7. Supplementary material

7.1. Sub-regions corresponding to system (9)

All sub-regions related to (9) can be defined as follows (Monfared & Durstewitz, 2020):

$$S_{\Omega^1} = \hat{S}_0 = \hat{S}_{(\underbrace{0\,0\,0\,\cdots\,0}_{M})_2^*} = \hat{S}_{\underbrace{0\,0\,0\,\cdots\,0}_{M}} \tag{18}$$
$$= \Big\{Z_t \in \mathbb{R}^M; z_{it} \leq 0,\, i = 1, 2, \cdots, M\Big\},$$
$$S_{\Omega^2} = \hat{S}_1 = \hat{S}_{(\underbrace{0\,0\,\cdots\,0\,1}_{M})_2^*} = \hat{S}_{\underbrace{1\,0\,0\,\cdots\,0}_{M}} \tag{19}$$
$$= \Big\{Z_t \in \mathbb{R}^M; z_{1t} > 0, z_{it} \leq 0,\, i \neq 1\Big\},$$
$$S_{\Omega^3} = \hat{S}_2 = \hat{S}_{(\underbrace{0\,\cdots\,0\,1\,0}_{M})_2^*} = \hat{S}_{\underbrace{0\,1\,0\,\cdots\,0}_{M}} \tag{20}$$
$$= \Big\{Z_t \in \mathbb{R}^M; z_{2t} > 0, z_{it} \leq 0,\, i \neq 2\Big\},$$
$$S_{\Omega^4} = \hat{S}_3 = \hat{S}_{(\underbrace{0\,\cdots\,0\,1\,1}_{M})_2^*} = \hat{S}_{\underbrace{1\,1\,0\,\cdots\,0}_{M}} \tag{21}$$
$$= \Big\{Z_t \in \mathbb{R}^M; z_{1t}, z_{2t} > 0, z_{it} \leq 0,\, i \neq 1, 2\Big\},$$
$$S_{\Omega^5} = \hat{S}_4 = \hat{S}_{(\underbrace{0\,\cdots\,1\,0\,0}_{M})_2^*} = \hat{S}_{\underbrace{0\,0\,1\,0\,\cdots\,0}_{M}} \tag{22}$$
$$= \Big\{Z_t \in \mathbb{R}^M; z_{3t} > 0, z_{it} \leq 0,\, i \neq 3\Big\},$$
$$\vdots \qquad \qquad \vdots$$
$$S_{\Omega^{2^M}} = \hat{S}_{2^M-1} = \hat{S}_{(\underbrace{1\,1\,1\,\cdots\,1}_{M})_2^*} = \hat{S}_{\underbrace{1\,1\,1\,\cdots\,1}_{M}} \tag{23}$$
$$= \Big\{Z_t \in \mathbb{R}^M; z_{it} > 0,\, i = 1, 2, \cdots, M\Big\}.$$

where each subindex d of $\hat{S}$, $0 \leq d \leq 2^M - 1$, is associated with a sequence $d_M\, d_{M-1}\, \cdots\, d_2\, d_1$ of binary digits. The notation $(d_1\, d_2\, \cdots\, d_M)_2^*$ in building each corresponding sequence stands for the mirror image of the binary representation of d with M digits. By mirror image here we mean writing digits $d_1\, d_2\, \cdots\, d_M$ from right to left, i.e. $d_M\, d_{M-1}\, \cdots\, d_2\, d_1$. For example, for $M = 2$ there are 4 sub-regions S_{Ω^k}, $k = 1, 2, 3, 4$, associated with 4 matrices $D_{\Omega^k} := \mathrm{diag}\big(d_2, d_1\big)$, where $d_2\, d_1 = (d_1\, d_2)_2^*$ and $d_i \in \{0, 1\}$ (Fig. S1).

