Supplementary Materials

A. Further Details on Policy Gradient

This paper considers the policy gradient algorithms that can adopt the following two types of trajectory gradients, namely REINFORCE (Williams, 1992) and G(PO)MDP (Baxter & Bartlett, 2001). We note that

$$\nabla J(\theta) = \nabla \mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta)}[\mathcal{R}(\tau)] = \mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta)}[\mathcal{R}(\tau)\nabla \log p(\tau|\theta)],$$

where $p(\tau|\theta) = \rho(s_0)\pi_\theta(a_0|s_0)\prod_{i=0}^{H-1}\mathcal{P}(s_{i+1}|s_i,a_i)\pi_\theta(a_{i+1}|s_{i+1})$. REINFORCE constructs the trajectory gradient as

$$g(\tau|\theta) = \underbrace{\Big(\sum_{t=0}^{H}\gamma^t\mathcal{R}(s_t,a_t) - b(s_t,a_t)\Big)\Big(\sum_{t=0}^{H}\nabla\log\pi_\theta(a_t|s_t)\Big)}_{\mathcal{R}(\tau)\nabla\log p(\tau|\theta)},$$

where $b : \mathcal{S}\times\mathcal{A}\to\mathcal{R}$ is a bias. G(PO)MDP enhances the trajectory gradient of REINFORCE by further utilizing the fact that the reward at time t does not depend on the action implemented after time t. Thus, G(PO)MDP constructs the trajectory gradient as

$$g(\tau|\theta) = \sum_{t=0}^{H}\big(\gamma^t\mathcal{R}(s_t,a_t) - b(s_t,a_t)\big)\sum_{i=0}^{t}\nabla\log\pi_\theta(a_i|s_i).$$

Note that REINFORCE and G(PO)MDP are both unbiased gradient estimators, i.e., $\mathbb{E}_{\tau\sim p(\cdot|\theta)}[g(\tau|\theta)] = \nabla J(\theta)$.

B. Further Specification of Experiments and Additional Results

B.1. Hyper-parameter Configuration of Algorithms for Nonconvex Optimization

To implement HSGD, we follow Zhou et al. 2018b and choose the linearly increasing mini-batch size at the t^{th} iteration to be $c_b(t+1)$, and tune c_b to the best. We set the epoch length $m = 10$ for all variance-reduced algorithms, because $m = 10$ works best for all variance-reduced algorithms for a fair comparison. ϵ is the target accuracy predetermined by users, typically dependent on specific applications. Specifically, we choose $\epsilon = 1e^{-3}$ for the logistic regression and $\epsilon = 1e^{-2}$ for the neural network training, respectively. We choose the batch size to be $\min\{n, c_1\epsilon^{-1}\}$ for SVRG+ and SpiderBoost, and $\min\{n, c_1\epsilon^{-1}, c_2\beta_s^{-1}\}$ for AbaSPIDER and AbaSVRG, where $\beta_s = \frac{1}{m}\sum_{t=1}^{m}\|v_{t-1}^{s-1}\|^2$ as given in Subsection 2.1.

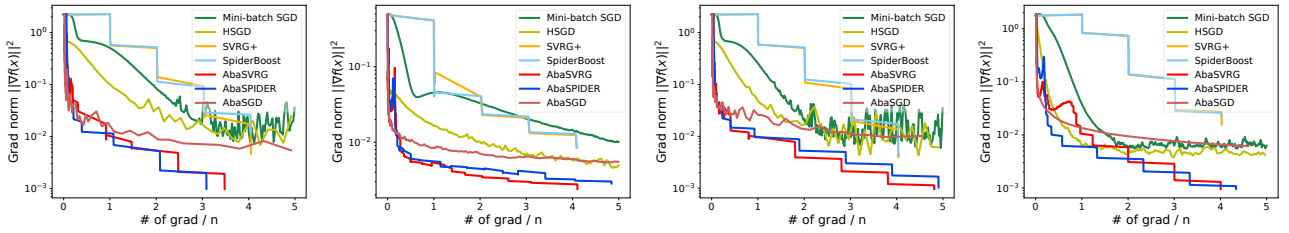
B.2. Additional Results for Nonconvex Logistic Regression

