
Model-free Reinforcement Learning in Infinite-horizon Average-reward Markov Decision Processes

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Abstract

Model-free reinforcement learning is known to be memory and computation efficient and more amendable to large scale problems. In this paper, two model-free algorithms are introduced for learning *infinite-horizon average-reward* Markov Decision Processes (MDPs). The first algorithm reduces the problem to the discounted-reward version and achieves $\mathcal{O}(T^{2/3})$ regret after T steps, under the minimal assumption of weakly communicating MDPs. To our knowledge, this is the first model-free algorithm for general MDPs in this setting. The second algorithm makes use of recent advances in adaptive algorithms for adversarial multi-armed bandits and improves the regret to $\mathcal{O}(\sqrt{T})$, albeit with a stronger ergodic assumption. This result significantly improves over the $\mathcal{O}(T^{3/4})$ regret achieved by the only existing model-free algorithm by Abbasi-Yadkori et al. (2019a) for ergodic MDPs in the infinite-horizon average-reward setting.

1. Introduction

Reinforcement learning (RL) refers to the problem of an agent interacting with an unknown environment with the goal of maximizing its cumulative reward through time. The environment is usually modeled as a Markov Decision Process (MDP) with an unknown transition kernel and/or an unknown reward function. The fundamental trade-off between exploration and exploitation is the key challenge for RL: should the agent exploit the available information to optimize the immediate performance, or should it explore the poorly understood states and actions to gather more information to improve future performance?

There are two broad classes of RL algorithms: *model-based*

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and *model-free*. Model-based algorithms maintain an estimate of the underlying MDP and use that to determine a policy during the learning process. Examples include UCRL2 (Jaksch et al., 2010), REGAL (Bartlett & Tewari, 2009), PSRL (Ouyang et al., 2017b), SCAL (Fruit et al., 2018b), UCBVI (Azar et al., 2017), EBF (Zhang & Ji, 2019) and EULER (Zanette & Brunskill, 2019). Model-based algorithms are well-known for their sample efficiency. However, there are two general disadvantages of model-based algorithms: First, model-based algorithms require large memory to store the estimate of the model parameters. Second, it is hard to extend model-based approaches to non-parametric settings, e.g., continuous state MDPs.

Model-free algorithms, on the other hand, try to resolve these issues by directly maintaining an estimate of the optimal Q-value function or the optimal policy. Examples include Q-learning (Watkins, 1989), Delayed Q-learning (Strehl et al., 2006), TRPO (Schulman et al., 2015), DQN (Mnih et al., 2013), A3C (Mnih et al., 2016), and more. Model-free algorithms are not only computation and memory efficient, but also easier to be extended to large scale problems by incorporating function approximation.

It was believed that model-free algorithms are less sample-efficient compared to model-based algorithms. However, recently Jin et al. (2018) showed that (model-free) Q-learning algorithm with UCB exploration achieves a nearly-optimal regret bound, implying the possibility of designing algorithms with advantages of both model-free and model-based methods. Jin et al. (2018) addressed the problem for episodic finite-horizon MDPs. Following this work, Dong et al. (2019) extended the result to the infinite-horizon discounted-reward setting.

However, Q-learning based model-free algorithms with low regret for *infinite-horizon average-reward* MDPs, an equally heavily-studied setting in the RL literature, remains unknown. Designing such algorithms has proven to be rather challenging since the Q-value function estimate may grow unbounded over time and it is hard to control its magnitude in a way that guarantees efficient learning. Moreover, techniques such as backward induction in the finite-horizon setting or contraction mapping in the infinite-horizon discounted setting can not be applied to the infinite-horizon

Table 1. Regret comparisons for RL algorithms in infinite-horizon average-reward MDPs with S states, A actions, and T steps. D is the diameter of the MDP, $\text{sp}(v^*) \leq D$ is the span of the optimal value function, $\mathbb{V}_{s,a}^* := \text{Var}_{s' \sim p(\cdot|s,a)}[v^*(s')] \leq \text{sp}(v^*)^2$ is the variance of the optimal value function, t_{mix} is the mixing time (Def 5.1), t_{hit} is the hitting time (Def 5.2), and $\rho \leq t_{\text{hit}}$ is some distribution mismatch coefficient (Eq. (5)). For more concrete definition of these parameters, see Sections 3-5.

	Algorithm	Regret	Comment
Model-based	REGAL (Bartlett & Tewari, 2009)	$\tilde{\mathcal{O}}(\text{sp}(v^*)\sqrt{SAT})$	no efficient implementation
	UCRL2 (Jaksch et al., 2010)	$\tilde{\mathcal{O}}(DS\sqrt{AT})$	-
	PSRL (Ouyang et al., 2017b)	$\tilde{\mathcal{O}}(\text{sp}(v^*)S\sqrt{AT})$	Bayesian regret
	OSP (Ortner, 2018)	$\tilde{\mathcal{O}}(\sqrt{t_{\text{mix}}SAT})$	ergodic assumption and no efficient implementation
	SCAL (Fruit et al., 2018b)	$\tilde{\mathcal{O}}(\text{sp}(v^*)S\sqrt{AT})$	-
	KL-UCRL (Talebi & Maillard, 2018)	$\tilde{\mathcal{O}}(\sqrt{S \sum_{s,a} \mathbb{V}_{s,a}^* T})$	-
	UCRL2B (Fruit et al., 2019)	$\tilde{\mathcal{O}}(S\sqrt{DAT})$	-
	EBF (Zhang & Ji, 2019)	$\tilde{\mathcal{O}}(\sqrt{DSAT})$	no efficient implementation
Model-free	POLITEX (Abbasi-Yadkori et al., 2019a)	$t_{\text{mix}}^3 t_{\text{hit}} \sqrt{SAT}^{\frac{3}{4}}$	ergodic assumption
	Optimistic Q-learning (this work)	$\tilde{\mathcal{O}}(\text{sp}(v^*)(SA)^{\frac{1}{3}}T^{\frac{2}{3}})$	-
	MDP-OOMD (this work)	$\tilde{\mathcal{O}}(\sqrt{t_{\text{mix}}^3 \rho AT})$	ergodic assumption
	lower bound (Jaksch et al., 2010)	$\Omega(\sqrt{DSAT})$	-

average-reward setting.

In this paper, we make significant progress in this direction and propose two model-free algorithms for learning infinite-horizon average-reward MDPs. The first algorithm, Optimistic Q-learning (Section 4), achieves a regret bound of $\tilde{\mathcal{O}}(T^{2/3})$ with high probability for the broad class of weakly communicating MDPs.¹ This is the first model-free algorithm in this setting under only the minimal weakly communicating assumption. The key idea of this algorithm is to artificially introduce a discount factor for the reward, to avoid the aforementioned unbounded Q-value estimate issue, and to trade-off this effect with the approximation introduced by the discount factor. We remark that this is very different from the R-learning algorithm of (Schwartz, 1993), which is a variant of Q-learning with no discount factor for the infinite-horizon average-reward setting.

The second algorithm, MDP-OOMD (Section 5), attains an improved regret bound of $\tilde{\mathcal{O}}(\sqrt{T})$ for the more restricted class of *ergodic* MDPs. This algorithm maintains an instance of a multi-armed bandit algorithm at each state to learn the best action. Importantly, the multi-armed bandit algorithm needs to ensure several key properties to achieve our claimed regret bound, and to this end we make use of the recent advances for adaptive adversarial bandit algorithms from (Wei & Luo, 2018) in a novel way.

¹Throughout the paper, we use the notation $\tilde{\mathcal{O}}(\cdot)$ to suppress log terms.

To the best of our knowledge, the only existing model-free algorithm for this setting is the POLITEX algorithm (Abbasi-Yadkori et al., 2019a;b), which achieves $\tilde{\mathcal{O}}(T^{3/4})$ regret for ergodic MDPs only. Both of our algorithms enjoy a better bound compared to POLITEX, and the first algorithm even removes the ergodic assumption completely.²

For comparisons with other existing model-based approaches for this problem, see Table 1. We also conduct experiments comparing our two algorithms. Details are deferred to Appendix D due to space constraints.

2. Related Work

We review the related literature with regret guarantees for learning MDPs with finite state and action spaces (there are many other works on asymptotic convergence or sample complexity, a different focus compared to our work). Three common settings have been studied: 1) finite-horizon episodic setting, 2) infinite-horizon discounted setting, and 3) infinite-horizon average-reward setting. For the first two settings, previous works have designed efficient algorithms with regret bound or sample complexity that is (almost) information-theoretically optimal, using either model-based approaches such as (Azar et al., 2017), or model-free approaches such as (Jin et al., 2018; Dong et al., 2019).

²POLITEX is studied in a more general setup with function approximation though. See the end of Section 5.1 for more comparisons.

For the infinite-horizon average-reward setting, many model-based algorithms have been proposed, such as (Auer & Ortner, 2007; Jaksch et al., 2010; Ouyang et al., 2017b; Agrawal & Jia, 2017; Talebi & Maillard, 2018; Fruit et al., 2018a;b). These algorithms either conduct *posterior sampling* or follow the *optimism in face of uncertainty* principle to build an MDP model estimate and then plan according to the estimate (hence model-based). They all achieve $\tilde{O}(\sqrt{T})$ regret, but the dependence on other parameters are suboptimal. Recent works made progress toward obtaining the optimal bound (Ortner, 2018; Zhang & Ji, 2019); however, their algorithms are not computationally efficient – the time complexity scales exponentially in the number of states. On the other hand, except for the naive approach of combining Q-learning with ϵ -greedy exploration (which is known to suffer regret exponential in some parameters (Osband et al., 2014)), the only existing model-free algorithm for this setting is POLITEX, which only works for ergodic MDPs.

Two additional works are closely related to our second algorithm MDP-OOMD: (Neu et al., 2013) and (Wang, 2017). They all belong to *policy optimization* method where the learner tries to learn the parameter of the optimal policy directly. Their settings are quite different from ours and the results are not comparable. We defer more detailed comparisons with these two works to the end of Section 5.1.

3. Preliminaries

An infinite-horizon average-reward Markov Decision Process (MDP) can be described by $(\mathcal{S}, \mathcal{A}, r, p)$ where \mathcal{S} is the state space, \mathcal{A} is the action space, $r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$ is the reward function and $p : \mathcal{S}^2 \times \mathcal{A} \rightarrow [0, 1]$ is the transition probability such that $p(s'|s, a) := \mathbb{P}(s_{t+1} = s' \mid s_t = s, a_t = a)$ for $s_t \in \mathcal{S}, a_t \in \mathcal{A}$ and $t = 1, 2, 3, \dots$. We assume that \mathcal{S} and \mathcal{A} are finite sets with cardinalities S and A , respectively. The average reward per stage of a deterministic/stationary policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$ starting from state s is defined as

$$J^\pi(s) := \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T r(s_t, \pi(s_t)) \mid s_1 = s \right]$$

where s_{t+1} is drawn from $p(\cdot | s_t, \pi(s_t))$. Let $J^*(s) := \max_{\pi \in \mathcal{A}^{\mathcal{S}}} J^\pi(s)$. A policy π^* is said to be optimal if it satisfies $J^{\pi^*}(s) = J^*(s)$ for all $s \in \mathcal{S}$.

We consider two standard classes of MDPs in this paper: (1) *weakly communicating* MDPs defined in Section 4 and (2) *ergodic* MDPs defined in Section 5. The weakly communicating assumption is weaker than the ergodic assumption, and is in fact known to be necessary for learning infinite-horizon MDPs with low regret (Bartlett & Tewari, 2009).

Standard MDP theory (Puterman, 2014) shows that for these two classes, there exist $q^* : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ (unique up to an

additive constant) and unique $J^* \in [0, 1]$ such that $J^*(s) = J^*$ for all $s \in \mathcal{S}$ and the following Bellman equation holds:

$$J^* + q^*(s, a) = r(s, a) + \mathbb{E}_{s' \sim p(\cdot | s, a)} [v^*(s')], \quad (1)$$

where $v^*(s) := \max_{a \in \mathcal{A}} q^*(s, a)$. The optimal policy is then obtained by $\pi^*(s) = \operatorname{argmax}_a q^*(s, a)$.

We consider a learning problem where \mathcal{S}, \mathcal{A} and the reward function r are known to the agent, but not the transition probability p (so one cannot directly solve the Bellman equation). The knowledge of the reward function is a typical assumption as in (Bartlett & Tewari, 2009; Gopalan & Mannor, 2015; Ouyang et al., 2017b), and can be removed at the expense of a constant factor for the regret bound.

Specifically, the learning protocol is as follows. An agent starts at an arbitrary state $s_1 \in \mathcal{S}$. At each time step $t = 1, 2, 3, \dots$, the agent observes state $s_t \in \mathcal{S}$ and takes action $a_t \in \mathcal{A}$ which is a function of the history $s_1, a_1, s_2, a_2, \dots, s_{t-1}, a_{t-1}, s_t$. The environment then determines the next state by drawing s_{t+1} according to $p(\cdot | s_t, a_t)$. The performance of a learning algorithm is evaluated through the notion of *cumulative regret*, defined as the difference between the total reward of the optimal policy and that of the algorithm:

$$R_T := \sum_{t=1}^T (J^* - r(s_t, a_t)).$$

Since $r \in [0, 1]$ (and subsequently $J^* \in [0, 1]$), the regret can at worst grow linearly with T . If a learning algorithm achieves sub-linear regret, then R_T/T goes to zero, i.e., the average reward of the algorithm converges to the optimal per stage reward J^* . The best existing regret bound is $\tilde{O}(\sqrt{DSAT})$ achieved by a model-based algorithm (Zhang & Ji, 2019) (where D is the diameter of the MDP) and it matches the lower bound of (Jaksch et al., 2010).

Throughout the paper, we assume that T is known. When it is unknown, one can simply apply the standard doubling trick to obtain the same regret bound up to a constant (see e.g., (Shalev-Shwartz, 2011, Section 2.3.1)).

4. Optimistic Q-Learning

In this section, we introduce our first algorithm, OPTIMISTIC Q-LEARNING (see Algorithm 1 for pseudocode). The algorithm works for any weakly communicating MDPs. An MDP is weakly communicating if its state space \mathcal{S} can be partitioned into two subsets: in the first subset, all states are transient under any stationary policy; in the second subset, every two states are accessible from each other under some stationary policy. It is well-known that the weakly communicating condition is necessary for ensuring low regret in this setting (Bartlett & Tewari, 2009).

Algorithm 1 OPTIMISTIC Q-LEARNING

Parameters: $H \geq 2$, confidence level $\delta \in (0, 1)$
Initialization: $\gamma = 1 - \frac{1}{H}$, $\forall s : \hat{V}_1(s) = H$
 $\forall s, a : Q_1(s, a) = \hat{Q}_1(s, a) = H$, $n_1(s, a) = 0$
Define: $\forall \tau, \alpha_\tau = \frac{H+1}{H+\tau}$, $b_\tau = 4 \text{sp}(v^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}}$
for $t = 1, \dots, T$ **do**

Take action

$$a_t = \operatorname{argmax}_{a \in \mathcal{A}} \hat{Q}_t(s_t, a). \quad (2)$$

 Observe s_{t+1} .

Update:

$$\begin{aligned} n_{t+1}(s_t, a_t) &\leftarrow n_t(s_t, a_t) + 1 \\ \tau &\leftarrow n_{t+1}(s_t, a_t) \\ Q_{t+1}(s_t, a_t) &\leftarrow (1 - \alpha_\tau)Q_t(s_t, a_t) \\ &\quad + \alpha_\tau \left[r(s_t, a_t) + \gamma \hat{V}_t(s_{t+1}) + b_\tau \right] \\ \hat{Q}_{t+1}(s_t, a_t) &\leftarrow \min \left\{ \hat{Q}_t(s_t, a_t), Q_{t+1}(s_t, a_t) \right\} \\ \hat{V}_{t+1}(s_t) &\leftarrow \max_{a \in \mathcal{A}} \hat{Q}_{t+1}(s_t, a). \end{aligned} \quad (3)$$

 (All other entries of $n_{t+1}, Q_{t+1}, \hat{Q}_{t+1}, \hat{V}_{t+1}$ remain the same as those in $n_t, Q_t, \hat{Q}_t, \hat{V}_t$.)

Define $\text{sp}(v^*) = \max_s v^*(s) - \min_s v^*(s)$ to be the span of the value function, which is known to be bounded for weakly communicating MDPs. In particular, it is bounded by the diameter of the MDP (see (Lattimore & Szepesvári, 2018, Lemma 38.1)). We assume that $\text{sp}(v^*)$ is known and use it to set the parameters. However, in the case when it is unknown, we can replace $\text{sp}(v^*)$ with any upper bound of it (e.g. the diameter) in both the algorithm and the analysis.

The key idea of Algorithm 1 is to solve the undiscounted problem via learning a discounted MDP (with the same states, actions, reward function, and transition), for some discount factor γ (defined in terms of a parameter H). Define V^* and Q^* to be the optimal value-function and Q-function of the discounted MDP, satisfying the Bellman equation:

$$\begin{aligned} \forall (s, a), \quad Q^*(s, a) &= r(s, a) + \gamma \mathbb{E}_{s' \sim p(\cdot | s, a)} [V^*(s')] \\ \forall s, \quad V^*(s) &= \max_{a \in \mathcal{A}} Q^*(s, a). \end{aligned}$$

The way we learn this discounted MDP is essentially the same as the algorithm of Dong et al. (2019), which itself is based on the idea from (Jin et al., 2018). Specifically, the algorithm maintains an estimate \hat{V}_t for the optimal value function V^* and \hat{Q}_t for the optimal Q-function Q^* , which itself is a clipped version of another estimate Q_t . Each time the algorithm takes a greedy action with the maximum estimated Q value (Eq. (2)). After seeing the next state, the

algorithm makes a stochastic update of Q_t based on the Bellman equation, importantly with an extra *bonus* term b_τ and a carefully chosen step size α_τ (Eq.(3)). Here, τ is the number of times the current state-action pair has been visited, and the bonus term b_τ scales as $\mathcal{O}(\sqrt{H/\tau})$, which encourages exploration since it shrinks every time a state-action pair is executed. The choice of the step size α_τ is also crucial as pointed out in (Jin et al., 2018) and determines a certain effective period of the history for the current update.

