Error-Bounded Correction of Noisy Labels
— Supplementary Material —

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1. Additional (Synthetic) Experiment for Validation of the Bound

In Section 2.3 of the submitted manuscript, we used the output of deep neural networks f as an approximation of η on the CIFAR10 dataset. We provided empirical estimates of the constants C and λ in the Tsybakov condition for η, as well as estimates of the probability $\Pr[\widetilde{y} = h^*(\boldsymbol{x}), f_{\widetilde{y}}(\boldsymbol{x}) < \Delta]$.

In this section, we provide additional experiments on a *synthetic data set* generated using a mixture-of-Gaussians distribution. In this ideal setting, we know η, τ_{01}, τ_{10}, $\widetilde{\eta}$ *exactly*. We can a) use $\widetilde{\eta}$ as the classifier and b) evaluate the constants in Tsybakov condition for η in order to evaluate the upper bound in Theorem 1.

Estimation of Tsybakov condition constants. We let $\Pr(\boldsymbol{x})$ be a mixture of Gaussian distribution in a 10 dimensional feature space, $\boldsymbol{x} \sim \frac{1}{2}\mathcal{N}(0, I_{10\times 10}) + \frac{1}{2}\mathcal{N}(1, I_{10\times 10})$. We sample from the two components with equal probability. If $\boldsymbol{x}$ comes from component $\mathcal{N}(0, I_{10\times 10})$, it is given label 0. Otherwise, if $\boldsymbol{x}$ comes from component $\mathcal{N}(1, I_{10\times 10})$, it is given label 1. The true conditional distribution is $\eta(\boldsymbol{x}) = \frac{\exp\left\{-\frac{1}{2}||\boldsymbol{x}-1||^2\right\}}{\exp\left\{-\frac{1}{2}||\boldsymbol{x}||^2\right\}+\exp\left\{-\frac{1}{2}||\boldsymbol{x}-1||^2\right\}}$.

Following the idea of our experiment on CIFAR10 in the manuscript (Section 2.4), we estimate $\Pr\left[|\eta(\boldsymbol{x}) - \frac{1}{2}| \leq t\right]$ for values of t sampled between 0 and 0.9 using the empirical frequency $p_t = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{|\eta(\boldsymbol{x})-1/2|\leq t\}}(\boldsymbol{x})$. Note that if the Tsybakov condition is tight, $\log(p_t)$ approximates $\log(Ct^\lambda)$. The samples for $\log(t)$ and correspondingly, $\log(Ct^\lambda) \approx \log(p_t)$ are drawn as blue dots in Figure 1(a). The ordinary least square (OLS) linear regression results is drawn as a red line. We found the estimated values of C and λ to be 0.58 and 1.27 respectively. The estimation is high is confidence: the determinant coefficient R^2 equals 0.904, and we have a p-value which is less than 10^{-4}.

Estimation of the error bound, and its tightness. We also introduce label noise using predefined transition probability τ_{01} and τ_{10}. We can estimate C and λ as mentioned above, and know $\tau_{01}, \tau_{10}, \eta(x)$, and thus, $\widetilde{\eta}(\boldsymbol{x})$. Therefore we can evaluate the error bound in Theorem 1. We plot the error bound as a function of ϵ in Figures 1(b) and (c) (drawn green curves).

Finally, we assume a perfect noisy classifier $f = \widetilde{\eta}$. In other words, $\epsilon = 0$. We empirically show that when $f(\boldsymbol{x}) < \Delta$, the probability of $\widetilde{y}$ being correct (i.e., $\widetilde{y} = h^*(\boldsymbol{x})$) is zero (blue lines in Figures 1(b) and (c)).

Validation of the label-correction algorithm. To the same synthetic dataset, we also apply our LRT-Correction algorithm and validate the bound in Corollary 1. Since we know $\widetilde{\eta}(\boldsymbol{x})$, τ_{01} and τ_{10}, we calculate the correction error bound of Corollary 1 in closed form. We draw the bound w.r.t. the error ϵ in orange curves in Figure 2. Finally, we run our label correction algorithm using the perfect noisy classifier $f = \widetilde{\eta}$ and validate that the corrected labels are very close to clean (the success rate is limited by the asymmetry level of the noise pattern). See blue lines in Figure 2.

