θ is needy at round t in the sense of determining outlier/normal arms, and analyze its properties.

Definition 3. At any time t , we separate arms into three different subsets as follows, according to the relation of their confidence intervals and unknown constants c_i^θ ,

$$\begin{aligned} S_{top,t}^\theta &= \{i \in [n] : L_{y_i,t} > c_1^\theta\}, \\ S_{bottom,t}^\theta &= \{i \in [n] : U_{y_i,t} < c_2^\theta\}, \\ S_{middle,t}^\theta &= \{i \in [n] : i \notin S_{top,t}^\theta \cup S_{bottom,t}^\theta\}. \end{aligned}$$

Then, we define the following NEEDY event for arm $i \in [n]$ in the sense of to be separated from θ

$$\text{NEEDY}_{i,t}^\theta = (i \in S_{middle,t}^\theta),$$

we also define the NEEDY event for the outlier threshold θ

$$\text{NEEDY}_{\theta,t}^\theta = (c_1^\theta \in \mathcal{I}_{\theta,t}) \cup (c_2^\theta \in \mathcal{I}_{\theta,t}).$$

Recall $\hat{S}_{o,t} = \{i \in [n] : \hat{y}_{i,t} > \hat{\theta}_t\}$ and $\hat{S}_{n,t} = [n] \setminus \hat{S}_{o,t}$. We define (breaking ties arbitrarily)

$$l_{\theta,t} = \arg \min_{a \in \hat{S}_{o,t}} \{L_{y_a,t}\}, \quad u_{\theta,t} = \arg \max_{a \in \hat{S}_{n,t}} \{U_{y_a,t}\}.$$

Lemma 11. If Algorithm 3 doesn't stop at time t , then there exists $a \in \{l_{\theta,t}, u_{\theta,t}, \theta\}$ such that $\text{NEEDY}_{a,t}^\theta$ holds.

Proof. We analyze this under the good event W where all confidence intervals are valid. If $\text{NEEDY}_{a,t}^\theta$ don't occur for all $a \in \{l_{\theta,t}, u_{\theta,t}, \theta\}$, we then show Algorithm 3 necessarily terminates as followings.

We first notice that when $\text{NEEDY}_{\theta,t}^\theta$ doesn't occur, we will have $\mathcal{I}_{\theta,t} \subseteq (c_2^\theta, c_1^\theta)$ according to the fact that $\theta \in \mathcal{I}_{\theta,t}$ and the definition of c_1^θ and c_2^θ . Secondly, if $\text{NEEDY}_{l_{\theta,t}}^\theta$ doesn't occur either, we necessarily have $L_{y_{l_{\theta,t}}} > c_1^\theta$ as $U_{y_{l_{\theta,t}}} < c_2^\theta$ cannot be true due to $\hat{y}_{l_{\theta,t}} > \hat{\theta}_t$; as a consequence, for any $i \in \hat{S}_{o,t}$, we have $L_{y_i,t} \geq L_{y_{l_{\theta,t}}} > c_1^\theta$. Similarly, we have $U_{y_i,t} < c_2^\theta$ for any $i \in \hat{S}_{n,t}$ if $\text{NEEDY}_{u_{\theta,t}}^\theta$ doesn't occur either. To summarize, we have $A_{L,t}^\theta = \emptyset$, which indicates the termination of Algorithm 3. \square

Lemma 12. If $\text{NEEDY}_{\theta,t}^\theta$ holds, then we have $\text{len}(\mathcal{I}_{\theta,t}) \geq \Delta_*^\theta/2$; furthermore, we have either $\text{len}(\mathcal{I}_{y_{(m)},t}) \geq \epsilon^{\text{median}}$ or $\text{len}(\mathcal{I}_{AD_{(m)},t}) \geq \epsilon^{\text{MAD}}$ with $\epsilon^{\text{median}} := \Delta_*^\theta/(2+8k)$ and $\epsilon^{\text{MAD}} := \Delta_*^\theta/(1/2+2k)$.

Proof. Notice that $\min\{c_1^\theta - \theta, \theta - c_2^\theta\} \geq \Delta_*^\theta/2$, thus the first statement is necessarily true to ensure $\theta \in \mathcal{I}_{\theta,t}$; the second statement need to be true as otherwise we will have $\text{len}(\mathcal{I}_{\theta,t}) < \Delta_*^\theta/2$ due to carefully chosen ϵ^{median} and ϵ^{MAD} such that $\epsilon^{\text{median}} + k \cdot \epsilon^{\text{MAD}} = \Delta_*^\theta/2$. \square

Remark 2. Here we deliberately chose $\epsilon^{\text{median}} = \epsilon^{\text{MAD}}/4$ here for convenience, while our analysis works as long as $\epsilon^{\text{median}} < \epsilon^{\text{MAD}}$. One may also optimize over ϵ^{median} and ϵ^{MAD} to get a slightly tighter, in terms of constant, sample complexity upper bound.

D.3. Analysis for Median Identification

In this section, we define $\text{NEEDY}_{i,t}^{\text{median}}$, which denotes the event arm i is needy at round t in the sense of shrinking the confidence interval of $y_{(m)}$, and analyze its properties.

Definition 4. At any time t , we separate arms into three different subsets as follows, according to the relation of their confidence intervals and unknown constants c_i^{median} ,

$$\begin{aligned} S_{top,t}^{\text{median}} &= \{i \in [n] : L_{y_i,t} > c_1^{\text{median}}\}, \\ S_{bottom,t}^{\text{median}} &= \{i \in [n] : U_{y_i,t} < c_2^{\text{median}}\}, \\ S_{middle,t}^{\text{median}} &= \{i \in [n] : i \notin S_{top,t}^{\text{median}} \cup S_{bottom,t}^{\text{median}}\}. \end{aligned}$$

Then, we define the following NEEDY event for arm $i \in [n]$ in the sense of shrinking the confidence interval of $y_{(m)}$

$$\text{NEEDY}_{i,t}^{\text{median}} = (i \in S_{\text{middle},t}^{\text{median}}) \cap (\beta_{N_{i,t}} \geq \epsilon^{\text{median}}/2).$$

Remark 3. Note that $S_{\text{middle},t}^{\text{median}}$ is equivalent to $\{i \in [n] : c_j^{\text{median}} \in \mathcal{I}_{y_i,t}\} \cup \{i \in [n] : y_i = y_{(m)}\}$ as arms in $\{i \in [n] : y_i = y_{(m)}\}$ cannot be in $S_{\text{top},t}^{\text{median}}$ or $S_{\text{bottom},t}^{\text{median}}$ under the good event W .

We perform LUCB at both $(m-1)$ -th and m -th locations, and aim at shrinking the confidence interval of the median below length ϵ^{median} , i.e., $\text{len}(\mathcal{I}_{y_{(m)},t}) < \epsilon^{\text{median}}$. Recall we set $\kappa_1 = m-1$ and $\kappa_2 = m$; and for $i \in \{1, 2\}$, we let $J_{\kappa_i,t}$ denote a subset of κ_i arms with the highest empirical rewards among $\{\hat{y}_i\}$, breaking ties arbitrarily. For $i = 1, 2$, we further define

$$l_{i,t} = \arg \min_{a \in J_{\kappa_i,t}} \{L_{y_a,t}\}, \quad u_{i,t} = \arg \max_{a \notin J_{\kappa_i,t}} \{U_{y_a,t}\} \quad (7)$$

to be the two critical arms from $J_{\kappa_i,t}$ and $(J_{\kappa_i,t})^c$ that are likely to be misclassified. Recall that, to simplify notations, we will ignore the second subscript t whenever $u_{i,t}$ or $l_{i,t}$ appears in the first subscript, which already indicates the time step t .