While the algorithmic idea is similar to (Dong et al., 2019), we emphasize that our analysis is different and novel:

- First, Dong et al. (2019) analyze the sample complexity of their algorithm while we analyze the regret.
- Second, we need to deal with the approximation effect due to the difference between the discounted MDP and the original undiscounted one (Lemma 2).
- Finally, part of our analysis improves over that of (Dong et al., 2019) (specifically our Lemma 3). Following the original analysis of (Dong et al., 2019) would lead to a worse bound here.

We now state the main regret guarantee of Algorithm 1.

Theorem 1. *If the MDP is weakly communicating, Algorithm 1 with $H = \min \left\{ \sqrt{\frac{\text{sp}(v^*)T}{SA}}, \left(\frac{T}{SA \ln \frac{4T}{\delta}} \right)^{\frac{1}{3}} \right\}$ ensures that with probability at least $1 - \delta$, R_T is of order*

$$\mathcal{O} \left(\sqrt{\text{sp}(v^*)SAT} + \text{sp}(v^*) \left(T^{\frac{2}{3}} \left(SA \ln \frac{T}{\delta} \right)^{\frac{1}{3}} + \sqrt{T \ln \frac{1}{\delta}} \right) \right).$$

Our regret bound scales as $\tilde{\mathcal{O}}(T^{2/3})$ and is suboptimal compared to model-based approaches with $\tilde{\mathcal{O}}(\sqrt{T})$ regret (such as UCRL2) that matches the information-theoretic lower bound (Jaksch et al., 2010). However, this is the first model-free algorithm with sub-linear regret (under only the weakly communicating condition), and how to achieve $\tilde{\mathcal{O}}(\sqrt{T})$ regret via model-free algorithms remains unknown. Also note that our bound depends on $\text{sp}(v^*)$ instead of the potentially much larger diameter of the MDP. To our knowledge, existing approaches that achieve $\text{sp}(v^*)$ dependence are all model-based (Bartlett & Tewari, 2009; Ouyang et al., 2017b; Fruit et al., 2018b) and use very different arguments.

4.1. Proof sketch of Theorem 1

The proof starts by decomposing the regret as

$$\begin{aligned} R_T &= \sum_{t=1}^T (J^* - r(s_t, a_t)) \\ &= \sum_{t=1}^T (J^* - (1 - \gamma)V^*(s_t)) \\ &\quad + \sum_{t=1}^T (V^*(s_t) - Q^*(s_t, a_t)) \\ &\quad + \sum_{t=1}^T (Q^*(s_t, a_t) - \gamma V^*(s_t) - r(s_t, a_t)). \end{aligned}$$

Each of these three terms are handled through Lemmas 2, 3 and 4 whose proofs are deferred to the appendix. Plugging in $\gamma = 1 - \frac{1}{H}$ and picking the optimal H finish the proof. One can see that the $\tilde{O}(T^{2/3})$ regret comes from the bound $\frac{T}{H}$ from the first term and the bound \sqrt{HT} from the second.

Lemma 2. *The optimal value function V^* of the discounted MDP satisfies*

1. $|J^* - (1 - \gamma)V^*(s)| \leq (1 - \gamma) \text{sp}(v^*), \forall s \in \mathcal{S}$,
2. $\text{sp}(V^*) \leq 2 \text{sp}(v^*)$.

This lemma shows that the difference between the optimal value in the discounted setting (scaled by $1 - \gamma$) and that of the undiscounted setting is small as long as γ is close to 1. The proof is by combining the Bellman equation of the these two settings and direct calculations.

Lemma 3. *With probability at least $1 - \delta$, we have*

$$\begin{aligned} &\sum_{t=1}^T (V^*(s_t) - Q^*(s_t, a_t)) \\ &\leq 4HSA + 24 \text{sp}(v^*) \sqrt{HSAT \ln \frac{2T}{\delta}}. \end{aligned}$$

This lemma is one of our key technical contributions. To prove this lemma one can write

$$\begin{aligned} &\sum_{t=1}^T (V^*(s_t) - Q^*(s_t, a_t)) \\ &= \sum_{t=1}^T (V^*(s_t) - \hat{V}_t(s_t)) + \sum_{t=1}^T (\hat{Q}_t(s_t, a_t) - Q^*(s_t, a_t)), \end{aligned}$$

using the fact that $\hat{V}_t(s_t) = \hat{Q}_t(s_t, a_t)$ by the greedy policy. The main part of the proof is to show that the second summation can in fact be bounded as $\sum_{t=2}^{T+1} (\hat{V}_t(s_t) - V^*(s_t))$ plus a small sub-linear term, which cancels with the first summation.

Lemma 4. *With probability at least $1 - \delta$,*

$$\begin{aligned} &\sum_{t=1}^T (Q^*(s_t, a_t) - \gamma V^*(s_t) - r(s_t, a_t)) \\ &\leq 2 \text{sp}(v^*) \sqrt{2T \ln \frac{1}{\delta}} + 2 \text{sp}(v^*). \end{aligned}$$

This lemma is proven via Bellman equation for the discounted setting and Azuma's inequality.

5. $\tilde{O}(\sqrt{T})$ Regret for Ergodic MDPs

In this section, we propose another model-free algorithm that achieves $\tilde{O}(\sqrt{T})$ regret bound for *ergodic* MDPs, a sub-class of weakly communicating MDPs. An MDP is ergodic if for any stationary policy π , the induced Markov chain is *irreducible* and *aperiodic*. Learning ergodic MDPs is arguably easier than the general case because the MDP is explorative by itself. However, achieving $\tilde{O}(\sqrt{T})$ regret bound in this case with model-free methods is still highly non-trivial and we are not aware of any such result in the literature. Below, we first introduce a few useful properties of ergodic MDPs, all of which can be found in (Puterman, 2014).

We use *randomized policies* in this approach. A randomized policy π maps every state s to a distribution over actions $\pi(\cdot|s) \in \Delta_A$, where $\Delta_A = \{x \in \mathbb{R}_+^A : \sum_a x(a) = 1\}$. In an ergodic MDP, any policy π induces a Markov chain with a unique stationary distribution $\mu^\pi \in \Delta_S$ satisfying $(\mu^\pi)^\top P^\pi = (\mu^\pi)^\top$, where $P^\pi \in \mathbb{R}^{S \times S}$ is the induced transition matrix defined as $P^\pi(s, s') = \sum_a \pi(a|s)p(s'|s, a)$. We denote the stationary distribution of the optimal policy π^* by μ^* .

For ergodic MDPs, the long-term average reward J^π of any fixed policy π is independent of the starting state and can be written as $J^\pi = (\mu^\pi)^\top r^\pi$ where $r^\pi \in [0, 1]^S$ is such that $r^\pi(s) := \sum_a \pi(a|s)r(s, a)$. For any policy π , the following Bellman equation has a solution $q^\pi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ that is unique up to an additive constant:

$$J^\pi + q^\pi(s, a) = r(s, a) + \mathbb{E}_{s' \sim p(\cdot|s, a)}[v^\pi(s')],$$

where $v^\pi(s) = \sum_a \pi(a|s)q^\pi(s, a)$. In this section, we impose an extra constraint: $\sum_s \mu^\pi(s)v^\pi(s) = 0$ so that q^π is indeed unique. In this case, it can be shown that v^π has the following form:

$$v^\pi(s) = \sum_{t=0}^{\infty} (\mathbf{e}_s^\top (P^\pi)^t - (\mu^\pi)^\top) r^\pi \quad (4)$$

where \mathbf{e}_s is the basis vector with 1 in coordinate s .

Furthermore, ergodic MDPs have finite *mixing time* and *hitting time*, defined as follows.

Algorithm 2 MDP-OOMD

Define: episode length $B = 16t_{\text{mix}}t_{\text{hit}}(\log_2 T)^2$ and number of episodes $K = T/B$

Initialize: $\pi'_1(a|s) = \pi_1(a|s) = \frac{1}{A}, \forall s, a.$

for $k = 1, 2, \dots, K$ **do**

for $t = (k-1)B + 1, \dots, kB$ **do**

 Draw $a_t \sim \pi_k(\cdot|s_t)$ and observe s_{t+1} .

 Define trajectory $\mathcal{T}_k =$

$$(s_{(k-1)B+1}, a_{(k-1)B+1}, \dots, s_{kB}, a_{kB}).$$

for all $s \in \mathcal{S}$ **do**

$\hat{\beta}_k(s, \cdot) = \text{ESTIMATEQ}(\mathcal{T}_k, \pi_k, s).$

$(\pi'_{k+1}(\cdot|s), \pi_{k+1}(\cdot|s)) =$

 OOMDUPDATE($\pi'_k(\cdot|s), \hat{\beta}_k(s, \cdot)$).

Definition 5.1 ((Levin & Peres, 2017; Wang, 2017)). *The mixing time of an ergodic MDP is defined as*

$$t_{\text{mix}} := \max_{\pi} \min \left\{ t \geq 1 \mid \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq \frac{1}{4}, \forall s \right\},$$

that is, the maximum time required for any policy starting at any initial state to make the state distribution $\frac{1}{4}$ -close (in ℓ_1 norm) to the stationary distribution.

Definition 5.2. *The hitting time of an ergodic MDP is defined as*

$$t_{\text{hit}} := \max_{\pi} \max_s \frac{1}{\mu^\pi(s)},$$

that is, the maximum inverse stationary probability of visiting any state under any policy.

Our regret bound also depends on the following *distribution mismatch coefficient*:

$$\rho := \max_{\pi} \sum_s \frac{\mu^*(s)}{\mu^\pi(s)} \quad (5)$$

which has been used in previous work (Kakade & Langford, 2002; Agarwal et al., 2019). Clearly, one has $\rho \leq t_{\text{hit}} \sum_s \mu^*(s) = t_{\text{hit}}$. Note that these quantities are all parameters of the MDP only and are considered as finite constants compared to the horizon T . We thus assume that T is large enough so that t_{mix} and t_{hit} are both smaller than $T/4$. Also, we assume that these quantities are known to the algorithm.

5.1. Policy Optimization via Optimistic OMD

The key to get $\tilde{\mathcal{O}}(\sqrt{T})$ bound is to learn the optimal policy π^* directly, by reducing the problem to solving an adversarial multi-armed bandit (MAB) (Auer et al., 2002) instance at each individual state.

The details of our algorithm MDP-OOMD is shown in Algorithm 2. It proceeds in episodes, and maintains an

Algorithm 3 ESTIMATEQ

Input: \mathcal{T}, π, s

\mathcal{T} : a state-action trajectory from t_1 to t_2

$$(s_{t_1}, a_{t_1}, \dots, s_{t_2}, a_{t_2})$$

π : a policy used to sample the trajectory \mathcal{T}

s : target state

Define: $N = 4t_{\text{mix}} \log_2 T$ (window length minus 1)

Initialize: $\tau \leftarrow t_1, i \leftarrow 0$

1 while $\tau \leq t_2 - N$ **do**

2 if $s_\tau = s$ **then**

3 $i \leftarrow i + 1$

4 Let $R = \sum_{t=\tau}^{\tau+N} r(s_t, a_t).$

5 Let $y_i(a) = \frac{R}{\pi(a|s)} \mathbf{1}[a_\tau = a], \forall a. \quad (y_i \in \mathbb{R}^A)$

6 $\tau \leftarrow \tau + 2N$

7 else

$\tau \leftarrow \tau + 1$

8 if $i \neq 0$ **then**

return $\frac{1}{i} \sum_{j=1}^i y_j.$

9 else

return 0.

Algorithm 4 OOMDUPDATE

Input: $\pi' \in \Delta_A, \hat{\beta} \in \mathbb{R}^A$

Define:

Regularizer $\psi(x) = \frac{1}{\eta} \sum_{a=1}^A \log \frac{1}{x(a)},$ for $x \in \mathbb{R}_+^A$

Bregman divergence associated with ψ :

$$D_\psi(x, x') = \psi(x) - \psi(x') - \langle \nabla \psi(x'), x - x' \rangle$$

Update:

$$\pi'_{\text{next}} = \operatorname{argmax}_{\pi \in \Delta_A} \left\{ \langle \pi, \hat{\beta} \rangle - D_\psi(\pi, \pi') \right\} \quad (6)$$

$$\pi_{\text{next}} = \operatorname{argmax}_{\pi \in \Delta_A} \left\{ \langle \pi, \hat{\beta} \rangle - D_\psi(\pi, \pi'_{\text{next}}) \right\} \quad (7)$$

return $(\pi'_{\text{next}}, \pi_{\text{next}}).$

independent copy of a specific MAB algorithm for each state. At the beginning of episode k , each MAB algorithm outputs an action distribution $\pi_k(\cdot|s)$ for the corresponding state s , which together induces a policy π_k . The learner then executes policy π_k throughout episode k . At the end of the episode, for every state s we feed a reward estimator $\hat{\beta}_k(s, \cdot) \in \mathbb{R}^A$ to the corresponding MAB algorithm, where $\hat{\beta}_k$ is constructed using the samples collected in episode

k (see Algorithm 3). Finally all MAB algorithms update their distributions and output π_{k+1} for the next episode (Algorithm 4).

The reward estimator $\widehat{\beta}_k(s, \cdot)$ is an almost unbiased estimator for

$$\beta^{\pi_k}(s, \cdot) := q^{\pi_k}(s, \cdot) + NJ^{\pi_k} \quad (8)$$

with negligible bias (N is defined in Algorithm 3). The term NJ^{π_k} is the same for all actions and thus the corresponding MAB algorithm is trying to learn the best action at state s in terms of the average of Q-value functions $q^{\pi_1}(s, \cdot), \dots, q^{\pi_K}(s, \cdot)$. To construct the reward estimator for state s , the sub-routine ESTIMATEQ collects non-overlapping intervals of length $N + 1 = \widetilde{O}(t_{\text{mix}})$ that start from state s , and use the standard inverse-propensity scoring to construct an estimator y_i for interval i (Line 5). In fact, to reduce the correlation among the non-overlapping intervals, we also make sure that these intervals are at least N steps apart from each other (Line 6). The final estimator $\widehat{\beta}_k(s, \cdot)$ is simply the average of all estimators y_i over these disjoint intervals. This averaging is important for reducing variance as explained later (see also Lemma 6).

The MAB algorithm we use is *optimistic online mirror descent* (OOMD) (Rakhlin & Sridharan, 2013) with *log-barrier* as the regularizer, analyzed in depth in (Wei & Luo, 2018). Here, optimism refers to something different from the optimistic exploration in Section 4. It corresponds to the fact that after a standard mirror descent update (Eq. (6)), the algorithm further makes a similar update using an optimistic prediction of the next reward vector, which in our case is simply the previous reward estimator (Eq. (7)). We refer the reader to (Wei & Luo, 2018) for more details, but point out that the optimistic prediction we use here is new.

It is clear that each MAB algorithm faces a non-stochastic problem (since π_k is changing over time) and thus it is important to deploy an adversarial MAB algorithm. The standard algorithm for adversarial MAB is EXP3 (Auer et al., 2002), which was also used for solving adversarial MDPs (Neu et al., 2013) (more comparisons with this to follow). However, there are several important reasons for our choice of the recently developed OOMD with log-barrier:

- First, the log-barrier regularizer produces a more exploratory distribution compared to EXP3 (as noticed in e.g. (Agarwal et al., 2017)), so we do not need an explicit exploration over the actions, which significantly simplifies the analysis compared to (Neu et al., 2013).
- Second, log-barrier regularizer provides more *stable* updates compared to EXP3 in the sense that $\pi_k(a|s)$ and $\pi_{k-1}(a|s)$ are within a multiplicative factor of each other (see Lemma 7). This implies that the corresponding policies and their Q-value functions are also stable, which is critical for our analysis.
- Finally, the optimistic prediction of OOMD, together with our particular reward estimator from ESTIMATEQ, provides a variance reduction effect that leads to a better regret bound in terms of ρ instead of t_{hit} . See Lemma 8 and Lemma 9.

The regret guarantee of our algorithm is shown below.

Theorem 5. *For ergodic MDPs, with an appropriate chosen learning rate η for Algorithm 4, MDP-OOMD achieves*

$$\mathbb{E}[R_T] = \widetilde{O}\left(\sqrt{t_{\text{mix}}^3 \rho AT}\right).$$

Note that in this bound, the dependence on the number of states S is hidden in ρ , since $\rho \geq \sum_s \frac{\mu^*(s)}{\mu^\pi(s)} = S$. Compared to the bound of Algorithm 1 or some other model-based algorithms such as UCRL2, this bound has an extra dependence on t_{mix} , a potentially large constant. As far as we know, all existing mirror-descent-based algorithms for the average-reward setting has the same issue (such as (Neu et al., 2013; Wang, 2017; Abbasi-Yadkori et al., 2019a)). The role of t_{mix} in our analysis is almost the same as that of $1/(1-\gamma)$ in the discounted setting (γ is the discount factor). Specifically, a small t_{mix} ensures 1) a short trajectory needed to approximate the Q-function with expected trajectory reward (in view of Eq. (12)) and 2) an upper bound for the magnitude of $q(s, a)$ and $v(s)$ (Lemma 14). For the discounted setting these are ensured by the discount factor already.

