A. Finite Element Analysis

The Finite Element Analysis (FEA) is arguably the most powerful approach known for the numerical solutions of problems characterized by partial differential equations (PDEs). We demonstrate the basic idea by considering the linear Poisson's equation as a model problem. As shown in Fig. 8, we start with the *strong formulation* of the problem and then introduce the *weak formulation* upon which the finite element approximation is built, namely, the *Galerkin weak formulation*. We further introduce the *minimization formulation* and the corresponding *Galerkin minimization formulation*, which is the cornerstone for the proposed amortized finite element analysis (AmorFEA).

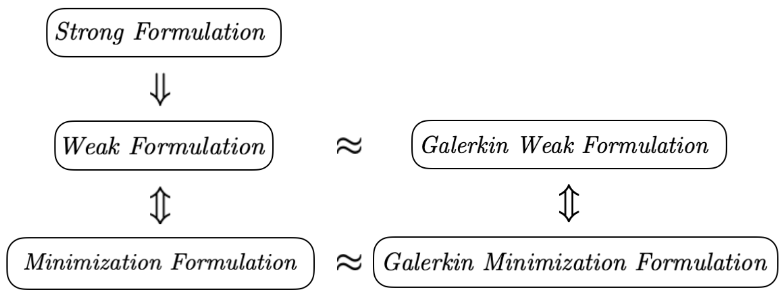

Figure 8. FEA roadmap.

The strong formulation of the Poisson's equation reads as

$$\begin{aligned} -\Delta u &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{28}$$

For simplicity, we have assumed homogenous Dirichlet boundary conditions. Instead of heuristically approximating the differential operator like the Finite Difference Method (FDM), FEA employs a more systematic approach by searching the best possible solution over a constructed finite-dimensional function space.

First, let us multiply a test function v for both sides of the PDE in Eq. 28, integrate over Ω, and use integration by parts to obtain the weak formulation: find $u \in \mathbb{V} := \{v \text{ sufficiently smooth} \mid v|_\Gamma = 0\}$ such that

$$a(u, v) = l(\lambda, v), \text{ for all } v \in \mathbb{V}, \tag{29}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad l(\lambda, v) = \int_\Omega \lambda v \, dx. \tag{30}$$

Second, we construct a finite-dimensional subspace $\mathbb{V}_h \subset \mathbb{V}$ and $\mathbb{V}_h = \text{span}\{\phi_1, ..., \phi_n\}$ is the piece-wise polynomial function space. Note that for any function $v \in \mathbb{V}_h$, there is a unique representation $v = \sum_{i=1}^n v_i \phi_i$. We thus can define an isomorphism $\mathbb{V}_h \cong \mathbb{R}^n$ by $v = \sum_{i=1}^n v_i \phi_i \leftrightarrow \boldsymbol{v} = (v_1, ..., v_n)^\top$.

The Galerkin weak formulation yields an approximation to Eq. 29 by solving the following system of linear equations:

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \tag{31}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is the stiffness matrix with $A_{ij} = a(\phi_i, \phi_j)$, $\boldsymbol{u} \in \mathbb{R}^n$ is the solution vector and $\boldsymbol{f} \in \mathbb{R}^n$ is the source vector with $f_i = l(\phi_i, \lambda)$. As a remark, if λ is also represented by piece-wise polynomial basis functions, we could further write $\boldsymbol{f} = \boldsymbol{B}\boldsymbol{\lambda}$ with $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ and $B_{ij} = l(\phi_i, \phi_j)$. It is not difficult to verify that $\boldsymbol{A}$ is symmetric and positive definite. By solving Eq. 31, the FEM gives the best possible solution we can find in $\mathbb{V}_h$.

The minimization formulation states that

$$\min_{u \in \mathbb{V}} \mathcal{L}(u), \tag{32}$$

where

$$\mathcal{L}(u) = \frac{1}{2} \int_\Omega \nabla u \cdot \nabla u \, dx - \int_\Omega \lambda u \, dx. \tag{33}$$

By setting the functional derivative to be zero, we recover the weak formulation.