Lemma 13. For any $i \in \{1, 2\}$, if $U_{y_{u_{i,t}}} - L_{y_{l_{i,t}}} \geq \epsilon^{\text{median}}$ holds, then $\text{NEEDY}_{k,t}^{\text{median}}$ holds for either $k = l_{i,t}$ or $k = u_{i,t}$, i.e., k satisfies

$$c_i^{\text{median}} \in \mathcal{I}_{y_k,t}, \text{ and } \beta_{N_{k,t}} \geq \epsilon^{\text{median}}/2. \quad (8)$$

Proof. The main idea of this proof comes from (Kalyanakrishnan et al., 2012); we provide the proof here for completeness.

We start arguing that $c_i^{\text{median}} \in \mathcal{I}_{y_{u_{i,t}}}$ or $c_i^{\text{median}} \in \mathcal{I}_{y_{l_{i,t}}}$ by arguing the following four exclusive cases cannot be true under event W .

Case 1. $c_i^{\text{median}} > U_{y_{u_{i,t}}}$ and $c_i^{\text{median}} > U_{y_{l_{i,t}}}$: This indicates there will be at least $n - \kappa_i + 1$ arms with means smaller than c_i^{median} as all $n - \kappa_i$ arms in $(J_{\kappa_i,t})^c$ have upper bounds smaller than c_i^{median} and at least one arm in $J_{\kappa_i,t}$, i.e., arm $L_{y_{i,t}}$, has upper bound smaller than c_i^{median} ; on the other side, we can have at most $n - \kappa_i$ arms with means smaller than c_i^{median} according to definition in Eq. (6), which leads to a contradiction.

Case 2. $c_i^{\text{median}} > U_{y_{u_{i,t}}}$ and $c_i^{\text{median}} < L_{y_{l_{i,t}}}$: This indicates $U_{y_{u_{i,t}}} < L_{y_{l_{i,t}}}$, which contradicts with the fact that $U_{y_{u_{i,t}}} - L_{y_{l_{i,t}}} \geq \epsilon^{\text{median}} > 0$.

Case 3. $c_i^{\text{median}} < L_{y_{u_{i,t}}}$ and $c_i^{\text{median}} > U_{y_{l_{i,t}}}$: This leads to the contradiction that $c_i^{\text{median}} < L_{y_{u_{i,t}}} \leq \hat{y}_{u_{i,t}} \leq \hat{y}_{l_{i,t}} \leq U_{y_{l_{i,t}}} < c_i^{\text{median}}$, where the third inequality comes from the fact $u_{i,t} \notin J_{\kappa_i,t}$ and $l_{i,t} \in J_{\kappa_i,t}$.

Case 4. $c_i^{\text{median}} < L_{y_{u_{i,t}}}$ and $c_i^{\text{median}} < L_{y_{l_{i,t}}}$: Similar to Case 1, Case 4 indicates there will be at least $\kappa_i + 1$ arms with mean greater than c_i^{median} , which contradicts with the fact that there can have at most κ_i such arms.

We next show Eq. (8) holds true by considering two situations: (1) c_i^{median} belongs to both $\mathcal{I}_{y_{u_{i,t}}}$ and $\mathcal{I}_{y_{l_{i,t}}}$; (2) c_i^{median} only belongs to one of $\mathcal{I}_{y_{u_{i,t}}}$ and $\mathcal{I}_{y_{l_{i,t}}}$.

In both situations, we notice that

$$\beta_{N_{u_{i,t}}} + \beta_{N_{l_{i,t}}} \geq \hat{y}_{u_{i,t}} + \beta_{N_{u_{i,t}}} - (\hat{y}_{l_{i,t}} - \beta_{N_{l_{i,t}}}) = U_{y_{u_{i,t}}} - L_{y_{l_{i,t}}} \geq \epsilon^{\text{median}}, \quad (9)$$

where the first inequality comes from $\hat{y}_{u_{i,t}} \leq \hat{y}_{l_{i,t}}$.

In the first situation: Since $c_i^{\text{median}} \in \mathcal{I}_{y_{u_{i,t}}}$ and $c_i^{\text{median}} \in \mathcal{I}_{y_{l_{i,t}}}$, and we also have either $\beta_{N_{u_{i,t}}} \geq \epsilon^{\text{median}}/2$ or $\beta_{N_{l_{i,t}}} \geq \epsilon^{\text{median}}/2$; thus Eq. (8) is satisfied.

In the second situation: We consider when c_i^{median} only belongs to one of the confidence intervals. Specifically, we consider the following four exclusive cases:

Case 1. $c_i^{\text{median}} \in \mathcal{I}_{y_{u_{i,t}}}$, and $c_i^{\text{median}} > U_{y_{l_{i,t}}} \implies \beta_{N_{u_{i,t}}} \geq \epsilon^{\text{median}}/2$:

This case indicates $\hat{y}_{u_{i,t}} + \beta_{N_{u_{i,t}}} \geq c_i^{\text{median}} > \hat{y}_{l_{i,t}} + \beta_{N_{l_{i,t}}}$; combine this with the fact that $\hat{y}_{u_{i,t}} \leq \hat{y}_{l_{i,t}}$, we further have

$$\beta_{N_{u_{i,t}}} \geq \beta_{N_{l_{i,t}}}. \quad (10)$$

Combine Eq. (10) with Eq. (9) leads to the desired result.

Case 2. $c_i^{\text{median}} \in \mathcal{I}_{y_{u_i,t}}$, and $c_i^{\text{median}} < L_{y_{l_i,t}} \implies \beta_{N_{u_i,t}} \geq \epsilon^{\text{median}}/2$:

This case indicates $\hat{y}_{u_i,t} - \beta_{N_{u_i,t}} \leq c_i^{\text{median}}$, combine this with the fact that

$$\hat{y}_{u_i,t} + \beta_{N_{u_i,t}} \geq \hat{y}_{l_i,t} - \beta_{N_{l_i,t}} + \epsilon^{\text{median}} = L_{y_{l_i,t}} + \epsilon^{\text{median}} > c_i^{\text{median}} + \epsilon^{\text{median}}$$

leads to the desired result.

Case 3. $c_i^{\text{median}} \in \mathcal{I}_{y_{l_i,t}}$, and $c_i^{\text{median}} > U_{y_{u_i,t}} \implies \beta_{N_{l_i,t}} \geq \epsilon^{\text{median}}/2$: The proof is similar to Case 2.

