# A. Convergence under Imperfect Feedback with Absolute Random Noise

In this section, we analyze the convergence of OGD-based learning on $\lambda$-cocercive games under imperfect feedback with absolute random noise (4). We first establish that OGD under noisy feedback converges almost surely in last-iterate to the set of Nash equilibria of a co-coercive game if $\sigma_t^2 \in (0, \sigma^2)$ for some $\sigma^2 < +\infty$ and the finite-time $O(1/\sqrt{T})$ convergence rate on $(1/T)\mathbb{E}[\sum_{t=0}^T \epsilon(\mathbf{x}_t)]$ under properly diminishing step-size sequences. We also present a finite-time convergence rate on $\mathbb{E}[\epsilon(\mathbf{x}_T)]$ if $\sigma_t^2$ satisfies certain conditions.

## A.1. Almost Sure Last-Iterate Convergence

We start by developing a key iterative formula for $\mathbb{E}[\epsilon(\mathbf{x}_t)]$ in the following lemma.

**Lemma A.1** *Fix a $\lambda$-cocoercive game $\mathcal{G}$ with a continuous action space, $\mathcal{G} = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^N \mathbb{R}^{n_i}, \{u_i\}_{i=1}^N)$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Under the noisy model (2) with absolute random noise (4) and letting the OGD-based learning run with a step-size sequence $\eta_t \in (0, \lambda)$, the noisy OGD iterate $\mathbf{x}_t$ satisfies*

$$\mathbb{E}[\epsilon(\mathbf{x}_{t+1})] \ \leq \ \mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{\sigma_t^2}{\lambda \eta_{t+1}}.$$

We are now ready to establish last-iterate convergence in a strong, almost sure sense. Note that the conditions imposed on $\sigma_t^2$ and $\eta_t$ are minimal.

**Theorem A.2** *Fix a $\lambda$-cocoercive game $\mathcal{G}$ with a continuous action space, $\mathcal{G} = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^N \mathbb{R}^{n_i}, \{u_i\}_{i=1}^N)$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Consider the noisy model (2) with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2]$ for some $\sigma^2 < +\infty$ and letting the OGD-based learning run with a step-size sequence satisfying*

$$\sum_{t=1}^{\infty} \eta_t \ = \ +\infty, \qquad \sum_{t=1}^{\infty} \eta_t^2 \ < \ +\infty.$$

*Then the noisy OGD iterate $\mathbf{x}_t$ converges to $\mathcal{X}^*$ almost surely.*

## A.2. Finite-Time Convergence Rate: Time-Average and Last-Iterate

For completeness, we characterize two types of rates: the time-average and last-iterate convergence rate, as formalized by the following theorems.

**Theorem A.3** *Fix a $\lambda$-cocoercive game $\mathcal{G}$ with a continuous action space, $\mathcal{G} = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^N \mathbb{R}^{n_i}, \{u_i\}_{i=1}^N)$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Under the noisy model (2) with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2]$ for some $\sigma^2 < +\infty$ and letting the OGD-based learning run with a step-size sequence $\eta_t = c/\sqrt{t}$ for some constant $c \in (0, \lambda)$, the noisy OGD iterate $\mathbf{x}_t$ satisfies*

$$\frac{1}{T+1}\left(\mathbb{E}\left[\sum_{t=0}^T \epsilon(\mathbf{x}_t)\right]\right) \ = \ O\left(\frac{\log(T)}{\sqrt{T}}\right).$$

Inspired by Lemma A.1, we impose an intuitive condition on the variance of noisy process $\{\sigma_t^2\}_{t \geq 0}$. More specifically, there exists a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(t) = o(1)$ and $a(t) = \Omega(1/t)$ such that

$$\frac{1}{T+1}\left(\sum_{t=0}^{T-1}(t+1)\sigma_t^2\right) \ = \ O(a(T)). \tag{15}$$

Under this condition, the noisy iterate generated by the OGD-based learning achieves the finite-time last-iterate convergence rate regardless of a sequence of possibly constant step-sizes $\eta_t$ satisfying $0 < \underline{\eta} \leq \eta_t \leq \overline{\eta} < \lambda$ for all $t \geq 1$.

**Theorem A.4** *Fix a $\lambda$-cocoercive game $\mathcal{G}$ with a continuous action space, $\mathcal{G} = (\mathcal{N}, \mathcal{X} = \prod_{i=1}^N \mathbb{R}^{n_i}, \{u_i\}_{i=1}^N)$, and let the set of Nash equilibria, $\mathcal{X}^*$, be nonempty. Under the noisy model (2) with absolute random noise (4) satisfying Eq. (15) and letting the OGD-based learning run with an nonincreasing step-size sequence satisfying $0 < \underline{\eta} \leq \eta_t \leq \overline{\eta} < \lambda$ for all $t \geq 1$, the noisy OGD iterate $\mathbf{x}_t$ satisfies*

$$\mathbb{E}[\epsilon(\mathbf{x}_T)] \ = \ O(a(T)).$$

## B. Proof of Lemma 3.1

Since $\mathcal{X}_i = \mathbb{R}^{n_i}$, we have

$$\begin{aligned}
& \|x_{i,t+2} - x_{i,t+1}\|^2 \\
= \;& \|x_{i,t+1} - x_{i,t} + \eta(v_i(\mathbf{x}_{t+1}) - v_i(\mathbf{x}_t))\|^2 \\
= \;& \|x_{i,t+1} - x_{i,t}\|^2 + 2\eta(x_{i,t+1} - x_{i,t})^\top (v_i(\mathbf{x}_{t+1}) - v_i(\mathbf{x}_t)) + \eta^2\|v_i(\mathbf{x}_{t+1}) - v_i(\mathbf{x}_t)\|^2.
\end{aligned}$$

Expanding the right-hand side of the above inequality and summing up the resulting inequality over $i \in \mathcal{N}$ yields that

$$\begin{aligned}
\|\mathbf{x}_{t+2} - \mathbf{x}_{t+1}\|^2 \;=\; & \sum_{i\in\mathcal{N}} \|x_{i,t+2} - x_{i,t+1}\|^2 \qquad (16) \\
\leq \; & \sum_{i\in\mathcal{N}} \left(\|x_{i,t+1} - x_{i,t}\|^2 + \eta^2\|v_i(\mathbf{x}_{t+1}) - v_i(\mathbf{x}_t)\|^2 + 2\eta(x_{i,t+1} - x_{i,t})^\top (v_i(\mathbf{x}_{t+1}) - v_i(\mathbf{x}_t))\right) \\
= \; & \|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2 + 2\eta(\mathbf{x}_{t+1} - \mathbf{x}_t)^\top(\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)) + \eta^2\|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2.
\end{aligned}$$