Denoting switching boundaries $\Sigma_{ij} = \bar{S}_{\Omega^i} \cap \bar{S}_{\Omega^j}$ between every pair of successive sub-regions S_{Ω^i} and S_{Ω^j} with $i, j \in \{1, 2, \cdots, 2^M\}$, we can rewrite map (7) as

$$Z_{t+1} = F(Z_t)$$
$$= \begin{cases} F_1(Z_t) = W_{\Omega^1} Z_t + h; & Z_t \in \bar{S}_{\Omega^1} \\ F_2(Z_t) = W_{\Omega^2} Z_t + h; & Z_t \in \bar{S}_{\Omega^2} \\ F_3(Z_t) = W_{\Omega^3} Z_t + h; & Z_t \in \bar{S}_{\Omega^3} \\ F_4(Z_t) = W_{\Omega^4} Z_t + h; & Z_t \in \bar{S}_{\Omega^4} \\ \vdots & \vdots \\ F_{2^M}(Z_t) = W_{\Omega^{2^M}} Z_t + h; & Z_t \in \bar{S}_{\Omega^{2^M}} \end{cases}. \tag{24}$$

7.2. Discontinuity boundaries

Consider map (7) and two sub-regions S_{Ω^i} and S_{Ω^j} ($i, j \in \{1, 2, \cdots, 2^M\}$) as defined in Section 4 (subsection 4.2). Suppose that subindices $i-1$ and $j-1$ of $\hat{S}_{i-1}$ and $\hat{S}_{j-1}$ are associated with $i - 1 = i_1\, i_2\, \cdots\, i_M$ and $j - 1 = j_1\, j_2\, \cdots\, j_M$. Then S_{Ω^i} and S_{Ω^j} are two successive sub-regions with the switching boundary $\Sigma_{ij} = \bar{S}_{\Omega^i} \cap \bar{S}_{\Omega^j}$, iff there is exactly one $1 \leq s \leq M$ such that for all $(z_{i_1 t}, \cdots, z_{i_M t})^T \in \mathring{S}_{\Omega^i}$ and $(z_{j_1 t}, \cdots, z_{j_M t})^T \in \mathring{S}_{\Omega^j}$

$$\begin{cases} z_{i_s t} \,.\, z_{j_s t} < 0 \\ z_{i_r t} \,.\, z_{j_r t} > 0, \;\; 1 \leq \underset{r \neq s}{r} \leq M \end{cases}. \tag{25}$$

Moreover, Σ_{ij} is a closed set ($\bar{\Sigma}_{ij} = \Sigma_{ij}$) and $\Sigma_{ij} = \mathring{\Sigma}_{ij} \cup \partial\Sigma_{ij}$ such that

$$\mathring{\Sigma}_{ij} = \Sigma_r^s = \Big\{Z_t \in \mathbb{R}^M; z_{st} = 0, \text{ and } sgn(z_{rt}) = sgn(z_{i_r t}) = sgn(z_{j_r t}),\; 1 \leq \underset{r \neq s}{r} \leq M\Big\}, \tag{26}$$

and $\partial\Sigma_{ij} = \bigcup_{\substack{s_m=1 \\ s_m \neq s}}^{M} \Sigma_\nu^{s, s_m}$ where

$$\Sigma_\nu^{s, s_m} = \Big\{Z_t \in \mathbb{R}^M; z_{s_m t} = z_{st} = 0, \text{ and } sgn(z_{\nu t}) = sgn(z_{rt}),\; 1 \leq \underset{\nu \neq s, s_m}{\nu} \leq M\Big\}. \tag{27}$$

Furthermore, it can be proven that

$$\bigcup_{i,j=1}^{2^M} \Sigma_{ij} = \bigcup_{l=1}^{M 2^{M-1}} \Sigma_l \subset \bigcup_{k=1}^{2^M} S_{\Omega^k} = \mathbb{R}^M. \tag{28}$$

7.3. Proof of theorem 3

(1) Without loss of generality let $t_0 = 0$. Assume that there exists an equivalent continuous-time system for (10)

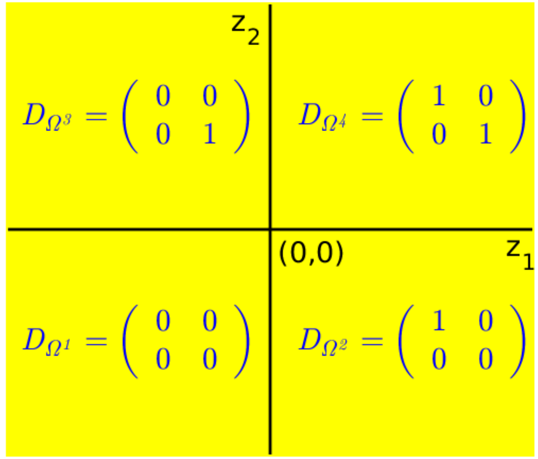

Figure S1. Example of subregions S_{Ω^k} and related matrices D_{Ω^k} for $M = 2$.