Case 4. $c_i^{\text{median}} \in \mathcal{I}_{y_{l_i,t}}$, and $c_i^{\text{median}} < L_{y_{u_i,t}} \implies \beta_{N_{l_i,t}} \geq \epsilon^{\text{median}}/2$: The proof is similar to Case 1. \square

Lemma 14. *If $\text{len}(\mathcal{I}_{y_{(m),t}}) \geq \epsilon^{\text{median}}$ and $U_{y_{u_i,t}} - L_{y_{l_i,t}} < \epsilon^{\text{median}}$ for both $i = 1, 2$, then there exists an arm $k \in \{l_{1,t}, l_{2,t}, u_{1,t}, u_{2,t}\}$ such that $\text{NEEDY}_{k,t}^{\text{median}}$ holds.*

Proof. We first notice $[L_{y_{(m),t}}, U_{y_{(m),t}}] \subseteq [L_{y_{l_{2,t}}}, U_{y_{u_{1,t}}}]$ based on the selection of $l_{2,t}$ and $u_{1,t}$ in Eq. (7). We then prove the Lemma by considering the following two exclusive cases:

Case 1. $U_{y_{u_{1,t}}} = U_{y_{u_{2,t}}}$ or $L_{y_{l_{1,t}}} = L_{y_{l_{2,t}}}$: We immediately have $\text{len}(\mathcal{I}_{y_{(m),t}}) \leq U_{y_{u_{1,t}}} - L_{y_{l_{2,t}}} < \epsilon^{\text{median}}$ according to $U_{y_{u_i,t}} - L_{y_{l_i,t}} < \epsilon^{\text{median}}$ for both $i = 1, 2$; but this contradicts with the assumption $\text{len}(\mathcal{I}_{y_{(m),t}}) \geq \epsilon^{\text{median}}$. Thus, this case cannot happen.

Case 2. $U_{u_{1,t}} \neq U_{u_{2,t}}$ and $L_{l_{1,t}} \neq L_{l_{2,t}}$: Let k be the index of the empirical median arm at time t according to the ranking, i.e., $k = \mathcal{J}_{\kappa_2,t} \setminus \mathcal{J}_{\kappa_1,t}$, we then know $u_{1,t} = l_{2,t} = k$. This further leads to $\text{len}(\mathcal{I}_{y_{k,t}}) \geq \text{len}(\mathcal{I}_{y_{(m),t}}) \geq \epsilon^{\text{median}}$ and thus $\beta_{N_{k,t}} \geq \epsilon^{\text{median}}/2$. We next show $\text{NEEDY}_{k,t}^{\text{median}}$ holds true by showing that we have $k \in S_{\text{middle},t}^{\text{median}}$ for either of the three sub-cases:

(1) if $y_k = y_{(m)}$, we have $k \in S_{\text{middle},t}^{\text{median}}$ according to Remark 3;

(2) if $y_k > y_{(m)}$, we know that $y_k \geq y_{(m-1)}$. Since $y_{(m)} \in \mathcal{I}_{y_{(m),t}} \subseteq \mathcal{I}_{y_{k,t}}$ and $y_k \in \mathcal{I}_{y_{k,t}}$, we then know $c_1^{\text{median}} = (y_{(m)} + y_{(m-1)})/2 \in \mathcal{I}_{y_{k,t}}$, which leads to $k \in S_{\text{middle},t}^{\text{median}}$;

(3) if $y_k < y_{(m)}$, similar to sub-case (2), we have $c_2^{\text{median}} \in \mathcal{I}_{y_{k,t}}$, which leads to $k \in S_{\text{middle},t}^{\text{median}}$. \square

Lemma 15. *If $\text{len}(\mathcal{I}_{y_{(m),t}}) \geq \epsilon^{\text{median}}$, then there exists an arm $k = l_{i,t}$ or $k = u_{i,t}$ such that $\text{NEEDY}_{k,t}^{\text{median}}$ holds.*

Proof. This Lemma is a direct consequence of the Lemma 13 and Lemma 14. \square

D.4. Analysis for MAD Identification

In this section, we define $\text{NEEDY}_{i,t}^{\text{MAD}}$, which denotes the event arm i is needy at round t in the sense of shrinking the confidence interval of $\text{AD}_{(m)}$, and analyze its properties.

Definition 5. *At any time t , we separate arms into three different subsets as follows, according to the relation of the confidence intervals of absolute deviations and unknown constants c_i^{MAD} ,*

$$\begin{aligned} S_{\text{top},t}^{\text{MAD}} &= \{i \in [n] : L_{\text{AD}_{i,t}} > c_1^{\text{MAD}}\}, \\ S_{\text{bottom},t}^{\text{MAD}} &= \{i \in [n] : U_{\text{AD}_{i,t}} < c_2^{\text{MAD}}\}, \\ S_{\text{middle},t}^{\text{MAD}} &= \{i \in [n] : i \notin S_{\text{top},t}^{\text{MAD}} \cup S_{\text{bottom},t}^{\text{MAD}}\}. \end{aligned}$$

Then, we define the following NEEDY event for arm $i \in [n]$ in the sense of shrinking the confidence interval of $\text{AD}_{(m)}$

$$\text{NEEDY}_{i,t}^{\text{MAD}} = (i \in S_{\text{middle},t}^{\text{MAD}}) \cap (\beta_{N_{i,t}} > \epsilon^{\text{MAD}}/4).$$

Remark 4. *Note that $S_{\text{middle},t}^{\text{MAD}}$ is equivalent to $\{i \in [n] : c_j^{\text{MAD}} \in \mathcal{I}_{\text{AD}_{i,t}}\} \cup \{i \in [n] : \text{AD}_i = \text{AD}_{(m)}\}$, as arms in $\{i \in [n] : \text{AD}_i = \text{AD}_{(m)}\}$ cannot be in set $S_{\text{top},t}^{\text{MAD}}$ or set $S_{\text{bottom},t}^{\text{MAD}}$ under the good event W .*

Algorithm 3 performs LUCB at both $(m-1)$ -th and m -th locations with respect to $\widehat{AD}_{i,t}$ and $\{L_{AD_{i,t}}, U_{AD_{i,t}}\}$, and aim at shrinking the length of $\mathcal{I}_{AD_{(m),t}}$ below ϵ^{MAD} . Recall we set $\kappa_1 = m-1$ and $\kappa_2 = m$; and for $i \in \{1, 2\}$, we let $J_{\kappa_i, t}^{\text{AD}}$ denote a subset of κ_i arms with the largest empirical absolute deviations among $\{\widehat{AD}_i\}$, breaking ties arbitrarily. For $i = 1, 2$, we further define

$$l_{i,t}^{\text{AD}} = \arg \min_{a \in J_{\kappa_i, t}^{\text{AD}}} \{L_{AD_{a,t}}\}, \quad u_{i,t}^{\text{AD}} = \arg \max_{a \notin J_{\kappa_i, t}^{\text{AD}}} \{U_{AD_{a,t}}\} \quad (11)$$

to be the two critical arms from $J_{\kappa_i, t}^{\text{AD}}$ and $(J_{\kappa_i, t}^{\text{AD}})^c$ that are likely to be misclassified. Recall that, to simplify notations, we will ignore the second subscript t whenever $u_{i,t}^{\text{AD}}$ or $l_{i,t}^{\text{AD}}$ appears in the first subscript.