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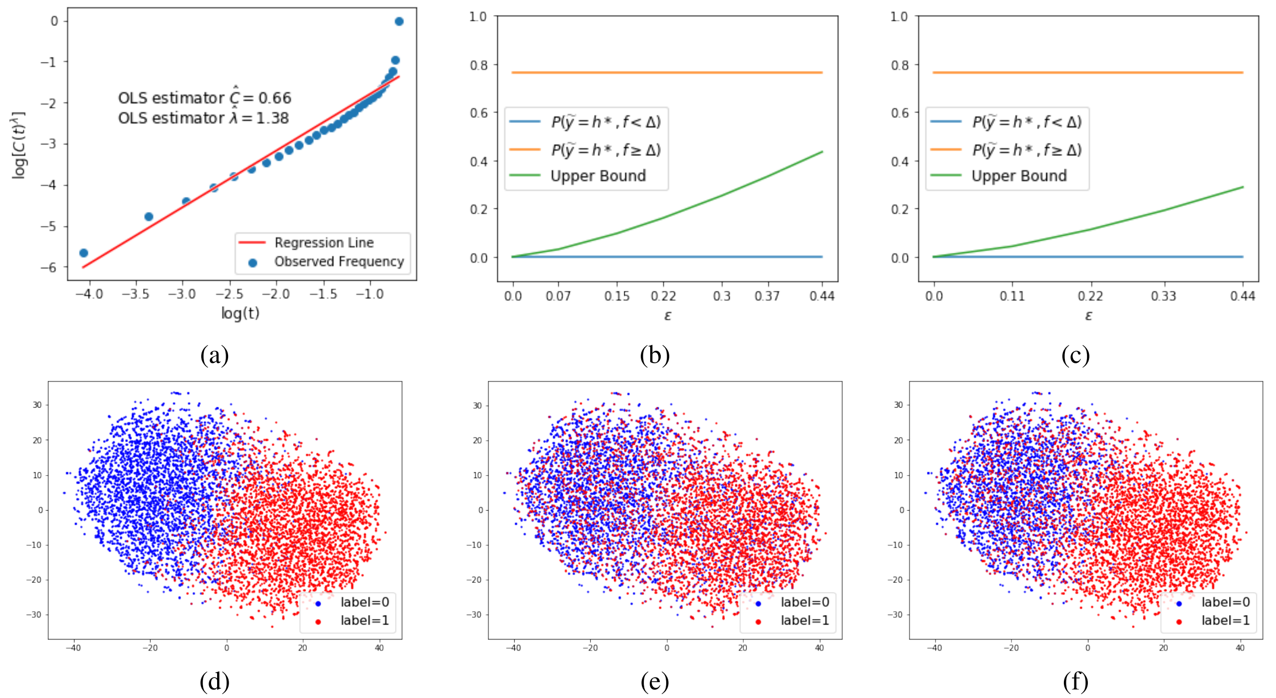


Figure 1. Synthetic experiment using Mixture of Gaussian at noise level 20%. (a): Check of Tsybakov condition using linear regression, where y-axis is the proportion of data points at distance t from decision boundary. (b): Proportion of labels that are not correct (not consistent with Bayes optimal decision rule) and the proposed upper bound. (c): Same as (b) but labels are corrupted with aysmmetric noise. (d): t-SNE of the clean data. (e): t-SNE of the data with symmetric noise. (f): t-SNE of the data with asymmetric noise.

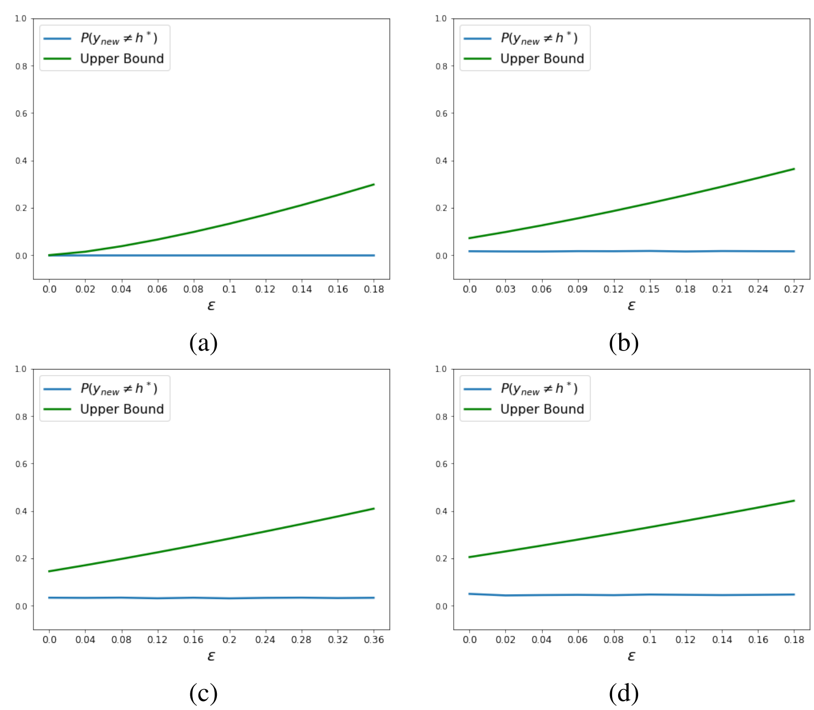
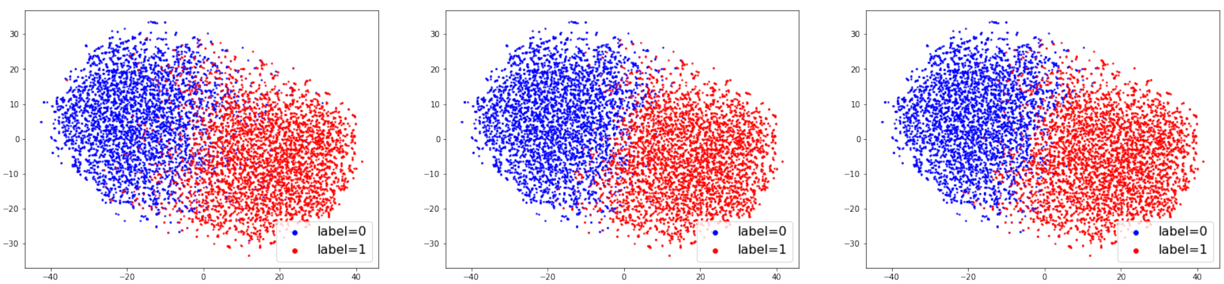