Since $\mathcal{G}$ is a $\lambda$-cocoercive game, we have

$$(\mathbf{x}_{t+1} - \mathbf{x}_t)^\top(\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)) \leq -\lambda\|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2.$$

Plugging the above equation into Eq. (16) together with the condition $\eta \in (0, \lambda]$ yields that

$$\|\mathbf{x}_{t+2} - \mathbf{x}_{t+1}\|^2 \;\leq\; \|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2.$$

Using the update formula in Eq. (1), we have $\|\mathbf{v}(\mathbf{x}_{t+1})\| \leq \|\mathbf{v}(\mathbf{x}_t)\|$ for all $t \geq 0$.

Then we proceed to bound $\sum_{t=0}^{+\infty} \|\mathbf{v}(\mathbf{x}_t)\|^2$. Indeed, for any $x_i \in \mathcal{X}_i$, we have

$$(x_i - x_{i,t+1})^\top(x_{i,t+1} - x_{i,t} - \eta v_i(\mathbf{x}_t)) \;=\; 0.$$

Applying the equality $a^\top b = (\|a + b\|^2 - \|a\|^2 - \|b\|^2)/2$ yields that

$$(x_{i,t+1} - x_i)^\top v_i(\mathbf{x}_t) \;=\; \frac{1}{2\eta}\left(\|x_{i,t} - x_{i,t+1}\|^2 + \|x_i - x_{i,t+1}\|^2 - \|x_i - x_{i,t}\|^2\right).$$

Summing up the resulting inequality over $i \in \mathcal{N}$ yields that

$$(\mathbf{x}_{t+1} - \mathbf{x})^\top\mathbf{v}(\mathbf{x}_t) \;=\; \frac{1}{2\eta}\left(\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2 + \|\mathbf{x} - \mathbf{x}_{t+1}\|^2 - \|\mathbf{x} - \mathbf{x}_t\|^2\right), \quad \forall \mathbf{x} \in \mathcal{X}.$$

Letting $\mathbf{x} = \mathbf{x}^* \in \mathcal{X}^*$, we have

$$(\mathbf{x}_{t+1} - \mathbf{x}^*)^\top\mathbf{v}(\mathbf{x}_t) \;=\; \frac{1}{2\eta}\left(\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2 + \|\mathbf{x}^* - \mathbf{x}_{t+1}\|^2 - \|\mathbf{x}^* - \mathbf{x}_t\|^2\right). \qquad (17)$$

Furthermore, we have

$$(\mathbf{x}_{t+1} - \mathbf{x}^*)^\top\mathbf{v}(\mathbf{x}_t) = (\mathbf{x}_t - \mathbf{x}^*)^\top\mathbf{v}(\mathbf{x}_t) + (\mathbf{x}_{t+1} - \mathbf{x}_t)^\top\mathbf{v}(\mathbf{x}_t).$$

Since $\mathcal{G}$ is a $\lambda$-cocoercive game and $\mathbf{v}(\mathbf{x}^*) = 0$, we have

$$(\mathbf{x}_t - \mathbf{x}^*)^\top\mathbf{v}(\mathbf{x}_t) \;=\; (\mathbf{x}_t - \mathbf{x}^*)^\top(\mathbf{v}(\mathbf{x}_t) - \mathbf{v}(\mathbf{x}^*)) \;\leq\; -\lambda\|\mathbf{v}(\mathbf{x}_t) - \mathbf{v}(\mathbf{x}^*)\|^2 \;=\; -\lambda\|\mathbf{v}(\mathbf{x}_t)\|^2.$$

By Young's inequality we have

$$(\mathbf{x}_{t+1} - \mathbf{x}_t)^\top\mathbf{v}(\mathbf{x}_t) \;\leq\; \frac{\lambda\|\mathbf{v}(\mathbf{x}_t)\|^2}{2} + \frac{\|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2}{2\lambda}.$$

Putting these pieces together yields that

$$(\mathbf{x}_{t+1} - \mathbf{x}^*)^\top\mathbf{v}(\mathbf{x}_t) \;\leq\; \frac{\|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2}{2\lambda} - \frac{\lambda\|\mathbf{v}(\mathbf{x}_t)\|^2}{2}. \qquad (18)$$

Plugging Eq. (18) into Eq. (17) together with the condition $\eta \in (0, \lambda]$ yields that

$$\lambda\|\mathbf{v}(\mathbf{x}_t)\|^2 \ \leq \ \frac{\|\mathbf{x}^* - \mathbf{x}_t\|^2 - \|\mathbf{x}^* - \mathbf{x}_{t+1}\|^2}{\eta}.$$

Summing up the above inequality over $t = 0, 1, 2, \ldots$ and using the boundedness of $\mathcal{X}$ yields that

$$\sum_{t=0}^{+\infty} \|\mathbf{v}(\mathbf{x}_t)\|^2 \ \leq \ \frac{\|\mathbf{x}^* - \mathbf{x}_0\|^2}{\eta\lambda}.$$

Note that $\mathbf{x}^* \in \mathcal{X}^*$ is chosen arbitrarily, we let $\mathbf{x}^* = \Pi_{\mathcal{X}^*}(\mathbf{x}_0)$ and conclude the desired inequality.

## C. Postponed Proofs in Section 4

In this section, we present the missing proofs in Section 4.