on $[0,\ \Delta t]$, in the form of equation (12). By equivalency in the sense of equation (13), we must have

$$Z_0 = \zeta(0), \qquad Z_1 = W_{\Omega^k} Z_0 + h = \zeta(\Delta t). \tag{29}$$

According to theorem 1, the solution of system (10) on $[0, \Delta t]$ is

$$\zeta(t) = e^{\tilde{W}_{\Omega^k} t}\, \zeta(0) + e^{\tilde{W}_{\Omega^k} t} \int_0^t e^{-\tilde{W}_{\Omega^k} \tau}\ \tilde{h}\, d\tau, \quad t \in [0, \Delta t]. \tag{30}$$

If $\tilde{W}_{\Omega^k}$ is invertible, then

$$\int_0^t e^{-\tilde{W}_{\Omega^k} \tau}\ \tilde{h}\, d\tau = -\tilde{W}_{\Omega^k}^{-1} \big(e^{-\tilde{W}_{\Omega^k} t} - I\big)\, \tilde{h}, \tag{31}$$

and thus

$$\zeta(t) = e^{\tilde{W}_{\Omega^k} t}\, \zeta(0) + \Big[e^{\tilde{W}_{\Omega^k} t}\ (-\tilde{W}_{\Omega^k}^{-1})\ e^{-\tilde{W}_{\Omega^k} t} - e^{\tilde{W}_{\Omega^k} t}\ (-\tilde{W}_{\Omega^k}^{-1})\Big]\tilde{h}. \tag{32}$$

Furthermore, since

$$(-\tilde{W}_{\Omega^k}^{-1})\ e^{-\tilde{W}_{\Omega^k} t} = e^{-\tilde{W}_{\Omega^k} t}\ (-\tilde{W}_{\Omega^k}^{-1}), \tag{33}$$

we have

$$\zeta(t) = e^{\tilde{W}_{\Omega^k} t}\, \zeta(0) + \Big[I - e^{\tilde{W}_{\Omega^k} t}\Big](-\tilde{W}_{\Omega^k}^{-1})\ \tilde{h} \qquad t \in [0, \Delta t]. \tag{34}$$

Putting conditions (29) in

$$\zeta(\Delta t) = e^{\tilde{W}_{\Omega^k} \Delta t}\, \zeta(0) + \Big[I - e^{\tilde{W}_{\Omega^k} \Delta t}\Big](-\tilde{W}_{\Omega^k}^{-1})\ \tilde{h}, \tag{35}$$

yields

$$W_{\Omega^k}\, Z_0 + h \ = \ e^{\tilde{W}_{\Omega^k} \Delta t}\, Z_0 + \Big[I - e^{\tilde{W}_{\Omega^k} \Delta t}\Big](-\tilde{W}_{\Omega^k}^{-1})\ \tilde{h}. \tag{36}$$

Equation (36) has to hold for all Z_0 including $Z_0 = 0$. Hence, it is deduced that

$$\begin{cases} W_{\Omega^k} \ = \ e^{\tilde{W}_{\Omega^k} \Delta t} \\ h \ = \ \big[I - e^{\tilde{W}_{\Omega^k} \Delta t}\big]\ (-\tilde{W}_{\Omega^k}^{-1})\ \tilde{h} \end{cases}. \tag{37}$$

According to (37), matrix W_{Ω^k} should be invertible and cannot have any zero eigenvalue. Also, since $\tilde{W}_{\Omega^k}$ is invertible, it does not have any zero eigenvalue, which implies that W_{Ω^k} has no eigenvalue equal to one. Then, $P_{W_{\Omega^k}}(1) \neq 0$, which means $\big[I - W_{\Omega^k}\big]$ is invertible and (37) becomes equivalent to (14).