Figure 2. Performance of LRT algorithm given $\widetilde{\eta}(\boldsymbol{x})$ v.s the proposed upper bound. (a): Symmetric noise ($\tau_{10} = \tau_{01} = 0.3$). (b): Asymmetric noise ($\tau_{10} = 0.2, \tau_{01} = 0.3$). (c): Asymmetric noise ($\tau_{10} = 0.1, \tau_{01} = 0.3$). (d): Asymmetric noise ($\tau_{10} = 0.3, \tau_{01} = 0$)



(a) (b) (c)

Figure 3. Label Correction Result Using LRT-Correct. (a): Clean data as it in Fig 1d. (b): Labels after correction for data in Fig 1e. (c): Labels after correction for data in Fig 1f.

2. Proof of Theorem 2

Define $m_{\boldsymbol{x}} := \arg\max_i f_i(\boldsymbol{x})$, $u_{\boldsymbol{x}} := \arg\max_i \eta_i(\boldsymbol{x})$ and $s_{\boldsymbol{x}} := \arg\max_{i \neq u_{\boldsymbol{x}}} \eta_i(\boldsymbol{x})$. Let $[Nc] := \{1, 2, \cdots, N_c\}$. Finally, define $\epsilon_i(\boldsymbol{x}) := |f_i(\boldsymbol{x}) - \widetilde{\eta}_i(\boldsymbol{x})|$ and $\epsilon := \max_{\boldsymbol{x}, i} \epsilon_i(\boldsymbol{x})$.

For multi-class scenario, we know $\forall i \in [N_c]$, $\widetilde{\eta}_i(\boldsymbol{x}) = \sum_{j \in [N_c]} \tau_{ji} \eta_j(\boldsymbol{x})$. We also restate the multi-class Tsybakov condition here:

Assumption 1 (Multi-class Tsybakov Condition)**.** *$\exists C, \lambda > 0$ and $t_0 \in (0, 1]$ such that for all $t \leq t_0$,*

$$\Pr\left[|\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})| \leq t\right] \leq C t^{\lambda}$$

Theorem 2. *Assume $\eta(\boldsymbol{x})$ fulfills multi-class Tsybakov condition for constant $C, \lambda > 0$ and $t_0 \in (0, 1]$. Assume that $\epsilon \leq t_0 \min_i \tau_{i,i}$. For $\Delta = \min\left[1, \min_{\boldsymbol{x}} [\tau_{\widetilde{y},\widetilde{y}} \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}} \eta_j(\boldsymbol{x})]\right]$:*

$$\Pr_{(x,y) \sim D}\left[\widetilde{y} = h^*(\boldsymbol{x}), f_{\widetilde{y}}(\boldsymbol{x}) < \Delta\right] \leq C\left[O(\epsilon)\right]^{\lambda}$$

Proof.

$$\begin{aligned}
&\Pr\left[\widetilde{y} = h^*(\boldsymbol{x}), f_{\widetilde{y}}(\boldsymbol{x}) < \Delta\right] = \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), f_{\widetilde{y}}(\boldsymbol{x}) < \Delta\right] \\
&\leq \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \widetilde{\eta}_{\widetilde{y}}(\boldsymbol{x}) < \Delta + \epsilon_{\widetilde{y}}\right] \\
&\leq \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \widetilde{\eta}_{\widetilde{y}}(\boldsymbol{x}) < \Delta + \epsilon\right] \\
&= \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \sum_{j \in [N_c]} \tau_{j,\widetilde{y}} \eta_{\widetilde{y}}(\boldsymbol{x}) < \Delta + \epsilon\right] \\
&= \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\Delta - \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}} \eta_j(\boldsymbol{x}) + \epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&= \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\Delta - \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}} \eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}} + \frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right]
\end{aligned} \tag{1}$$

Remember that $\Delta = \min\left[1, \min_{\boldsymbol{x}} [\tau_{\widetilde{y},\widetilde{y}} \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}} \eta_j(\boldsymbol{x})]\right] \leq \tau_{\widetilde{y},\widetilde{y}} \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}} \eta_j(\boldsymbol{x})$. Then if we substitute Δ in (1) with $\tau_{\widetilde{y},\widetilde{y}} \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}} \eta_j(\boldsymbol{x})$, continuing the derivation of (1), we will end up with:

$$\begin{aligned}
&\Pr\left[\widetilde{y}=h^*(\boldsymbol{x}), f_{\widetilde{y}}(\boldsymbol{x})<\Delta\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\Delta - \sum\limits_{j\neq\widetilde{y}} \tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}} + \frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \leq C\left(\frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right)^{\lambda}
\end{aligned}$$

Notice that Tsybakov condition holds here because $\epsilon \leq t_0 \min_i \tau_{i,i}$, which implies that $\frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}} \leq t_0$. This complete the proof for this case.