### C.1. Proof of Lemma 4.1

Using the update formula of $x_{i,t+1}$ in Eq. (1), we have the following for any $x_i^* \in \mathcal{X}_i^*$:

$$\|x_{i,t+1} - x_i^*\|^2 \ = \ \|x_{i,t} + \eta_{t+1}\hat{v}_{i,t+1} - x_i^*\|^2.$$

which implies that

$$\|x_{i,t+1} - x_i^*\|^2 \ = \ \|x_{i,t} - x_i^*\|^2 + \eta_{t+1}^2\|\hat{v}_{i,t+1}\|^2 + 2\eta_{t+1}(x_{i,t} - x_i^*)^\top \hat{v}_{i,t+1}.$$

Summing up the above inequality over $i \in \mathcal{N}$ and rearranging yields that

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \ = \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 + 2\eta_{t+1}(\mathbf{x}_t - \mathbf{x}^*)^\top \hat{v}_{t+1} + \eta_{t+1}^2\|\hat{v}_{t+1}\|^2. \tag{19}$$

Using Young's inequality, we have

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 + 2\eta_{t+1}^2\|\xi_{t+1}\|^2 + 2\eta_{t+1}^2\|\mathbf{v}(\mathbf{x}_t)\|^2 + 2\eta_{t+1}(\mathbf{x}_t - \mathbf{x}^*)^\top(\mathbf{v}(\mathbf{x}_t) + \xi_{t+1}).$$

Since $\mathbf{x}^* \in \mathcal{X}^*$ and $\mathcal{G}$ is a $\lambda$-cocoercive game, we have $\mathbf{v}(\mathbf{x}^*) = 0$ and

$$(\mathbf{x}_t - \mathbf{x}^*)^\top \mathbf{v}(\mathbf{x}_t) \ = \ (\mathbf{x}_t - \mathbf{x}^*)^\top(\mathbf{v}(\mathbf{x}_t) - \mathbf{v}(\mathbf{x}^*)) \ \leq \ -\lambda\|\mathbf{v}(\mathbf{x}_t) - \mathbf{v}(\mathbf{x}^*)\|^2 \ = \ -\lambda\|\mathbf{v}(\mathbf{x}_t)\|^2.$$

Putting these pieces yields the desired inequality.

### C.2. Proof of Lemma 4.2

Using the same argument as in Lemma 4.1, we have

$$\mathbb{E}[\epsilon(\mathbf{x}_{t+1}) \mid \mathcal{F}_t] - \epsilon(\mathbf{x}_t) \ \leq \ \frac{\mathbb{E}[\|\xi_{t+1}\|^2 \mid \mathcal{F}_t]}{\lambda\eta_{t+1}} + \left(1 - \frac{\lambda}{\eta_{t+1}}\right)\mathbb{E}\left[\|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2 \mid \mathcal{F}_t\right].$$

Since the noisy model (2) is with relative random noise (4), we have $\mathbb{E}[\|\xi_{t+1}\|^2 \mid \mathcal{F}_t] \leq \tau_t\|\mathbf{v}(\mathbf{x}_t)\|^2$. Also, $\eta_t \in (0, \lambda)$ for all $t \geq 1$. Therefore, we conclude that

$$\mathbb{E}[\epsilon(\mathbf{x}_{t+1}) \mid \mathcal{F}_t] - \epsilon(\mathbf{x}_t) \ \leq \ \frac{\tau_t\|\mathbf{v}(\mathbf{x}_t)\|^2}{\lambda\eta_{t+1}}.$$

Taking an expectation of both sides yields the desired inequality.

### C.3. Proof of Theorem 4.4

Using the same argument as in Theorem 4.3, we obtain that

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \mid \mathcal{F}_t] \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 - 2(\lambda - \overline{\eta} - \tau\overline{\eta})\eta_{t+1}\|\mathbf{v}(\mathbf{x}_t)\|^2. \tag{20}$$

Taking an expectation of both sides of Eq. (20) and rearranging yields that

$$\mathbb{E}[\epsilon(\mathbf{x}_t)] \ \leq \ \frac{1}{2(\lambda - \overline{\eta} - \tau\overline{\eta})\eta_{t+1}} \left(\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] - \mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2]\right). \tag{21}$$

Summing up the above inequality over $t = 0, 1, \ldots, T$ yields that

$$\mathbb{E}\left[\sum_{t=0}^T \epsilon(\mathbf{x}_t)\right] \ \leq \ \sum_{t=1}^T \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \frac{\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2]}{2(\lambda - \overline{\eta} - \tau\overline{\eta})} + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta} - \tau\overline{\eta})\eta_1}.$$

On the other hand, we have $\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2] \ \leq \ \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2]$. This implies that $\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] \leq \|\mathbf{x}_0 - \mathbf{x}^*\|^2$ for all $t \geq 1$. Therefore, we conclude that

$$\begin{aligned} \mathbb{E}\left[\sum_{t=0}^T \epsilon(\mathbf{x}_t)\right] \ &\leq \ \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta} - \tau\overline{\eta})\eta_1} + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta} - \tau\overline{\eta})} \sum_{t=1}^T \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \\ &= \ \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta} - \tau\overline{\eta})\eta_1} + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta} - \tau\overline{\eta})\eta_{T+1}} \\ &\leq \ \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{(\lambda - \overline{\eta} - \tau\overline{\eta})\underline{\eta}} \ = \ O(1). \end{aligned}$$

This completes the proof.

### C.4. Proof of Theorem 4.7

Since the step-size sequence $\{\eta_t\}_{t \geq 1}$ is decreasing and converges to zero, we define the first iconic time in our analysis as follows,

$$t^* \ = \ \max\left\{t \geq 0 \mid \eta_{t+1} > \frac{\lambda}{2(1+\tau)}\right\} \ < \ +\infty.$$

First, we claim that $\mathbb{E}[\|\mathbf{x}_t - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\|] \leq D$ where $D = \max_{1 \leq t \leq t^*} \mathbb{E}[\|\mathbf{x}_t - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\|]$. Indeed, it suffices to show that $\mathbb{E}[\|\mathbf{x}_t - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\|] \leq \mathbb{E}[\|\mathbf{x}_{t^*} - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\|]$ holds for $t > t^*$. By the definition of $t^*$, we have $\eta_{t+1} < \lambda/(1+\tau)$ for all $t > t^*$. The desired inequality follows from Eq. (21) and the fact that $\mathbb{E}[\epsilon(\mathbf{x}_t)] \geq 0$ for all $t > t^*$.