Lemma 16. *Assume $\text{len}(\mathcal{I}_{y_{(m),t})} < \epsilon^{\text{median}}$. If $U_{AD_{u_{i,t}^{\text{AD}}}} - L_{AD_{l_{i,t}^{\text{AD}}}} \geq \epsilon^{\text{MAD}}$ holds, then either $k = l_{i,t}^{\text{AD}}$ or $k = u_{i,t}^{\text{AD}}$ and satisfies*

$$c_i^{\text{MAD}} \in \mathcal{I}_{AD_{k,t}}, \text{ and } \beta_{N_{k,t}} > \epsilon^{\text{MAD}}/4. \quad (12)$$

Proof. Since we deliberately define select \widehat{AD}_i to be the median point of its confidence interval, i.e., $U_{AD_{i,t}} - \widehat{AD}_{i,t} = \widehat{AD}_{i,t} - L_{AD_{i,t}}$, similar to the proof of Lemma 13,⁷ we have either $k = l_{i,t}^{\text{AD}}$ or $k = u_{i,t}^{\text{AD}}$ satisfies

$$c_i^{\text{MAD}} \in \mathcal{I}_{AD_{k,t}}, \text{ and } \text{len}(\mathcal{I}_{AD_{k,t}}) \geq \epsilon^{\text{MAD}}.$$

By assumption $\text{len}(\mathcal{I}_{y_{(m),t})} < \epsilon^{\text{median}} = \epsilon^{\text{MAD}}/4$, we further obtain the following equation after invoking Lemma 10

$$c_i^{\text{MAD}} \in \mathcal{I}_{AD_{k,t}}, \text{ and } \beta_{N_{k,t}} > 3\epsilon^{\text{MAD}}/4 > \epsilon^{\text{MAD}}/4.$$

□

Lemma 17. *Assume $\text{len}(\mathcal{I}_{y_{(m),t})} < \epsilon^{\text{median}}$. If $\text{len}(\mathcal{I}_{AD_{(m),t})} \geq \epsilon^{\text{MAD}}$ and $U_{AD_{u_{i,t}^{\text{AD}}}} - L_{AD_{l_{i,t}^{\text{AD}}}} < \epsilon^{\text{MAD}}$ for both $i \in \{1, 2\}$, then there exists an arm $k \in \{l_{1,t}^{\text{AD}}, l_{2,t}^{\text{AD}}, u_{1,t}^{\text{AD}}, u_{2,t}^{\text{AD}}\}$ such that $\text{NEEDY}_{k,t}^{\text{MAD}}$ holds.*

Proof. We first notice $[L_{AD_{(m),t}}, U_{AD_{(m),t}}] \subseteq [L_{AD_{l_{2,t}^{\text{AD}}}}, U_{AD_{u_{1,t}^{\text{AD}}}}]$ based on the selection of $l_{2,t}^{\text{AD}}$ and $u_{1,t}^{\text{AD}}$ in Eq. (11). We then prove the Lemma by considering the following two exclusive cases:

Case 1. $U_{AD_{u_{1,t}^{\text{AD}}}} = U_{AD_{u_{2,t}^{\text{AD}}}}$ or $L_{AD_{l_{1,t}^{\text{AD}}}} = L_{AD_{l_{2,t}^{\text{AD}}}}$: We immediately have $\text{len}(\mathcal{I}_{AD_{(m),t})} \leq U_{AD_{u_{1,t}^{\text{AD}}}} - L_{AD_{l_{2,t}^{\text{AD}}}} < \epsilon^{\text{MAD}}$ according to $U_{AD_{u_{i,t}^{\text{AD}}}} - L_{AD_{l_{i,t}^{\text{AD}}}} < \epsilon^{\text{MAD}}$ for both $i \in \{1, 2\}$; but this contradicts with the assumption that $\text{len}(\mathcal{I}_{AD_{(m),t})} \geq \epsilon^{\text{MAD}}$. Thus, this case cannot happen.

Case 2. $U_{AD_{u_{1,t}^{\text{AD}}}} \neq U_{AD_{u_{2,t}^{\text{AD}}}}$ and $L_{AD_{l_{1,t}^{\text{AD}}}} \neq L_{AD_{l_{2,t}^{\text{AD}}}}$: Let k be the index associated with the empirical median absolute deviation at time t according to the ranking, i.e., $k = J_{\kappa_2, t}^{\text{AD}} \setminus J_{\kappa_1, t}^{\text{AD}}$, we then know $u_{1,t}^{\text{AD}} = l_{2,t}^{\text{AD}} = k$, which leads to $\text{len}(\mathcal{I}_{AD_{k,t}}) \geq \text{len}(\mathcal{I}_{AD_{(m),t})} \geq \epsilon^{\text{MAD}}$. Since we have $\text{len}(\mathcal{I}_{y_{(m),t})} < \epsilon^{\text{median}} = \epsilon^{\text{MAD}}/4$, we further have $\beta_{N_{k,t}} > 3\epsilon^{\text{MAD}}/8 > \epsilon^{\text{MAD}}/4$ according to Lemma 10. $\text{NEEDY}_{k,t}^{\text{MAD}}$ holds true as we have $k \in S_{\text{middle}, t}^{\text{MAD}}$ for either of the three sub-cases:

(1) if $AD_k = AD_{(m)}$, we have $k \in S_{\text{middle}, t}^{\text{MAD}}$ according to Remark 4;

(2) if $AD_k > AD_{(m)}$, we know that $AD_k \geq AD_{(m-1)}$. Since $AD_{(m)} \in \mathcal{I}_{AD_{(m),t}} \subseteq \mathcal{I}_{AD_{k,t}}$ and $AD_k \in \mathcal{I}_{AD_{k,t}}$, we then know $c_1^{\text{MAD}} = (AD_{(m)} + AD_{(m-1)})/2 \in \mathcal{I}_{AD_{k,t}}$, which leads to $k \in S_{\text{middle}, t}^{\text{MAD}}$;

(3) if $AD_k < AD_{(m)}$, similar to sub-case (2), we have $c_2^{\text{MAD}} \in \mathcal{I}_{k,t}$, which leads to $k \in S_{\text{middle}, t}^{\text{MAD}}$. □

Lemma 18. *Assume $\text{len}(\mathcal{I}_{y_{(m),t})} < \epsilon^{\text{median}}$. If $\text{len}(\mathcal{I}_{AD_{(m),t})} \geq \epsilon^{\text{MAD}}$, then there exists an arm $k = l_{i,t}^{\text{AD}}$ or $k = u_{i,t}^{\text{AD}}$ such that $\text{NEEDY}_{k,t}^{\text{MAD}}$ holds.*

Proof. This Lemma is a direct consequence of Lemma 16 and Lemma 17. □

⁷Note that in Lemma 13, in terms of the relation between confidence interval and the empirical value, we only use the property that $\hat{y}_{i,t}$ is the median point of the $\mathcal{I}_{y_i,t}$.

D.5. Sample Complexity Analysis

We analyze the sample complexity upper bound of ROAILUCB in this section. To reduce the clutter, we will sometimes use the notation $[a \vee b] = \max\{a, b\}$, and use \vee to represent **or**.

Lemma 19. *There exists a universal constant C , for any $k \in [n]$, if*

$$N_{k,t} \geq \lambda_k^\theta := C \frac{1}{(\Delta_k^\theta)^2} \log \left(\frac{n}{\delta \Delta_k^\theta} \right),$$

then $\text{NEEDY}_{k,t}^\theta$ cannot happen.