□

3. Proof of Theorem 3

Lemma 1. *(Algorithm Multiclass-Theorem Guarantee). Assume $\eta(\boldsymbol{x})$ fulfills multi-class Tsybakov condition for constant $C>0$, $\lambda>0$ and $t_0\in(0,1]$. Assume that $\epsilon \leq t_0 \min_i \tau_{ii}$. Let $\widetilde{y}_{new}$ denote the output of the* `LRT-Correction` *with $\boldsymbol{x}$, $\widetilde{y}_{\boldsymbol{x}}$, f, and the given δ, then:*

1. *Sensitivity Optimized Critical Value. Let $\delta = \min_{\boldsymbol{x}}\left[\frac{\tau_{\widetilde{y},\widetilde{y}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j\neq\widetilde{y}}\tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\right]$ then :*

$$\Pr_{(x,y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}\right] \leq C\left[O(\epsilon)\right]^{\lambda} + \Pr_{(\boldsymbol{x},y)\sim D}\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right]$$

2. *Specificity Optimized Critical Value. Let $\delta = \max_{\boldsymbol{x}}\left[\frac{f_{\widetilde{y}}(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j\neq m_{\boldsymbol{x}}}\tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}\right]$ then :*

$$\Pr_{(x,y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}\right] \leq C\left[O(\epsilon)\right]^{\lambda} + \Pr_{(\boldsymbol{x},y)\sim D}\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right]$$

Proof. First look at cases where $\widetilde{y}$ is rejected.

$$\begin{aligned}
&\Pr\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}\right] \\
&= \Pr\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] \\
&= \Pr\left[\widetilde{y}_{new} = m_{\boldsymbol{x}} \neq h^*(\boldsymbol{x}) = \widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] + \Pr\left[\widetilde{y}_{new} = m_{\boldsymbol{x}} \neq h^*(\boldsymbol{x}) = u_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] \\
&\leq \Pr\left[h^*(\boldsymbol{x}) = \widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] + \Pr\left[\widetilde{y}_{new} = m_{\boldsymbol{x}} \neq h^*(\boldsymbol{x}) = u_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] \qquad (2)
\end{aligned}$$

For the first term in (2), we have:

$$\begin{aligned}
&\Pr\left[h^*(\boldsymbol{x}) = \widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] = \Pr\left[h^*(\boldsymbol{x}) = \widetilde{y}, f_{\widetilde{y}}(\boldsymbol{x}) < \delta f_{m_{\boldsymbol{x}}}(\boldsymbol{x})\right] \\
&\leq \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \widetilde{\eta}_{\widetilde{y}}(\boldsymbol{x}) - \epsilon < \delta f_{m_{\boldsymbol{x}}}(\boldsymbol{x})\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\delta f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \sum\limits_{j\neq\widetilde{y}}\tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}} + \frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \qquad (3)
\end{aligned}$$

We substitute δ in (3) with $\frac{\tau_{\widetilde{y},\widetilde{y}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})+\sum_{j\neq\widetilde{y}}\tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}$ and continue the calculation:

$$\begin{aligned}
&\Pr\left[h^*(\boldsymbol{x})=\widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}<\delta\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\delta f_{m_{\boldsymbol{x}}}(\boldsymbol{x})-\sum_{j\neq\widetilde{y}}\tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}}+\frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) \leq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})+\frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&\leq C\left(\frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right)^{\lambda}
\end{aligned} \tag{4}$$

In (4), the Tsybakov condition holds here because $\epsilon \leq t_0 \min_i \tau_{ii}$, which implies $\frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}} \leq t_0$.

For the second term in (2), we have:

$$\Pr\left[\widetilde{y}_{new}=m_{\boldsymbol{x}}\neq h^*(\boldsymbol{x})=u_{\boldsymbol{x}}, u_{\boldsymbol{x}}\neq\widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}<\delta\right] \leq \Pr\left[u_{\boldsymbol{x}}\neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}}\neq\widetilde{y}\right] \tag{5}$$

for which our algorithm currently doesn't have a good way to deal with and we will leave it as future research problem.

