Supplementary File for "A Markov Decision Process Model for Socio-Economic Systems Impacted by Climate Change"

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Proof of Theorem 1

Nondecreasing and Concave:

We will first show that if $V(s_n, \ell_n)$ is nondecreasing and concave in ℓ_n, then so is

$$F_m(s_n, \ell_n) = \mathsf{E}\left[m\alpha - \beta y_n + z_n + a_g V(s_{n-1} + m, \ell_{n-1} + r_n)\right],$$

for $m = 0, 1, \ldots, q$. Assume

- $\frac{\partial}{\partial \ell_n} V(s_n, \ell_n) \geq 0$ *(nondecreasing)*,
- $\frac{\partial^2}{\partial \ell_n^2} V(s_n, \ell_n) < 0$ *(concavity)*.

Using the nondecreasing monotonicity of $V(s_n, \ell_n)$ we can write

$$\frac{\partial}{\partial \ell_n} F_m(s_n, \ell_n) = \frac{\eta a \ell_{n-1}^{a-1}}{(1-k)s_{n-1}^b} + a_g \mathsf{E}\left[\frac{\partial}{\partial \ell_n} V(s_n, \ell_n)\right] \geq 0,$$

where the derivative can be brought inside the integral due to the monotone convergence theorem. Since $0 < a < 1$, for the second derivative we have

$$\frac{\partial^2}{\partial \ell_n^2} F_m(s_n, \ell_n) = \frac{\eta a(a-1) \ell_{n-1}^{a-2}}{(1-k)s_{n-1}^b} + a_g \mathsf{E}\left[\frac{\partial^2}{\partial \ell_n^2} V(s_n, \ell_n)\right] < 0.$$

Hence, it is sufficient to show that $V(s_n, \ell_n)$ is nondecreasing and concave.

Finding the value function iteratively (i.e., value iteration) is a common approach which is known to converge (Sutton & Barto, 2018): $\lim_{i \to \infty} V_i(s, \ell) = V(s, \ell)$. For brevity, we drop the time index from now on. We will next prove that $V(s, \ell)$ is nondecreasing and concave iteratively. Initializing all the state values as zero, i.e., $V^0(s, \ell) = 0, \forall s, \ell$, after the first iteration we get

$$\begin{aligned} V_1(s, \ell) &= \min_x \Big\{ \mathsf{E}[\alpha x - \beta y(x, z) + z(s, \ell) + a_g V^0(s + x, \ell + r)] \Big\}, \\ &= \mathsf{E}[z(s, \ell)] = \theta + \frac{\sigma}{1-k} = \theta + \frac{\eta \ell^a}{(1-k)s^b}, \end{aligned}$$

where we used the fact that $\mathsf{E}[y] = 0$ when $x = 0$ for all states. Differentiating with respect to ℓ, we get

$$\begin{aligned} \frac{\partial}{\partial \ell} V_1(s, \ell) &= \eta a \frac{\ell^{a-1}}{(1-k)s^b} \geq 0, \ \forall s, \\ \frac{\partial^2}{\partial \ell^2} V_1(s, \ell) &= \eta a(a-1) \frac{\ell^{a-2}}{(1-k)s^b} < 0, \ \forall s, \end{aligned} \tag{S1}$$

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since $\eta > 0, a \in (0,1), b > 0, k < 0$. Thus,$V_1(s,\ell)$ is nondecreasing and concave in ℓ for all s. Next, value function after the second iteration becomes

$$\begin{aligned} V_2(s,\ell) &= \min_x \left\{ \mathsf{E}[\alpha x - \beta y(x,z) + z(s,\ell) + a_g V_1(s+x, \ell+r)] \right\} \\ &= \min_x \left\{ \mathsf{E}[\alpha x - \beta y(x,z)] + \theta + \frac{\eta \ell^a}{(1-k)s^b} + a_g\theta + a_g \mathsf{E}\left[\frac{\eta(\ell+r)^a}{(1-k)(s+x)^b} \right] \right\}. \end{aligned}$$