For logistic regression, we use four datasets: a8a ($n = 22696, d = 123$), a9a ($n = 32561, d = 123$), w8a ($n = 43793, d = 300$) and ijcnn1 ($n = 49990, d = 22$). We select the stepsize η from $\{0.1k, k = 1, 2, ..., 15\}$ and the mini-batch size B from $\{10, 28, 64, 128, 256, 512, 1024\}$ for all algorithms, and we present the best performance among these parameters. For all variance-reduced algorithms, we select constants c_1 and c_2 from $\{1, 2, 3, ..., 10\}$, and present the best performance among these parameters. For HSGD algorithm, we select c_b in its linearly increasing batch size $c_b(t+1)$ from $\{1, 5, 10, 40, 100, 400, 1000\}$, and present the best performance among these parameters. For AbaSGD, we set its batch size as $\min\Big\{\frac{c_\beta}{\sum_{i=1}^{5}\|v_{t-i}\|^2/5}, \frac{c_\epsilon}{\epsilon}, n\Big\}$, and select the best c_β and c_ϵ from $\{1, 2, 3, ..., 10\}$.

As shown in Fig. 4, AbaSVRG and AbaSPIDER converge much faster than all other algorithms in terms of the total number of gradient evaluations on all four datasets. It can be seen that both of them take the advantage of sample-efficient SGD-like updates (due to the small batch size) at the initial stage and attain high accuracy provided by variance-reduced methods at the final stage. This is consistent with the choice of our batch-size adaptation scheme.

B.3. Results for Training Multi-Layer Neural Networks

In this subsection, we compare our proposed algorithms with other competitive algorithms as specified in Section 4.1 for training a three-layer ReLU neural network with a cross entropy loss on the dataset of MNIST ($n = 60000, d = 780$). The neural network has a size of $(d_{\text{in}}, 100, 100, d_{\text{out}})$, where d_{in} and d_{out} are the input and output dimensions and 100 is the



(a) dataset: a8a (b) dataset: ijcnn1 (c) dataset: a9a (d) dataset: w8a

Figure 4. Comparison of different algorithms for logistic regression problem on four datasets. All figures plot gradient norm v.s. # of gradient evaluations.

number of neurons in the two hidden layers. We select the stepsize η from $\{10^{-4}k, k = 1, 2, ..., 15\}$ and the mini-batch size B from $\{64, 96, 128, 256, 512\}$ for all algorithms, and we present the best performance among these parameters. For all variance-reduced algorithms, we set $c_1 = 1$ and select the best c_2 from $\{10^3, 5 \times 10^3, 10^4\}$. For HSGD algorithm, we select c_b from $\{1, 10, 50, 100, 500, 1000\}$, and present the best performance among these parameters.

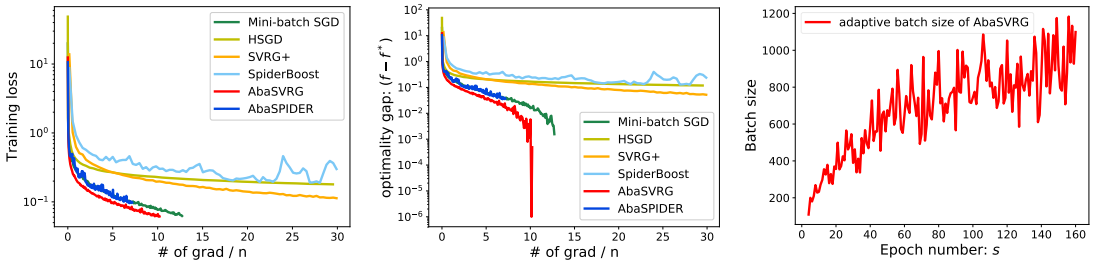


Figure 5. Comparison of various algorithms for training a three-layer neural network on MNIST.