Now, considering $\tilde{W}_{\Omega^k}$ and $\tilde{h}$ as in (14), we can obtain the desired equivalent continuous-time system (12) for (10) on $[0,\ \Delta t]$. It is just required to prove that every fixed point Z^* of map (10) is also an equilibrium point of system (12), and (14) is a solution of (36) for all Z^*. For this purpose, let Z^* be a fixed point of (10), then

$$F(Z^*) = W_{\Omega^k} Z^* + h \ = \ Z^*. \tag{38}$$

Z^* must be an equilibrium of (12), i.e.

$$G(Z^*) = \tilde{W}_{\Omega^k} Z^* + \tilde{h} \ = \ 0. \tag{39}$$

From (38) and (39) it is concluded that

$$h \ = \ \big[I - W_{\Omega^k}\big]\ (-\tilde{W}_{\Omega^k}^{-1})\ \tilde{h}, \tag{40}$$

which shows that (37) or, equivalently, (14) is a solution of (36) for all Z^* satisfying both relations (38) and (39). Finally, let each Jordan block of W_{Ω^k} associated with a negative eigenvalue occur an even number of times. Then, by theorem (2), the logarithm of real matrix W_{Ω^k}, i.e. the matrix $\tilde{W}_{\Omega^k}$ defined in (14), will be real.

(2) Let W_{Ω^k} be diagonalizable, then

$$W_{\Omega^k} \;=\; V\, E_k\, V^{-1}, \tag{41}$$

where $E_k = diag(\lambda_1^k, \lambda_2^k, \cdots, \lambda_M^k)$ and V is the matrix of eigenvectors of W_{Ω^k}. Since W_{Ω^k} is also invertible, by (14)

$$\begin{aligned}\tilde{W}_{\Omega^k} \;&=\; \frac{1}{\Delta t}\, log(W_{\Omega^k}) \;=\; \frac{1}{\Delta t}\, log(V\, E_k\, V^{-1}) \\ &=\; V \frac{1}{\Delta t}\, log(E_k)\, V^{-1}, \end{aligned}\tag{42}$$

such that

$$log(E_k) = diag\big(log(\lambda_1^k), log(\lambda_2^k), \cdots, log(\lambda_M^k)\big),$$

which completes the proof.

Remark. Due to (37) and (14), one can see that system (12) is homogeneous ($\tilde{h} = 0$) if and only if system (10) is homogeneous ($h = 0$).

7.4. Proof of theorem 4

Again we prove the theorem for $t_0 = 0$ without loss of generality. Suppose that there is the equivalent continuous-time system (12) for (10) with non-invertible and diagonalizable matrix $\tilde{W}_{\Omega^k}$. Similar to the proof of the previous theorem, relations (29) and (30) must hold for (10) and (12). On the other hand, non-invertibility and diagonalizability of $\tilde{W}_{\Omega^k}$ demand that it has at least one eigenvalue equal to zero and

$$\tilde{W}_{\Omega^k} \;=\; V \begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & C \end{pmatrix} V^{-1}, \tag{43}$$

where $\mathrm{O}_{n\times n}$ is a zero matrix corresponding to zero eigenvalues (n denotes the number of zero eigenvalues) and C is an invertible matrix corresponding to nonzero eigenvalues of $\tilde{W}_{\Omega^k}$. Therefore, for relation (31) we obtain

$$\int_0^t e^{-\tilde{W}_{\Omega^k}\tau}\; \tilde{h}\, d\tau = $$
$$V \begin{pmatrix} \begin{pmatrix} t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t \end{pmatrix}_{n\times n} & 0 \\ 0 & -C^{-1}\big(e^{-Ct} - I\big) \end{pmatrix} V^{-1}\, \tilde{h}. \tag{44}$$

In this case, relation (35) becomes

$$\zeta(\Delta t) = e^{\tilde{W}_{\Omega^k}\Delta t}\, \zeta(0) + e^{\tilde{W}_{\Omega^k}\Delta t}\, V \times$$
$$\begin{pmatrix} \begin{pmatrix} \Delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta t \end{pmatrix}_{n\times n} & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} V^{-1}\tilde{h}. \tag{45}$$