Comparisons. Neu et al. (2013) considered learning ergodic MDPs with *known* transition kernel and *adversarial* rewards, a setting incomparable to ours. Their algorithm maintains a copy of EXP3 for each state, but the reward estimators fed to these algorithms are constructed using the knowledge of the transition kernel and are very different from ours. They proved a regret bound of order $\widetilde{O}\left(\sqrt{t_{\text{mix}}^3 t_{\text{hit}} AT}\right)$, which is worse than ours since $\rho \leq t_{\text{hit}}$.

In another recent work, (Wang, 2017) considered learning ergodic MDPs under the assumption that the learner is provided with a generative model (an oracle that takes in a state-action pair and output a sample of the next state). They derived a sample-complexity bound of order $\widetilde{O}\left(\frac{t_{\text{mix}}^2 \tau^2 SA}{\epsilon^2}\right)$ for finding an ϵ -optimal policy, where $\tau = \max\left\{\max_s \left(\frac{\mu^*(s)}{1/S}\right)^2, \max_{s', \pi} \left(\frac{1/S}{\mu^\pi(s')}\right)^2\right\}$, which is at least $\max_\pi \max_{s, s'} \frac{\mu^*(s)}{\mu^\pi(s')}$ by AM-GM inequality. This result is again incomparable to ours, but we point out that our distribution mismatch coefficient ρ is always bounded by τS , while τ can be much larger than ρ on the other hand.

Finally, Abbasi-Yadkori et al. (2019a) considers a more general setting with function approximation, and their algorithm POLITEX maintains a copy of the standard exponential

weight algorithm for each state, very similar to (Neu et al., 2013). When specified to our tabular setting, one can verify (according to their Theorem 5.2) that POLITEX achieves $t_{\text{mix}}^3 t_{\text{hit}} \sqrt{SAT}^{\frac{3}{4}}$ regret, which is significantly worse than ours in terms of all parameters.

5.2. Proof sketch of Theorem 5

We first decompose the regret as follows:

$$\begin{aligned} R_T &= \sum_{t=1}^T J^* - r(s_t, a_t) \\ &= B \sum_{k=1}^K (J^* - J^{\pi_k}) + \sum_{k=1}^K \sum_{t \in \mathcal{I}_k} (J^{\pi_k} - r(s_t, a_t)), \quad (9) \end{aligned}$$

where $\mathcal{I}_k := \{(k-1)B + 1, \dots, kB\}$ is the set of time steps for episode k . Using the *reward difference lemma* (Lemma 15 in the appendix), the first term of Eq. (9) can be written as

$$B \sum_s \mu^*(s) \left[\sum_{k=1}^K \sum_a (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right],$$

where the term in the square bracket can be recognized as exactly the regret of the MAB algorithm for state s and is analyzed in Lemma 8 of Section 5.3. Combining the regret of all MAB algorithms, Lemma 9 then shows that in expectation the first term of Eq. (9) is at most

$$\tilde{\mathcal{O}} \left(\frac{BA}{\eta} + \frac{\eta TN^3 \rho}{B} + \eta^3 TN^6 \right). \quad (10)$$

On the other hand, the expectation of the second term in

Eq.(9) can be further written as

$$\begin{aligned} & \mathbb{E} \left[\sum_{k=1}^K \sum_{t \in \mathcal{I}_k} (J^{\pi_k} - r(s_t, a_t)) \right] \\ &= \mathbb{E} \left[\sum_{k=1}^K \sum_{t \in \mathcal{I}_k} (\mathbb{E}_{s' \sim p(\cdot | s_t, a_t)} [v^{\pi_k}(s')] - q^{\pi_k}(s_t, a_t)) \right] \\ & \hspace{15em} \text{(Bellman equation)} \\ &= \mathbb{E} \left[\sum_{k=1}^K \sum_{t \in \mathcal{I}_k} (\mathbb{E}_{s' \sim p(\cdot | s_t, a_t)} [v^{\pi_k}(s')] - v^{\pi_k}(s_{t+1})) \right] \\ & \quad + \mathbb{E} \left[\sum_{k=1}^K \sum_{t \in \mathcal{I}_k} (v^{\pi_k}(s_t) - q^{\pi_k}(s_t, a_t)) \right] \\ & \quad + \mathbb{E} \left[\sum_{k=1}^K \sum_{t \in \mathcal{I}_k} (v^{\pi_k}(s_{t+1}) - v^{\pi_k}(s_t)) \right] \\ &= \mathbb{E} \left[\sum_{k=1}^K (v^{\pi_k}(s_{kB+1}) - v^{\pi_k}(s_{(k-1)B+1})) \right] \\ & \hspace{15em} \text{(the first two terms above are zero)} \\ &= \mathbb{E} \left[\sum_{k=1}^{K-1} (v^{\pi_k}(s_{kB+1}) - v^{\pi_{k+1}}(s_{kB+1})) \right] \\ & \quad + \mathbb{E} [v^{\pi_K}(s_{KB+1}) - v^{\pi_1}(s_1)]. \quad (11) \end{aligned}$$

The first term in the last expression can be bounded by $\mathcal{O}(\eta N^3 K) = \mathcal{O}(\eta N^3 T/B)$ due to the stability of OOMUPDATE (Lemma 7) and the second term is at most $\mathcal{O}(t_{\text{mix}})$ according to Lemma 14 in the appendix.

Combining these facts with $N = \tilde{\mathcal{O}}(t_{\text{mix}})$, $B = \tilde{\mathcal{O}}(t_{\text{mix}} t_{\text{hit}})$, Eq. (9) and Eq. (10) and choosing the optimal η , we arrive at

$$\begin{aligned} \mathbb{E}[R_T] &= \tilde{\mathcal{O}} \left(\frac{BA}{\eta} + \eta \frac{t_{\text{mix}}^3 \rho T}{B} + \eta^3 t_{\text{mix}}^6 T \right) \\ &= \tilde{\mathcal{O}} \left(\sqrt{t_{\text{mix}}^3 \rho AT} + (t_{\text{mix}}^3 t_{\text{hit}} A)^{\frac{3}{4}} T^{\frac{1}{4}} + t_{\text{mix}}^2 t_{\text{hit}} A \right). \end{aligned}$$

5.3. Auxiliary Lemmas

To analyze the regret, we establish several useful lemmas, whose proofs can be found in the Appendix. First, we show that $\hat{\beta}_k(s, a)$ is an almost unbiased estimator for $\beta^{\pi_k}(s, a)$.

Lemma 6. *Let $\mathbb{E}_k[x]$ denote the expectation of a random variable x conditioned on all history before episode k . Then for any k, s, a (recall β defined in Eq. (8)),*

$$\left| \mathbb{E}_k \left[\hat{\beta}_k(s, a) \right] - \beta^{\pi_k}(s, a) \right| \leq \mathcal{O} \left(\frac{1}{T} \right), \quad (12)$$

$$\mathbb{E}_k \left[\left(\hat{\beta}_k(s, a) - \beta^{\pi_k}(s, a) \right)^2 \right] \leq \mathcal{O} \left(\frac{N^3 \log T}{B \pi_k(a|s) \mu^{\pi_k}(s)} \right). \quad (13)$$

The next lemma shows that in OOMD, π_k and π_{k-1} are close in a strong sense, which further implies the stability for several other related quantities.

Lemma 7. For any k, s, a ,

$$\begin{aligned} |\pi_k(a|s) - \pi_{k-1}(a|s)| &\leq \mathcal{O}(\eta N \pi_{k-1}(a|s)), \\ |J^{\pi_k} - J^{\pi_{k-1}}| &\leq \mathcal{O}(\eta N^2), \\ |v^{\pi_k}(s) - v^{\pi_{k-1}}(s)| &\leq \mathcal{O}(\eta N^3), \\ |q^{\pi_k}(s, a) - q^{\pi_{k-1}}(s, a)| &\leq \mathcal{O}(\eta N^3), \\ |\beta^{\pi_k}(s, a) - \beta^{\pi_{k-1}}(s, a)| &\leq \mathcal{O}(\eta N^3). \end{aligned} \quad (14)$$

The next lemma shows the regret bound of OOMD based on an analysis similar to (Wei & Luo, 2018).

Lemma 8. For a specific state s , we have

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^K \sum_a (\pi^*(a|s) - \pi_k(a|s)) \widehat{\beta}_k(s, a) \right] &\leq \mathcal{O} \left(\frac{A \ln T}{\eta} \right. \\ &\left. + \eta \mathbb{E} \left[\sum_{k=1}^K \sum_a \pi_k(a|s)^2 \left(\widehat{\beta}_k(s, a) - \widehat{\beta}_{k-1}(s, a) \right)^2 \right] \right), \end{aligned}$$

where we define $\widehat{\beta}_0(s, a) = 0$ for all s and a .

Finally, we state a key lemma for proving Theorem 5.

Lemma 9. MDP-OOMD ensures

$$\begin{aligned} \mathbb{E} \left[B \sum_{k=1}^K \sum_s \sum_a \mu^*(s) (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right] \\ = \mathcal{O} \left(\frac{BA \ln T}{\eta} + \eta \frac{TN^3 \rho}{B} + \eta^3 TN^6 \right). \end{aligned}$$

6. Conclusions

In this work we propose two model-free algorithms for learning infinite-horizon average-reward MDPs. They are based on different ideas: one reduces the problem to the discounted version, while the other optimizes the policy directly via a novel application of adaptive adversarial multi-armed bandit algorithms. The main open question is how to achieve the information-theoretically optimal regret bound via a model-free algorithm, if it is possible at all. We believe that the techniques we develop in this work would be useful in answering this question.

We also remark that to run our algorithms, prior knowledge on parameters such as $\text{sp}(v^*)$, t_{hit} , and t_{mix} (or their upper bounds) is required. In practice, they can be viewed as hyperparameters and tuned with standard techniques; in theory, this kind of assumption is made in almost all previous works on average-reward MDPs, except for some attempts in (Bartlett & Tewari, 2009) (unfortunately, their algorithm is not computationally tractable). Thus, how to learn an average-reward MDP without knowing the problem-dependent quantities still largely remains open.

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A. Omitted Proofs in Section 4

In this section, we provide detailed proof for the lemmas used in Section 4. Recall that the learning rate $\alpha_\tau = \frac{H+1}{H+\tau}$ is similar to the one used by (Jin et al., 2018). For notational convenience, let

$$\alpha_\tau^0 := \prod_{j=1}^{\tau} (1 - \alpha_j), \quad \alpha_\tau^i := \alpha_i \prod_{j=i+1}^{\tau} (1 - \alpha_j). \quad (15)$$

It can be verified that $\alpha_\tau^0 = 0$ for $\tau \geq 1$ and we define $\alpha_0^0 = 1$. These quantities are used in the proof of Lemma 3 and have some nice properties summarized in the following lemma.

Lemma 10 ((Jin et al., 2018)). *The following properties hold for α_τ^i :*

1. $\frac{1}{\sqrt{\tau}} \leq \sum_{i=1}^{\tau} \frac{\alpha_\tau^i}{\sqrt{i}} \leq \frac{2}{\sqrt{\tau}}$ for every $\tau \geq 1$.
2. $\sum_{i=1}^{\tau} (\alpha_\tau^i)^2 \leq \frac{2H}{\tau}$ for every $\tau \geq 1$.
3. $\sum_{i=1}^{\tau} \alpha_\tau^i = 1$ for every $\tau \geq 1$ and $\sum_{\tau=i}^{\infty} \alpha_\tau^i = 1 + \frac{1}{H}$ for every $i \geq 1$.

Also recall the well-known Azuma's inequality:

Lemma 11 (Azuma's inequality). *Let X_1, X_2, \dots be a martingale difference sequence with $|X_i| \leq c_i$ for all i . Then, for any $0 < \delta < 1$,*

$$\mathbb{P} \left(\sum_{i=1}^T X_i \geq \sqrt{2\bar{c}_T^2 \ln \frac{1}{\delta}} \right) \leq \delta,$$

where $\bar{c}_T^2 := \sum_{i=1}^T c_i^2$.

A.1. Proof of Lemma 2

Lemma 2 (Restated). Let V^* be the optimal value function in the discounted MDP with discount factor γ and v^* be the optimal value function in the undiscounted MDP. Then,

1. $|J^* - (1 - \gamma)V^*(s)| \leq (1 - \gamma) \text{sp}(v^*), \forall s \in \mathcal{S}$,
2. $\text{sp}(V^*) \leq 2 \text{sp}(v^*)$.

Proof. 1. Let π^* and π_γ be the optimal policy under undiscounted and discounted settings, respectively. By Bellman's equation, we have

$$v^*(s) = r(s, \pi^*(s)) - J^* + \mathbb{E}_{s' \sim p(\cdot | s, \pi^*(s))} v^*(s').$$

Consider a state sequence s_1, s_2, \dots generated by π^* . Then, by sub-optimality of π^* for the discounted setting, we have

$$\begin{aligned} V^*(s_1) &\geq \mathbb{E} \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, \pi^*(s_t)) \mid s_1 \right] \\ &= \mathbb{E} \left[\sum_{t=1}^{\infty} \gamma^{t-1} (J^* + v^*(s_t) - v^*(s_{t+1})) \mid s_1 \right] \\ &= \frac{J^*}{1 - \gamma} + v^*(s_1) - \mathbb{E} \left[\sum_{t=2}^{\infty} (\gamma^{t-2} - \gamma^{t-1}) v^*(s_t) \mid s_1 \right] \\ &\geq \frac{J^*}{1 - \gamma} + \min_s v^*(s) - \max_s v^*(s) \sum_{t=2}^{\infty} (\gamma^{t-2} - \gamma^{t-1}) \\ &= \frac{J^*}{1 - \gamma} - \text{sp}(v^*), \end{aligned}$$

where the first equality is by the Bellman equation for the undiscounted setting.

Similarly, for the other direction, let s_1, s_2, \dots be generated by π_γ . We have

$$\begin{aligned}
 V^*(s_1) &= \mathbb{E} \left[\sum_{t=1}^{\infty} \gamma^{t-1} r(s_t, \pi_\gamma(s_t)) \mid s_1 \right] \\
 &\leq \mathbb{E} \left[\sum_{t=1}^{\infty} \gamma^{t-1} (J^* + v^*(s_t) - v^*(s_{t+1})) \mid s_1 \right] \\
 &= \frac{J^*}{1-\gamma} + v^*(s_1) - \mathbb{E} \left[\sum_{t=2}^{\infty} (\gamma^{t-2} - \gamma^{t-1}) v^*(s_t) \mid s_1 \right] \\
 &\leq \frac{J^*}{1-\gamma} + \max_s v^*(s) - \min_s v^*(s) \sum_{t=2}^{\infty} (\gamma^{t-2} - \gamma^{t-1}) \\
 &= \frac{J^*}{1-\gamma} + \text{sp}(v^*),
 \end{aligned}$$

where the first inequality is by sub-optimality of π_γ for the undiscounted setting.

2. Using previous part, for any $s_1, s_2 \in \mathcal{S}$, we have

$$|V^*(s_1) - V^*(s_2)| \leq \left| V^*(s_1) - \frac{J^*}{1-\gamma} \right| + \left| V^*(s_2) - \frac{J^*}{1-\gamma} \right| \leq 2 \text{sp}(v^*).$$

Thus, $\text{sp}(V^*) \leq 2 \text{sp}(v^*)$. □

A.2. Proof of Lemma 3

Lemma 3. With probability at least $1 - \delta$,

$$\sum_{t=1}^T (V^*(s_t) - Q^*(s_t, a_t)) \leq 4HSA + 24 \text{sp}(v^*) \sqrt{HSAT \ln \frac{2T}{\delta}}.$$

Proof. We condition on the statement of Lemma 12, which happens with probability at least $1 - \delta$. Let $n_t \geq 1$ denote $n_{t+1}(s_t, a_t)$, that is, the total number of visits to the state-action pair (s_t, a_t) for the first t rounds (including round t). Also let $t_i(s, a)$ denote the timestep at which (s, a) is visited the i -th time. Recalling the definition of $\alpha_{n_t}^i$ in Eq. (15), we have

$$\sum_{t=1}^T (\hat{V}_t(s_t) - V^*(s_t)) + \sum_{t=1}^T (V^*(s_t) - Q^*(s_t, a_t)) \tag{16}$$

$$= \sum_{t=1}^T (\hat{Q}_t(s_t, a_t) - Q^*(s_t, a_t)) \tag{because } a_t = \text{argmax}_a \hat{Q}_t(s_t, a)$$

$$= \sum_{t=1}^T (\hat{Q}_{t+1}(s_t, a_t) - Q^*(s_t, a_t)) + \sum_{t=1}^T (\hat{Q}_t(s_t, a_t) - \hat{Q}_{t+1}(s_t, a_t)) \tag{17}$$

$$\leq 12 \text{sp}(v^*) \sum_{t=1}^T \sqrt{\frac{H}{n_t} \ln \frac{2T}{\delta}} + \gamma \sum_{t=1}^T \sum_{i=1}^{n_t} \alpha_{n_t}^i \left[\hat{V}_{t_i(s_t, a_t)}(s_{t_i(s_t, a_t)+1}) - V^*(s_{t_i(s_t, a_t)+1}) \right] + SAH. \tag{18}$$

Here, we apply Lemma 12 to bound the first term of Eq. (17) (note $\alpha_{n_t}^0 = 0$ by definition since $n_t \geq 1$), and also bound the second term of Eq. (17) by SAH since for each fixed (s, a) , $\hat{Q}_t(s, a)$ is non-increasing in t and overall cannot decrease by more than H (the initial value).