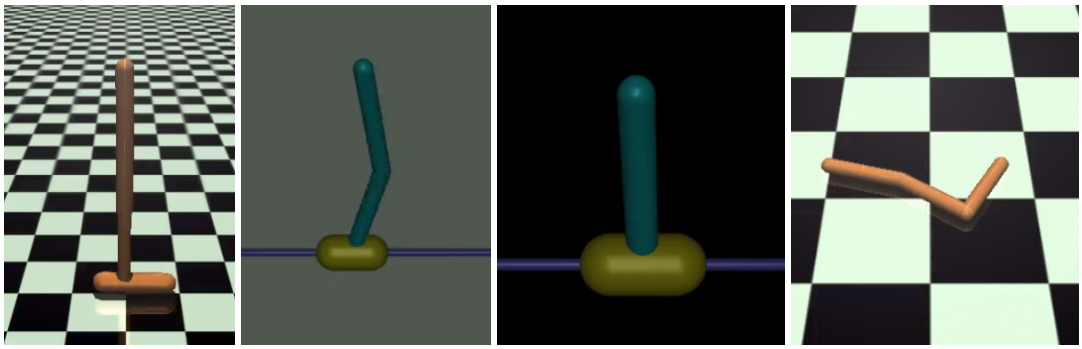
As shown in Fig. 5, our AbaSVRG achieves the best performance among all competing algorithms, and AbaSPIDER performs similarly to mini-batch SGD for decreasing training loss, but converges faster in terms of gradient norm. Interestingly, the batch-size adaptation used by AbaSVRG increases the batch size slower than both exponential and linear increase of the batch size, and its scaling is close to the *logarithmical* increase as shown in the right-most plot in Figure 5. Such an observation further demonstrates that our gradient-based batch-size adaptation scheme can also adapt to the neural network landscape with a differently (i.e., more slowly) increased batch size from that for nonconvex regression problem over a9a and w8a datasets.

B.4. Experimental Details for Reinforcement Learning

The hyper-parameters listed in Table 1 are the same among all methods on each task. For the proposed AbaSVRPG and AbaSPIDER-PG, we adopt the same hyper-parameter of $\alpha\sigma^2 = 1$ and $\beta = 1000$ in all experiments.

Table 1. Parameters used in the RL experiments

Task	InvertedPendulum	InvertedDoublePendulum	Swimmer	Hopper
Horizon	500	500	500	500
Discount Factor γ	0.99	0.99	0.99	0.99
q	10	10	10	10
N	100	100	50	50
B	20	20	20	20
ϵ	0.01	0.01	0.01	0.01
Step Size	0.001	0.001	0.0001	0.001
NN Hidden Weights	16×16	16×16	32×32	64×64
NN Activation	tanh	tanh	tanh	tanh
Baseline	No	No	Yes	Yes



(a) Hopper (b) Inverteddoublependulum (c) Invertedpendulum (d) Swimmer

Figure 6. Task Environments.

Figure 6 illustrates all task environments. The problem setup regarding each task is summarized as follows:

1. *InvertedPendulum*: A cart is moving along a track with zero friction and a pole is attached through an un-actuated joint. The pendulum is balanced by controlling the velocity of the cart. The action space is continuous with $a \in [-1, 1]$ (with -1 for pushing cart to the left and 1 for pushing cart to the right). For a single episode with time step h enumerated from 1 to 500, the episode is terminated when the pole angle $\theta_h > 0.2rad$, and otherwise a reward of value 1 is awarded.

2. *InvertedDoublePendulum*: The setup of this task is similar to that at the InvertedPendulum. The only difference is that another pendulum is added to the end of the previous pendulum through an unactuated rotational joint.

3. *Swimmer*: The agent is a 3-link robot defined in Mujoco with the state-space dimension of 13. It is actuated by two joints to swim in a viscous fluid. For a single episode with time step h enumerated from 1 to 500, the reward function encourages the agent to move forward as fast as possible while maintaining energy efficiency. That is, given forward velocity v_x and joint action a, $r(v_x, a) = v_x^2 - 10^{-4}\|a\|_2^2$.

