A. Mass Matrix

For a rigid body i with rotation $\mathbf{r} = (\phi, \theta, \psi)^\top$ and translation $\mathbf{t} = (t_x, t_y, t_z)^\top$, the generalized coordinates are $\mathbf{q} = [\mathbf{r}^\top, \mathbf{t}^\top]^\top \in \mathbb{R}^6$. Its kinetic energy can be computed by $\mathbf{E} = \frac{1}{2}\dot{\mathbf{q}}\hat{\mathbf{M}}\dot{\mathbf{q}}$, where $\hat{\mathbf{M}}$ is a diagonal blocked mass matrix composed of angular and linear inertia,

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathcal{I}_a & \mathbf{0} \\ \mathbf{0} & \mathcal{I}_l \end{bmatrix}. \tag{16}$$

The linear inertia $\mathcal{I}_l$ is simply $m\mathbf{I}_{3\times3}$, where m is the total mass of the rigid body.

When the rigid body's distribution is approximated by a set of particles, an angular inertial is given by,

$$\mathcal{I}' = \sum_i m_i(\mathbf{p}_i^\top \mathbf{p}_i \mathbf{I}_{3\times3} - \mathbf{p}_i \mathbf{p}_i^\top), \tag{17}$$

where m_i is the mass of the particle i; $\mathbf{p}_i$ is the vector from the center of mass to this particle. Note that this angular inertia corresponds to the axis-angle velocity ω in the *world frame*. The angular momentum is $\mathcal{I}'\omega$. However, We cannot directly use this formula since we have to represent the angular velocity in terms of velocities of Euler angles.

We used the RPY convention for Euler angle representation: given the Euler angle $\mathbf{r} = (\phi, \theta, \psi)^\top$, the rigid body will first rotate about the Z axis by ψ, then rotate about the new Y' axis by θ, and finally rotate about the new X'' axis by ϕ.

By the above definition, the angular velocity vector Ω can be represented by Euler angles,

$$\Omega = \dot{\psi}\mathbf{e}_Z + \dot{\theta}\mathbf{e}_{Y'} + \dot{\phi}\mathbf{e}_{X''} \tag{18}$$

We convert the angular velocity in local frame back to the world frame so that it matches with the angular inertia in the world frame:

$$\begin{aligned} \Omega = &(\cos\theta\cos\psi\dot{\phi} - \sin\psi\dot{\theta})\mathbf{e}_X \\ &+ (\cos\theta\cos\psi\dot{\phi} + \cos\psi\dot{\theta})\mathbf{e}_Y. \\ &+ (-\sin\theta\dot{\phi} + \dot{\psi})\mathbf{e}_Z \end{aligned} \tag{19}$$

The matrix form is given as:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta\sin\psi & \cos\psi & 0 \\ -\sin\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{20}$$

We denote this transformation as $\omega = \mathbf{T}\dot{\mathbf{r}}$, so the angular momentum is reformed as $\mathcal{I}'\omega = \mathcal{I}'\mathbf{T}\dot{\mathbf{r}}$. Therefore, in the Euler angle representation, the angular inertia becomes

$$\mathcal{I}_a = \mathbf{T}^\top \mathcal{I}' \mathbf{T}. \tag{21}$$

The new mass matrix $\hat{\mathbf{M}}$ for the generalized coordinates is,

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{T}^\top \mathcal{I}' \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{3\times3} \end{bmatrix}. \tag{22}$$

B. Represent a Vertex using Generalized Coordinates

For a vertex p attached to the rigid body, its coordinate in the body frame is $\mathbf{p}_0$. The origin of the body frame is set to be the center of mass (COM). $\mathbf{p}_0 = (p_x, p_y, p_z)^\top$ is the relative displacement of p w.r.t. the COM in the first frame of simulation. The coordinates of the vertex in the world frame are then given by,

$$\mathbf{p} = \boldsymbol{f}(\mathbf{q}) = [\mathbf{r}]\mathbf{p}_0 + \mathbf{t}, \tag{23}$$

where $[\mathbf{r}]$ is a rotation matrix represented by the Euler angle $\mathbf{r} = (\phi, \theta, \psi)^\top$. The corresponding rotation matrix $[\mathbf{r}] = \mathbf{R}_{3\times3}$ is computed by