Denoting the optimum action with $\bar{x}$ we will show that $V_2(s,\ell)$ is nondecreasing and concave for any $\bar{x}$. Moreover, the pointwise minimum of nondecreasing and concave functions is also nondecreasing and concave. The residents' probability of support $\mathsf{E}[y(x,z)]$ depends on past values of x and z, but not ℓ directly, so taking the derivative with respect to ℓ we get

$$\begin{aligned} \frac{\partial}{\partial \ell} V_2(s,\ell) &= \frac{\partial}{\partial \ell} \left\{ \frac{\eta \ell^a}{(1-k)s^b} + a_g \frac{\eta \mathsf{E}[(\ell+r)^a]}{(1-k)(s+\bar{x})^b} \right\} \\ &= \eta a \frac{\ell^{a-1}}{(1-k)s^b} + a_g \eta a \frac{\mathsf{E}[(\ell+r)^{a-1}]}{(1-k)(s+\bar{x})^b} \geq 0, \;\; \forall s \\ \frac{\partial^2}{\partial \ell^2} V_2(s,\ell) &= \eta a(a-1) \frac{\ell^{a-2}}{(1-k)s^b} + a_g \eta a(a-1) \frac{\mathsf{E}[(\ell+r)^{a-2}]}{(1-k)(s+\bar{x})^b} < 0, \;\; \forall s. \end{aligned}$$

Hence, $V_2(s,\ell)$ is nondecreasing and concave. Now, for any i, given that $V_{i-1}(s,\ell)$ is nondecreasing and concave, we can write

$$\begin{aligned} \frac{\partial}{\partial \ell} V_i(s,\ell) &= \eta a \frac{\ell^{a-1}}{(1-k)s^b} + a_g \mathsf{E}\left[\frac{\partial}{\partial \ell} V_{i-1}(s+\bar{x},\ell) \right] \geq 0, \;\; \forall s \\ \frac{\partial^2}{\partial \ell^2} V_i(s,\ell) &= \eta a(a-1) \frac{\ell^{a-2}}{(1-k)s^b} + a_g \mathsf{E}\left[\frac{\partial^2}{\partial \ell^2} V_{i-1}(s+\bar{x},\ell) \right] < 0, \;\; \forall s. \end{aligned} \tag{S2}$$

Consequently, by mathematical induction, $V(s,\ell)$ is nondecreasing and concave.

Comparison of Derivatives:

Similarly, if we show that

$$\frac{\partial}{\partial \ell} V(s+m,\ell) < \frac{\partial}{\partial \ell} V(s+m-1,\ell),$$

we can conclude that $\frac{\partial}{\partial \ell} F_m(s,\ell) < \frac{\partial}{\partial \ell} F_{m-1}(s,\ell)$ since

$$\begin{aligned} \frac{\partial}{\partial \ell_n} F_m(s_n,\ell_n) &= \frac{\eta a \ell_{n-1}^{a-1}}{(1-k)s_{n-1}^b} + a_g \mathsf{E}\left[\frac{\partial}{\partial \ell_n} V(s_{n-1}+m,\ell_n) \right] \\ \frac{\partial}{\partial \ell_n} F_{m-1}(s_n,\ell_n) &= \frac{\eta a \ell_{n-1}^{a-1}}{(1-k)s_{n-1}^b} + a_g \mathsf{E}\left[\frac{\partial}{\partial \ell_n} V(s_{n-1}+m-1,\ell_n) \right]. \end{aligned}$$

Starting again with $V_0(s,\ell) = 0, \forall s,\ell$, from (S1) we can write the following inequality for the first iteration

$$\frac{\partial}{\partial \ell} V_1(s+m,\ell) = \eta a \frac{\ell^{a-1}}{(1-k)(s+m)^b} < \frac{\partial}{\partial \ell} V_1(s+m-1,\ell) = \eta a \frac{\ell^{a-1}}{(1-k)(s+m-1)^b}.$$

For any i, given that $\frac{\partial}{\partial \ell} V_{i-1}(s+m,\ell) < \frac{\partial}{\partial \ell} V_{i-1}(s+m-1,\ell)$, from (S2) we have

$$\begin{aligned} \frac{\partial}{\partial \ell} V_i(s+m,\ell) = \eta a \frac{\ell^{a-1}}{(1-k)(s+m)^b} + a_g \mathsf{E}\left[\frac{\partial}{\partial \ell} V_{i-1}(s+m+\bar{x},\ell) \right] < \\ \frac{\partial}{\partial \ell} V_i(s+m-1,\ell) = \eta a \frac{\ell^{a-1}}{(1-k)(s+m-1)^b} + a_g \mathsf{E}\left[\frac{\partial}{\partial \ell} V_{i-1}(s+m-1+\bar{x},\ell) \right]. \end{aligned}$$

As a result, by mathematical induction we can conclude that $\frac{\partial}{\partial \ell} V(s+m,\ell) < \frac{\partial}{\partial \ell} V(s+m-1,\ell)$.

References

Sutton, R. S. and Barto, A. G. *Reinforcement learning: An introduction*. MIT press, 2018.