Finally, summarize every piece and we finished the proof for cases where $\widetilde{y}$ is rejected:

$$\begin{aligned}
&\Pr\left[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}\right] \leq (2) \\
&\leq (4)+(5) \\
&\leq C\left[\frac{\epsilon}{\tau_{u_{\boldsymbol{x}},u_{\boldsymbol{x}}}}\right]^{\lambda}+\Pr\left[u_{\boldsymbol{x}}\neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}}\neq\widetilde{y}\right] \\
&= C\left[O(\epsilon)\right]^{\lambda}+\Pr\left[u_{\boldsymbol{x}}\neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}}\neq\widetilde{y}\right]
\end{aligned}$$

For cases where $\widetilde{y}$ is accepted:

$$\begin{aligned}
&\Pr\left[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}\right] = \Pr\left[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\geq\delta\right] \\
&= \Pr\left[\widetilde{y}_{new}=\widetilde{y}\neq h^*(\boldsymbol{x})=m_{\boldsymbol{x}}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\geq\delta\right]+\Pr\left[\widetilde{y}_{new}=\widetilde{y}\neq h^*(\boldsymbol{x}), m_{\boldsymbol{x}}\neq h^*(\boldsymbol{x}), \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\geq\delta\right] \\
&= \Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x})\geq\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), f_{m_{\boldsymbol{x}}}(\boldsymbol{x})\leq f_{\widetilde{y}}(\boldsymbol{x})/\delta\right]+\Pr\left[u_{\boldsymbol{x}}\neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}}\neq\widetilde{y}\right]
\end{aligned} \tag{6}$$

For the first term in (6), we have:

$$\begin{aligned}
&\Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x})\geq\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), f_{m_{\boldsymbol{x}}}(\boldsymbol{x})\leq f_{\widetilde{y}}(\boldsymbol{x})/\delta\right] \leq \Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x})\geq\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \widetilde{\eta}_{m_{\boldsymbol{x}}}(\boldsymbol{x})-\epsilon\leq f_{\widetilde{y}}(\boldsymbol{x})/\delta\right] \\
&= \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})\leq\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x})\leq\frac{f_{\widetilde{y}}(\boldsymbol{x})/\delta-\sum_{j\neq m_{\boldsymbol{x}}}\tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}+\frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right]
\end{aligned} \tag{7}$$

Firstly, observe that if $\delta>1$, then $\Pr\left[\widetilde{y}_{new}=\widetilde{y}\neq h^*(\boldsymbol{x}), \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\geq\delta\right]=0$ due to the definition of $m_{\boldsymbol{x}}$.

Then notice that $\delta=\max_{\boldsymbol{x}}\frac{f_{\widetilde{y}}(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})+\sum_{j\neq m_{\boldsymbol{x}}}\tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}\geq\frac{f_{\widetilde{y}}(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})+\sum_{j\neq m_{\boldsymbol{x}}}\tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}$. If we substitute δ in (7) with

$\frac{f_{\widetilde{y}}(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x})+\sum\limits_{j\neq m_{\boldsymbol{x}}}\tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}$ and continuing the calculation, we will have:

$$\begin{aligned}
&\Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq f_{\widetilde{y}}(\boldsymbol{x})/\delta\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \frac{f_{\widetilde{y}}(\boldsymbol{x})/\delta - \sum\limits_{j\neq m_{\boldsymbol{x}}}\tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&\leq C\left[\frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right]^{\lambda} \qquad (8)
\end{aligned}$$

For the second term in (6), our algorithm cannot deal with it properly. We will leave it as the future research problem.

Now we summarize all pieces and we get:

$$\begin{aligned}
&\Pr\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}\right] = (6) \\
&\leq (8) + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right] \\
&\leq C\left[\frac{\epsilon}{\tau_{u_{\boldsymbol{x}},u_{\boldsymbol{x}}}}\right]^{\lambda} + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right]
\end{aligned}$$

which compete the proof for cases that are accepted. □

We give following several facts based on our theorem:

1. For binary case, if we set $\delta = \frac{1-|\tau_{10}-\tau_{01}|}{1+|\tau_{10}-\tau_{01}|}$ and further assume $\epsilon \leq t_0(1-\tau_{10}-\tau_{01}) - \frac{|\tau_{10}-\tau_{01}|}{2}$, we have:

$$\Pr_{(\boldsymbol{x},y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x})\right] \leq C\left[\left|\frac{\tau_{10}-\tau_{01}}{2(1-\tau_{10}-\tau_{01})}\right| + \frac{\epsilon}{1-\tau_{10}-\tau_{01}}\right]^{\lambda}$$