Furthermore, we derive an upper bound for the term $\sum_{t=0}^T \|\mathbf{v}(\mathbf{x}_t)\|^2$. Using the update formula (cf. Eq. (1)) to obtain that

$$(\mathbf{x}_{t+1} - \mathbf{x}^*)^\top(\mathbf{v}(\mathbf{x}_t) + \xi_{t+1}) = \frac{1}{2\eta_{t+1}}\left(\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2 + \|\mathbf{x}^* - \mathbf{x}_{t+1}\|^2 - \|\mathbf{x}^* - \mathbf{x}_t\|^2\right).$$

Recall that $\mathcal{G}$ is $\lambda$-cocoercive and the noisy model is defined with relative random noise, we have

$$\mathbb{E}[(\mathbf{x}_t - \mathbf{x}^*)^\top(\mathbf{v}(\mathbf{x}_t) + \xi_{t+1}) \mid \mathcal{F}_t] \ \geq \ \lambda\|\mathbf{v}(\mathbf{x}_t)\|^2.$$

Using Young's inequality, we have

$$\mathbb{E}[(\mathbf{x}_{t+1} - \mathbf{x}_t)^\top(\mathbf{v}(\mathbf{x}_t) + \xi_{t+1}) \mid \mathcal{F}_t] \ \geq \ -\frac{\lambda\|\mathbf{v}(\mathbf{x}_t)\|^2}{2} - \frac{(1+\tau)\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2 \mid \mathcal{F}_t]}{\lambda}.$$

Putting these pieces together and taking an expectation yields that

$$\lambda\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2] \ \leq \ \mathbb{E}\left[\frac{\|\mathbf{x}^* - \mathbf{x}_t\|^2 - \|\mathbf{x}^* - \mathbf{x}_{t+1}\|^2}{\eta_{t+1}}\right] + \mathbb{E}\left[\left(\frac{2(1+\tau)}{\lambda} - \frac{1}{\eta_{t+1}}\right)\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2\right]. \tag{22}$$

Recall that the step-size sequence $\{\eta_t\}_{t\geq 1}$ is nonincreasing and $\|\mathbf{x}_t - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\| \leq D$, we let $\mathbf{x}^* = \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})$ in Eq. (22) and obtain that

$$\sum_{t=0}^T \lambda\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2] \ \leq \ \mathbb{E}\left[\frac{D^2}{\eta_{T+1}}\right] + \sum_{t=0}^T \mathbb{E}\left[\left(\frac{2(1+\tau)}{\lambda} - \frac{1}{\eta_{t+1}}\right)\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2\right].$$

To proceed, we define the second iconic time as

$$t_1^* \ = \ \max\left\{t \geq 0 \mid \eta_{t+1} > \frac{\lambda}{4(1+\tau)D^2 + 2(1+\tau)}\right\} \ > \ t^*.$$

It is clear that $t_1^* < +\infty$ and $\eta_{t+1} \leq \lambda/(2+2\tau)$ for all $t > t_1^*$ which implies that $(2+2\tau)/\lambda - 1/\eta_{t+1} \leq 0$. Assume $T$ sufficiently large without loss of generality, we have

$$\sum_{t=0}^T \lambda\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2] \ \leq \ \mathbb{E}\left[\frac{D^2}{\eta_{T+1}}\right] + \frac{2(1+\tau)}{\lambda}\left(\sum_{t=0}^{t_1^*}\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2]\right) \ = \ \mathrm{I} + \mathrm{II}.$$

We also use Lemmas A.1 and A.2 from Bach & Levy (2019) to bound terms I and II. For convenience, we present these two lemmas here:

**Lemma C.1** *For a sequence of numbers $a_0, a_1, \ldots, a_n \in [0, a]$ and $b \geq 0$, the following inequality holds:*

$$\sqrt{b + \sum_{i=0}^{n-1} a_i} - \sqrt{b} \ \leq \ \sum_{i=0}^n \frac{a_i}{\sqrt{b + \sum_{j=0}^{i-1} a_j}} \ \leq \ \frac{2a}{\sqrt{b}} + 3\sqrt{a} + 3\sqrt{b + \sum_{i=0}^{n-1} a_i}.$$

**Bounding I:** We derive from the definition of $\eta_t$ and Jensen's inequality that

$$\mathrm{I} \ \leq \ D^2\sqrt{\beta + \log(T+1) + \sum_{j=0}^{T-1}\mathbb{E}\left[\frac{\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2}{\eta_{j+1}^2}\right]}.$$

Since $\mathbb{E}[\|\mathbf{x}_t - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\|] \leq D$ for all $t \geq 0$ and the notion of $\lambda$-cocercivity implies the notion of $(1/\lambda)$-Lipschiz continuity, we have

$$\mathbb{E}\left[\frac{\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2}{\eta_{t+1}^2}\right] \ \leq \ (2+2\tau)\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2] \ \leq \ \frac{(2+2\tau)\|\mathbf{x}_t - \Pi_{\mathcal{X}^*}(\mathbf{x}_{t^*})\|^2}{\lambda^2} \ \leq \ \frac{(2+2\tau)D^2}{\lambda^2}.$$

Using the first inequality in Lemma C.1, we have

$$\begin{aligned}
\mathrm{I} \ & \leq \ D^2\sqrt{\beta + \log(T+1)} + \sum_{t=0}^T \frac{D^2\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2/\eta_{t+1}^2]}{\sqrt{\beta + \log(T+1) + \sum_{j=0}^{t-1}\mathbb{E}[\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2/\eta_{j+1}^2]}} \\
& \leq \ D^2\sqrt{\beta + \log(T+1)} + \sum_{t=0}^{t_1^*} \frac{D^2\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2/\eta_{t+1}^2]}{\sqrt{\beta + \log(T+1) + \sum_{j=0}^{t-1}\mathbb{E}[\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2/\eta_{j+1}^2]}} \\
& \quad + \sum_{t=t_1^*+1}^T D^2\eta_{t+1}\mathbb{E}\left[\frac{\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2}{\eta_{t+1}^2}\right] \\
& \leq \ D^2\sqrt{\beta + \log(T+1)} + \sum_{t=0}^{t_1^*} \frac{D^2\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2/\eta_{t+1}^2]}{\sqrt{\beta + \log(T+1) + \sum_{j=0}^{t-1}\mathbb{E}[\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2/\eta_{j+1}^2]}} \\
& \quad + \sum_{t=t_1^*+1}^T (2+2\tau)D^2\eta_{t+1}\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2]. \qquad (23)
\end{aligned}$$

Since $\eta_{t+1} \leq \lambda/[4(1+\tau)D^2]$ for all $t > t_1^*$, we have

$$\sum_{t=t_1^*+1}^{T} (2+2\tau)D^2\eta_{t+1}\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2] \;\leq\; \sum_{t=t_1^*+1}^{T} \frac{\lambda\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2]}{2}. \tag{24}$$