Proof. According to Remark 1, we see there exists a universal constant C such that when $N_{k,t} \geq \lambda_k^\theta$, we have $\beta_{N_{k,t}} < \Delta_k^\theta/4$. Since $\min_{i=1,2} \{|c_i^\theta - y_k|\} \geq \Delta_k^\theta/2$ by definition, we then know $c_i^\theta \notin \mathcal{I}_{k,t}$ when $N_{k,t} \geq \lambda_k^\theta$, which indicates that $\text{NEEDY}_{k,t}^\theta$ cannot happen. \square

Lemma 20. *There exists a universal constant C , for any $k \in [n]$, if*

$$N_{k,t} \geq \lambda_k^{\text{median}} := C \frac{1}{[\Delta_k^{\text{median}} \vee \epsilon^{\text{median}}]^2} \log \left(\frac{n}{\delta [\Delta_k^{\text{median}} \vee \epsilon^{\text{median}}]} \right),$$

then $\text{NEEDY}_{k,t}^{\text{median}}$ cannot happen.

Proof. According to Remark 1, we see there exist a universal constant C such that when $N_{k,t} \geq \lambda_k^\theta$, we have $\beta_{N_{k,t}} < \max\{\Delta_k^{\text{median}}/4, \epsilon^{\text{median}}/2\}$.⁸ We consider the following two cases:

Case 1. $\beta_{N_{k,t}} \leq \epsilon^{\text{median}}/2$: We directly know that $\text{NEEDY}_{k,t}^{\text{median}}$ cannot happen according to Definition 4.

Case 2. $\beta_{N_{k,t}} \leq \Delta_k^{\text{median}}/4$: Since $\min_{i=1,2} \{|c_i^{\text{median}} - y_k|\} \geq \Delta_k^{\text{median}}/2$ by definition, we then know $c_i^{\text{median}} \notin \mathcal{I}_{k,t}$ when $N_{k,t} \geq \lambda_k^{\text{median}}$, which indicates that $\text{NEEDY}_{k,t}^{\text{median}}$ cannot happen. \square

Lemma 21. *Assume $\text{len}(\mathcal{I}_{y(m),t}) < \epsilon^{\text{median}} = \epsilon^{\text{MAD}}/4$. There exists a universal constant C , for any $k \in [n]$, if*

$$N_{k,t} \geq \lambda_k^{\text{MAD}} := C \frac{1}{[\Delta_k^{\text{MAD}} \vee \epsilon^{\text{MAD}}]^2} \log \left(\frac{n}{\delta [\Delta_k^{\text{MAD}} \vee \epsilon^{\text{MAD}}]} \right),$$

then $\text{NEEDY}_{k,t}^{\text{MAD}}$ cannot happen.

Proof. According to Remark 1, it's easy to see that there exist a universal constant C such that when $N_{k,t} \geq \lambda_k^\theta$, we have $\beta_{N_{k,t}} < \max\{\Delta_k^{\text{MAD}}/8, \epsilon^{\text{MAD}}/4\} \leq \max\{\Delta_k^{\text{MAD}}/4 - \epsilon^{\text{MAD}}/4, \epsilon^{\text{MAD}}/4\}$, where the second inequality is a mathematical fact obtained by comparing $\Delta_k^{\text{MAD}}/8$ and $\epsilon^{\text{MAD}}/4$ and also noticing $\Delta_k^{\text{MAD}}/4 - \epsilon^{\text{MAD}}/4 = \Delta_k^{\text{MAD}}/8 + \Delta_k^{\text{MAD}}/8 - \epsilon^{\text{MAD}}/4$. As before, we consider the following two cases.

Case 1. $\beta_{N_{k,t}} < \epsilon^{\text{MAD}}/4$: We directly know that $\text{NEEDY}_{k,t}^{\text{MAD}}$ cannot happen according to Definition 5.

Case 2. $\beta_{N_{k,t}} < \Delta_k^{\text{MAD}}/4 - \epsilon^{\text{MAD}}/4$: If $N_{k,t} \geq \lambda_k^{\text{MAD}}$ and $\text{len}(\mathcal{I}_{y(m),t}) \leq \epsilon^{\text{median}} = \epsilon^{\text{MAD}}/4$, we then have

$$\text{len}(\mathcal{I}_{\text{AD}_k,t}) \leq \text{len}(\mathcal{I}_{k,t}) + \text{len}(\mathcal{I}_{y(m),t}) < 2(\Delta_k^{\text{MAD}}/4 - \epsilon^{\text{MAD}}/4) + \epsilon^{\text{MAD}}/4 < \Delta_k^{\text{MAD}}/2$$

according to Lemma 10. Since $\min_{i=1,2} \{|c_i^{\text{MAD}} - \text{AD}_k|\} \geq \Delta_k^{\text{MAD}}/2$ by definition, we then know $c_i^{\text{MAD}} \notin \mathcal{I}_{\text{AD}_k,t}$, which indicates that $\text{NEEDY}_{k,t}^{\text{MAD}}$ cannot happen. \square

The sample complexity of ROAILUCB is characterized in the following theorem.

⁸Note that $\max\{\Delta_k^{\text{median}}/4, \epsilon^{\text{median}}/2\}$ and $\max\{\Delta_k^{\text{median}}, \epsilon^{\text{median}}\}$ are in the same order.

Theorem 5. *With probability of at least $1 - \delta$, the sample complexity of ROAILUCB is upper bounded by*

$$O\left(\sum_{i=1}^n \frac{\log\left(\frac{n}{\delta \tilde{\Delta}_i^*}\right)}{(\tilde{\Delta}_i^*)^2}\right),$$

where

$$\tilde{\Delta}_i^* = \max\{\Delta_*^\theta/(1+k), \min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}\}.$$

Proof. We only need to upper bound the total number of rounds performed by Algorithm 3 before termination as Algorithm 3 only plays a constant number of arms at each round. To simplify the analysis, we first define the following notations:

$$\text{NT}_t = (\text{Algorithm 3 doesn't terminate at round } t),$$

$$A_t = (\text{len}(\mathcal{I}_{y(m),t}) \geq \epsilon^{\text{median}}),$$

$$B_t = (\text{len}(\mathcal{I}_{\text{AD}(m),t}) \geq \epsilon^{\text{MAD}}),$$

$$C_t = \text{NT}_t \cap \text{NEEDY}_{\theta,t}^\theta.$$

The total number of rounds up to time T is

$$\begin{aligned} \#\text{rounds}(T) &= \sum_{t=1}^T \mathbf{1}[\text{NT}_t] \\ &= \sum_{t=1}^T \mathbf{1}\left[\left(\text{NT}_t \cap \neg \text{NEEDY}_{\theta,t}^\theta\right) \cup \left(\text{NT}_t \cap \text{NEEDY}_{\theta,t}^\theta\right)\right] \\ &= \sum_{t=1}^T \mathbf{1}\left[\left(\text{NEEDY}_{l_{\theta,t}}^\theta \cup \text{NEEDY}_{u_{\theta,t}}^\theta\right) \cup \left(C_t \cap A_t\right) \cup \left(C_t \cap \neg A_t \cap B_t\right) \cup \left(C_t \cap \neg A_t \cap \neg B_t\right)\right] \end{aligned} \quad (13)$$

$$\leq \sum_{t=1}^T \mathbf{1}\left[\left(\text{NEEDY}_{l_{\theta,t}}^\theta \cup \text{NEEDY}_{u_{\theta,t}}^\theta\right) \cup \left(A_t\right) \cup \left(\neg A_t \cap B_t\right)\right] \quad (14)$$