To bound the third term of Eq. (18) we write:

$$\begin{aligned}
 & \gamma \sum_{t=1}^T \sum_{i=1}^{n_t} \alpha_{n_t}^i \left[\hat{V}_{t_i(s_t, a_t)}(s_{t_i(s_t, a_t)+1}) - V^*(s_{t_i(s_t, a_t)+1}) \right] \\
 &= \gamma \sum_{t=1}^T \sum_{s, a} \mathbb{1}_{[s_t=s, a_t=a]} \sum_{i=1}^{n_{t+1}(s, a)} \alpha_{n_{t+1}(s, a)}^i \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right] \\
 &= \gamma \sum_{s, a} \sum_{j=1}^{n_{T+1}(s, a)} \sum_{i=1}^j \alpha_j^i \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right].
 \end{aligned}$$

By changing the order of summation on i and j , the latter is equal to

$$\begin{aligned}
 & \gamma \sum_{s, a} \sum_{i=1}^{n_{T+1}(s, a)} \sum_{j=i}^{n_{T+1}(s, a)} \alpha_j^i \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right] \\
 &= \gamma \sum_{s, a} \sum_{i=1}^{n_{T+1}(s, a)} \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right] \sum_{j=i}^{n_{T+1}(s, a)} \alpha_j^i
 \end{aligned}$$

Now, we can upper bound $\sum_{j=i}^{n_{T+1}(s, a)} \alpha_j^i$ by $\sum_{j=i}^{\infty} \alpha_j^i$ where the latter is equal to $1 + \frac{1}{H}$ by Lemma 10. Since $\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \geq 0$ (by Lemma 12), we can write:

$$\begin{aligned}
 & \gamma \sum_{s, a} \sum_{i=1}^{n_{T+1}(s, a)} \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right] \sum_{j=i}^{n_{T+1}(s, a)} \alpha_j^i \\
 & \leq \gamma \sum_{s, a} \sum_{i=1}^{n_{T+1}(s, a)} \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right] \sum_{j=i}^{\infty} \alpha_j^i \\
 &= \gamma \sum_{s, a} \sum_{i=1}^{n_{T+1}(s, a)} \left[\hat{V}_{t_i(s, a)}(s_{t_i(s, a)+1}) - V^*(s_{t_i(s, a)+1}) \right] \left(1 + \frac{1}{H} \right) \\
 &= \left(1 + \frac{1}{H} \right) \gamma \sum_{t=1}^T \left[\hat{V}_t(s_{t+1}) - V^*(s_{t+1}) \right] \\
 &= \left(1 + \frac{1}{H} \right) \gamma \sum_{t=1}^T \left[\hat{V}_{t+1}(s_{t+1}) - V^*(s_{t+1}) \right] + \left(1 + \frac{1}{H} \right) \sum_{t=1}^T \left[\hat{V}_t(s_{t+1}) - \hat{V}_{t+1}(s_{t+1}) \right] \\
 & \leq \sum_{t=2}^{T+1} \left[\hat{V}_t(s_t) - V^*(s_t) \right] + \left(1 + \frac{1}{H} \right) SH.
 \end{aligned}$$

The last inequality is because $(1 + \frac{1}{H})\gamma \leq 1$ and that for any state s , $\hat{V}_t(s) \geq \hat{V}_{t+1}(s)$ and the value can decrease by at most H (the initial value). Substituting in Eq. (18) and telescoping with the left hand side, we have

$$\begin{aligned}
 \sum_{t=1}^T (V^*(s_t) - Q^*(s_t, a_t)) & \leq 12 \text{sp}(v^*) \sum_{t=1}^T \sqrt{\frac{H}{n_t} \ln \frac{2T}{\delta}} + \left(\hat{V}_{T+1}(s_{T+1}) - V^*(s_{T+1}) \right) + \left(1 + \frac{1}{H} \right) SH + SAH \\
 & \leq 12 \text{sp}(v^*) \sum_{t=1}^T \sqrt{\frac{H}{n_t} \ln \frac{2T}{\delta}} + 4SAH.
 \end{aligned}$$

Moreover, $\sum_{t=1}^T \frac{1}{\sqrt{n_t}} \leq 2\sqrt{SAT}$ because

$$\sum_{t=1}^T \frac{1}{\sqrt{n_{t+1}(s_t, a_t)}} = \sum_{t=1}^T \sum_{s, a} \frac{\mathbb{1}_{[s_t=s, a_t=a]}}{\sqrt{n_{t+1}(s, a)}} = \sum_{s, a} \sum_{j=1}^{n_{T+1}(s, a)} \frac{1}{\sqrt{j}} \leq \sum_{s, a} 2\sqrt{n_{T+1}(s, a)} \leq 2\sqrt{SA \sum_{s, a} n_{T+1}(s, a)} = 2\sqrt{SAT},$$

where the last inequality is by Cauchy-Schwarz inequality. This finishes the proof. \square

Lemma 12. *With probability at least $1 - \delta$, for any $t = 1, \dots, T$ and state-action pair (s, a) , the following holds*

$$0 \leq \hat{Q}_{t+1}(s, a) - Q^*(s, a) \leq H\alpha_\tau^0 + \gamma \sum_{i=1}^{\tau} \alpha_\tau^i \left[\hat{V}_{t_i}(s_{t_i+1}) - V^*(s_{t_i+1}) \right] + 12 \text{sp}(v^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}},$$

where $\tau = n_{t+1}(s, a)$ (i.e., the total number of visits to (s, a) for the first t timesteps), α_τ^i is defined by (15), and $t_1, \dots, t_\tau \leq t$ are the timesteps on which (s, a) is taken.

Proof. Recursively substituting $Q_t(s, a)$ in Eq. (3) of the algorithm, we have

$$Q_{t+1}(s, a) = H\alpha_\tau^0 + \sum_{i=1}^{\tau} \alpha_\tau^i \left[r(s, a) + \gamma \hat{V}_{t_i}(s_{t_i+1}) \right] + \sum_{i=1}^{\tau} \alpha_\tau^i b_i.$$

Moreover, since $\sum_{i=1}^{\tau} \alpha_\tau^i = 1$ (Lemma 10), By Bellman equation we have

$$Q^*(s, a) = \alpha_\tau^0 Q^*(s, a) + \sum_{i=1}^{\tau} \alpha_\tau^i \left[r(s, a) + \gamma \mathbb{E}_{s' \sim p(\cdot | s, a)} V^*(s') \right].$$

Taking their difference and adding and subtracting a term $\gamma \sum_{i=1}^{\tau} \alpha_\tau^i V^*(s_{t_i+1})$ lead to:

$$\begin{aligned} Q_{t+1}(s, a) - Q^*(s, a) &= \alpha_\tau^0 (H - Q^*(s, a)) + \gamma \sum_{i=1}^{\tau} \alpha_\tau^i \left[\hat{V}_{t_i}(s_{t_i+1}) - V^*(s_{t_i+1}) \right] \\ &\quad + \gamma \sum_{i=1}^{\tau} \alpha_\tau^i \left[V^*(s_{t_i+1}) - \mathbb{E}_{s' \sim p(\cdot | s, a)} V^*(s') \right] + \sum_{i=1}^{\tau} \alpha_\tau^i b_i. \end{aligned}$$

The first term is upper bounded by $\alpha_\tau^0 H$ clearly and lower bounded by 0 since $Q^*(s, a) \leq \sum_{i=0}^{\infty} \gamma^i = \frac{1}{1-\gamma} = H$.

The third term is a martingale difference sequence with each term bounded in $[-\gamma \alpha_\tau^i \text{sp}(V^*), \gamma \alpha_\tau^i \text{sp}(V^*)]$. Therefore, by Azuma's inequality (Lemma 11), its absolute value is bounded by $\gamma \text{sp}(V^*) \sqrt{2 \sum_{i=1}^{\tau} (\alpha_\tau^i)^2 \ln \frac{2T}{\delta}} \leq 2\gamma \text{sp}(V^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}} \leq 4\gamma \text{sp}(v^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}}$ with probability at least $1 - \frac{\delta}{T}$, where the first inequality is by Lemma 10 and the last inequality is by Lemma 2. Note that when t varies from 1 to T and (s, a) varies over all possible state-action pairs, the third term only takes T different forms. Therefore, by taking a union bound over these T events, we have: with probability $1 - \delta$, the third term is bounded by $4\gamma \text{sp}(v^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}}$ in absolute value for all t and (s, a) .

The fourth term is lower bounded by $4 \text{sp}(v^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}}$ and upper bounded by $8 \text{sp}(V^*) \sqrt{\frac{H}{\tau} \ln \frac{2T}{\delta}}$, by Lemma 10.

Combining all aforementioned upper bounds and the fact $\hat{Q}_{t+1}(s, a) = \min \left\{ \hat{Q}_t(s, a), Q_{t+1}(s, a) \right\} \leq Q_{t+1}(s, a)$ we prove the upper bound in the lemma statement. To prove the lower bound, further note that the second term can be written as $\gamma \sum_{i=1}^{\tau} \alpha_\tau^i \left[\max_a \hat{Q}_{t_i}(s_{t_i+1}, a) - \max_a Q^*(s_{t_i+1}, a) \right]$. Using a direct induction with all aforementioned lower bounds and the fact $\hat{Q}_{t+1}(s, a) = \min \left\{ \hat{Q}_t(s, a), Q_{t+1}(s, a) \right\}$ we prove the lower bound in the lemma statement as well. \square

A.3. Proof of Lemma 4

Lemma 4. With probability at least $1 - \delta$,

$$\sum_{t=1}^T (Q^*(s_t, a_t) - \gamma V^*(s_t) - r(s_t, a_t)) \leq 2 \text{sp}(v^*) \sqrt{2T \ln \frac{1}{\delta}} + 2 \text{sp}(v^*).$$

Proof. By Bellman equation for the discounted problem, we have $Q^*(s_t, a_t) - \gamma V^*(s_t) - r(s_t, a_t) = \gamma (\mathbb{E}_{s' \sim p(\cdot | s_t, a_t)} [V^*(s')] - V^*(s_t))$. Adding and subtracting $V^*(s_{t+1})$ and summing over t we will get

$$\sum_{t=1}^T (Q^*(s_t, a_t) - \gamma V^*(s_t) - r(s_t, a_t)) = \gamma \sum_{t=1}^T (\mathbb{E}_{s' \sim p(\cdot | s_t, a_t)} [V^*(s')] - V^*(s_{t+1})) + \gamma \sum_{t=1}^T (V^*(s_{t+1}) - V^*(s_t))$$

The summands of the first term on the right hand side constitute a martingale difference sequence. Thus, by Azuma's inequality (Lemma 11) and the fact that $\text{sp}(V^*) \leq 2 \text{sp}(v^*)$ (Lemma 2), this term is upper bounded by $2\gamma \text{sp}(v^*) \sqrt{2T \ln \frac{1}{\delta}}$, with probability at least $1 - \delta$. The second term is equal to $\gamma(V^*(s_{T+1}) - V^*(s_1))$ which is upper bounded by $2\gamma \text{sp}(v^*)$. Recalling $\gamma < 1$ completes the proof. \square

B. Omitted Proofs in Section 5 — Proofs for Lemma 6 and Lemma 7

B.1. Auxiliary Lemmas

In this subsection, we state several lemmas that will be helpful in the analysis.

Lemma 13 ((Levin & Peres, 2017, Section 4.5)). *Define*

$$t_{\text{mix}}(\epsilon) := \max_{\pi} \min \left\{ t \geq 1 \mid \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq \epsilon, \forall s \right\},$$

so that $t_{\text{mix}} = t_{\text{mix}}(\frac{1}{4})$. We have

$$t_{\text{mix}}(\epsilon) \leq \left\lceil \log_2 \frac{1}{\epsilon} \right\rceil t_{\text{mix}}$$

for any $\epsilon \in (0, \frac{1}{2}]$.

Corollary 13.1. *For an ergodic MDP with mixing time t_{mix} , we have*

$$\|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq 2 \cdot 2^{-\frac{t}{t_{\text{mix}}}}, \quad \forall \pi, s$$

for all π and all $t \geq 2t_{\text{mix}}$.

Proof. Lemma 13 implies for any $\epsilon \in (0, \frac{1}{2}]$, as long as $t \geq \lceil \log_2(1/\epsilon) \rceil t_{\text{mix}}$, we have

$$\|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq \epsilon.$$

This condition can be satisfied by picking $\log_2(1/\epsilon) = \frac{t}{t_{\text{mix}}} - 1$, which leads to $\epsilon = 2 \cdot 2^{-\frac{t}{t_{\text{mix}}}}$. \square

Corollary 13.2. *Let $N = 4t_{\text{mix}} \log_2 T$. For an ergodic MDP with mixing time $t_{\text{mix}} < T/4$, we have for all π :*

$$\sum_{t=N}^{\infty} \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq \frac{1}{T^3}.$$

Proof. By Corollary 13.1,

$$\sum_{t=N}^{\infty} \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq \sum_{t=N}^{\infty} 2 \cdot 2^{-\frac{t}{t_{\text{mix}}}} = \frac{2 \cdot 2^{-\frac{N}{t_{\text{mix}}}}}{1 - 2^{-\frac{1}{t_{\text{mix}}}}} \leq \frac{2t_{\text{mix}}}{\ln 2} \cdot 2 \cdot 2^{-\frac{N}{t_{\text{mix}}}} = \frac{2t_{\text{mix}}}{\ln 2} \cdot 2 \cdot \frac{1}{T^4} \leq \frac{1}{T^3}.$$

\square

Lemma 14 (Stated in (Wang, 2017) without proof). *For an ergodic MDP with mixing time t_{mix} , and any π, s, a ,*

$$\begin{aligned} |v^\pi(s)| &\leq 5t_{\text{mix}}, \\ |q^\pi(s, a)| &\leq 6t_{\text{mix}}. \end{aligned}$$

Proof. Using the identity of Eq. (4) we have

$$\begin{aligned}
 |v^\pi(s)| &= \left| \sum_{t=0}^{\infty} ((P^\pi)^t(s, \cdot) - \mu^\pi)^\top r^\pi \right| \\
 &\leq \sum_{t=0}^{\infty} \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \|r^\pi\|_\infty \\
 &\leq \sum_{t=0}^{2t_{\text{mix}}-1} \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 + \sum_{i=2}^{\infty} \sum_{t=it_{\text{mix}}}^{(i+1)t_{\text{mix}}-1} \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \\
 &\leq 4t_{\text{mix}} + \sum_{i=2}^{\infty} 2 \cdot 2^{-i} t_{\text{mix}} \quad (\text{by } \|(P^\pi)^t(s, \cdot) - \mu^\pi\|_1 \leq 2 \text{ and Corollary 13.1}) \\
 &\leq 5t_{\text{mix}},
 \end{aligned}$$

and thus

$$|q^\pi(s, a)| = |r(s, a) + \mathbb{E}_{s' \sim p(\cdot|s, a)}[v^\pi(s')]| \leq 1 + 5t_{\text{mix}} \leq 6t_{\text{mix}}.$$

□

Lemma 15 ((Neu et al., 2013, Lemma 2)). *For any two policies $\pi, \tilde{\pi}$,*

$$J^{\tilde{\pi}} - J^\pi = \sum_s \sum_a \mu^{\tilde{\pi}}(s) (\tilde{\pi}(a|s) - \pi(a|s)) q^\pi(s, a).$$

Proof. Using Bellman equation we have

$$\begin{aligned}
 &\sum_s \sum_a \mu^{\tilde{\pi}}(s) \tilde{\pi}(a|s) q^\pi(s, a) \\
 &= \sum_s \sum_a \mu^{\tilde{\pi}}(s) \tilde{\pi}(a|s) \left(r(s, a) - J^\pi + \sum_{s'} p(s'|s, a) v^\pi(s') \right) \\
 &= J^{\tilde{\pi}} - J^\pi + \sum_{s'} \mu^{\tilde{\pi}}(s') v^\pi(s') \\
 &= J^{\tilde{\pi}} - J^\pi + \sum_s \mu^{\tilde{\pi}}(s) v^\pi(s) \\
 &= J^{\tilde{\pi}} - J^\pi + \sum_s \sum_a \mu^{\tilde{\pi}}(s) \pi(a|s) q^\pi(s, a),
 \end{aligned}$$

where the second equality uses the facts $J^{\tilde{\pi}} = \sum_s \sum_a \mu^{\tilde{\pi}}(s) \tilde{\pi}(a|s) r(s, a)$ and $\sum_{s, a} \mu^{\tilde{\pi}}(s) \tilde{\pi}(a|s) p(s'|s, a) = \mu^{\tilde{\pi}}(s')$. Rearranging gives the desired equality. □

Lemma 16. *Let $\mathcal{I} = \{t_1+1, t_1+2, \dots, t_2\}$ be a certain period of an episode k of Algorithm 2 with $|\mathcal{I}| \geq N = 4t_{\text{mix}} \log_2 T$. Then for any s , the probability that the algorithm never visits s in \mathcal{I} is upper bounded by*

$$\left(1 - \frac{3\mu^{\pi_k}(s)}{4} \right)^{\lfloor \frac{|\mathcal{I}|}{N} \rfloor}.$$