Using the second inequality in Lemma 3.5, we have

$$\begin{aligned} & \sum_{t=0}^{t_1^*} \frac{D^2\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2/\eta_{t+1}^2]}{\sqrt{\beta + \log(T+1) + \sum_{j=0}^{t-1}\mathbb{E}[\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2/\eta_{j+1}^2]}} \\ \leq \; & \frac{(4+4\tau)D^2}{\lambda^2\sqrt{\beta+\log(T+1)}} + \frac{3D\sqrt{2+2\tau}}{\lambda} + 3\sqrt{\beta + \log(T+1) + \sum_{j=0}^{t_1^*-1}\mathbb{E}\left[\frac{\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2}{\eta_{j+1}^2}\right]}. \end{aligned} \tag{25}$$

By the definition of $\eta_t$, we have

$$\begin{aligned} & \sqrt{\beta + \log(T+1) + \sum_{j=0}^{t_1^*-1}\mathbb{E}\left[\frac{\|\mathbf{x}_j - \mathbf{x}_{j+1}\|^2}{\eta_{j+1}^2}\right]} \\ \leq \; & \frac{1}{\eta_{t_1^*+1}} + \sqrt{\log(T+1)} \;<\; \frac{4(1+\tau)D^2 + 2(1+\tau)}{\lambda} + \sqrt{\log(T+1)}. \end{aligned} \tag{26}$$

Putting Eq. (24)-(26) together yields that

$$\begin{aligned} \mathrm{I} \;\leq\; & D^2\sqrt{\beta+\log(T+1)} + \frac{(4+4\tau)D^2}{\lambda^2\sqrt{\beta+\log(T+1)}} + \frac{3D\sqrt{2+2\tau}}{\lambda} + \frac{12(1+\tau)D^2 + 6(1+\tau)}{\lambda} \\ & + \sqrt{\log(T+1)} + \sum_{t=t_1^*+1}^{T}\frac{\lambda\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2]}{2}. \end{aligned}$$

**Bounding II:** Recalling that

$$\mathbb{E}\left[\frac{\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2}{\eta_{t+1}^2}\right] \;\leq\; \frac{(2+2\tau)D^2}{\lambda^2},$$

and $\eta_t \leq 1/\beta$ for all $t \geq 1$, we have

$$\mathbb{E}\left[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2\right] \;\leq\; \frac{(2+2\tau)D^2}{\lambda^2\beta^2},$$

Putting these pieces together yields that

$$\mathrm{II} \;=\; \frac{2(1+\tau)}{\lambda}\left(\sum_{t=0}^{t_1^*}\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2]\right) \;\leq\; \frac{4(1+\tau)^2D^2t_1^*}{\lambda^3\beta^2}.$$

Therefore, we have

$$\begin{aligned} \sum_{t=0}^{T}\frac{\lambda\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2]}{2} \;\leq\; & D^2\sqrt{\beta+\log(T+1)} + \sqrt{\log(T+1)} + \frac{(4+4\tau)D^2}{\lambda^2\sqrt{\beta+\log(T+1)}} \\ & + \frac{3D\sqrt{2+2\tau}}{\lambda} + \frac{12(1+\tau)D^2+6(1+\tau)}{\lambda} + \frac{4(1+\tau)^2D^2t_1^*}{\lambda^3\beta^2}. \end{aligned}$$

By the definition, we have $t_1^* < +\infty$ is uniformly bounded. To this end, we conclude that $\sum_{t=0}^{T}\mathbb{E}[\|\mathbf{v}(\mathbf{x}_t)\|^2] \leq C_1 + C_2\sqrt{\log(T+1)}$, where $C_1 > 0$ and $C_2 > 0$ are universal constants.

Finally, we proceed to bound the term $\epsilon(\mathbf{x}_T)$. Without loss of generality, we can start the sequence at a later index $t_1^*$ since $t_1^* < +\infty$. This implies that $\eta_{t+1} \leq \lambda/2(1+\tau)$. Using the last equation in the proof of Lemma 4.2, we have

$$\mathbb{E}[\epsilon(\mathbf{x}_T)] \ \leq \ \mathbb{E}[\epsilon(\mathbf{x}_t)] + \sum_{j=t}^{T-1} \frac{\tau_j}{\lambda}\mathbb{E}\left[\frac{\|\mathbf{v}(\mathbf{x}_j)\|^2}{\eta_{j+1}}\right].$$

Summing up the above inequality over $t = t_1^*, \ldots, T+$ yields

$$(T - t_1^* + 1)\mathbb{E}[\epsilon(\mathbf{x}_T)] \ \leq \ \sum_{t=t_1^*}^{T} \mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{1}{\lambda}\left(\sum_{t=t_1^*}^{T-1}\sum_{j=t}^{T-1} \tau_j \mathbb{E}\left[\frac{\|\mathbf{v}(\mathbf{x}_j)\|^2}{\eta_{j+1}}\right]\right).$$

Since $\{\tau_t\}_{t \geq 0}$ is an nonincreasing sequence, we have

$$\sum_{t=t_1^*}^{T-1}\sum_{j=t}^{T-1} \tau_j \mathbb{E}\left[\frac{\|\mathbf{v}(\mathbf{x}_j)\|^2}{\eta_{j+1}}\right] \ \leq \ \left(\sum_{t=0}^{T-1} \tau_j\right)\left(\sum_{t=t_1^*}^{T-1} \mathbb{E}\left[\frac{\|\mathbf{v}(\mathbf{x}_j)\|^2}{\eta_{j+1}}\right]\right).$$

Using Eq. (20) and $\eta_{t+1} \leq \lambda/2(1+\tau)$ for all $t > t_1^*$, we have

$$\mathbb{E}\left[\frac{\|\mathbf{v}(\mathbf{x}_j)\|^2}{\eta_{j+1}}\right] \ \leq \ \mathbb{E}\left[\frac{\|\mathbf{x}_j - \mathbf{x}^*\|^2 - \|\mathbf{x}_{j+1} - \mathbf{x}^*\|^2}{\eta_{j+1}^2}\right].$$