$$\leq \sum_{t=1}^T \mathbf{1}\left[\bigcup_{a \in [n]} \left(\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap \text{NEEDY}_{a,t}^\theta\right) \cup \left((a = l_{i,t} \vee u_{i,t}) \cap \text{NEEDY}_{a,t}^{\text{median}}\right)\right)\right] \quad (15)$$

$$\begin{aligned} &\quad \cup \left(\neg A_t \cap \left(a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}\right) \cap \text{NEEDY}_{a,t}^{\text{MAD}}\right)\right] \\ &\leq \sum_{t=1}^T \sum_{a \in [n]} \mathbf{1}\left[\left(\left(a = l_{\theta,t} \vee u_{\theta,t}\right) \cap \text{NEEDY}_{a,t}^\theta\right) \cup \left(\left(a = l_{i,t} \vee u_{i,t}\right) \cap \text{NEEDY}_{a,t}^{\text{median}}\right)\right. \\ &\quad \left. \cup \left(\neg A_t \cap \left(a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}\right) \cap \text{NEEDY}_{a,t}^{\text{MAD}}\right)\right] \\ &\leq \sum_{t=1}^T \sum_{a \in [n]} \mathbf{1}\left[\left(\left(a = l_{\theta,t} \vee u_{\theta,t}\right) \cap N_{a,t} \leq \lambda_a^\theta\right) \cup \left(\left(a = l_{i,t} \vee u_{i,t}\right) \cap N_{a,t} \leq \lambda_a^{\text{median}}\right)\right. \end{aligned} \quad (16)$$

$$\begin{aligned} &\quad \left. \cup \left(\neg A_t \cap \left(a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}\right) \cap N_{a,t} \leq \lambda_a^{\text{MAD}}\right)\right] \\ &\leq \sum_{a \in [n]} \sum_{t=1}^T \mathbf{1}\left[\left(\left(a = l_{\theta,t} \vee u_{\theta,t}\right) \cap N_{a,t} \leq \lambda_a^\theta\right) \cup \left(\left(a = l_{i,t} \vee u_{i,t}\right) \cap N_{a,t} \leq \lambda_a^{\text{median}}\right)\right. \end{aligned}$$

$$\begin{aligned} & \bigcup \left((a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}) \cap N_{a,t} \leq \lambda_a^{\text{MAD}} \right) \\ & \leq \sum_{a \in [n]} \max_{j \in \{\theta, \text{median}, \text{MAD}\}} \{\lambda_a^j\}. \end{aligned}$$

where the Eq. (13) comes from Lemma 11 and Lemma 12; the Eq. (14) is derived by noticing $C_t \cap \neg A_t \cap \neg B_t = \emptyset$; the Eq. (15) comes from Lemma 15 and Lemma 18; and the Eq. (16) comes from Lemma 19, Lemma 20 and Lemma 21.

Notice the fact that $\epsilon^{\text{median}} = \epsilon^{\text{MAD}}/4 = \Theta(\Delta_*^\theta/(1+k))$, and for $\forall i \in [n]$, $\Delta_i^\theta \geq \Delta_*^\theta/(1+k)$. An analysis with respect to all possible orderings over $\{\Delta_*^\theta/(1+k), \Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}$ leads to the following result

$$\sum_{a \in [n]} \max_{j \in \{\theta, \text{median}, \text{MAD}\}} \{\lambda_a^j\} \leq O \left(\sum_{i=1}^n \frac{\log(n/\delta \tilde{\Delta}_i^*)}{(\tilde{\Delta}_i^*)^2} \right).$$

□

E. Lower Bound: Proof of Theorem 3

Our proof of lower bounds rely on the *change of measure* lemma proved in (Kaufmann et al., 2016), which will be restated shortly for completeness. Recall that we use $D_y = (D_{y_1}, \dots, D_{y_n})$ to represent a bandit instance, and assume each arm follows the distribution $D_{y_i} := \mathcal{N}(y_i, 1)$. $\mathbb{E}_y(\cdot)$ is used to represent the expectation with respect to the bandit instance D_y and randomness in the algorithm. We will use $\text{KL}(P, Q)$ to denote the KL-divergence between distribution P and Q . Based on the calculation of KL-divergence between two Gaussian distributions, we also have

$$\text{KL}(D_{y_1}, D_{y_2}) = \text{KL}(\mathcal{N}(y_1, 1), \mathcal{N}(y_2, 1)) = (y_1 - y_2)^2/2.$$

Lemma 22. ((Kaufmann et al., 2016)) *Let D_y and $D_{y'}$ be two bandit instances with n arms such that for all i , the distributions D_{y_i} and $D_{y'_i}$ are mutually absolutely continuous. Let τ be a stopping time with respect to filtration $\{\mathcal{F}_t\}$. For any event $A \in \mathcal{F}_\tau$, we have*

$$\sum_{i=1}^n \mathbb{E}_y[N_{i,\tau}] \cdot \text{KL}(D_{y_i}, D_{y'_i}) \geq d(\mathbb{P}_y(A), \mathbb{P}_{y'}(A)),$$

where $d(x, y) = x \log\left(\frac{x}{y}\right) + (1-x) \log\left(\frac{1-x}{1-y}\right)$, with the convention that $d(0, 0) = d(1, 1) = 0$.

For any $D_y \in \mathcal{M}_{n,\rho}$, we assume $y_i \geq y_{i+1}$ and use $S_o = \{1, \dots, n_1\}$ to denote the subset of outlier arms. We use θ as derived from Eq. (1). Before get into our proof, we first define some mapping functions for each arm as followings:

$$\psi_\rho^\theta(y_i) = \begin{cases} \theta - \rho & \text{if } y_i > \theta \\ \theta + \rho & \text{if } y_i < \theta \end{cases},$$

$$\psi_\rho^{\text{median}}(y_i) = \begin{cases} y_{(m)} - \rho & \text{if } y_i \geq y_{(m)} \\ y_{(m)} + \rho & \text{if } y_i < y_{(m)} \end{cases},$$

and

$$\psi_\rho^{\text{MAD}}(y_i) = \begin{cases} y_{(m)} - \rho & \text{if } y_i = y_{(m)} \\ y_i - \text{AD}_i + \text{AD}_{(m)} - \rho & \text{if } y_i > y_{(m)} \text{ and } \text{AD}_i \geq \text{AD}_{(m)} \\ y_i - \text{AD}_i + \text{AD}_{(m)} + \rho & \text{if } y_i > y_{(m)} \text{ and } \text{AD}_i < \text{AD}_{(m)} \\ y_i + \text{AD}_i - \text{AD}_{(m)} + \rho & \text{if } y_i < y_{(m)} \text{ and } \text{AD}_i \geq \text{AD}_{(m)} \\ y_i + \text{AD}_i - \text{AD}_{(m)} - \rho & \text{if } y_i < y_{(m)} \text{ and } \text{AD}_i < \text{AD}_{(m)} \end{cases}.$$

We could see that these mappings essentially map value y_i to another value with deviations closely related to $\Delta_{i,\rho}^\theta$, $\Delta_{i,\rho}^{\text{median}}$

or $\Delta_{i,\rho}^{\text{MAD}}$, where we define

$$\begin{aligned}\Delta_{i,\rho}^{\theta} &= \Delta_i^{\theta} + \rho, \\ \Delta_{i,\rho}^{\text{median}} &= \Delta_i^{\text{median}} + \rho, \\ \Delta_{i,\rho}^{\text{MAD}} &= \Delta_i^{\text{MAD}} + \rho.\end{aligned}$$

Next lemma shows how changing the mean of one single arm changes the output decision of the outlier arm identification problem.