Proof. Consider a subset of \mathcal{I} : $\{t_1 + N, t_1 + 2N, \dots\}$ which consists of at least $\lfloor \frac{t_2 - t_1}{N} \rfloor$ rounds that are at least N -step away from each other. By Corollary 13.1, we have for any i ,

$$\left| \Pr[s_{t_1+iN} = s \mid s_{t_1+(i-1)N}] - \mu^{\pi_k}(s) \right| \leq 2 \cdot 2^{-\frac{N}{t_{\text{mix}}}} \leq 2 \cdot 2^{-4 \log_2 T} \leq \frac{2}{T^4},$$

that is, conditioned on the state at time $t_1 + (i-1)N$, the state distribution at time $t_1 + iN$ is close to the stationary distribution induced by π_k . Therefore we further have $\Pr[s_{t_1+iN} = s \mid s_{t_1+(i-1)N}] \geq \mu^{\pi_k}(s) - \frac{2}{T^4} \geq \frac{3}{4} \mu^{\pi_k}(s)$, where

the last step uses the fact $\mu^{\pi_k}(s) \geq \frac{1}{t_{\text{hit}}} \geq \frac{4}{T}$. The probability that the algorithm does not visit s in any of the rounds $\{t_1 + N, t_1 + 2N, \dots\}$ is then at most

$$\left(1 - \frac{3\mu^{\pi_k}(s)}{4}\right)^{\lfloor \frac{t_2 - t_1}{N} \rfloor} = \left(1 - \frac{3\mu^{\pi_k}(s)}{4}\right)^{\lfloor \frac{|Z|}{N} \rfloor},$$

finishing the proof. \square

B.2. Proof for Lemma 6

Proof for Eq.(12). In this proof, we consider a specific episode k and a specific state s . For notation simplicity, we use π for π_k throughout this proof, and all the expectations or probabilities are conditioned on the history before episode k . Suppose that when Algorithm 2 calls ESTIMATEQ in episode k for state s , it finds M disjoint intervals that starts from s . Denote the reward estimators corresponding to the i -th interval as $\widehat{\beta}_{k,i}(s, \cdot)$ (i.e., the $y_i(\cdot)$ in Algorithm 3), and the time when the i -th interval starts as τ_i (thus $s_{\tau_i} = s$). Then by the algorithm, we have

$$\widehat{\beta}_k(s, a) = \begin{cases} \frac{\sum_{i=1}^M \widehat{\beta}_{k,i}(s, a)}{M} & \text{if } M > 0, \\ 0 & \text{if } M = 0. \end{cases} \quad (19)$$

Since each $\widehat{\beta}_{k,i}(s, a)$ is constructed by a length- $(N + 1)$ trajectory starting from s at time $\tau_i \leq kB - N$, we can calculate its *conditional expectation* as follows:

$$\begin{aligned} & \mathbb{E} \left[\widehat{\beta}_{k,i}(s, a) \middle| s_{\tau_i} = s \right] \\ &= \Pr[a_{\tau_i} = a \mid s_{\tau_i} = s] \times \frac{r(s, a) + \mathbb{E} \left[\sum_{t=\tau_i+1}^{\tau_i+N} r(s_t, a_t) \mid (s_{\tau_i}, a_{\tau_i}) = (s, a) \right]}{\pi(a|s)} \\ &= r(s, a) + \sum_{s'} p(s'|s, a) \mathbb{E} \left[\sum_{t=\tau_i+1}^{\tau_i+N} r(s_t, a_t) \middle| s_{\tau_i+1} = s' \right] \\ &= r(s, a) + \sum_{s'} p(s'|s, a) \sum_{j=0}^{N-1} \mathbf{e}_{s'}^\top (P^\pi)^j r^\pi \\ &= r(s, a) + \sum_{s'} p(s'|s, a) \sum_{j=0}^{N-1} (\mathbf{e}_{s'}^\top (P^\pi)^j - (\mu^\pi)^\top) r^\pi + NJ^\pi && \text{(because } \mu^{\pi^\top} r^\pi = J^\pi) \\ &= r(s, a) + \sum_{s'} p(s'|s, a) v^\pi(s') + NJ^\pi - \sum_{s'} p(s'|s, a) \sum_{j=N}^{\infty} (\mathbf{e}_{s'}^\top (P^\pi)^j - (\mu^\pi)^\top) r^\pi && \text{(By Eq. (4))} \\ &= q^\pi(s, a) + NJ^\pi - \delta(s, a) \\ &= \beta^\pi(s, a) - \delta(s, a), \end{aligned} \quad (20)$$

where $\delta(s, a) \triangleq \sum_{s'} p(s'|s, a) \sum_{j=N}^{\infty} (\mathbf{e}_{s'}^\top (P^\pi)^j - (\mu^\pi)^\top) r^\pi$. By Corollary 13.2,

$$|\delta(s, a)| \leq \frac{1}{T^3}. \quad (21)$$

Thus,

$$\left| \mathbb{E} \left[\widehat{\beta}_{k,i}(s, a) \middle| s_{\tau_i} = s \right] - \beta^\pi(s, a) \right| \leq \frac{1}{T^3}.$$

This shows that $\widehat{\beta}_{k,i}(s, a)$ is an *almost unbiased* estimator for β^π conditioned on all history before τ_i . Also, by our selection of the episode length, $M > 0$ will happen with very high probability according to Lemma 16. These facts seem to indicate that $\widehat{\beta}_k(s, a)$ – an average of several $\widehat{\beta}_{k,i}(s, a)$ – will also be an almost unbiased estimator for $\beta^\pi(s, a)$ with small error.

However, a caveat here is that the quantity M in Eq.(19) is random, and it is not independent from the reward estimators $\sum_{i=1}^M \hat{\beta}_{k,i}(s, a)$. Therefore, to argue that the expectation of $\mathbb{E}[\hat{\beta}_k(s, a)]$ is close to $\beta^\pi(s, a)$, more technical work is needed. Specifically, we use the following two steps to argue that $\mathbb{E}[\hat{\beta}_k(s, a)]$ is close to $\beta^\pi(s, a)$.

Step 1. Construct an *imaginary world* where $\hat{\beta}_k(s, a)$ is an almost unbiased estimator of $\beta^\pi(s, a)$.

Step 2. Argue that the expectation of $\hat{\beta}_k(s, a)$ in the real world and the expectation of $\hat{\beta}_k(s, a)$ in the imaginary world are close.

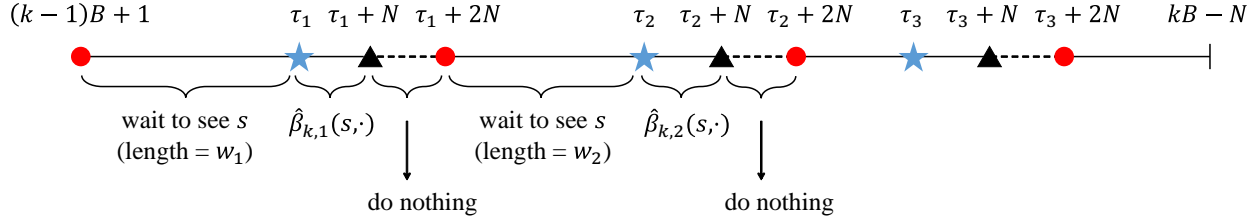


Figure 1. An illustration for the sub-algorithm ESTIMATEQ with target state s (best viewed in color). The red round points indicate that the algorithm “starts to wait” for a visit to s . When the algorithm reaches s (the blue stars) at time τ_i , it starts to record the sum of rewards in the following $N + 1$ steps, i.e. $\sum_{t=\tau_i}^{\tau_i+N} r(s_t, a_t)$. This is used to construct $\hat{\beta}_{k,i}(s, \cdot)$. The next point the algorithm “starts to wait for s ” would be $\tau_i + 2N$ if this is still no later than $kB - N$.

Step 1. We first examine what ESTIMATEQ sub-algorithm does in an episode k for a state s . The goal of this sub-algorithm is to collect disjoint intervals of length $N + 1$ that start from s , calculate a reward estimator from each of them, and finally average the estimators over all intervals to get a good estimator for $\beta^\pi(s, \cdot)$. However, after our algorithm collects an interval $[\tau, \tau + N]$, it *rests* for another N steps before starting to find the next visit to s – i.e., it restarts from $\tau + 2N$ (see Line 6 in ESTIMATEQ (Algorithm 3), and also the illustration in Figure 1).

The goal of doing this is to de-correlate the observed reward and the number of collected intervals: as shown in Eq.(19), these two quantities affect the numerator and the denominator of $\hat{\beta}_k(s, \cdot)$ respectively, and if they are highly correlated, then $\hat{\beta}_k(s, \cdot)$ may be heavily biased from $\beta^\pi(s, \cdot)$. On the other hand, if we introduce the “rest time” after we collect each interval (i.e., the dashed segments in Figure 1), then since the length of the rest time (N) is longer than the mixing time, the process will almost totally “forget” about the reward estimators collected before. In Figure 1, this means that the state distributions at the red round points (except for the left most one) will be close to μ^π when conditioned on all history that happened N rounds ago.

We first argue that if the process can indeed “reset its memory” at those red round points in Figure 1 (except for the left most one), then we get almost unbiased estimators for $\beta^\pi(s, \cdot)$. That is, consider a process like in Figure 2 where everything remains same as in ESTIMATEQ except that after every rest interval, the state distribution is directly reset to the stationary distribution μ^π .

Below we calculate the expectation of $\hat{\beta}_k(s, a)$ in this imaginary world. As specified in Figure 2, we use τ_i to denote the i -th time ESTIMATEQ starts to record an interval (therefore $s_{\tau_i} = s$), and let $w_i = \tau_i - (\tau_{i-1} + 2N)$ for $i > 1$ and $w_1 = \tau_1 - ((k-1)B + 1)$ be the “wait time” before starting the i -th interval. Note the following facts in the imaginary world:

1. M is determined by the sequence w_1, w_2, \dots because all other segments in the figures have fixed length.
2. w_1 only depends on $s_{(k-1)B+1}$ and P^π , and w_i only depends on the stationary distribution μ^π and P^π because of the reset.

The above facts imply that in the imaginary world, w_1, w_2, \dots , as well as M , are all independent from

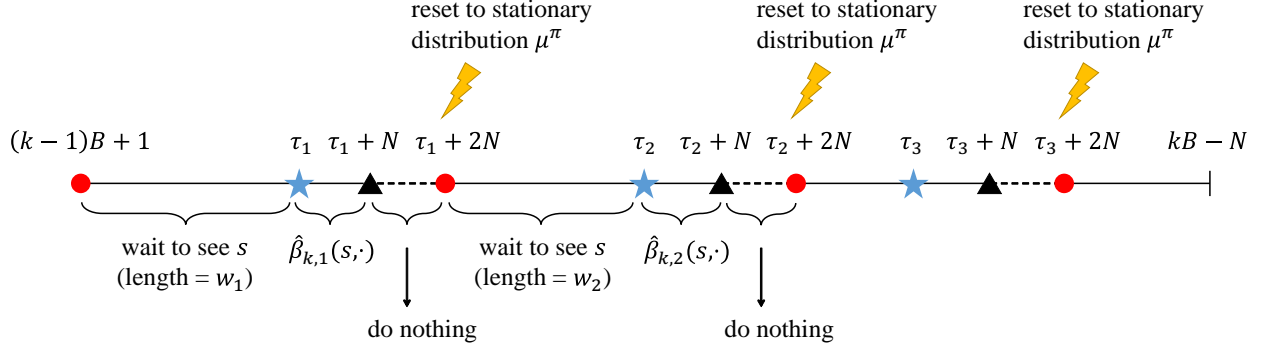


Figure 2. The imaginary world (best viewed in color)

$\widehat{\beta}_{k,1}(s, a), \widehat{\beta}_{k,2}(s, a), \dots$. Let \mathbb{E}' denote the expectation in the imaginary world. Then

$$\begin{aligned}
 \mathbb{E}' \left[\widehat{\beta}_k(s, a) \right] &= \Pr[w_1 \leq B - N] \times \mathbb{E}'_{\{w_i\}} \left[\frac{1}{M} \sum_{i=1}^M \mathbb{E}' \left[\widehat{\beta}_{k,i}(s, a) \mid \{w_i\} \right] \Big| w_1 \leq B - N \right] + \Pr[w_1 > B - N] \times 0 \\
 &= \Pr[w_1 \leq B - N] \times \mathbb{E}'_{\{w_i\}} \left[\frac{1}{M} \left(\sum_{i=1}^M \beta^\pi(s, a) - \delta(s, a) \right) \right] \quad (\text{by the same calculation as in (20)}) \\
 &= \Pr[w_1 \leq B - N] \times (\beta^\pi(s, a) - \delta'(s, a)) \\
 &= \beta^\pi(s, a) - \delta'(s, a), \tag{22}
 \end{aligned}$$

where $\mathbb{E}'_{\{w_i\}}$ denotes the expectation over the randomness of w_1, w_2, \dots , and $\delta'(s, a) = (1 - \Pr[w_1 \leq B - N]) (\beta^\pi(s, a) - \delta(s, a)) + \delta(s, a)$. By Lemma 16, we have $\Pr[w_1 \leq B - N] \geq 1 - \left(1 - \frac{3}{4t_{\text{hit}}}\right)^{\frac{B-N}{N}} = 1 - \left(1 - \frac{3}{4t_{\text{hit}}}\right)^{4t_{\text{hit}} \log_2 T^{-1}} \geq 1 - \frac{1}{T^3}$. Together with Eq. (21) and Lemma 14, we have

$$|\delta'(s, a)| \leq \frac{1}{T^3} (|\beta^\pi(s, a)| + |\delta(s, a)|) + |\delta(s, a)| \leq \frac{1}{T^3} (6t_{\text{mix}} + N + \frac{1}{T^3}) + \frac{1}{T^3} = \mathcal{O}\left(\frac{1}{T^2}\right),$$

and thus

$$\left| \mathbb{E}' \left[\widehat{\beta}_k(s, a) \right] - \beta^\pi(s, a) \right| = \mathcal{O}\left(\frac{1}{T^2}\right). \tag{23}$$

Step 2. Note that $\widehat{\beta}_k(s, a)$ is a deterministic function of $X = (M, \tau_1, \mathcal{T}_1, \tau_2, \mathcal{T}_2, \dots, \tau_M, \mathcal{T}_M)$, where $\mathcal{T}_i = (a_{\tau_i}, s_{\tau_i+1}, a_{\tau_i+1}, \dots, s_{\tau_i+N}, a_{\tau_i+N})$. We use $\widehat{\beta}_k(s, a) = f(X)$ to denote this mapping. To say $\mathbb{E}[\widehat{\beta}_k(s, a)]$ and $\mathbb{E}'[\widehat{\beta}_k(s, a)]$ are close, we bound their ratio:

$$\frac{\mathbb{E}[\widehat{\beta}_k(s, a)]}{\mathbb{E}'[\widehat{\beta}_k(s, a)]} = \frac{\sum_X f(X) \mathbb{P}(X)}{\sum_X f(X) \mathbb{P}'(X)} \leq \max_X \frac{\mathbb{P}(X)}{\mathbb{P}'(X)}, \tag{24}$$

where we use \mathbb{P} and \mathbb{P}' to denote the probability mass function in the real world and the imaginary world respectively, and in the last inequality we use the non-negativeness of $f(X)$.