Note that $\{\eta_t\}_{t \geq 0}$ is an nonnegative and nonincreasing sequence and $\mathbb{E}[\|\mathbf{x}_{j+1} - \mathbf{x}^*\|^2] \leq D^2$. Putting these pieces together yields that

$$\begin{aligned}
\sum_{t=t_1^*}^{T-1} \mathbb{E}\left[\frac{\|\mathbf{v}(\mathbf{x}_j)\|^2}{\eta_{j+1}}\right] \ &\leq \ \mathbb{E}\left[\frac{D^2}{\eta_T^2}\right] \ \leq \ D^2\left(\beta + \log(T) + \sum_{t=0}^{T} \mathbb{E}\left[\frac{\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2}{\eta_{t+1}^2}\right]\right) \\
&\leq \ D^2\left(\beta + \log(T) + 2(1+\tau)\sum_{t=0}^{T} \mathbb{E}\left[\|\mathbf{v}(\mathbf{x}_t)\|^2\right]\right) \\
&= \ O(\log(T)).
\end{aligned}$$

Therefore, we conclude that

$$\mathbb{E}[\epsilon(\mathbf{x}_T)] \ \leq \ \frac{\sum_{t=t_1^*}^{T} \mathbb{E}[\epsilon(\mathbf{x}_t)]}{T - t_1^* + 1} + \frac{C\log(T+1)}{\lambda(T - t_1^* + 1)}\left(\sum_{t=0}^{T-1} \tau_t\right) \ \text{ for some } C > 0.$$

This completes the proof.

## D. Postponed Proofs in Section A

In this section, we present the missing proofs in Section A.

### D.1. Proof of Lemma A.1

By the definition of $\epsilon(\mathbf{x})$, we have

$$\begin{aligned}
\epsilon(\mathbf{x}_{t+1}) - \epsilon(\mathbf{x}_t) \ &= \ \|\mathbf{v}(\mathbf{x}_{t+1})\|^2 - \|\mathbf{v}(\mathbf{x}_t)\|^2 \ = \ (\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t))^\top(\mathbf{v}(\mathbf{x}_{t+1}) + \mathbf{v}(\mathbf{x}_t)) \\
&= \ 2(\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t))^\top \mathbf{v}(\mathbf{x}_t) + \|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2.
\end{aligned}$$

Using the update formula in Eq. (1), it holds that $\mathbf{v}(\mathbf{x}_t) = \eta_{t+1}^{-1}(\mathbf{x}_{t+1} - \mathbf{x}_t) - \xi_{t+1}$. Therefore, we have

$$\epsilon(\mathbf{x}_{t+1}) - \epsilon(\mathbf{x}_t) \ = \ \frac{2}{\eta_{t+1}}\left((\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t))^\top(\mathbf{x}_{t+1} - \mathbf{x}_t) - (\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t))^\top \xi_{t+1}\right) + \|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2.$$

Since $\mathcal{G}$ is a $\lambda$-cocoercive game, we have

$$(\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t))^\top (\mathbf{x}_{t+1} - \mathbf{x}_t) \ \leq \ -\lambda \|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2.$$

Using Young's inequality, we have

$$-(\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t))^\top \xi_{t+1} \ \leq \ \frac{\lambda \|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2}{2} + \frac{\|\xi_{t+1}\|^2}{2\lambda}.$$

Putting these pieces together yields that

$$\epsilon(\mathbf{x}_{t+1}) - \epsilon(\mathbf{x}_t) \ \leq \ \frac{\|\xi_{t+1}\|^2}{\lambda \eta_{t+1}} + \left(1 - \frac{\lambda}{\eta_{t+1}}\right) \|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2. \tag{27}$$

Taking an expectation of Eq. (27) conditioned on $\mathcal{F}_t$ yields that

$$\mathbb{E}[\epsilon(\mathbf{x}_{t+1}) \mid \mathcal{F}_t] - \epsilon(\mathbf{x}_t) \ \leq \ \frac{\mathbb{E}[\|\xi_{t+1}\|^2 \mid \mathcal{F}_t]}{\lambda \eta_{t+1}} + \left(1 - \frac{\lambda}{\eta_{t+1}}\right) \mathbb{E}\left[\|\mathbf{v}(\mathbf{x}_{t+1}) - \mathbf{v}(\mathbf{x}_t)\|^2 \mid \mathcal{F}_t\right].$$

Since the noisy model (2) is with absolute random noise (4), we have $\mathbb{E}[\|\xi_{t+1}\|^2 \mid \mathcal{F}_t] \leq \sigma_t^2$. Also, $\eta_t \in (0, \lambda)$ for all $t \geq 1$. Therefore, we conclude that

$$\mathbb{E}[\epsilon(\mathbf{x}_{t+1}) \mid \mathcal{F}_t] - \epsilon(\mathbf{x}_t) \ \leq \ \frac{\sigma_t^2}{\lambda \eta_{t+1}}.$$

Taking the expectation of both sides yields the desired inequality.

### D.2. Proof of Theorem A.2

Recalling Eq. (11) (cf. Lemma 4.1), we take the expectation of both sides conditioned on $\mathcal{F}_t$ to obtain

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \mid \mathcal{F}_t] \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 - (2\lambda - 2\eta_{t+1})\eta_{t+1}\|\mathbf{v}(\mathbf{x}_t)\|^2 + 2\eta_{t+1}^2 \mathbb{E}[\|\xi_{t+1}\|^2 \mid \mathcal{F}_t] + 2\eta_{t+1}\mathbb{E}[(\mathbf{x}_t - \mathbf{x}^*)^\top \xi_{t+1} \mid \mathcal{F}_t].$$

Since the noisy model (2) is with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2]$ for some $\sigma^2 < +\infty$, we have $\mathbb{E}[(\mathbf{x}_t - \mathbf{x}^*)^\top \xi_{t+1} \mid \mathcal{F}_t] = 0$ and $\mathbb{E}[\|\xi_{t+1}\|^2 \mid \mathcal{F}_t] \leq \sigma^2$. Therefore, we have

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \mid \mathcal{F}_t] \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 - (2\lambda - 2\eta_{t+1})\eta_{t+1}\|\mathbf{v}(\mathbf{x}_t)\|^2 + 2\eta_{t+1}^2\sigma^2.$$

Since $\sum_{t=1}^\infty \eta_t^2 < \infty$, we have $\eta_t \to 0$ as $t \to +\infty$. Without loss of generality, we assume $\eta_t \leq \lambda$ for all $t$. Then we have

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \mid \mathcal{F}_t] \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 + 2\eta_{t+1}^2\sigma^2. \tag{28}$$

We let $M_t = \|\mathbf{x}_t - \mathbf{x}^*\|^2 + 2\sigma^2(\sum_{j>t}\eta_j^2)$ and obtain that $M_t$ is an nonnegative supermartingale. Then Doob's martingale convergence theorem shows that $M_n$ converges to an nonnegative and integrable random variable almost surely. Let $M_\infty = \lim_{t \to +\infty} M_t$, it suffices to show that $M_\infty = 0$ almost surely.