Inserting conditions (29) into (45) gives

$$W_{\Omega^k}\, Z_0 + h \;=\; e^{\tilde{W}_{\Omega^k}\Delta t}\, Z_0 + e^{\tilde{W}_{\Omega^k}\Delta t}\, V \times$$
$$\begin{pmatrix} \begin{pmatrix} \Delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta t \end{pmatrix}_{n\times n} & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} V^{-1}\tilde{h}. \tag{46}$$

Denoting

$$H \;=\; \begin{pmatrix} \Delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta t \end{pmatrix}_{n\times n}, \tag{47}$$

and considering equality (46) for all Z_0, particularly for $Z_0 = 0$, yields

$$\begin{cases} W_{\Omega^k} \;=\; e^{\tilde{W}_{\Omega^k}\Delta t} \\ h \;=\; e^{\tilde{W}_{\Omega^k}\Delta t}\, V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} V^{-1}\tilde{h} \end{cases}. \tag{48}$$

Since

$$e^{\tilde{W}_{\Omega^k}\Delta t} \;=\; V \begin{pmatrix} I & 0 \\ 0 & e^{C\Delta t} \end{pmatrix} V^{-1}, \tag{49}$$

we can simplify h in (48) and rewrite it as

$$\begin{cases} W_{\Omega^k} \;=\; e^{\tilde{W}_{\Omega^k}\Delta t} \\ h \;=\; V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}\big(I - e^{C\Delta t}\big) \end{pmatrix} V^{-1}\tilde{h}, \end{cases} \tag{50}$$

or equivalently

$$\begin{cases} \tilde{W}_{\Omega^k} \;=\; \frac{1}{\Delta t}\, log(W_{\Omega^k}) \\ \tilde{h} \;=\; V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}\big(I - e^{C\Delta t}\big) \end{pmatrix}^{-1} V^{-1} h. \end{cases} \tag{51}$$

In addition, we can write

$$\tilde{h} = V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}\big(I - e^{C\Delta t}\big) \end{pmatrix}^{-1} V^{-1} h$$

$$= V \left(\begin{matrix} \begin{pmatrix} \frac{1}{\Delta t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\Delta t} \end{pmatrix}_{n\times n} & 0 \\ 0 & C\big(e^{C\Delta t} - I\big)^{-1} \end{matrix} \right) V^{-1} h$$

$$= V \left[\frac{1}{\Delta t} \begin{pmatrix} I_n & 0 \\ 0 & \mathrm{O} \end{pmatrix} + \begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & C \end{pmatrix} \right] \times \begin{pmatrix} I_n & 0 \\ 0 & \big(e^{C\Delta t} - I\big)^{-1} \end{pmatrix} V^{-1} h$$

$$= V \left[\frac{1}{\Delta t} \begin{pmatrix} I_n & 0 \\ 0 & \mathrm{O} \end{pmatrix} + V^{-1} \tilde{W}_{\Omega^k} V \right] \times \left[\begin{pmatrix} I_n & 0 \\ 0 & e^{C\Delta t} \end{pmatrix} - \begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & I \end{pmatrix} \right]^{-1} V^{-1} h$$

$$= \left[\frac{1}{\Delta t} \begin{pmatrix} I_n & 0 \\ 0 & \mathrm{O} \end{pmatrix} + \tilde{W}_{\Omega^k} \right] \times \left[e^{\tilde{W}_{\Omega^k}\Delta t} - \begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & I \end{pmatrix} \right]^{-1} h. \tag{52}$$

Therefore

$$\tilde{W}_{\Omega^k} = \frac{1}{\Delta t} \, log(W_{\Omega^k}),$$

$$\tilde{h} = \left[-\frac{1}{\Delta t} \begin{pmatrix} I_n & 0 \\ 0 & \mathrm{O} \end{pmatrix} - \tilde{W}_{\Omega^k} \right] \times \left[\begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & I \end{pmatrix} - e^{\tilde{W}_{\Omega^k}\Delta t} \right]^{-1} h \tag{53}$$

which is equivalent to (15).