For a fixed sequence of X , the probability of generating X in the real world is

$$\begin{aligned}
 \mathbb{P}(X) &= \mathbb{P}(\tau_1) \times \mathbb{P}(\mathcal{T}_1 | \tau_1) \times \mathbb{P}(\tau_2 | \tau_1, \mathcal{T}_1) \times \mathbb{P}(\mathcal{T}_2 | \tau_2) \times \dots \times \mathbb{P}(\tau_M | \tau_{M-1}, \mathcal{T}_{M-1}) \\
 &\quad \times \mathbb{P}(\mathcal{T}_M | \tau_M) \times \Pr \left[s_t \neq s, \forall t \in [\tau_M + 2N, kB - N] \mid \tau_M, \mathcal{T}_M \right]. \tag{25}
 \end{aligned}$$

In the imaginary world, it is

$$\begin{aligned} \mathbb{P}'(X) &= \mathbb{P}(\tau_1) \times \mathbb{P}(\mathcal{T}_1|\tau_1) \times \mathbb{P}'(\tau_2|\tau_1, \mathcal{T}_1) \times \mathbb{P}(\mathcal{T}_2|\tau_2) \times \cdots \times \mathbb{P}'(\tau_M|\tau_{M-1}, \mathcal{T}_{M-1}) \\ &\quad \times \mathbb{P}(\mathcal{T}_M|\tau_M) \times \Pr \left[s_t \neq s, \forall t \in [\tau_M + 2N, kB - N] \middle| \tau_M, \mathcal{T}_M \right]. \end{aligned} \quad (26)$$

Their difference only comes from $\mathbb{P}(\tau_{i+1}|\tau_i, \mathcal{T}_i) \neq \mathbb{P}'(\tau_{i+1}|\tau_i, \mathcal{T}_i)$ because of the reset. Note that

$$\mathbb{P}(\tau_{i+1}|\tau_i, \mathcal{T}_i) = \sum_{s' \neq s} \mathbb{P}(s_{\tau_i+2N} = s' | \tau_i, \mathcal{T}_i) \times \Pr \left[s_t \neq s, \forall t \in [\tau_i + 2N, \tau_{i+1} - 1], s_{\tau_{i+1}} = s \middle| s_{\tau_i+2N} = s' \right], \quad (27)$$

$$\mathbb{P}'(\tau_{i+1}|\tau_i, \mathcal{T}_i) = \sum_{s' \neq s} \mathbb{P}'(s_{\tau_i+2N} = s' | \tau_i, \mathcal{T}_i) \times \Pr \left[s_t \neq s, \forall t \in [\tau_i + 2N, \tau_{i+1} - 1], s_{\tau_{i+1}} = s \middle| s_{\tau_i+2N} = s' \right]. \quad (28)$$

Because of the reset in the imaginary world, $\mathbb{P}'(s_{\tau_i+2N} = s' | \tau_i, \mathcal{T}_i) = \mu^\pi(s')$ for all s' ; in the real world, since at time $\tau_i + 2N$, the process has proceeded N steps from $\tau_i + N$ (the last step of \mathcal{T}_i), by Corollary 13.1 we have

$$\frac{\mathbb{P}(s_{\tau_i+2N} = s' | \tau_i, \mathcal{T}_i)}{\mathbb{P}'(s_{\tau_i+2N} = s' | \tau_i, \mathcal{T}_i)} = 1 + \frac{\mathbb{P}(s_{\tau_i+2N} = s' | \tau_i, \mathcal{T}_i) - \mu^\pi(s')}{\mu^\pi(s')} \leq 1 + \frac{2}{T^4 \mu^\pi(s')} \leq 1 + \frac{1}{T^3} \quad \text{for all } s',$$

which implies $\frac{\mathbb{P}(\tau_{i+1}|\tau_i, \mathcal{T}_i)}{\mathbb{P}'(\tau_{i+1}|\tau_i, \mathcal{T}_i)} \leq 1 + \frac{1}{T^3}$ by (27) and (28). This further implies $\frac{\mathbb{P}(X)}{\mathbb{P}'(X)} \leq \left(1 + \frac{1}{T^3}\right)^M \leq e^{\frac{M}{T^3}} \leq e^{\frac{1}{T^2}} \leq 1 + \frac{2}{T^2}$ by (25) and (26). From (24), we then have

$$\frac{\mathbb{E}[\widehat{\beta}_k(s, a)]}{\mathbb{E}'[\widehat{\beta}_k(s, a)]} \leq 1 + \frac{2}{T^2}.$$

Thus, using the bound from Eq. (23) we have

$$\mathbb{E}[\widehat{\beta}_k(s, a)] \leq \left(1 + \frac{2}{T^2}\right) \mathbb{E}'[\widehat{\beta}_k(s, a)] \leq \left(1 + \frac{2}{T^2}\right) \left(\beta_k(s, a) + \mathcal{O}\left(\frac{1}{T^2}\right)\right) \leq \beta_k(s, a) + \mathcal{O}\left(\frac{1}{T}\right).$$

Similarly we can prove the other direction: $\beta_k(s, a) \leq \mathbb{E}[\widehat{\beta}_k(s, a)] + \mathcal{O}\left(\frac{1}{T}\right)$, finishing the proof. \square

Proof for Eq.(13). We use the same notations, and the similar approach as in the previous proof for Eq. (12). That is, we first bound the expectation of the desired quantity in the imaginary world, and then argue that the expectation in the imaginary world and that in the real world are close.

Step 1. Define $\Delta_i = \widehat{\beta}_{k,i}(s, a) - \beta^\pi(s, a) + \delta(s, a)$. Then $\mathbb{E}'[\Delta_i | \{w_i\}] = 0$ by Eq.(20). Thus in the imaginary world,

$$\begin{aligned}
 & \mathbb{E}' \left[\left(\widehat{\beta}_k(s, a) - \beta^\pi(s, a) \right)^2 \right] \\
 &= \mathbb{E}' \left[\left(\frac{1}{M} \sum_{i=1}^M \left(\widehat{\beta}_{k,i}(s, a) - \beta^\pi(s, a) \right) \right)^2 \mathbf{1}[M > 0] + \beta^\pi(s, a)^2 \mathbf{1}[M = 0] \right] \\
 &= \mathbb{E}' \left[\left(\frac{1}{M} \sum_{i=1}^M \Delta_i - \delta(s, a) \right)^2 \mathbf{1}[M > 0] + \beta^\pi(s, a)^2 \mathbf{1}[M = 0] \right] \\
 &\leq \mathbb{E}' \left[\left(2 \left(\frac{1}{M} \sum_{i=1}^M \Delta_i \right)^2 + 2\delta(s, a)^2 \right) \mathbf{1}[M > 0] + \beta^\pi(s, a)^2 \mathbf{1}[M = 0] \right] \quad (\text{using } (a-b)^2 \leq 2a^2 + 2b^2) \\
 &\leq \Pr[w_1 \leq B - N] \times \mathbb{E}'_{\{w_i\}} \left[\mathbb{E}' \left[2 \left(\frac{1}{M} \sum_{i=1}^M \Delta_i \right)^2 + 2\delta(s, a)^2 \mid \{w_i\} \right] \mid w_1 \leq B - N \right] + \Pr[w_1 > B - N] \times (N + 6t_{\text{mix}})^2 \\
 &\hspace{15em} (\beta^\pi(s, a) \leq N + 6t_{\text{mix}} \text{ by Lemma 14}) \\
 &\leq \mathbb{E}'_{\{w_i\}} \left[\mathbb{E}' \left[2 \left(\frac{1}{M} \sum_{i=1}^M \Delta_i \right)^2 \mid \{w_i\} \right] \mid w_1 \leq B - N \right] + \mathcal{O} \left(\frac{1}{T} \right) \\
 &\hspace{15em} (\text{using Lemma 16: } \Pr[w_1 > B - N] \leq \left(1 - \frac{3}{4t_{\text{hit}}} \right)^{\frac{B-N}{N}} \leq \frac{1}{T^3}.) \\
 &\leq \mathbb{E}'_{\{w_i\}} \left[\frac{2}{M^2} \sum_{i=1}^M \mathbb{E}' [\Delta_i^2 | \{w_i\}] \mid w_1 \leq B - N \right] + \mathcal{O} \left(\frac{1}{T} \right) \\
 &\hspace{15em} (\Delta_i \text{ is zero-mean and independent of each other conditioned on } \{w_i\}) \\
 &\leq \mathbb{E}'_{\{w_i\}} \left[\frac{2}{M^2} \cdot M \times \frac{\mathcal{O}(N^2)}{\pi(a|s)} \mid w_1 \leq B - N \right] + \mathcal{O} \left(\frac{1}{T} \right) \\
 &\hspace{15em} (\mathbb{E}'[\Delta_i^2] \leq \pi(a|s) \frac{\mathcal{O}(N^2)}{\pi(a|s)^2} = \frac{\mathcal{O}(N^2)}{\pi(a|s)} \text{ by definition of } \widehat{\beta}_k(s, a), \text{ Lemma 14, and Eq. (21)}) \\
 &\leq \frac{\mathcal{O}(N^2)}{\pi(a|s)} \mathbb{E}' \left[\frac{1}{M} \mid w_1 \leq B - N \right] + \mathcal{O} \left(\frac{1}{T} \right). \tag{29}
 \end{aligned}$$

Since $\Pr'[M = 0] \leq \frac{1}{T^3}$ by Lemma 16, we have $\Pr'[w_1 \leq B - N] = \Pr'[M > 0] \geq 1 - \frac{1}{T^3}$. Also note that if

$$M < M_0 := \frac{B - N}{2N + \frac{4N \log T}{\mu^\pi(s)}},$$

then there exists at least one waiting interval (i.e., w_i) longer than $\frac{4N \log T}{\mu^\pi(s)}$ (see Figure 1 or 2). By Lemma 16, this happens with probability smaller than $\left(1 - \frac{3\mu^\pi(s)}{4} \right)^{\frac{4 \log T}{\mu^\pi(s)}} \leq \frac{1}{T^3}$.

Therefore,

$$\begin{aligned}
 \mathbb{E}' \left[\frac{1}{M} \mid M > 0 \right] &= \frac{\sum_{m=1}^{\infty} \frac{1}{m} \Pr'[M = m]}{\Pr'[M > 0]} \leq \frac{1 \times \Pr'[M < M_0] + \frac{1}{M_0} \times \Pr'[M \geq M_0]}{\Pr'[M > 0]} \\
 &\leq \frac{1 \times \frac{1}{T^3} + \frac{2N + \frac{4N \log T}{\mu^\pi(s)}}{B - N}}{1 - \frac{1}{T^3}} \leq \mathcal{O} \left(\frac{N \log T}{B \mu^\pi(s)} \right).
 \end{aligned}$$

Combining with (29), we get

$$\mathbb{E}' \left[\left(\widehat{\beta}_k(s, a) - \beta^\pi(s, a) \right)^2 \right] \leq \mathcal{O} \left(\frac{N^3 \log T}{B \pi(a|s) \mu^\pi(s)} \right).$$

Step 2. By the same argument as in the ‘‘Step 2’’ of the previous proof for Eq. (12), we have

$$\mathbb{E} \left[\left(\widehat{\beta}_k(s, a) - \beta^\pi(s, a) \right)^2 \right] \leq \left(1 + \frac{2}{T^2} \right) \mathbb{E}' \left[\left(\widehat{\beta}_k(s, a) - \beta^\pi(s, a) \right)^2 \right] \leq \mathcal{O} \left(\frac{N^3 \log T}{B\pi(a|s)\mu^\pi(s)} \right),$$

which finishes the proof. \square

B.3. Proof for Lemma 7

Proof. We defer the proof of Eq. (14) to Lemma 17 and prove the rest of the statements assuming Eq. (14). First, we have

$$\begin{aligned} |J^{\pi_k} - J^{\pi_{k-1}}| &= \left| \sum_s \sum_a \mu^{\pi_k}(s) (\pi_k(a|s) - \pi_{k-1}(a|s)) q^{\pi_{k-1}}(s, a) \right| && \text{(By Lemma 15)} \\ &\leq \sum_s \sum_a \mu^{\pi_k}(s) |(\pi_k(a|s) - \pi_{k-1}(a|s))| |q^{\pi_{k-1}}(s, a)| \\ &= \mathcal{O} \left(\sum_s \sum_a \mu^{\pi_k}(s) N \eta \pi_{k-1}(a|s) t_{\text{mix}} \right) && \text{(By Eq. (14) and Lemma 14)} \\ &= \mathcal{O}(\eta t_{\text{mix}} N) = \mathcal{O}(\eta N^2). && (30) \end{aligned}$$

Next, to prove a bound on $|v^{\pi_k}(s) - v^{\pi_{k-1}}(s)|$, first note that for any policy π ,

$$\begin{aligned} v^\pi(s) &= \sum_{n=0}^{\infty} (\mathbf{e}_s^\top (P^\pi)^n - (\mu^\pi)^\top) r^\pi && \text{(By Eq. (4))} \\ &= \sum_{n=0}^{N-1} (\mathbf{e}_s^\top (P^\pi)^n - (\mu^\pi)^\top) r^\pi + \sum_{n=N}^{\infty} (\mathbf{e}_s^\top (P^\pi)^n - (\mu^\pi)^\top) r^\pi \\ &= \sum_{n=0}^{N-1} \mathbf{e}_s^\top (P^\pi)^n r^\pi - N J^\pi + \text{error}^\pi(s), && (J^\pi = (\mu^\pi)^\top r^\pi) \end{aligned}$$

where $\text{error}^\pi(s) := \sum_{n=N}^{\infty} (\mathbf{e}_s^\top (P^\pi)^n - \mu^\pi)^\top r^\pi$. By Corollary 13.2, $|\text{error}^\pi(s)| \leq \frac{1}{T^2}$. Thus

$$\begin{aligned} |v^{\pi_k}(s) - v^{\pi_{k-1}}(s)| &= \left| \sum_{n=0}^{N-1} \mathbf{e}_s^\top ((P^{\pi_k})^n - (P^{\pi_{k-1}})^n) r^{\pi_k} + \sum_{n=0}^{N-1} \mathbf{e}_s^\top (P^{\pi_{k-1}})^n (r^{\pi_k} - r^{\pi_{k-1}}) - N J^{\pi_k} + N J^{\pi_{k-1}} \right| + \frac{2}{T^2} \\ &\leq \sum_{n=0}^{N-1} \|((P^{\pi_k})^n - (P^{\pi_{k-1}})^n) r^{\pi_k}\|_\infty + \sum_{n=0}^{N-1} \|r^{\pi_k} - r^{\pi_{k-1}}\|_\infty + N |J^{\pi_k} - J^{\pi_{k-1}}| + \frac{2}{T^2}. \end{aligned} \quad (31)$$

Below we bound each individual term above (using notation $\pi' := \pi_k, \pi := \pi_{k-1}, P' := P^{\pi_k}, P := P^{\pi_{k-1}}, r' := r^{\pi_k}, r := r^{\pi_{k-1}}, \mu := \mu^{\pi_{k-1}}$ for simplicity). The first term can be bounded as

$$\begin{aligned} &\|(P'^n - P^n) r'\|_\infty \\ &= \|(P'(P'^{n-1} - P^{n-1}) + (P' - P)P^{n-1}) r'\|_\infty \\ &\leq \|P'(P'^{n-1} - P^{n-1}) r'\|_\infty + \|(P' - P)P^{n-1} r'\|_\infty \\ &\leq \|(P'^{n-1} - P^{n-1}) r'\|_\infty + \|(P' - P)P^{n-1} r'\|_\infty && \text{(because every row of } P' \text{ sums to 1)} \\ &= \|(P'^{n-1} - P^{n-1}) r'\|_\infty + \max_s |\mathbf{e}_s^\top (P' - P)P^{n-1} r'| \\ &\leq \|(P'^{n-1} - P^{n-1}) r'\|_\infty + \max_s \|\mathbf{e}_s^\top (P' - P)P^{n-1}\|_1, \end{aligned}$$

where the last term can be further bounded by

$$\begin{aligned}
 \max_s \|\mathbf{e}_s^\top (P' - P)P^{n-1}\|_1 &\leq \max_s \|\mathbf{e}_s^\top (P' - P)\|_1 \\
 &= \max_s \left(\sum_{s'} \left| \sum_a (\pi'(a|s) - \pi(a|s))p(s'|s, a) \right| \right) \\
 &\leq \mathcal{O} \left(\max_s \left(\sum_{s'} \sum_a \eta N \pi(a|s)p(s'|s, a) \right) \right) \quad (\text{By Eq. (14)}) \\
 &= \mathcal{O}(\eta N).
 \end{aligned}$$

Repeatedly applying this bound we arrive at $\|(P'^n - P^n)r'\|_\infty \leq \mathcal{O}(\eta N^2)$, and therefore,

$$\sum_{n=0}^{N-1} \|((P'^n - P^n)r')\|_\infty \leq \mathcal{O}(\eta N^3).$$

The second term in Eq. (31) can be bounded as (by Eq. (14) again)

$$\sum_{n=0}^{N-1} \|r' - r\|_\infty = \sum_{n=0}^{N-1} \max_s \left| \sum_a (\pi'(a|s) - \pi(a|s))r(s, a) \right| \leq \mathcal{O} \left(\sum_{n=0}^{N-1} \max_s \sum_a \eta N \pi(a|s) \right) = \mathcal{O}(\eta N^2),$$

and the third term in Eq. (31) is bounded via the earlier proof (for bounding $|J^{\pi_k} - J^{\pi_{k-1}}|$):

$$N |J^{\pi_k} - J^{\pi_{k-1}}| = \mathcal{O}(\eta N^3). \quad (\text{Eq.(30)})$$

Plugging everything into Eq.(31), we prove $|v^{\pi_k}(s) - v^{\pi_{k-1}}(s)| = \mathcal{O}(\eta N^3)$.

Finally, it is straightforward to prove the rest of the two statements:

$$\begin{aligned}
 |q^{\pi_k}(s, a) - q^{\pi_{k-1}}(s, a)| &= |r(s, a) + \mathbb{E}_{s' \sim p(\cdot|s, a)}[v^{\pi_k}(s')] - r(s, a) - \mathbb{E}_{s' \sim p(\cdot|s, a)}[v^{\pi_{k-1}}(s')]| \\
 &= |\mathbb{E}_{s' \sim p(\cdot|s, a)}[v^{\pi_k}(s') - v^{\pi_{k-1}}(s')]| = \mathcal{O}(\eta N^3).
 \end{aligned}$$

$$|\beta^{\pi_k}(s, a) - \beta^{\pi_{k-1}}(s, a)| \leq |q^{\pi_k}(s, a) - q^{\pi_{k-1}}(s, a)| + N |J^{\pi_k} - J^{\pi_{k-1}}| = \mathcal{O}(\eta N^3).$$

This completes the proof. \square

C. Analyzing Optimistic Online Mirror Descent with Log-barrier Regularizer — Proofs for Eq.(14), Lemma 8, and Lemma 9

In this section, we derive the *stability property* (Eq.(14)) and the regret bound (Lemma 8 and Lemma 9) for optimistic online mirror descent with the log-barrier regularizer. Most of the analysis is similar to that in (Wei & Luo, 2018; Bubeck et al., 2019). Since in our MDP-OOMD algorithm, we run optimistic online mirror descent independently on each state, the analysis in this section only focuses on a specific state s . We simplify our notations using $\pi_k(\cdot) := \pi_k(\cdot|s)$, $\pi'_k(\cdot) := \pi'_k(\cdot|s)$, $\widehat{\beta}_k(\cdot) := \widehat{\beta}_k(s, \cdot)$ throughout the whole section.