We first claim that *every neighborhood $U$ of $\mathcal{X}^*$ is recurrent*: there exists a subsequence $\mathbf{x}_{t_k}$ of $\mathbf{x}_t$ such that $\mathbf{x}_{t_k} \to \mathcal{X}^*$ almost surely. Equivalently, there exists a Nash equilibria $\mathbf{x}^* \in \mathcal{X}^*$ such that $\|\mathbf{x}_{t_k} - \mathbf{x}^*\|^2 \to 0$ almost surely. To this end, we can define $M_t$ with such Nash equilibria. Since $\sum_{t=1}^\infty \eta_t^2 < \infty$, we have $\sum_{j>t}\eta_j^2 \to 0$ as $t \to +\infty$ and the following statement holds almost surely:

$$\lim_{k \to +\infty} M_{t_k} \ = \ \lim_{k \to +\infty} \|\mathbf{x}_{t_k} - \mathbf{x}^*\|^2 \ = \ 0.$$

Since the whole sequence converges to $M_\infty$ almost surely, we conclude that $M_\infty = 0$ almost surely.

**Proof of the recurrence claim:** Let $U$ be a neighborhood of $\mathcal{X}^*$ and assume to the contrary that, $\mathbf{x}_t \notin U$ for sufficiently large $t$ with positive probability. By starting the sequence at a later index if necessary and noting that $\sum_{t=1}^\infty \eta_t^2 < \infty$, we may assume that $\mathbf{x}_t \notin U$ and $\eta_t \leq \lambda/2$ for all $t$ without loss of generality. Thus, there exists some $c > 0$ such that $\|\mathbf{v}(\mathbf{x}_t)\|^2 \geq c$ for all $t$. As a result, for all $\mathbf{x}^* \in \mathcal{X}^*$, we let $\psi_{t+1} = (\mathbf{x}_t - \mathbf{x}^*)^\top \xi_{t+1}$ and have

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 - \lambda c \eta_{t+1} + 2\eta_{t+1}\psi_{t+1} + 2\eta_{t+1}^2\|\xi_{t+1}\|^2.$$

Summing up the above inequality over $t = 0, 1, \ldots, T$ together with $\theta_t = \sum_{j=1}^t \eta_j$ yields that

$$\|\mathbf{x}_{T+1} - \mathbf{x}^*\|^2 \ \leq \ \|\mathbf{x}_0 - \mathbf{x}^*\|^2 - \lambda c \theta_{T+1} + 2\theta_{T+1}\left[\frac{\sum_{t=1}^{T+1} \eta_t \psi_t}{\theta_{T+1}} + \frac{\sum_{t=1}^{T+1} \eta_t^2 \|\xi_t\|^2}{\theta_{T+1}}\right]. \tag{29}$$

Since the noisy model (2) is with absolute random noise (4) satisfying $\sigma_t^2 \in (0, \sigma^2]$ for some $\sigma^2 < +\infty$, we have $\mathbb{E}[\psi_{t+1} \mid \mathcal{F}_t] = 0$. Furthermore, we obtain by taking the expectation of both sides of Eq. (28) that

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2] \ \leq \ \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] + 2\eta_{t+1}^2 \sigma^2,$$

and the following inequality holds true for all $t \geq 1$:

$$\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] \leq \|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\sigma^2 \sum_{j=1}^{\infty} \eta_j^2 \ < \ +\infty.$$

Since $\|\mathbf{x}_t - \mathbf{x}^*\|^2 \geq 0$, we have $\|\mathbf{x}_t - \mathbf{x}^*\|^2 < +\infty$ almost surely. Therefore, $\mathbb{E}[\|\psi_{t+1}\|^2 \mid \mathcal{F}_t] \leq \sigma^2 \|\mathbf{x}_t - \mathbf{x}^*\|^2 < +\infty$. Then the law of large numbers for martingale differences yields that $\theta_{T+1}^{-1}(\sum_{t=1}^{T+1} \eta_t \psi_t) \to 0$ almost surely (Hall & Heyde, 2014, Theorem 2.18). Furthermore, let $R_t = \sum_{j=1}^t \eta_j^2 \|\xi_j\|^2$, then $R_t$ is a submartingale and

$$\mathbb{E}[R_t] \ \leq \ \sigma^2 \sum_{j=1}^t \eta_j^2 < \sigma^2 \sum_{j=1}^{\infty} \eta_j^2 \ < \ +\infty.$$

From Doob's martingale convergence theorem, $R_t$ converges to some random, finite value almost surely (Hall & Heyde, 2014, Theorem 2.5). Putting these pieces together with Eq. (29) yields that $\|\mathbf{x}_t - \mathbf{x}^*\|^2 \sim -\lambda c \tau_t \to -\infty$ almost surely, a contradiction. Therefore, we conclude that every neighborhood of $\mathcal{X}^*$ is recurrent.

### D.3. Proof of Theorem A.3

Since $\eta_t = c/\sqrt{t}$ for all $t \geq 1$, we have $\eta_t \to 0$ and $\eta_t \leq c$ for all $t \geq 1$. This implies that

$$\lambda \eta_{t+1} - \eta_{t+1}^2 \ \geq \ (\lambda - c)\eta_{t+1}. \tag{30}$$

Plugging Eq. (30) into Eq. (11) (cf. Lemma 4.1) yields that

$$\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 - 2(\lambda - c)\eta_{t+1}\|\mathbf{v}(\mathbf{x}_t)\|^2 + 2\eta_{t+1}(\mathbf{x}_t - \mathbf{x}^*)^\top \xi_{t+1} + 2\eta_{t+1}^2 \|\xi_{t+1}\|^2.$$

Using the same argument as in Theorem A.2, we have

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2 \mid \mathcal{F}_t] \ \leq \ \|\mathbf{x}_t - \mathbf{x}^*\|^2 - 2(\lambda - c)\eta_{t+1}\|\mathbf{v}(\mathbf{x}_t)\|^2 + 2\eta_{t+1}^2 \sigma^2. \tag{31}$$