Finally, we introduce the Galerkin minimization formulation, which is the discretized version of the minimization formulation above. Replace $u \in \mathbb{V}$ in Eq. 32 with $u = \sum_{i=1}^{n} u_i \phi_i \in \mathbb{V}_h$, we get the following finite-dimensional optimization problem:

$$\min_{\boldsymbol{u} \in \mathbb{R}^n} \mathcal{L}(\boldsymbol{u}), \tag{34}$$

where

$$\mathcal{L}(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u} - \boldsymbol{f}^\top \boldsymbol{u}. \tag{35}$$

The quadratic programming problem yields the same solution as Eq. 31. AmorFEA is based on the Galerkin minimization formulation. Note that for a nonlinear problem, we may have a more complicated nonlinear optimization problem rather than the quadratic programming.

For further details regarding FEA and its variational formulation, we refer readers to Brezzi & Fortin (2012).

B. Proofs

Lemma 1. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = f(\boldsymbol{x} + t\boldsymbol{y})$ is convex (as a univariate function) for all $\boldsymbol{x}$ in domain of f and all $\boldsymbol{y} \in \mathbb{R}^n$. (The domain of g here is all t for which $\boldsymbol{x} + t\boldsymbol{y}$ is in the domain of f.)*

Proof. This is straightforward from the definition of convexity of multivariable functions. □

Proposition 3. *The empirical loss function $\overline{\mathcal{L}}_a(\boldsymbol{W})$ of AmorFEA in the linear model is convex.*

Proof. We use Lemma 1 to show the function $\overline{\mathcal{L}}_a : \mathbb{R}^{n\times n} \to \mathbb{R}$ is convex. For any fixed $\boldsymbol{W}_1 \in \mathbb{R}^{n\times n}$ and $\boldsymbol{W}_2 \in \mathbb{R}^{n\times n}$, let

$$\begin{aligned} g(t) =& \overline{\mathcal{L}}_a(\boldsymbol{W}_1 + t\boldsymbol{W}_2) \\ =& \frac{1}{K}\sum_{k=1}^{K}\Big(\frac{1}{2}\big((\boldsymbol{W}_1 + t\boldsymbol{W}_2)\boldsymbol{f}_k\big)^\top \boldsymbol{A}\big((\boldsymbol{W}_1 + t\boldsymbol{W}_2)\boldsymbol{f}_k\big) - \boldsymbol{f}_k^\top\big((\boldsymbol{W}_1 + t\boldsymbol{W}_2)\boldsymbol{f}_k\big)\Big) \\ =& \Big(\frac{1}{2K}\sum_{k=1}^{K}\boldsymbol{f}_k^\top \boldsymbol{W}_2^\top \boldsymbol{A}\boldsymbol{W}_2\boldsymbol{f}_k\Big)t^2 + \Big(\frac{1}{2K}\sum_{k=1}^{K}\big(\boldsymbol{f}_k^\top \boldsymbol{W}_2^\top \boldsymbol{A}\boldsymbol{W}_1\boldsymbol{f}_k + \boldsymbol{f}_k^\top \boldsymbol{W}_1^\top \boldsymbol{A}\boldsymbol{W}_2\boldsymbol{f}_k - 2\boldsymbol{f}_k^\top \boldsymbol{W}_2\boldsymbol{f}_k\big)\Big)t \\ &- \Big(\frac{1}{2K}\sum_{k=1}^{K}\boldsymbol{f}_k^\top \boldsymbol{W}_1\boldsymbol{f}_k\Big)\Big) \end{aligned} \tag{36}$$

Since $\boldsymbol{A}$ is symmetric and positive definite, the coefficient of the quadratic term is always positive:

$$\frac{1}{2K}\sum_{k=1}^{K}\boldsymbol{f}_k^\top \boldsymbol{W}_2^\top \boldsymbol{A}\boldsymbol{W}_2\boldsymbol{f}_k = \frac{1}{2K}\sum_{k=1}^{K}\|\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{f}_k\|_2^2 \geq 0. \tag{37}$$

This shows that $g(t)$ is always convex. □

Proposition 4. *The empirical loss function $\overline{\mathcal{L}}_s(\boldsymbol{W})$ of supervised learning in the linear model is convex.*