Our MDP-OOMD algorithm is effectively running Algorithm 5 on each state. We first verify that the condition in Line 1 of Algorithm 5 indeed holds in our MDP-OOMD algorithm. Recall that in ESTIMATEQ (Algorithm 3) we collect trajectories in every episode for every state. Suppose for episode k and state s it collects M trajectories that start from time τ_1, \dots, τ_M and has total reward R_1, \dots, R_M respectively. Let $m_a = \sum_{i=1}^M \mathbf{1}[a_{\tau_i} = a]$, then we have $\sum_a m_a = M$. By our way of constructing $\widehat{\beta}_k(s, \cdot)$, we have

$$\widehat{\beta}_k(s, a) = \sum_{i=1}^M \frac{R_i \mathbf{1}[a_{\tau_i} = a]}{M \pi_k(a|s)}$$

when $M > 0$. Thus we have $\sum_a \pi_k(a|s) \widehat{\beta}_k(s, a) = \sum_a \sum_{i=1}^M \frac{R_i \mathbf{1}[a_{\tau_i} = a]}{M} = \sum_{i=1}^M \frac{R_i}{M} \leq (N + 1)$ because every R_i is the total reward for an interval of length $N + 1$. This verifies the condition in Line 1 for the case $M > 0$. When $M = 0$, ESTIMATEQ sets $\widehat{\beta}(s, \cdot)$ to zero so the condition clearly still holds.

Algorithm 5 Optimistic Online Mirror Descent (OOMD) with log-barrier regularizer

Define:
 $C := N + 1$

 Regularizer $\psi(x) = \frac{1}{\eta} \sum_{a=1}^A \log \frac{1}{x(a)}$, for $x \in \mathbb{R}_+^A$

 Bregman divergence associated with ψ :

$$D_\psi(x, x') = \psi(x) - \psi(x') - \langle \nabla \psi(x'), x - x' \rangle$$

Initialization: $\pi'_1 = \pi_1 = \frac{1}{A} \mathbf{1}$
for $k = 1, \dots, K$ **do**
1 | Receive $\widehat{\beta}_k \in \mathbb{R}_+^A$ for which $\sum_a \pi_k(a) \widehat{\beta}_k(a) \leq C$.

2 | Update

$$\pi'_{k+1} = \operatorname{argmax}_{\pi \in \Delta_A} \left\{ \langle \pi, \widehat{\beta}_k \rangle - D_\psi(\pi, \pi'_k) \right\}$$

$$\pi_{k+1} = \operatorname{argmax}_{\pi \in \Delta_A} \left\{ \langle \pi, \widehat{\beta}_k \rangle - D_\psi(\pi, \pi'_{k+1}) \right\}$$

C.1. The stability property of Algorithm 5 — Proof of Eq.(14)

The statement and the proofs of Lemmas 17 and 18 are almost identical to those of Lemma 9 and 10 in (Bubeck et al., 2019).

Lemma 17. *In Algorithm 5, if $\eta \leq \frac{1}{270C} = \frac{1}{270(N+1)}$, then*

$$|\pi_{k+1}(a) - \pi_k(a)| \leq 120\eta C \pi_k(a).$$

 To prove this lemma we make use of the following auxiliary result, where we use the notation $\|a\|_M = \sqrt{a^\top M a}$ for a vector $a \in \mathbb{R}^A$ and a positive semi-definite matrix $M \in \mathbb{R}^{A \times A}$.

Lemma 18. *For some arbitrary $b_1, b_2 \in \mathbb{R}^A$, $a_0 \in \Delta_A$ with $\eta \leq \frac{1}{270C}$, define*

$$\begin{cases} a_1 = \operatorname{argmin}_{a \in \Delta_A} F_1(a), & \text{where } F_1(a) \triangleq \langle a, b_1 \rangle + D_\psi(a, a_0), \\ a_2 = \operatorname{argmin}_{a \in \Delta_A} F_2(a), & \text{where } F_2(a) \triangleq \langle a, b_2 \rangle + D_\psi(a, a_0). \end{cases}$$

(ψ and D_ψ are defined in Algorithm 5). Then as long as $\|b_1 - b_2\|_{\nabla^{-2}\psi(a_1)} \leq 12\sqrt{\eta}C$, we have for all $i \in [A]$, $|a_{2,i} - a_{1,i}| \leq 60\eta C a_{1,i}$.
Proof of Lemma 18. First, we prove $\|a_1 - a_2\|_{\nabla^2\psi(a_1)} \leq 60\sqrt{\eta}C$ by contradiction. Assume $\|a_1 - a_2\|_{\nabla^2\psi(a_1)} > 60\sqrt{\eta}C$. Then there exists some a'_2 lying in the line segment between a_1 and a_2 such that $\|a_1 - a'_2\|_{\nabla^2\psi(a_1)} = 60\sqrt{\eta}C$. By Taylor's theorem, there exists \bar{a} that lies in the line segment between a_1 and a'_2 such that

$$\begin{aligned} F_2(a'_2) &= F_2(a_1) + \langle \nabla F_2(a_1), a'_2 - a_1 \rangle + \frac{1}{2} \|a'_2 - a_1\|_{\nabla^2 F_2(\bar{a})}^2 \\ &= F_2(a_1) + \langle b_2 - b_1, a'_2 - a_1 \rangle + \langle \nabla F_1(a_1), a'_2 - a_1 \rangle + \frac{1}{2} \|a'_2 - a_1\|_{\nabla^2\psi(\bar{a})}^2 \\ &\geq F_2(a_1) - \|b_2 - b_1\|_{\nabla^{-2}\psi(a_1)} \|a'_2 - a_1\|_{\nabla^2\psi(a_1)} + \frac{1}{2} \|a'_2 - a_1\|_{\nabla^2\psi(\bar{a})}^2 \\ &\geq F_2(a_1) - 12\sqrt{\eta}C \times 60\sqrt{\eta}C + \frac{1}{2} \|a'_2 - a_1\|_{\nabla^2\psi(\bar{a})}^2 \end{aligned} \tag{32}$$

 where in the first inequality we use Hölder inequality and the first-order optimality condition $\langle \nabla F_1(a_1), a'_2 - a_1 \rangle \geq 0$, and in the last inequality we use the conditions $\|b_1 - b_2\|_{\nabla^{-2}\psi(a_1)} \leq 12\sqrt{\eta}C$ and $\|a_1 - a'_2\|_{\nabla^2\psi(a_1)} = 60\sqrt{\eta}C$. Note that

$\nabla^2\psi(x)$ is a diagonal matrix and $\nabla^2\psi(x)_{ii} = \frac{1}{\eta} \frac{1}{x_i^2}$. Therefore for any $i \in [A]$,

$$60\sqrt{\eta}C = \|a'_2 - a_1\|_{\nabla^2\psi(a_1)} = \sqrt{\sum_{j=1}^A \frac{(a'_{2,j} - a_{1,j})^2}{\eta a_{1,j}^2}} \geq \frac{|a'_{2,i} - a_{1,i}|}{\sqrt{\eta}a_{1,i}}$$

and thus $\frac{|a'_{2,i} - a_{1,i}|}{a_{1,i}} \leq 60\eta C \leq \frac{2}{9}$, which implies $\max\left\{\frac{a'_{2,i}}{a_{1,i}}, \frac{a_{1,i}}{a'_{2,i}}\right\} \leq \frac{9}{7}$. Thus the last term in (32) can be lower bounded by

$$\begin{aligned} \|a'_2 - a_1\|_{\nabla^2\psi(\bar{a})}^2 &= \frac{1}{\eta} \sum_{i=1}^A \frac{1}{\bar{a}_i^2} (a'_{2,i} - a_{1,i})^2 \geq \frac{1}{\eta} \left(\frac{7}{9}\right)^2 \sum_{i=1}^A \frac{1}{a_{1,i}^2} (a'_{2,i} - a_{1,i})^2 \\ &\geq 0.6 \|a'_2 - a_1\|_{\nabla^2\psi(a_1)}^2 = 0.6 \times (60\sqrt{\eta}C)^2 = 2160\eta C^2. \end{aligned}$$

Combining with (32) gives

$$F_2(a'_2) \geq F_2(a_1) - 720\eta C^2 + \frac{1}{2} \times 2160\eta C^2 > F_2(a_1).$$

Recall that a'_2 is a point in the line segment between a_1 and a_2 . By the convexity of F_2 , the above inequality implies $F_2(a_1) < F_2(a_2)$, contradicting the optimality of a_2 .

Thus we conclude $\|a_1 - a_2\|_{\nabla^2\psi(a_1)} \leq 60\sqrt{\eta}C$. Since $\|a_1 - a_2\|_{\nabla^2\psi(a_1)} = \sqrt{\sum_{j=1}^A \frac{(a_{1,j} - a_{2,j})^2}{\eta a_{1,j}^2}} \geq \frac{|a_{2,i} - a_{1,i}|}{\sqrt{\eta}a_{1,i}}$ for all i , we get $\frac{|a_{2,i} - a_{1,i}|}{\sqrt{\eta}a_{1,i}} \leq 60\sqrt{\eta}C$, which implies $|a_{2,i} - a_{1,i}| \leq 60\eta C a_{1,i}$. \square

Proof of Lemma 17. We prove the following stability inequalities

$$|\pi_k(a) - \pi'_{k+1}(a)| \leq 60\eta C \pi_k(a), \quad (33)$$

$$|\pi'_{k+1}(a) - \pi_{k+1}(a)| \leq 60\eta C \pi_k(a). \quad (34)$$

Note that (33) and (34) imply

$$|\pi_k(a) - \pi_{k+1}(a)| \leq 120\eta C \pi_k(a), \quad (35)$$

which is the inequality we want to prove.

We use induction on k to prove (33) and (34). Note that (33) implies

$$\pi'_{k+1}(a) \leq \pi_k(a) + 60\eta C \pi_k(a) \leq \pi_k(a) + \frac{60}{270} \pi_k(a) \leq 2\pi_k(a), \quad (36)$$

and (35) implies

$$\pi_{k+1}(a) \leq \pi_k(a) + 120\eta C \pi_k(a) \leq \pi_k(a) + \frac{120}{270} \pi_k(a) \leq 2\pi_k(a). \quad (37)$$

Thus, (36) and (37) are also inequalities we may use in the induction process.

Base case. For the case $k = 1$, note that

$$\begin{cases} \pi_1 = \operatorname{argmin}_{\pi \in \Delta_A} D_\psi(\pi, \pi'_1), & (\text{because } \pi_1 = \pi'_1) \\ \pi'_2 = \operatorname{argmin}_{\pi \in \Delta_A} \langle \pi, -\hat{\beta}_1 \rangle + D_\psi(\pi, \pi'_1). \end{cases}$$

To apply Lemma 18 and obtain (33), we only need to show $\|\hat{\beta}_1\|_{\nabla^{-2}\psi(\pi_1)} \leq 12\sqrt{\eta}C$. Recall $\nabla^2\psi(u)_{ii} = \frac{1}{\eta} \frac{1}{u_i^2}$ and $\nabla^{-2}\psi(u)_{ii} = \eta u_i^2$. Thus,

$$\|\hat{\beta}_1\|_{\nabla^{-2}\psi(\pi_1)}^2 \leq \sum_{a=1}^A \eta \pi_1(a)^2 \hat{\beta}_1(a)^2 \leq \eta C^2$$

because $\sum_a \pi_1(a)^2 \widehat{\beta}_1(a)^2 \leq \left(\sum_a \pi_1(a) \widehat{\beta}_1(a)\right)^2 \leq C^2$ by the condition in Line 1 of Algorithm 5. This proves (33) for the base case.

Now we prove (34) of the base case. Note that

$$\begin{cases} \pi'_2 = \operatorname{argmin}_{\pi \in \Delta_A} D_\psi(\pi, \pi'_2), \\ \pi_2 = \operatorname{argmin}_{\pi \in \Delta_A} \langle \pi, -\widehat{\beta}_1 \rangle + D_\psi(\pi, \pi'_2). \end{cases} \quad (38)$$

Similarly, with the help of Lemma 18, we only need to show $\|\widehat{\beta}_1\|_{\nabla^{-2}\psi(\pi'_2)} \leq 12\sqrt{\eta}C$. This can be verified by

$$\|\widehat{\beta}_1\|_{\nabla^{-2}\psi(\pi'_2)}^2 \leq \sum_{a=1}^A \eta \pi'_2(a)^2 \widehat{\beta}_1(a)^2 \leq 4 \sum_{a=1}^A \eta \pi_1(a)^2 \widehat{\beta}_1(a)^2 \leq 4\eta C^2,$$

where the second inequality uses (36) for the base case (implied by (33) for the base case, which we just proved).

Induction. Assume (33) and (34) hold before k . To prove (33), observe that

$$\begin{cases} \pi_k = \operatorname{argmin}_{\pi \in \Delta_A} \langle \pi, -\widehat{\beta}_{k-1} \rangle + D_\psi(\pi, \pi'_k), \\ \pi'_{k+1} = \operatorname{argmin}_{\pi \in \Delta_A} \langle \pi, -\widehat{\beta}_k \rangle + D_\psi(\pi, \pi'_k). \end{cases} \quad (39)$$

To apply Lemma 18 and obtain (33), we only need to show $\|\widehat{\beta}_k - \widehat{\beta}_{k-1}\|_{\nabla^{-2}\psi(\pi_k)} \leq 12\sqrt{\eta}C$. This can be verified by

$$\begin{aligned} \|\widehat{\beta}_k - \widehat{\beta}_{k-1}\|_{\nabla^{-2}\psi(\pi_k)}^2 &\leq \sum_{a=1}^A \eta \pi_k(a)^2 \left(\widehat{\beta}_k(a) - \widehat{\beta}_{k-1}(a)\right)^2 \\ &\leq 2\eta \sum_{a=1}^A \pi_k(a)^2 \left(\widehat{\beta}_k(a)^2 + \widehat{\beta}_{k-1}(a)^2\right) \\ &\leq 2\eta \sum_{a=1}^A \pi_k(a)^2 \widehat{\beta}_k(a)^2 + 2\eta \sum_{a=1}^A 4\pi_{k-1}(a)^2 \widehat{\beta}_{k-1}(a)^2 \\ &\leq 10\eta C^2, \end{aligned}$$

where the third inequality uses (37) for $k-1$.

To prove (34), we observe:

$$\begin{cases} \pi'_{k+1} = \operatorname{argmin}_{\pi \in \Delta_A} D_\psi(\pi, \pi'_{k+1}), \\ \pi_{k+1} = \operatorname{argmin}_{\pi \in \Delta_A} \langle \pi, -\widehat{\beta}_k \rangle + D_\psi(\pi, \pi'_{k+1}). \end{cases} \quad (40)$$

Similarly, with the help of Lemma 18, we only need to show $\|\widehat{\beta}_k\|_{\nabla^{-2}\psi(\pi'_{k+1})} \leq 12\sqrt{\eta}C$. This can be verified by

$$\|\widehat{\beta}_k\|_{\nabla^{-2}\psi(\pi'_{k+1})}^2 \leq \sum_{a=1}^A \eta \pi'_{k+1}(a)^2 \widehat{\beta}_k(a)^2 \leq 4 \sum_{a=1}^A \eta \pi_k(a)^2 \widehat{\beta}_k(a)^2 \leq 4\eta C^2,$$

where in the second inequality we use (36) (implied by (33), which we just proved). This finishes the proof. \square

C.2. The regret bound of Algorithm 5 — Proof of Lemma 8

Proof of Lemma 8. By standard analysis for optimistic online mirror descent (e.g. (Wei & Luo, 2018, Lemma 6), (Chiang et al., 2012, Lemma 5)), we have (recall $\widehat{\beta}_0$ is the all-zero vector)

$$\langle \tilde{\pi} - \pi_k, \widehat{\beta}_k \rangle \leq D_\psi(\tilde{\pi}, \pi'_k) - D_\psi(\tilde{\pi}, \pi'_{k+1}) + \langle \pi_k - \pi'_{k+1}, \widehat{\beta}_{k-1} - \widehat{\beta}_k \rangle \quad (41)$$

for any $\tilde{\pi} \in \Delta_A$. Summing over k and telescoping give

$$\sum_{k=1}^K \langle \tilde{\pi} - \pi_k, \widehat{\beta}_k \rangle \leq D_\psi(\tilde{\pi}, \pi'_1) - D_\psi(\tilde{\pi}, \pi'_{K+1}) + \sum_{k=1}^K \langle \pi_k - \pi'_{k+1}, \widehat{\beta}_{k-1} - \widehat{\beta}_k \rangle \leq D_\psi(\tilde{\pi}, \pi'_1) + \sum_{k=1}^K \langle \pi_k - \pi'_{k+1}, \widehat{\beta}_{k-1} - \widehat{\beta}_k \rangle.$$

As in (Wei & Luo, 2018), we pick $\tilde{\pi} = (1 - \frac{1}{T}) \pi^* + \frac{1}{TA} \mathbf{1}_A$, and thus

$$\begin{aligned} D_\psi(\tilde{\pi}, \pi'_1) &= \psi(\tilde{\pi}) - \psi(\pi'_1) - \langle \nabla \psi(\pi'_1), \tilde{\pi} - \pi'_1 \rangle \\ &= \psi(\tilde{\pi}) - \psi(\pi'_1) && (\nabla \psi(\pi'_1) = -\frac{A}{\eta} \mathbf{1} \text{ and } \langle \mathbf{1}, \tilde{\pi} - \pi'_1 \rangle = 0) \\ &= \frac{1}{\eta} \sum_{a=1}^A \log \frac{1}{\tilde{\pi}(a)} - \frac{1}{\eta} \sum_{a=1}^A \log \frac{1}{\pi'_1(a)} \\ &\leq \frac{A \log(AT)}{\eta} - \frac{A \log A}{\eta} = \frac{A \ln T}{\eta}. \end{aligned}$$

On the other hand, to bound $\langle \pi_k - \pi'_{k+1}, \widehat{\beta}_{k-1} - \widehat{\beta}_k \rangle$, we follow the same approach as in (Wei & Luo, 2018, Lemma 14): define $F_k(\pi) = \langle \pi, -\widehat{\beta}_{k-1} \rangle + D_\psi(\pi, \pi'_k)$ and $F'_{k+1}(\pi) = \langle \pi, -\widehat{\beta}_k \rangle + D_\psi(\pi, \pi'_k)$. Then by definition we have $\pi_k = \operatorname{argmin}_{\pi \in \Delta_A} F_k(\pi)$ and $\pi'_{k+1} = \operatorname{argmin}_{\pi \in \Delta_A} F'_{k+1}(\pi)$.