Taking the expectation of both sides of Eq. (31) and rearranging yields that

$$\mathbb{E}[\epsilon(\mathbf{x}_t)] \ \leq \ \frac{1}{2(\lambda - c)\eta_{t+1}}\left(\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] - \mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2]\right) + \frac{\eta_{t+1}\sigma^2}{\lambda - c}.$$

Summing up the above inequality over $t = 0, 1, \ldots, T$ and using $\eta_t = c/\sqrt{T+1}$ yields that

$$\mathbb{E}\left[\sum_{t=0}^T \epsilon(\mathbf{x}_t)\right] \ \leq \ \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - c)\eta_1} + \sum_{t=1}^T \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right)\frac{\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2]}{2(\lambda - c)} + \frac{\sigma^2}{\lambda - c}\left(\sum_{t=1}^{T+1} \eta_t\right).$$

On the other hand, we have

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2] \ \leq \ \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] + 2\eta_{t+1}^2 \sigma^2.$$

This implies that the following inequality holds for all $t \geq 1$:

$$\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] \ \leq \ \|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\sigma^2 \left(\sum_{j=1}^t \eta_j^2\right) \ \leq \ \|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\sigma^2 c^2 \log(t+1).$$

Therefore, we conclude that

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{T} \epsilon(\mathbf{x}_t)\right] &\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\sigma^2 c^2 \log(T+1)}{2(\lambda - c)} \sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - c)\eta_1} + \frac{\sigma^2}{\lambda - c}\left(\sum_{t=1}^{T+1} \eta_t\right) \\
&\leq \frac{\sqrt{T+1}(\|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\sigma^2 c^2 \log(T+1))}{2c(\lambda - c)} + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2c(\lambda - c)} + \frac{\sigma^2 c\sqrt{T+1}}{\lambda - c} \\
&= O\left(\sqrt{T+1}\log(T+1)\right).
\end{aligned}$$

This completes the proof.

### D.4. Proof of Theorem A.4

Using Lemma A.1, we have

$$\mathbb{E}[\epsilon(\mathbf{x}_T)] \;\leq\; \mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{1}{\lambda}\left(\sum_{j=t}^{T-1} \frac{\sigma_j^2}{\eta_{j+1}}\right) \;\leq\; \mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{1}{\lambda\underline{\eta}}\left(\sum_{j=t}^{T-1} \sigma_j^2\right).$$

Summing up the above inequality over $t = 0, 1, \ldots, T$ yields that

$$(T+1)\mathbb{E}[\epsilon(\mathbf{x}_T)] \;\leq\; \sum_{t=0}^{T} \mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{1}{\lambda}\left(\sum_{t=0}^{T-1}\sum_{j=t}^{T-1} \frac{\sigma_j^2}{\eta_{j+1}}\right) \;\leq\; \sum_{t=0}^{T} \mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{1}{\lambda\underline{\eta}}\left(\sum_{t=0}^{T} (t+1)\sigma_t^2\right).$$

On the other hand, the derivation in Theorem A.3 implies that

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{T} \epsilon(\mathbf{x}_t)\right] &\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta})\eta_1} + \sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \frac{\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2]}{2(\lambda - \overline{\eta})} + \frac{1}{\lambda - \overline{\eta}}\left(\sum_{t=1}^{T+1} \eta_t \sigma_t^2\right) \\
&\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta})\eta_1} + \sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) \frac{\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2]}{2(\lambda - \overline{\eta})} + \frac{\overline{\eta}}{\lambda - \overline{\eta}}\left(\sum_{t=1}^{T+1} \sigma_t^2\right).
\end{aligned}$$

On the other hand, we have

$$\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}^*\|^2] \;\leq\; \mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] + 2\eta_{t+1}^2 \sigma_t^2.$$

This implies that the following inequality holds for all $t \geq 1$:

$$\mathbb{E}[\|\mathbf{x}_t - \mathbf{x}^*\|^2] \;\leq\; \|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\overline{\eta}^2\left(\sum_{j=1}^{t} \sigma_j^2\right).$$

Therefore, we conclude that

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=0}^{T} \epsilon(\mathbf{x}_t)\right] &\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\overline{\eta}^2(\sum_{t=1}^{T} \sigma_t^2)}{2(\lambda - \overline{\eta})} \sum_{t=1}^{T} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta})\eta_1} + \frac{\overline{\eta}}{\lambda - \overline{\eta}}\left(\sum_{t=1}^{T+1} \sigma_t^2\right) \\
&\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2 + 2\overline{\eta}^2(\sum_{t=1}^{T} \sigma_t^2)}{2(\lambda - \overline{\eta})\underline{\eta}} + \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{2(\lambda - \overline{\eta})\underline{\eta}} + \frac{\overline{\eta}}{\lambda - \overline{\eta}}\left(\sum_{t=1}^{T+1} \sigma_t^2\right) \\
&\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{(\lambda - \overline{\eta})\underline{\eta}} + \left(1 + \frac{\overline{\eta}}{\underline{\eta}}\right) \frac{\overline{\eta}}{\lambda - \overline{\eta}}\left(\sum_{t=1}^{T+1} \sigma_t^2\right) \\
&\leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{(\lambda - \overline{\eta})\underline{\eta}} + \left(1 + \frac{\overline{\eta}}{\underline{\eta}}\right) \frac{\overline{\eta}}{\lambda - \overline{\eta}}\left(\sum_{t=1}^{T+1} (t+1)\sigma_t^2\right).
\end{aligned}$$

Putting these pieces together yields that

$$\begin{aligned}
\mathbb{E}[\epsilon(\mathbf{x}_T)] &\leq \frac{1}{T+1}\left[\sum_{t=0}^{T}\mathbb{E}[\epsilon(\mathbf{x}_t)] + \frac{1}{\lambda\underline{\eta}}\left(\sum_{t=0}^{T-1}(t+1)\sigma_t^2\right)\right] \\
&\leq \frac{1}{T+1}\left(\frac{\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{(\lambda - \overline{\eta})\underline{\eta}} + \left(\frac{1}{\lambda\underline{\eta}} + \left(1 + \frac{\overline{\eta}}{\underline{\eta}}\right)\frac{\overline{\eta}}{\lambda - \overline{\eta}}\right)\left(\sum_{t=1}^{T+1}(t+1)\sigma_t^2\right)\right) \\
&\overset{\text{Eq. (15)}}{=} O(a(T)).
\end{aligned}$$

This completes the proof.