Lemma 23. *Suppose $D_y = (D_{y_1}, D_{y_2}, \dots, D_{y_n}) \in \mathcal{M}_{n,\rho}$, then for any $i \in [n]$ and any $a \in \{\theta, \text{median}, \text{MAD}\}$, the subset of outlier arms in the following instance*

$$D_{y'} = (D_{y_1}, \dots, D_{y_{i-1}}, D_{\psi_{\rho}^a(y_i)}, D_{y_{i+1}}, \dots, D_{y_n})$$

is not $S_o = [n_1]$ anymore.

Proof. We prove this lemma by showing that it holds for all three cases; and we use θ' , $y'_{(m)}$ and $\text{AD}'_{(m)}$ to represent, respectively, the outlier threshold, the median and the median absolute deviation with respect to (expected rewards of) bandit instance $D_{y'}$.

Case 1: $a = \theta$. If $|\theta' - \theta| < \rho$, then arm i is removed or added to S_o in instance $D_{y'}$; otherwise, at least one of $\{u_1, u_2, l_1, l_2\}$ is removed or added to S_o .

Case 2: $a = \text{median}$. According to definition of η and the fact $\rho < \eta$, arm i becomes the unique median arm after mapping y_i to $\psi_{\rho}^{\text{median}}(y_i)$; we thus have $|y'_{(m)} - y_{(m)}| = \rho$ and $\text{AD}'_i = 0$. Furthermore, we have $|\text{AD}'_j - \text{AD}_j| = \rho$ for all $j \in [n] \setminus i$ as the median value is changed by ρ and $\min\{y_{(m)} - y_{(m+1)}, y_{(m-1)} - y_{(m)}\} \geq 2\eta > 2\rho$ by Definition 1.

If $\text{AD}_i < \text{AD}_{(m)}$, since $\min\{\text{AD}_{(m)} - \text{AD}_{(m+1)}, \text{AD}_{(m-1)} - \text{AD}_{(m)}\} \geq 2\eta > 2\rho$ by Definition 1, we know that the arm associated with the MAD value in D_y is still associated with the MAD value in $D_{y'}$. We thus have $|\text{AD}_{(m)} - \text{AD}'_{(m)}| = \rho$. Since $k \geq 2$ by definition, we have $|\theta' - \theta| \geq \rho$, resulting in at least one of $\{u_1, u_2, l_1, l_2\}$ being removed or added to S_o .

If $\text{AD}_i \geq \text{AD}_{(m)}$, we know there exists an arm p such that $\text{AD}_p \leq \text{AD}_{(m+1)}$ is now associated with $\text{AD}'_{(m)}$ in $D_{y'}$ (since AD'_i becomes 0 in $D_{y'}$). Since $\min\{\text{AD}_{(m)} - \text{AD}_{(m+1)}\} \geq 2\eta > 2\rho$ by Definition 1, we thus have $|\text{AD}_{(m)} - \text{AD}'_{(m)}| \geq \rho$ as $\text{AD}'_{(m)} = \text{AD}'_p \leq \text{AD}_p + \rho \leq \text{AD}_{(m+1)} + \rho$. We then have $|\theta' - \theta| \geq \rho$ as $k \geq 2$ by definition; this further results in at least one of $\{u_1, u_2, l_1, l_2\}$ being removed or added to S_o .

Case 3: $a = \text{MAD}$. If $y_i = y_{(m)}$, same analysis appears in Case 2 applies here and leads to at least one of $\{u_1, u_2, l_1, l_2\}$ will be removed or added to S_o .

If $y_i \neq y_{(m)}$, we have $y'_{(m)} = y_{(m)}$ (notice that $\text{AD}_{(m)} \geq 2\eta > 2\rho$ by assumption) and $|\text{AD}'_{(m)} - \text{AD}_{(m)}| = \rho$ by the construction of $\psi_{\rho}^{\text{MAD}}(y_i)$, which results in $|\theta' - \theta| \geq \rho$ and thus at least one of $\{u_1, u_2, l_1, l_2\}$ being removed or added to S_o . \square

Now we restated Theorem 3 and provide the proof.

Theorem 3. *Suppose bandit instance $D_y \in \mathcal{M}_{n,\rho}$. Then for $\delta \leq 0.15$, any δ -PAC outlier arm identification algorithm \mathcal{A} with outlier threshold constructed as in Eq. (1) and an almost surely finite stopping time τ , we have that*

$$\mathbb{E}_y[\tau] \geq \sum_{i \in [n]} \frac{1}{5(\Delta_i^*)^2} \log\left(\frac{1}{2.4\delta}\right).$$

Proof. For any $i \in [n]$ and any $a \in \{\theta, \text{median}, \text{MAD}\}$, we construct

$$D_{y'} = (D_{y_1}, \dots, D_{y_{i-1}}, D_{\psi_{\rho}^a(y_i)}, D_{y_{i+1}}, \dots, D_{y_n}).$$

We also define the event $A = \{\hat{S}_o = [n_1]\}$, which is measurable with respect to \mathcal{F}_{τ} . For any δ -PAC algorithm, according to its definition and Lemma 23, we have $\mathbb{P}_y(A) \geq 1 - \delta$ and $\mathbb{P}_{y'}(A) \leq \delta$. Thus, according to Lemma 22, we have

$$\mathbb{E}_y[N_{i,\tau}] \cdot \text{KL}(D_{y_i}, D_{y'_i}) \geq d(1 - \delta, \delta) \geq \log\left(\frac{1}{2.4\delta}\right), \quad (17)$$

where we use the property that for $x \in [0, 1]$, $d(x, 1 - x) \geq \log\left(\frac{1}{2.4\delta}\right)$ for the last inequality. Eq. (17) further gives us

$$\mathbb{E}_y[N_{i,\tau}] \geq \frac{1}{\text{KL}(D_{y_i}, D_{\psi_\rho^a(y_i)})} \log\left(\frac{1}{2.4\delta}\right). \quad (18)$$

Combining Eq. (18) with $\text{KL}(D_{y_i}, D_{\psi_\rho^a(y_i)}) = 2/(\Delta_{i,\rho}^a)^2$ for $a \in \{\theta, \text{median}, \text{MAD}\}$ and $\Delta_{i,\rho}^* = \min\{\Delta_{i,\rho}^\theta, \Delta_{i,\rho}^{\text{median}}, \Delta_{i,\rho}^{\text{MAD}}\} \geq \rho$, we have

$$\mathbb{E}_y[N_{i,\tau}] \geq \frac{2}{(\Delta_{i,\rho}^*)^2} \log\left(\frac{1}{2.4\delta}\right). \quad (19)$$

For any arm i such that $\min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\} \neq 0$, we have $\min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\} \geq \rho/2$ according to the construction of $\mathcal{M}_{n,\rho}$. Thus, $2/(\Delta_{i,\rho}^*)^2 = 2/(\min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\} + \rho)^2 \geq 2/(3 \min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\})^2 \geq 2/(3\Delta_i^*)^2$.