Finally, from $W_{\Omega^k} = e^{\tilde{W}_{\Omega^k}\Delta t}$ it is deduced that W_{Ω^k} is invertible. It is only necessary to prove that for every point Z^* satisfying both equations (38) and (39), i.e. equations

$$\begin{cases} \big(W_{\Omega^k} - I\big) Z^* = -h \\ \tilde{W}_{\Omega^k} Z^* = -\tilde{h} \end{cases}, \tag{54}$$

relation (15) is a solution of (46). Note that here we cannot simplify (54) to find some equation similar to (40), as neither $\big(W_{\Omega^k} - I\big)$ or $\tilde{W}_{\Omega^k}$ is invertible. Hence, we show that (54) fulfills solution (15) or, identically, solution (50). Thus, inserting $\tilde{h} = -\tilde{W}_{\Omega^k} Z^*$ in (50), we have

$$h = e^{\tilde{W}_{\Omega^k}\Delta t}\, V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} V^{-1} \tilde{h}$$

$$= e^{\tilde{W}_{\Omega^k}\Delta t}\, V \begin{pmatrix} H & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} V^{-1} (-\tilde{W}_{\Omega^k} Z^*)$$

$$= e^{\tilde{W}_{\Omega^k}\Delta t}\, V \begin{pmatrix} H & 0 \\ 0 & -\big(e^{-C\Delta t} - I\big)C^{-1} \end{pmatrix} V^{-1} V \times \begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & -C \end{pmatrix} V^{-1} Z^*$$

$$= e^{\tilde{W}_{\Omega^k}\Delta t}\, V \begin{pmatrix} \mathrm{O}_{n\times n} & 0 \\ 0 & \big(e^{-C\Delta t} - I\big) \end{pmatrix} V^{-1} Z^*$$

$$= e^{\tilde{W}_{\Omega^k}\Delta t}\, V \left[\begin{pmatrix} I_n & 0 \\ 0 & e^{-C\Delta t} \end{pmatrix} - \begin{pmatrix} I_n & 0 \\ 0 & I \end{pmatrix} \right] V^{-1} Z^*$$

$$= \Big(I - e^{\tilde{W}_{\Omega^k}\Delta t}\Big) Z^* = \big(I - W_{\Omega^k}\big) Z^*, \tag{55}$$

which demonstrates that (54) meets solution (50).

If every Jordan block of W_{Ω^k} associated with a negative eigenvalue occurs an even number of times, then theorem (2) guarantees that $\tilde{W}_{\Omega^k}$ will be real. Also, similar to the proof of theorem 3, it is easy to see that $\tilde{W}_{\Omega^k}$ will be diagonalizable when W_{Ω^k} has no negative real eigenvalues.

7.5. Proof of theorem 5

Let $t_0 = 0$ without loss of generality and assume there exists the equivalent continuous-time system (12) for (10), for which matrix W_{Ω^k} is non-invertible. Then, relations (29) and (30) must hold for (10) and (12), analogously to the proofs of the previous theorems. Also, by similar reasoning we have

$$\zeta(\Delta t) = e^{\tilde{W}_{\Omega^k}\Delta t}\, \zeta(0) + \left[e^{\tilde{W}_{\Omega^k}\Delta t} \int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau \right] \tilde{h}. \tag{56}$$

Inserting conditions (29) in equation (56) and solving the resulting equation for all Z_0, including $Z_0 = 0$, yields

$$\begin{cases} W_{\Omega^k} = e^{\tilde{W}_{\Omega^k}\Delta t} \\ h = e^{\tilde{W}_{\Omega^k}\Delta t} \left(\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau \right) \tilde{h}. \end{cases} \tag{57}$$

Now let

$$\lambda \in \text{Spectrum}(\tilde{W}_{\Omega^k}) \;\Rightarrow\; \lambda\Delta t \notin 2i\pi\mathbb{Z}^*. \tag{58}$$