$$\mathbf{R}_{11} = \cos\theta\cos\psi$$

$$\mathbf{R}_{12} = -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi$$

$$\mathbf{R}_{13} = \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi$$

$$\mathbf{R}_{21} = \cos\theta\sin\psi$$

$$\mathbf{R}_{22} = \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi$$

$$\mathbf{R}_{23} = -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi$$

$$\mathbf{R}_{31} = -\sin\theta$$

$$\mathbf{R}_{32} = \sin\phi\cos\theta$$

$$\mathbf{R}_{33} = \cos\phi\cos\theta$$

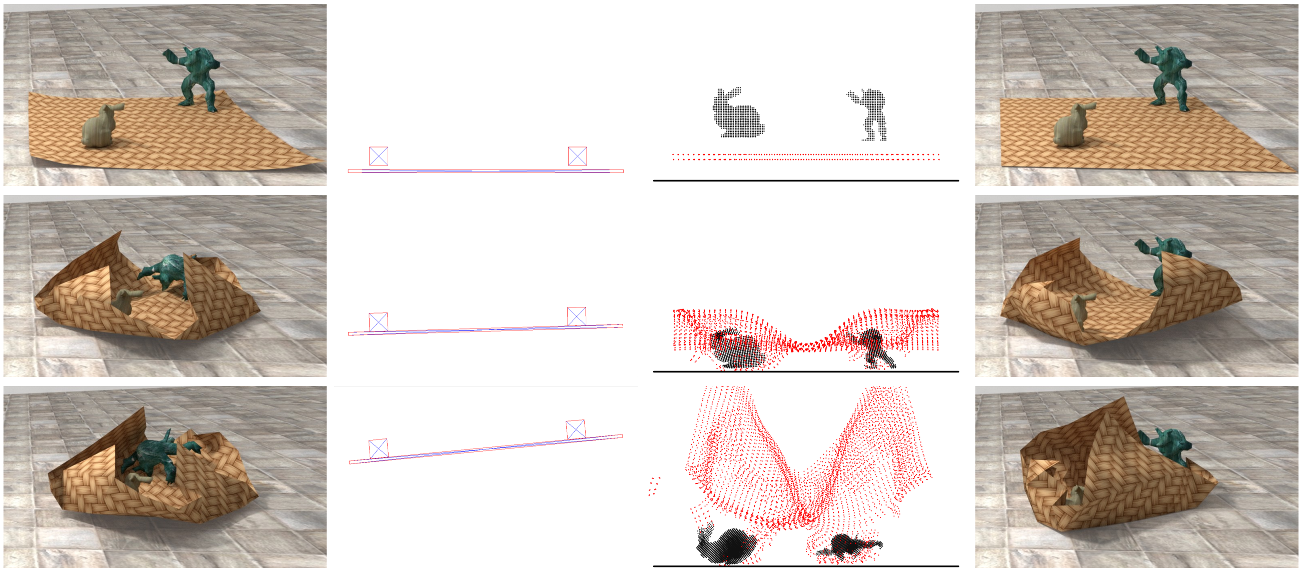
C. Computation of Derivatives

To backpropagate the gradients at vertex $\mathbf{p}$ to the generalized coordinates $\mathbf{q}$, we need to compute the derivatives $\partial\boldsymbol{f}(\mathbf{q})/\partial\mathbf{q}$.

The coordinates of a vertex $(x, y, z)^\top = \boldsymbol{f}([\phi, \theta, \psi, t_x, t_y, t_z]^\top)$ are given by Equation 23. Therefore, the Jacobian matrix for the partial derivatives is:

$$\nabla\boldsymbol{f} = \begin{bmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\psi} & 1 & 0 & 0 \\ \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\psi} & 0 & 1 & 0 \\ \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\psi} & 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

Assuming that the relative displacement of p w.r.t. the COM in the first frame is $\mathbf{p}_0 = (p_x, p_y, p_z)$, the corresponding elements in the Jacobian matrix are:



(a) Ours (b) E2E (c) ChainQueen (d) DiffCloth

Figure 11. Qualitative comparison on complex models and interactions. In this scene, we simulate two complex rigid bodies, bunny and armadillo on a cloth. (a) is the simulation result of our method, and the comparisons, (b) is E2E (de Avila Belbute-Peres et al., 2018); (c) is ChainQueen (Hu et al., 2019); (d) is DiffCloth (Liang et al., 2019). Our method gets the most reasonable results because we can deal with both rigid body and cloth dynamics. Meanwhile, the mesh representation enables us to model different objects precisely.

$\frac{\partial x}{\partial \psi} = -p_x \cos\theta \sin\psi + p_y(-\cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi) + p_z(\sin\phi\cos\psi - \cos\phi\sin\theta\sin\psi)$

$\frac{\partial x}{\partial \theta} = -p_x \sin\theta\cos\psi + p_y \sin\phi\cos\theta\cos\psi + p_z \cos\phi\cos\theta\cos\psi$

$\frac{\partial x}{\partial \phi} = p_y \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi + p_z(\cos\phi\sin\psi - \sin\phi\sin\theta\cos\psi)$

$\frac{\partial y}{\partial \psi} = p_x \cos\theta\cos\psi + p_y(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi) + p_z(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)$

$\frac{\partial y}{\partial \theta} = -p_x \sin\theta\sin\psi + p_y \sin\phi\cos\theta\sin\psi + p_z \cos\phi\cos\theta\sin\psi$

$\frac{\partial y}{\partial \phi} = p_y(-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi) + p_z(-\cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi)$

$\frac{\partial z}{\partial \psi} = 0$

$\frac{\partial z}{\partial \theta} = -p_x \cos\theta - p_y \sin\phi\sin\theta - p_z \cos\phi\sin\theta$

$\frac{\partial z}{\partial \phi} = p_y \cos\phi\cos\theta - p_z \sin\phi\cos\theta$

D. Qualitative Comparisons

Figure 11 shows qualitative comparisons of two complex rigid bodies interacting with a soft cloth.

In this comparison, our differentiable physics gets the most realistic results, that both the bunny and armadillo are lifted from the table. E2E cannot simulate the detailed motion of the objects because it only supports basic primitives. ChainQueen cannot preserve the *rigid* property of bunny and armadillo because these objects are composed of small particles during the simulation and obey deformable body dynamics. Moreover, the cloth would break up because the MPM method cannot keep connectivity information faithfully during the simulation. Diffcloth is able to model the motion of cloth, but the cloth cannot have an impact on the bunny and armadillo, therefore they can not be lifted.