Observe that

$$\begin{aligned} F'_{k+1}(\pi_k) - F'_{k+1}(\pi'_{k+1}) &= (\pi_k - \pi'_{k+1})^\top (\widehat{\beta}_{k-1} - \widehat{\beta}_k) + F_k(\pi_k) - F_k(\pi'_{k+1}) \\ &\leq (\pi_k - \pi'_{k+1})^\top (\widehat{\beta}_{k-1} - \widehat{\beta}_k) && \text{(by the optimality of } \pi_k) \\ &\leq \|\pi_k - \pi'_{k+1}\|_{\nabla^2 \psi(\pi_k)} \left\| \widehat{\beta}_{k-1} - \widehat{\beta}_k \right\|_{\nabla^{-2} \psi(\pi_k)}. \end{aligned} \quad (42)$$

On the other hand, for some ξ that lies on the line segment between π_k and π'_{k+1} , we have by Taylor's theorem and the optimality of π'_{k+1} ,

$$\begin{aligned} F'_{k+1}(\pi_k) - F'_{k+1}(\pi'_{k+1}) &= \nabla F'_{k+1}(\pi'_{k+1})^\top (\pi_k - \pi'_{k+1}) + \frac{1}{2} \|\pi_k - \pi'_{k+1}\|_{\nabla^2 F'_{k+1}(\xi)}^2 \\ &\geq \frac{1}{2} \|\pi_k - \pi'_{k+1}\|_{\nabla^2 \psi(\xi)}^2 && \text{(by the optimality of } \pi'_{k+1} \text{ and that } \nabla^2 F'_{k+1} = \nabla^2 \psi) \end{aligned} \quad (43)$$

By Eq.(33) we know $\pi'_{k+1}(a) \in [\frac{1}{2}\pi_k(a), 2\pi_k(a)]$, and hence $\xi(a) \in [\frac{1}{2}\pi_k(a), 2\pi_k(a)]$ holds as well, because ξ is in the line segment between π_k and π'_{k+1} . This implies for any x ,

$$\|x\|_{\nabla^2 \psi(\xi)} = \sqrt{\sum_{a=1}^A \frac{x(a)^2}{\eta \xi(a)^2}} \geq \frac{1}{2} \sqrt{\sum_{a=1}^A \frac{x(a)^2}{\eta \pi_k(a)^2}} = \frac{1}{2} \|x\|_{\nabla^2 \psi(\pi_k)}.$$

Combine this with (42) and (43), we get

$$\|\pi_k - \pi'_{k+1}\|_{\nabla^2 \psi(\pi_k)} \left\| \widehat{\beta}_{k-1} - \widehat{\beta}_k \right\|_{\nabla^{-2} \psi(\pi_k)} \geq \frac{1}{8} \|\pi_k - \pi'_{k+1}\|_{\nabla^2 \psi(\pi_k)}^2,$$

which implies $\|\pi_k - \pi'_{k+1}\|_{\nabla^2 \psi(\pi_k)} \leq 8 \left\| \widehat{\beta}_{k-1} - \widehat{\beta}_k \right\|_{\nabla^{-2} \psi(\pi_k)}$. Hence we can bound the third term in (41) by

$$\|\pi_k - \pi'_{k+1}\|_{\nabla^2 \psi(\pi_k)} \left\| \widehat{\beta}_{k-1} - \widehat{\beta}_k \right\|_{\nabla^{-2} \psi(\pi_k)} \leq 8 \left\| \widehat{\beta}_{k-1} - \widehat{\beta}_k \right\|_{\nabla^{-2} \psi(\pi_k)}^2 = 8\eta \sum_a \pi_k(a)^2 \left(\widehat{\beta}_{k-1}(a) - \widehat{\beta}_k(a) \right)^2.$$

Finally, combining everything we have

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{k=1}^K \langle \pi^* - \pi_k, \widehat{\beta}_k \rangle \right] \\
 &= \mathbb{E} \left[\sum_{k=1}^K \langle \pi^* - \tilde{\pi}, \widehat{\beta}_k \rangle + \langle \tilde{\pi} - \pi_k, \widehat{\beta}_k \rangle \right] \\
 &\leq \left[\frac{1}{T} \sum_{k=1}^K \left\langle \pi^* - \frac{1}{A} \mathbf{1}, \widehat{\beta}_k \right\rangle \right] + \mathcal{O} \left(\frac{A \log T}{\eta} + \eta \sum_{k=1}^K \sum_a \pi_k(a)^2 \left(\widehat{\beta}_{k-1}(a) - \widehat{\beta}_k(a) \right)^2 \right),
 \end{aligned}$$

where the expectation of the first term is bounded by $\mathcal{O} \left(\frac{KN}{T} \right) = \mathcal{O}(1)$ by the fact $\mathbb{E}[\widehat{\beta}_k(s)] = \mathcal{O}(N)$ (implied by Lemma 6 and Lemma 14). This completes the proof. \square

C.3. Proof for Lemma 9

Lemma 19 (Restatement of Lemma 9).

$$\begin{aligned}
 & \mathbb{E} \left[B \sum_{k=1}^K \sum_s \sum_a \mu^*(s) (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right] \\
 &= \tilde{\mathcal{O}} \left(\frac{BA \ln T}{\eta} + \eta \frac{TN^3 \rho}{B} + \eta^3 TN^6 \right).
 \end{aligned}$$

With the choice of $\eta = \min \left\{ \frac{1}{270(N+1)}, \frac{B\sqrt{A}}{\sqrt{\rho TN^3}}, \frac{\sqrt[4]{BA}}{\sqrt[4]{TN^6}} \right\}$, the bound becomes

$$\tilde{\mathcal{O}} \left(\sqrt{N^3 \rho AT} + (BAN^2)^{\frac{3}{4}} T^{\frac{1}{4}} + BNA \right) = \tilde{\mathcal{O}} \left(\sqrt{t_{\text{mix}}^3 \rho AT} + (t_{\text{mix}}^3 t_{\text{hit}} A)^{\frac{3}{4}} T^{\frac{1}{4}} + t_{\text{mix}}^2 t_{\text{hit}} A \right).$$

Proof. For any s ,

$$\begin{aligned}
 & \mathbb{E} \left[\sum_{k=1}^K \sum_a (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right] \\
 &= \mathbb{E} \left[\sum_{k=1}^K \sum_a (\pi^*(a|s) - \pi_k(a|s)) \beta^{\pi_k}(s, a) \right] \quad (\text{by the definition of } \beta^{\pi_k} \text{ and that } \sum_a (\pi^*(a|s) - \pi_k(a|s)) J^{\pi_k} = 0) \\
 &\leq \mathbb{E} \left[\sum_{k=1}^K \sum_a (\pi^*(a|s) - \pi_k(a|s)) \mathbb{E}_k [\widehat{\beta}_k(s, a)] \right] + \mathcal{O} \left(\frac{K}{T} \right) \quad (\text{by Eq. (12)}) \\
 &= \mathcal{O} \left(\frac{A \ln T}{\eta} \right) + \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=1}^K \sum_a \pi_k(a|s)^2 (\widehat{\beta}_k(s, a) - \widehat{\beta}_{k-1}(s, a))^2 \right] \right) \quad (\text{by Lemma 8}) \\
 &\leq \mathcal{O} \left(\frac{A \ln T}{\eta} + \eta N^2 \right) + \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 (\widehat{\beta}_k(s, a) - \beta^{\pi_k}(s, a))^2 \right] \right) \\
 &\quad + \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 (\beta^{\pi_k}(s, a) - \beta^{\pi_{k-1}}(s, a))^2 \right] \right) \\
 &\quad + \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 (\beta^{\pi_{k-1}}(s, a) - \widehat{\beta}_{k-1}(s, a))^2 \right] \right), \tag{44}
 \end{aligned}$$

where the last line uses the fact $(z_1 + z_2 + z_3)^2 \leq 3z_1^2 + 3z_2^2 + 3z_3^2$. The second term in (44) can be bounded using Eq. (13):

$$\begin{aligned} & \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 (\hat{\beta}_k(s, a) - \beta^{\pi_k}(s, a))^2 \right] \right) \\ &= \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 \frac{N^3 \log T}{B \pi_k(a|s) \mu^{\pi_k}(s)} \right] \right) \\ &= \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \frac{N^3 \log T}{B \mu^{\pi_k}(s)} \right] \right). \end{aligned}$$

The fourth term in (44) can be bounded similarly, except that we first use Lemma 17 to upper bound $\pi_k(a|s)$ by $2\pi_{k-1}(a|s)$. Eventually this term is upper bounded by $\mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \frac{N^3 \log T}{B \mu^{\pi_{k-1}}(s)} \right] \right) = \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=1}^K \frac{N^3 \log T}{B \mu^{\pi_k}(s)} \right] \right)$.

The third term in (44) can be bounded using Lemma 7:

$$\begin{aligned} & \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 (\beta^{\pi_k}(s, a) - \beta^{\pi_{k-1}}(s, a))^2 \right] \right) \\ &= \mathcal{O} \left(\eta \mathbb{E} \left[\sum_{k=2}^K \sum_a \pi_k(a|s)^2 (\eta N^3)^2 \right] \right) \\ &= \mathcal{O} (\eta^3 K N^6). \end{aligned}$$

Combining all these bounds in (44), we get

$$\mathbb{E} \left[\sum_{k=1}^K \sum_a (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right] = \mathcal{O} \left(\frac{A \ln T}{\eta} + \eta \mathbb{E} \left[\sum_{k=1}^K \frac{N^3 \log T}{B \mu^{\pi_k}(s)} \right] + \eta^3 K N^6 \right).$$

Now multiplying both sides by $B \mu^*(s)$ and summing over s we get

$$\begin{aligned} \mathbb{E} \left[B \sum_{k=1}^K \sum_s \sum_a \mu^*(s) (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right] &= \mathcal{O} \left(\frac{BA \ln T}{\eta} + \eta \mathbb{E} \left[\sum_{k=1}^K \sum_s \frac{N^3 (\log T) \mu^*(s)}{\mu^{\pi_k}(s)} \right] + \eta^3 B K N^6 \right) \\ &\leq \mathcal{O} \left(\frac{BA \ln T}{\eta} + \eta \rho K N^3 (\log T) + \eta^3 B K N^6 \right) \\ &= \tilde{\mathcal{O}} \left(\frac{BA}{\eta} + \eta \rho \frac{TN^3}{B} + \eta^3 T N^6 \right) \quad (T = BK) \end{aligned}$$

Choosing $\eta = \min \left\{ \frac{1}{270(N+1)}, \frac{B\sqrt{A}}{\sqrt{\rho T N^3}}, \frac{\sqrt[4]{BA}}{\sqrt[4]{T N^6}} \right\}$ ($\eta \leq \frac{1}{270(N+1)}$ is required by Lemma 17), we finally obtain

$$\begin{aligned} \mathbb{E} \left[B \sum_{k=1}^K \sum_s \sum_a \mu^*(s) (\pi^*(a|s) - \pi_k(a|s)) q^{\pi_k}(s, a) \right] &= \tilde{\mathcal{O}} \left(\sqrt{N^3 \rho A T} + (BAN^2)^{\frac{3}{4}} T^{\frac{1}{4}} + BNA \right) \\ &= \tilde{\mathcal{O}} \left(\sqrt{t_{\text{mix}}^3 \rho A T} + (t_{\text{mix}}^3 t_{\text{hit}} A)^{\frac{3}{4}} T^{\frac{1}{4}} + t_{\text{mix}}^2 t_{\text{hit}} A \right). \end{aligned}$$

□

D. Experiments

In this section, we compare the performance of our proposed algorithms and previous model-free algorithms. We note that model-based algorithms (UCRL2, PSRL, ...) typically have better performance in terms of regret but require more memory. For a fair comparison, we restrict our attention to model-free algorithms.

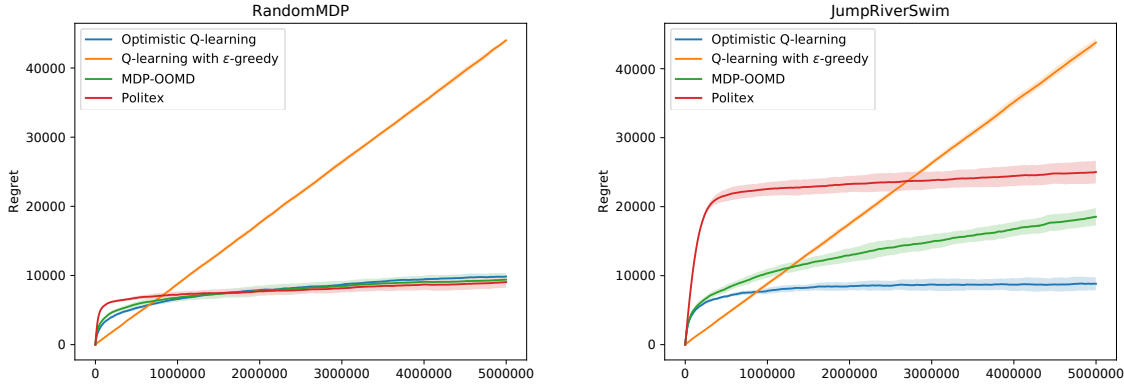


Figure 3. Performance of model-free algorithms on random MDP (left) and JumpRiverSwim (right). The standard Q-learning algorithm with ϵ -greedy exploration suffers from linear regret. The OPTIMISTIC Q-LEARNING and MDP-OODM algorithms achieve sub-linear regret. The shaded area denotes the standard deviation of regret over multiple runs.

Table 2. Hyper parameters used in the experiments. These hyper parameters are optimized to perform the best possible result for all the algorithms. All the experiments are averaged over 10 independent runs for a horizon of 5×10^6 . For the POLITEX algorithm, τ and τ' are the lengths of the two stages defined in Figure 3 of (Abbasi-Yadkori et al., 2019a).

	Algorithm	Parameters
Random MDP	Q-learning with ϵ -greedy	$\epsilon = 0.05$
	Optimistic Q-learning	$H = 100, c = 1, b_\tau = c\sqrt{H/\tau}$
	MDP-OODM	$N = 2, B = 4, \eta = 0.01$
	POLITEX	$\tau = 1000, \tau' = 1000, \eta = 0.2$
JumpRiverSwim	Q-learning with ϵ -greedy	$\epsilon = 0.03$
	Optimistic Q-learning	$H = 100, c = 1, b_\tau = c\sqrt{H/\tau}$
	MDP-OODM	$N = 10, B = 30, \eta = 0.01$
	POLITEX	$\tau = 3000, \tau' = 3000, \eta = 0.2$

Two environments are considered: a randomly generated MDP and JumpRiverSwim. Both of the environments consist of 6 states and 2 actions. The reward function and the transition kernel of the random MDP are chosen uniformly at random. The JumpRiverSwim environment is a modification of the RiverSwim environment (Strehl & Littman, 2008; Ouyang et al., 2017a) with a small probability of jumping to an arbitrary state at each time step.

The standard RiverSwim models a swimmer who can choose to swim either left or right in a river. The states are arranged in a chain and the swimmer starts from the leftmost state ($s = 1$). If the swimmer chooses to swim left, i.e., the direction of the river current, he is always successful. If he chooses to swim right, he may fail with a certain probability. The reward function is: $r(1, \text{left}) = 0.2, r(6, \text{right}) = 1$ and $r(s, a) = 0$ for all other states and actions. The optimal policy is to always swim right to gain the maximum reward of state $s = 6$. The standard RiverSwim is not an ergodic MDP and does not satisfy the assumption of the MDP-OODM algorithm. To handle this issue, we consider the JumpRiverSwim environment which has a small probability 0.01 of moving to an arbitrary state at each time step. This small modification provides an ergodic environment.

We compare our algorithms with two benchmark model-free algorithms. The first benchmark is the standard Q-learning with ϵ -greedy exploration. Figure 3 shows that this algorithm suffers from linear regret, indicating that the naive ϵ -greedy exploration is not efficient. The second benchmark is the POLITEX algorithm by Abbasi-Yadkori et al. (2019a). The implementation of POLITEX is based on the variant designed for the tabular case, which is presented in their Appendix F

and Figure 3. POLITEX usually requires longer episode length than MDP-OOMD (see Table 2) because in each episode it needs to accurately estimate the Q-function, rather than merely getting an unbiased estimator of it as in MDP-OOMD. Figure 3 shows that the proposed OPTIMISTIC Q-LEARNING, MDP-OOMD algorithms, and the POLITEX algorithm by Abbasi-Yadkori et al. (2019a) all achieve similar performance in the RandomMDP environment. In the JumpRiverSwim environment, the Optimistic Q-learning algorithm outperforms the other three algorithms. Although the regret upper bound for OPTIMISTIC Q-LEARNING scales as $\tilde{O}(T^{2/3})$ (Theorem 1), which is worse than that of MDP-OOMD (Theorem 5), Figure 3 suggests that in the environments that lack good mixing properties, OPTIMISTIC Q-LEARNING algorithm may perform better. The detail of the experiments is listed in Table 2.