Proof. For binary case, we have:

$$\begin{aligned}
&\Pr_{(\boldsymbol{x},y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x})\right] = \Pr_{(\boldsymbol{x},y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}\right] + \Pr_{(\boldsymbol{x},y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}\right] \\
&= \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) > \frac{1}{2}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \delta\right] + \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \leq \frac{1}{2}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} \geq \delta\right] \\
&\leq \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) > \frac{1}{2}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{1-f_{\widetilde{y}}(\boldsymbol{x})} < \delta\right] + \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \leq \frac{1}{2}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{1-f_{\widetilde{y}}(\boldsymbol{x})} \geq \delta\right] \\
&\leq \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) > \frac{1}{2}, \widetilde{\eta}_{\widetilde{y}}(\boldsymbol{x}) < \frac{\delta}{1+\delta} + \epsilon\right] + \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \leq \frac{1}{2}, \widetilde{\eta}_{\widetilde{y}}(\boldsymbol{x}) \geq \frac{\delta}{1+\delta} - \epsilon\right] \\
&= \Pr\left[\frac{1}{2} < \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\frac{\delta}{1+\delta} - \tau_{1-\widetilde{y},\widetilde{y}}}{1-\tau_{10}-\tau_{01}} + \frac{\epsilon}{1-\tau_{10}-\tau_{01}}\right] + \Pr\left[\frac{\frac{\delta}{1+\delta} - \tau_{1-\widetilde{y},\widetilde{y}}}{1-\tau_{10}-\tau_{01}} - \frac{\epsilon}{1-\tau_{10}-\tau_{01}} \leq \eta_{\widetilde{y}}(\boldsymbol{x}) \leq \frac{1}{2}\right] \quad (9)
\end{aligned}$$

Observe that $\delta = \frac{1-|\tau_{10}-\tau_{01}|}{1+|\tau_{10}-\tau_{01}|} \leq \frac{1-\tau_{\widetilde{y},1-\widetilde{y}}+\tau_{1-\widetilde{y},\widetilde{y}}}{1+\tau_{\widetilde{y},1-\widetilde{y}}-\tau_{1-\widetilde{y},\widetilde{y}}}$. We also have $\frac{\delta}{1+\delta} = \frac{1-|\tau_{10}-\tau_{01}|}{2} \leq \frac{1}{2}$. Now we substitute $\delta = \frac{1-\tau_{\widetilde{y},1-\widetilde{y}}+\tau_{1-\widetilde{y},\widetilde{y}}}{1+\tau_{\widetilde{y},1-\widetilde{y}}-\tau_{1-\widetilde{y},\widetilde{y}}}$ in the first term of (9) and substitute $\frac{\delta}{1+\delta}$ with $\frac{1}{2}$ in the second term of (9), by algebra we know that :

$$\begin{aligned}
&\Pr_{(\boldsymbol{x},y)\sim D}[\widetilde{y}_{new}\neq h^*(\boldsymbol{x})] = \Pr_{(\boldsymbol{x},y)\sim D}[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}] + \Pr_{(\boldsymbol{x},y)\sim D}[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}] \\
&\leq \Pr\left[\frac{1}{2} < \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{1}{2} + \frac{\epsilon}{1-\tau_{10}-\tau_{01}}\right] + \Pr\left[\frac{1/2-\max(\tau_{10},\tau_{01})}{1-\tau_{10}-\tau_{01}} - \frac{\epsilon}{1-\tau_{10}-\tau_{01}} \leq \eta_{\widetilde{y}}(\boldsymbol{x}) \leq \frac{1}{2}\right] \\
&\leq C\left[\left|\frac{\tau_{10}-\tau_{01}}{2(1-\tau_{10}-\tau_{01})}\right| + \frac{\epsilon}{1-\tau_{10}-\tau_{01}}\right]^{\lambda}
\end{aligned}$$

Tsybakov assumption holds because $\frac{\epsilon}{1-\tau_{10}-\tau_{01}} + \frac{|\tau_{10}-\tau_{01}|}{2(1-\tau_{10}-\tau_{01})} \leq \frac{t_0(1-\tau_{10}-\tau_{01}) - \frac{|\tau_{10}-\tau_{01}|}{2}}{1-\tau_{10}-\tau_{01}} + \frac{|\tau_{10}-\tau_{01}|}{2(1-\tau_{10}-\tau_{01})} \leq t_0$. □

2. For symmetric noise $\tau_{ij} = \tau_{ji} = \tau, \forall i,j \in [N_c]$ and further assume (besides the assumption we made in Lemma 1) $\epsilon \leq \frac{1}{2}\min_{\boldsymbol{x}}[\widetilde{\eta}_{u_{\boldsymbol{x}}}(\boldsymbol{x}) - \widetilde{\eta}_{s_{\boldsymbol{x}}}(\boldsymbol{x})]$, we have:

   (a) Sensitivity Optimized Critical Value. Let $\delta = \min_{\boldsymbol{x}}\left[\frac{\tau_{\widetilde{y},\widetilde{y}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j\neq\widetilde{y}}\tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\right]$ then :

$$\Pr_{(x,y)\sim D}[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}] \leq C\left[O(\epsilon)\right]^{\lambda}$$

   (b) Specificity Optimized Critical Value. Let $\delta = \max_{\boldsymbol{x}}\left[\frac{f_{\widetilde{y}}(\boldsymbol{x})}{(\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}} - \tau_{\widetilde{y},m_{\boldsymbol{x}}})\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \tau_{\widetilde{y},m_{\boldsymbol{x}}}}\right]$ then :

$$\Pr_{(x,y)\sim D}[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}] \leq C\left[O(\epsilon)\right]^{\lambda}$$