Then, by proposition 1, $\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau$ is invertible and so

$$\begin{cases} \tilde{W}_{\Omega^k} \,=\, \frac{1}{\Delta t}\, log(W_{\Omega^k}) \\ \tilde{h} \,=\, \left(\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau\right)^{-1} \; e^{-\tilde{W}_{\Omega^k}\Delta t}\; h \end{cases}, \tag{59}$$

which is equal to equation (17). The last point which still has to be proven is that equation (54) meets solution (17) or, identically, (57), for every Z^*. Since $\tilde{W}_{\Omega^k}$ is non-invertible, it can be written in the following Jordan form:

$$\tilde{W}_{\Omega^k} \,=\, U \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} U^{-1}, \tag{60}$$

where B is a strictly upper triangular matrix and C is an invertible matrix. Then

$$e^{-\tilde{W}_{\Omega^k}\Delta t} \,=\, U \begin{pmatrix} e^{-B\Delta t} & 0 \\ 0 & e^{-C\Delta t} \end{pmatrix} U^{-1}, \tag{61}$$

$$\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, \tilde{h}\, d\tau = \\ U \begin{pmatrix} \int_0^{\Delta t} e^{-B\tau} d\tau & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} U^{-1}. \tag{62}$$

Now, substituting $\tilde{h} \,=\, -\tilde{W}_{\Omega^k} Z^*$ in (57) we have

$$\begin{aligned} h \,&=\, e^{\tilde{W}_{\Omega^k}\Delta t} \left(\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau\right) \tilde{h} \\ &= e^{\tilde{W}_{\Omega^k}\Delta t}\, U \begin{pmatrix} \int_0^{\Delta t} e^{-B\tau}\, d\tau & 0 \\ 0 & -C^{-1}\big(e^{-C\Delta t} - I\big) \end{pmatrix} \\ &\quad \times U^{-1}(-\tilde{W}_{\Omega^k} Z^*) \\ &= e^{\tilde{W}_{\Omega^k}\Delta t}\, U \begin{pmatrix} \int_0^{\Delta t} e^{-B\tau}\, d\tau & 0 \\ 0 & -\big(e^{-C\Delta t} - I\big)\, C^{-1} \end{pmatrix} \\ &\quad \times U^{-1}\, U \begin{pmatrix} -B & 0 \\ 0 & -C \end{pmatrix} U^{-1} Z^* \\ &= e^{\tilde{W}_{\Omega^k}\Delta t}\, U \begin{pmatrix} \int_0^{\Delta t} -B\, e^{-B\tau}\, d\tau & 0 \\ 0 & \big(e^{-C\Delta t} - I\big) \end{pmatrix} \\ &\quad \times U^{-1} Z^* \\ &= e^{\tilde{W}_{\Omega^k}\Delta t}\, U \begin{pmatrix} \big(e^{-B\Delta t} - I\big) & 0 \\ 0 & \big(e^{-C\Delta t} - I\big) \end{pmatrix} U^{-1} Z^* \\ &= \Big(I - e^{\tilde{W}_{\Omega^k}\Delta t}\Big)\, Z^* = (I - W_{\Omega^k})\, Z^*, \end{aligned} \tag{63}$$

which completes the proof.

Finally, due to theorem (2), $\tilde{W}_{\Omega^k}$ will be real, provided that each Jordan block of W_{Ω^k} related to a negative eigenvalue occurs an even number of times.

Remark. In theorem 5, by (62) we have

$$\left(\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau\right)^{-1} \,=\, \tag{64}$$

$$U \begin{pmatrix} \left(\int_0^{\Delta t} e^{-B\tau}\, d\tau\right)^{-1} & 0 \\ 0 & \big(I - e^{-C\Delta t}\big)^{-1} C \end{pmatrix} U^{-1} h. \tag{65}$$

On the other hand, since C is invertible, $\det\big(I - e^{-C\Delta t}\big) \neq 0$. Therefore, $\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau$ is invertible if and only if $\int_0^{\Delta t} e^{-B\tau}\, d\tau$ is invertible. Thus, for invertibility of $\int_0^{\Delta t} e^{-\tilde{W}_{\Omega^k}\tau}\, d\tau$, it is required that relation (58) holds only for any pair of eigenvalues of B.