4. *Hopper*: A two-dimensional single-legged robot is trained to hop forward. The system has the state-space dimension of 11 and action space dimension of 3. For a single episode with time step h enumerated from 1 to 600, we have forward velocity v_x and commanded action a. The episode terminates early (before h reaches 500) when the tilting angle of upper body or the height position for center of mass drops below a certain preset threshold. The reward function encourages the agent to move forward as fast as possible in an energy efficient manner. It also gets one alive bonus for every step it survives without triggering any of the termination threshold.

For the tasks of Swimmer and Hopper, we also include the linear baseline for value function approximation (Duan et al., 2016).

C. Convergence of AbaSVRG and AbaSPIDER under Local PL Geometry

Many nonconvex machine learning problems (e.g., phase retrieval (Zhou et al., 2016)) and deep learning (e.g., neural networks (Zhong et al., 2017; Zhou & Liang, 2017)) problems have been shown to satisfy the following Polyak-Łojasiewicz (PL) (i.e., gradient dominance) condition in local regions near local or global minimizers.

Definition 4 ((Polyak, 1963; Nesterov & Polyak, 2006)). *Let $x^* = \arg\min_{x\in\mathbb{R}^d} f(x)$. Then, the function f is said to be τ-gradient dominated if for any $x \in \mathbb{R}^d$, $f(x) - f(x^*) \leq \tau\|\nabla f(x)\|^2$.*

In this section, we explore whether our proposed AbaSVRG and AbaSPIDER with batch size adaptation can attain a faster linear convergence rate if the iterate enters the local PL regions. All the proofs are provided in Appendix H.

C.1. AbaSVRG: Convergence under PL Geometry without Restart

The following theorem provides the convergence and complexity for AbaSVRG under the PL condition.

Theorem 5. *Let* $\eta = \frac{1}{c_\eta L}$, $B = m^2$ *with* $\frac{8L\tau}{c_\eta - 2} \le m < 4L\tau$, $\beta_1 \le \epsilon\big(\frac{1}{\gamma}\big)^{m(S-1)}$, *and* $c_\beta = c_\epsilon = \Big(2\tau + \frac{2\tau}{1-\exp(\frac{-4}{c_\eta(c_\eta-2)})}\Big) \vee \frac{16 c_\eta L\tau}{m}$, *where constants* $c_\eta > 4$ *and* $\gamma = 1 - \frac{1}{8L\tau} < 1$. *Then under the PL condition, the final iterate* $\tilde{x}^S$ *of AbaSVRG satisfies*

$$\mathbb{E}(f(\tilde{x}^S) - f(x^*)) \le \gamma^K (f(x_0) - f(x^*)) + \frac{\epsilon}{2}.$$

To obtain an ϵ*-accurate solution* $\tilde{x}^S$, *the total number of SFO calls is given by*

$$\sum_{s=1}^{S} \min\left\{ \frac{c_\beta \sigma^2}{\sum_{t=1}^m \|v_{t-1}^{s-1}\|^2/m}, c_\epsilon \sigma^2 \epsilon^{-1}, n \right\} + KB \le \mathcal{O}\Big(\Big(\frac{\tau}{\epsilon} \wedge n\Big) \log\frac{1}{\epsilon} + \tau^3 \log\frac{1}{\epsilon} \Big). \tag{3}$$

Our proof of Theorem 5 is different from and more challenging than the previous techniques developed in Reddi et al. 2016a;b; Li & Li 2018 for SVRG-type algorithms, because we need to handle the adaptive batch size N_s with the dependencies on the iterations at the previous epoch. In addition, we do not need *extra* assumptions for proving the convergence under PL condition, whereas Reddi et al. 2016b and Li & Li 2018 require $\tau \ge n^{1/3}$ and $\tau \ge n^{1/2}$, respectively. As a result, Theorem 5 can be applied to any condition number regime. For the small condition number regime where $1 \le \tau \le \Theta(n^{1/3})$, the worst-case complexity of AbaSVRG outperforms the result achieved by SVRG (Reddi et al., 2016b). Furthermore, the actual complexity of our AbaSVRG can be much lower than the worst-case complexity due to the adaptive batch size.

C.2. AbaSPIDER: Convergence under PL Geometry without Restart

The following theorem shows that AbaSPIDER achieves a linear convergence rate under the PL condition without restart. Our analysis can be of independent interest for other SPIDER-type methods.