For any arm i such that $\min\{\Delta_i^\theta, \Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\} = 0$, we have $\Delta_i^* \geq \rho/2$ according to the construction of $\mathcal{M}_{n,\rho}$. Thus, $2/(\Delta_{i,\rho}^*)^2 = 2/\rho^2 \geq 2/(2\Delta_i^*)^2 = 2/(2\Delta_i^*)^2$.

Combining Eq. (19) with the above analysis, we have

$$\mathbb{E}_y[N_{i,\tau}] \geq \frac{1}{5(\Delta_i^*)^2} \log\left(\frac{1}{2.4\delta}\right).$$

Summing over all $i \in [n]$ yields the desired bound in Theorem 3. \square

F. Heuristic to Reduce Sample Complexity: Proof of Corollary 1

Corollary 1. *Suppose we run Algorithm 3 with $y_{(m)}$, $\text{AD}_{(m)}$ and θ constructed using arms in $\Omega \subseteq [n]$. Then, with probability at least $1 - \delta$, the sample complexity is upper bounded by*

$$Ck^2 \sum_{i \in \Omega} \frac{\log(nk/(\delta\Delta_i^*))}{(\Delta_i^*)^2} + C \sum_{i \notin \Omega} \frac{\log(n/(\delta\Delta_i^\theta))}{(\Delta_i^\theta)^2},$$

where $\Delta_i^* = \max\{\Delta_i^\theta, \min\{\Delta_i^{\text{median}}, \Delta_i^{\text{MAD}}\}\}$ and C is a universal constant.

Proof. The subsampling algorithm is implemented as in Algorithm 3, with some notational changes to adapt to the subset Ω , as described here. We still assume the total number of arms is n , but set $|\Omega| = 2m - 1$.⁹ $y_{(m)}$, AD_i , $\text{AD}_{(m)}$ and θ are all calculated with respect to arms in Ω . We use the notation $J_{\kappa_i,t}$ to denote κ_i arms in Ω with the largest empirical means $\{\hat{y}_{i,t}\}$, and $J_{\kappa_i,t}^{\text{AD}}$ to denote the κ_i arms in Ω with the largest empirical absolute deviations $\{\widehat{\text{AD}}_{i,t}\}$. Since we are mainly interested in shrinking confidence intervals around the median, we set $\kappa_1 = m - 1$ and $\kappa_2 = m$.

The $\text{NEEDY}_{i,t}^\theta$ event remains the same for all arms in $[n]$; however, we will have $\text{NEEDY}_{i,t}^{\text{median}}$ and $\text{NEEDY}_{i,t}^{\text{MAD}}$ events only for arms in Ω as arms outside Ω are not involved in the construction of the outlier threshold. Eq. (6) and lemmas in Appendix D.3 and Appendix D.4 can be adapted to the subset Ω . Since Algorithm 3 pulls a constant number of arms each round, we only need to upper bound the total number of rounds up to time T . Similar to the analysis in the proof of Theorem 5, we have

$$\#\text{rounds}(T) = \sum_{t=1}^T \mathbf{1}[\text{NT}_t]$$

⁹Recall we simply choose the median as m if $|\Omega| = 2m$.

$$\begin{aligned}
 &= \sum_{t=1}^T \mathbb{1} \left[\left(\text{NT}_t \cap \neg \text{NEEDY}_{\theta,t}^\theta \right) \cup \left(\text{NT}_t \cap \text{NEEDY}_{\theta,t}^\theta \right) \right] \\
 &= \sum_{t=1}^T \mathbb{1} \left[\left(\text{NEEDY}_{l_{\theta,t}}^\theta \cup \text{NEEDY}_{u_{\theta,t}}^\theta \right) \cup \left(C_t \cap A_t \right) \cup \left(C_t \cap \neg A_t \cap B_t \right) \cup \left(C_t \cap \neg A_t \cap \neg B_t \right) \right] \\
 &\leq \sum_{t=1}^T \mathbb{1} \left[\left(\text{NEEDY}_{l_{\theta,t}}^\theta \cup \text{NEEDY}_{u_{\theta,t}}^\theta \right) \cup \left(A_t \right) \cup \left(\neg A_t \cap B_t \right) \right] \\
 &\leq \sum_{t=1}^T \mathbb{1} \left[\bigcup_{a \in \Omega} \left(\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap \text{NEEDY}_{a,t}^\theta \right) \cup \left((a = l_{i,t} \vee u_{i,t}) \cap \text{NEEDY}_{a,t}^{\text{median}} \right) \right. \right. \\
 &\quad \left. \left. \cup \left(\neg A_t \cap (a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}) \cap \text{NEEDY}_{a,t}^{\text{MAD}} \right) \right) \cup \left(\bigcup_{i \notin \Omega} \left((a = l_{\theta,t} \vee u_{\theta,t}) \cap \text{NEEDY}_{a,t}^\theta \right) \right) \right] \\
 &\leq \sum_{t=1}^T \sum_{a \in \Omega} \mathbb{1} \left[\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap \text{NEEDY}_{a,t}^\theta \right) \cup \left((a = l_{i,t} \vee u_{i,t}) \cap \text{NEEDY}_{a,t}^{\text{median}} \right) \right. \\
 &\quad \left. \cup \left(\neg A_t \cap (a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}) \cap \text{NEEDY}_{a,t}^{\text{MAD}} \right) \right] + \sum_{t=1}^T \sum_{a \notin \Omega} \mathbb{1} \left[\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap \text{NEEDY}_{a,t}^\theta \right) \right] \\
 &\leq \sum_{t=1}^T \sum_{a \in \Omega} \mathbb{1} \left[\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap N_{a,t} \leq \lambda_a^\theta \right) \cup \left((a = l_{i,t} \vee u_{i,t}) \cap N_{a,t} \leq \lambda_a^{\text{median}} \right) \right. \\
 &\quad \left. \cup \left(\neg A_t \cap (a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}) \cap N_{a,t} \leq \lambda_a^{\text{MAD}} \right) \right] + \sum_{t=1}^T \sum_{a \notin \Omega} \mathbb{1} \left[\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap N_{a,t} \leq \lambda_a^\theta \right) \right] \\
 &\leq \sum_{a \in \Omega} \sum_{t=1}^T \mathbb{1} \left[\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap N_{a,t} \leq \lambda_a^\theta \right) \cup \left((a = l_{i,t} \vee u_{i,t}) \cap N_{a,t} \leq \lambda_a^{\text{median}} \right) \right. \\
 &\quad \left. \cup \left((a = l_{i,t}^{\text{AD}} \vee u_{i,t}^{\text{AD}}) \cap N_{a,t} \leq \lambda_a^{\text{MAD}} \right) \right] + \sum_{a \notin \Omega} \sum_{t=1}^T \mathbb{1} \left[\left((a = l_{\theta,t} \vee u_{\theta,t}) \cap N_{a,t} \leq \lambda_a^\theta \right) \right] \\
 &\leq \sum_{a \in \Omega} \max_{j \in \{\theta, \text{median}, \text{MAD}\}} \{\lambda_a^j\} + \sum_{a \notin \Omega} \lambda_a^\theta.
 \end{aligned}$$

The rest of the proof is the similar to that in Theorem 5. □