Proof. We use Lemma 1 to show the function $\overline{\mathcal{L}}_s : \mathbb{R}^{n\times n} \to \mathbb{R}$ is convex. For any fixed $\boldsymbol{W}_1 \in \mathbb{R}^{n\times n}$ and $\boldsymbol{W}_2 \in \mathbb{R}^{n\times n}$, let

$$\begin{aligned} g(t) =& \overline{\mathcal{L}}_s(\boldsymbol{W}_1 + t\boldsymbol{W}_2) \\ =& \frac{1}{K}\sum_{k=1}^{K}\Big(\frac{1}{2}\big((\boldsymbol{W}_1+t\boldsymbol{W}_2)\boldsymbol{f}_k - \boldsymbol{A}^{-1}\boldsymbol{f}_k\big)^\top\big((\boldsymbol{W}_1+t\boldsymbol{W}_2)\boldsymbol{f}_k - \boldsymbol{A}^{-1}\boldsymbol{f}_k\big)\Big) \\ =& \Big(\frac{1}{2K}\sum_{k=1}^{K}\boldsymbol{f}_k^\top\boldsymbol{W}_2^\top\boldsymbol{W}_2\boldsymbol{f}_k\Big)t^2 - \Big(\frac{1}{2K}\sum_{k=1}^{K}\big(\boldsymbol{f}_k^\top(\boldsymbol{W}_1-\boldsymbol{A}^{-1})^\top\boldsymbol{W}_2\boldsymbol{f}_k + \boldsymbol{f}_k^\top\boldsymbol{W}_2^\top(\boldsymbol{W}_1-\boldsymbol{A}^{-1})\boldsymbol{f}_k\big)\Big)t \\ &+ \Big(\frac{1}{2K}\sum_{k=1}^{K}\boldsymbol{f}_k^\top(\boldsymbol{W}_1-\boldsymbol{A}^{-1})^\top(\boldsymbol{W}_1-\boldsymbol{A}^{-1})\boldsymbol{f}_k\Big) \end{aligned} \tag{38}$$

The coefficient of the quadratic term is always positive:

$$\frac{1}{2K}\sum_{k=1}^{K}\boldsymbol{f}_k^\top\boldsymbol{W}_2^\top\boldsymbol{W}_2\boldsymbol{f}_k = \frac{1}{2K}\sum_{k=1}^{K}\|\boldsymbol{W}_2\boldsymbol{f}_k\|_2^2 \geq 0. \tag{39}$$

This shows that $g(t)$ is always convex. □

C. Detailed formulations

C.1. Source Field Finding

The potential energy is

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\int_\Omega \|\nabla u\|_2^2 dx + \frac{1}{2}\int_\Omega (10u^2 + 5u^4)dx - \int_\Omega \lambda u dx. \tag{40}$$

The governing PDE (Eq. 23) can be derived by minimizing Eq. 40.

We set $\lambda(x) = 100\exp(\frac{\|x-(0.1,0.1)\|_2^2}{0.02})$, solve the governing PDE by FEA and set the solution to be the desired field $u_d(x)$ for Eq. 22.

C.2. Inverse Kinematics of a Soft Robot

We model the soft robot with a neo-Hookean hyperelastic solid (Ogden, 1997). The total potential energy is

$$\mathcal{L}(u,\lambda) = \int_\Omega W dx, \tag{41}$$

where the energy density W is defined for material bulk and shear moduli μ and κ as:

$$W = \frac{\mu}{2}\Big((\det F)^{-2/3}\text{tr}(FF^T) - 3\Big) + \frac{\kappa}{2}(\det F - 1)^2, \tag{42}$$

and F is the deformation gradient,

$$F = \nabla u + I. \tag{43}$$

Since our problem is in 2d, we have assumed plain strain condition. The governing PDE (Eq. 27) can be derived by minimization Eq. 41. The stress tensor P in Eq. 27 is known as the first Piola-Kirchoff stress and can be obtained by

$$P = \frac{\partial W}{\partial F}. \tag{44}$$

The boundary constraint $r(u,\lambda)$ in Eq. 27 is expressed over the bottom side boundary Γ_b, top side boundary Γ_t and the side boundaries Γ_s respectively as

$$u = 0 \quad \text{on } \Gamma_b, \tag{45}$$

$$P\cdot N = 0 \quad \text{on } \Gamma_t, \tag{46}$$

$$\Lambda = h(\lambda) \quad \text{on } \Gamma_s, \tag{47}$$

where N is the normal vector to the boundary, Λ is the stretch ratio (see definition in (Holzapfel, 2000)) and $h(\lambda) = \frac{1}{1+e^{-\lambda}} + \frac{1}{2}$ forces the range of Λ to $(0.5, 1.5)$ in order to avoid extreme and unrealistic deformations.

References

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Holzapfel, A. G. Nonlinear solid mechanics ii. 2000.

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