Proof. Observe that under symmetric noise scenario, $\forall i \in [N_c]$, $\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \eta_i(\boldsymbol{x})$ will implies that $\widetilde{\eta}_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \widetilde{\eta}_i(\boldsymbol{x})$, i.e. $h^*(\boldsymbol{x}) = \widetilde{h}^*(\boldsymbol{x})$. To show this:

$$\begin{aligned}
&\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \eta_i(\boldsymbol{x}) \\
\iff &[1-N_c\tau]\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq [1-N_c\tau]\eta_{u_i}(\boldsymbol{x}) \\
\iff &[1-(N_c-1)\tau]\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) - \tau\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq [1-(N_c-1)\tau]\eta_i(\boldsymbol{x}) - \tau\eta_i(\boldsymbol{x}) \\
\iff &[1-(N_c-1)\tau]\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) + \tau\eta_i(\boldsymbol{x}) \geq [1-(N_c-1)\tau]\eta_i(\boldsymbol{x}) + \tau\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \\
\iff &[1-(N_c-1)\tau]\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) + \tau\eta_i(\boldsymbol{x}) + \tau\sum_{j\neq u_{\boldsymbol{x}}, j\neq i}\eta_j(\boldsymbol{x}) \geq [1-(N_c-1)\tau]\eta_i(\boldsymbol{x}) + \tau\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) + \tau\sum_{j\neq u_{\boldsymbol{x}}, j\neq i}\eta_j(\boldsymbol{x}) \\
\iff &[1-(N_c-1)\tau]\eta_{u_{\boldsymbol{x}}}(\boldsymbol{x}) + \tau\sum_{j\neq u_{\boldsymbol{x}}}\eta_j(\boldsymbol{x}) \geq [1-(N_c-1)\tau]\eta_i(\boldsymbol{x}) + \tau\sum_{j\neq i}\eta_j(\boldsymbol{x}) \\
\iff &\sum_{j\in[Nc]}\tau_{j,u_{\boldsymbol{x}}}\eta_j(\boldsymbol{x}) \geq \sum_{j\in[N_c]}\tau_{ji}\eta_i(\boldsymbol{x}) \\
\iff &\widetilde{\eta}_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \widetilde{\eta}_i(\boldsymbol{x})
\end{aligned}$$

Since $\widetilde{\eta}_{u_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \widetilde{\eta}_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + 2\epsilon$, then $\widetilde{\eta}_{u_{\boldsymbol{x}}}(\boldsymbol{x}) - \epsilon \geq \widetilde{\eta}_i(\boldsymbol{x}) + \epsilon$ and thus $f_{u_{\boldsymbol{x}}} \geq f_i(\boldsymbol{x})\ \forall i \in [N_c]$, which implies $f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) = f_{u_{\boldsymbol{x}}}(\boldsymbol{x})$. As a result, second term in (2) and second term in (6) will be 0. □

Theorem 3. *Assume η and f satisfy the same conditions as Lemma 1. Also assume $\xi < \delta$ and further assume that $\epsilon \leq \min\left(\frac{t_0\delta^2\min_i\tau_{ii} - \xi^2 - \xi}{\delta^2}, (t_0-\xi)\min_i\tau_{ii}\right)$. Let $\widetilde{y}_{new}$ be the output of the* `LRT-Correction` *with $(\boldsymbol{x}, \widetilde{y})$, f, and the approximate $\hat{\delta}$. Then:*

1. *Sensitivity Optimized Critical Value. Let $\delta = \min_{\boldsymbol{x}}\left[\frac{\tau_{\widetilde{y},\widetilde{y}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j\neq\widetilde{y}}\tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})}\right]$ then :*

$$\Pr_{(x,y)\sim D}[\widetilde{y}_{new}\neq h^*(\boldsymbol{x}), \widetilde{y} \textit{ is rejected}] \leq C\left[O(\max(\epsilon,\xi))\right]^{\lambda} + \Pr\left[u_{\boldsymbol{x}}\neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}}\neq\widetilde{y}\right]$$

2. *Specificity Optimized Critical Value. Let* $\delta = \max_{\boldsymbol{x}} \frac{f_{\widetilde{y}}(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \sum_{j \neq m_{\boldsymbol{x}}} \tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}$ *then :*

$$\Pr_{(x,y)\sim D}\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}\right] \leq C\left[O(\max(\epsilon,\xi))\right]^{\lambda} + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right]$$

Proof. The proof will be similar to the proof of Lemma 1, but we need to adjust the error introduced by picking $\hat{\delta}$. Recall that ξ and ϵ are both less than one.