7.6. Grazing bifurcation

Here we investigate a grazing bifurcation of periodic orbits for the continuous PLRNN derived from the van-der-Pol oscillator (Example 1). For this purpose, we consider the converted continuous-time system locally in the neighborhood of only one border

$$\Sigma = \Big\{\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_{10})^T \in \mathbb{R}^{10} \,|\, H(\zeta) = \zeta_2 = 0\Big\},$$

where the scalar function $H: \mathbb{R}^{10} \to \mathbb{R}$ defines the border and has non-vanishing gradient. According to (di Bernardo & Hogani, 2010; Monfared et al., 2017), a periodic orbit $\hat{\zeta}(t)$ undergoes a grazing bifurcation for some critical value of a bifurcation parameter, if it is a grazing orbit for some $t = t^*$. This means $\hat{\zeta}(t)$ hits Σ tangentially at the grazing point $\hat{\zeta}^* = \hat{\zeta}(t^*)$ and satisfies the following conditions:

$$\begin{aligned} & H(\hat{\zeta}^*) \,=\, \hat{\zeta}_2^* = 0, \\ & \nabla H(\hat{\zeta}^*) \,=\, (0,1,0,\cdots,0)^T \neq 0, \\ & \big\langle \nabla H(\hat{\zeta}^*),\, \tilde{W}_{\Omega^1}\, \hat{\zeta}^* + \tilde{h}_1 \big\rangle \,=\, \sum_{\substack{j=1 \\ j\neq 2}}^{10} \tilde{w}_{2j}^{(1)}\, \hat{\zeta}_j^* \,+\, \tilde{h}_{12} = 0, \\ & \big\langle \nabla H(\hat{\zeta}^*),\, \tilde{W}_{\Omega^2}\, \hat{\zeta}^* + \tilde{h}_2 \big\rangle \,=\, \sum_{\substack{j=1 \\ j\neq 2}}^{10} \tilde{w}_{2j}^{(2)}\, \hat{\zeta}_j^* \,+\, \tilde{h}_{22} = 0, \end{aligned}$$

$$\big\langle \nabla H(\hat{\zeta}^*),\, \tilde{W}^2_{\Omega^1}\hat{\zeta}^* + \tilde{W}_{\Omega^1}\tilde{h}_1 \big\rangle + \big\langle \nabla^2 H(\hat{\zeta}^*)(\tilde{W}_{\Omega^1}\,\hat{\zeta}^* + \tilde{h}_1),\, \tilde{W}_{\Omega^1}\,\hat{\zeta}^* + \tilde{h}_1 \big\rangle = \sum_{\substack{j=1\\ j\neq 2}}^{10} \tilde{v}^{(1)}_{2j}\hat{\zeta}^*_j + \sum_{\substack{j=1\\ j\neq 2}}^{10} \tilde{w}^{(1)}_{2j}\,\tilde{h}_{1j} = 0,$$

$$\big\langle \nabla H(\hat{\zeta}^*),\, \tilde{W}^2_{\Omega^2}\hat{\zeta}^* + \tilde{W}_{\Omega^2}\tilde{h}_2 \big\rangle + \big\langle \nabla^2 H(\hat{\zeta}^*)(\tilde{W}_{\Omega^2}\,\hat{\zeta}^* + \tilde{h}_2),\, \tilde{W}_{\Omega^2}\,\hat{\zeta}^* + \tilde{h}_2 \big\rangle = \sum_{\substack{j=1\\ j\neq 2}}^{10} \tilde{v}^{(2)}_{2j}\hat{\zeta}^*_j + \sum_{\substack{j=1\\ j\neq 2}}^{10} \tilde{w}^{(2)}_{2j}\,\tilde{h}_{2j} = 0,$$

where $\tilde{W}_{\Omega^1} = [\tilde{w}^{(1)}_{ij}]$, $\tilde{W}_{\Omega^2} = [\tilde{w}^{(2)}_{ij}]$, $\tilde{W}^2_{\Omega^1} = [\tilde{v}^{(1)}_{ij}]$ and $\tilde{W}^2_{\Omega^2} = [\tilde{v}^{(2)}_{ij}]$.

In this case the periodic orbit $\hat{\zeta}(t)$ crosses Σ transversally as the bifurcation parameter passes through the bifurcation value. The grazing bifurcation leads to a transition or a sudden jump in the system's response by the dis-/appearance of a tangential intersection between the trajectory and the switching boundary. The occurrence of a grazing bifurcation in the continuous PLRNN is illustrated in Fig. 2.