If we pick $\hat{\delta}$ instead of δ, then for (3) in Lemma 1, we have:

$$\begin{aligned}
&\Pr\left[h^*(\boldsymbol{x}) = \widetilde{y}, \frac{f_{\widetilde{y}}(\boldsymbol{x})}{f_{m_{\boldsymbol{x}}}(\boldsymbol{x})} < \hat{\delta}\right] = \Pr\left[h^*(\boldsymbol{x}) = \widetilde{y}, f_{\widetilde{y}}(\boldsymbol{x}) < \hat{\delta} f_{m_{\boldsymbol{x}}}(\boldsymbol{x})\right] \\
&\leq \Pr\left[\eta_{\widetilde{y}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \widetilde{\eta}_{\widetilde{y}}(\boldsymbol{x}) - \epsilon < \hat{\delta} f_{m_{\boldsymbol{x}}}(\boldsymbol{x})\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\hat{\delta} f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}} + \frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{(\delta + \xi) f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}} + \frac{\epsilon}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{\widetilde{y}}(\boldsymbol{x}) < \frac{\delta f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \sum_{j \neq \widetilde{y}} \tau_{j,\widetilde{y}}\eta_j(\boldsymbol{x})}{\tau_{\widetilde{y},\widetilde{y}}} + \frac{\epsilon + \xi}{\tau_{\widetilde{y},\widetilde{y}}}\right] \\
&\leq C\left[\frac{\epsilon + \xi}{\tau_{\widetilde{y},\widetilde{y}}}\right]^{\lambda}
\end{aligned} \tag{10}$$

The same upper bound holds for (5) with the same reason. Then:

$$\begin{aligned}
&\Pr\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is rejected}\right] \leq (10) + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right] \\
&= C\left[O(\max(\epsilon,\xi))\right]^{\lambda} + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right]
\end{aligned}$$

We next analyze (7) in Lemma 1:

$$\begin{aligned}
&\Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq f_{\widetilde{y}}(\boldsymbol{x})/\hat{\delta}\right] \leq \Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), \widetilde{\eta}_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \epsilon \leq f_{\widetilde{y}}(\boldsymbol{x})/\hat{\delta}\right] \\
&= \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \frac{f_{\widetilde{y}}(\boldsymbol{x})/\hat{\delta} - \sum_{j \neq m_{\boldsymbol{x}}} \tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&\leq \Pr\left[\eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \frac{f_{\widetilde{y}}(\boldsymbol{x})/(\delta - \xi) - \sum_{j \neq m_{\boldsymbol{x}}} \tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&= \Pr\left[0 < \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) < \frac{f_{\widetilde{y}}(\boldsymbol{x})/\delta - \sum_{j \neq m_{\boldsymbol{x}}} \tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\frac{\xi f_{\widetilde{y}}(\boldsymbol{x})}{\delta(\delta-\xi)}}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right]
\end{aligned}$$

Observe that $\frac{\xi}{\delta(\delta-\xi)} = \frac{\delta}{(\delta-\xi)}\frac{\xi}{\delta^2} = [1 + O(\xi)]\frac{\xi}{\delta^2}$, where second equality comes from Taylor expansion. Then we substitute the δ as what we did in Lemma 1 and continue the calculation:

$$\begin{aligned}
&\Pr\left[\eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \geq \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}), f_{m_{\boldsymbol{x}}}(\boldsymbol{x}) \leq f_{\widetilde{y}}(\boldsymbol{x})/\hat{\delta}\right] \\
&\leq \Pr\left[0 < \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) < \frac{f_{\widetilde{y}}(\boldsymbol{x})/\delta - \sum\limits_{j \neq m_{\boldsymbol{x}}} \tau_{j,m_{\boldsymbol{x}}}\eta_j(\boldsymbol{x})}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) + \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\frac{\xi f_{\widetilde{y}}(\boldsymbol{x})}{\delta(\delta-\xi)}}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&\leq \Pr\left[0 \leq \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi f_{\widetilde{y}}(\boldsymbol{x})}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi O(\xi) f_{\widetilde{y}}(\boldsymbol{x})}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&\leq \Pr\left[0 \leq \eta_{m_{\boldsymbol{x}}}(\boldsymbol{x}) - \eta_{s_{\boldsymbol{x}}}(\boldsymbol{x}) \leq \frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi^2}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right] \\
&\leq C\left[\frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi^2}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}}\right]^{\lambda} \qquad (11)
\end{aligned}$$

Here Tsybakove condition hold, because $\frac{\epsilon}{\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi^2}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} \leq \frac{t_0\delta^2 \min\limits_i \tau_{ii} - \xi^2 - \xi}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} + \frac{\xi^2}{\delta^2\tau_{m_{\boldsymbol{x}},m_{\boldsymbol{x}}}} \leq t_0$. As a result:

$$\begin{aligned}
&\Pr\left[\widetilde{y}_{new} \neq h^*(\boldsymbol{x}), \widetilde{y} \text{ is accepted}\right] \\
&\leq (11) + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right] \\
&\leq C\left[O(\max(\epsilon, \xi))\right]^{\lambda} + \Pr\left[u_{\boldsymbol{x}} \neq m_{\boldsymbol{x}}, u_{\boldsymbol{x}} \neq \widetilde{y}\right]
\end{aligned}$$

which compete the proof for cases that are accepted.

Other terms will not be affected by the choice of δ. By now we completes the proof. $\square$