Theorem 6. *Let* $\eta = \frac{1}{c_\eta L}$, $B = m$ *with* $\frac{8L\tau}{c_\eta - 2} \le m < 4L\tau$, $\beta_1 \le \epsilon\big(\frac{1}{\gamma}\big)^{m(S-1)}$, *and* $c_\beta = c_\epsilon = \Big(2\tau + \frac{2\tau}{1-\exp(\frac{-4}{c_\eta(c_\eta-2)})}\Big) \vee \frac{16 c_\eta L\tau}{m}$, *where constants* $c_\eta > 4$ *and* $\gamma = 1 - \frac{1}{8L\tau}$. *Then under the PL condition, the final iterate* $\tilde{x}^S$ *of AbaSPIDER satisfies*

$$\mathbb{E}(f(\tilde{x}^S) - f(x^*)) \le \gamma^K \mathbb{E}(f(x_0) - f(x^*)) + \frac{\epsilon}{2}.$$

To obtain an ϵ*-accurate solution* $\tilde{x}^S$, *the total number of SFO calls is given by*

$$\sum_{s=1}^{S} \min\left\{ \frac{c_\beta \sigma^2}{\sum_{t=1}^m \|v_{t-1}^{s-1}\|^2/m}, c_\epsilon \sigma^2 \epsilon^{-1}, n \right\} + KB \le \mathcal{O}\left(\Big(\frac{\tau}{\epsilon} \wedge n\Big) \log\frac{1}{\epsilon} + \tau^2 \log\frac{1}{\epsilon} \right).$$

As shown in Theorem 6, AbaSPIDER achieves a lower worst-case SFO complexity than AbaSVRG by a factor of $\Theta(\tau)$, and matches the best result provided by Prox-SpiderBoost-gd (Wang et al., 2019). However, Prox-SpiderBoost-gd is a variant of Prox-SpiderBoost with algorithmic modification, and has not been shown to achieve the near-optimal complexity for general nonconvex optimization. In addition, AbaSPIDER has a much lower complexity in practice due to the adaptive batch size.

D. An analysis for SGD with Adaptive Mini-Batch Size

Recently, Sievert & Charles 2019 proposed an improved SGD algorithm by adapting the batch size to the gradient norms in preceding steps. However, they do not show performance guarantee for their proposed algorithm. In this section, we aim to fill this gap by providing an analysis for adaptive batch size SGD (AbaSGD) with mini-batch size depending on the stochastic gradients in the preceding m steps. as shown in Algorithm 5. To simplify notations, we set norms of the stochastic gradients before the algorithm starts to be $\|\mathbf{v}_{-1}\| = \|\mathbf{v}_{-2}\| = \cdots = \|\mathbf{v}_{-m}\| = \alpha_0$ and let $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \,|\, \mathbf{x}_0, ..., \mathbf{x}_t)$.

Algorithm 5 AbaSGD

Input: $\mathbf{x}_0$, stepsize η, $m > 0$, $\alpha_0 > 0$.
for $t = 0, 1, ..., T$ **do**
  Set $|B_t| = \min\left\{\frac{2\sigma^2}{\sum_{i=1}^m \|\mathbf{v}_{t-i}\|^2/m}, \frac{24\sigma^2}{\epsilon}, n\right\}$.
  if $|B_t| = n$ **then**
    Compute $\mathbf{v}_t = \nabla f(\mathbf{x}_t)$
  else
    Sample B_t from $[n]$ with replacement, and compute $\mathbf{v}_t = \nabla f_{B_t}(\mathbf{x}_t)$
  end if
  $\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \mathbf{v}_t$
end for
Output: choose $\mathbf{x}_\zeta$ from $\{\mathbf{x}_i\}_{i=0,...,T}$ uniformly at random

Theorem 7. *Let Assumption 1 hold, $\epsilon > 0$ and choose a stepsize η such that*

$$\phi = \eta - \frac{L\eta^2}{2} > 0.$$

Then, the output $\mathbf{x}_\zeta$ returned by AbaSGD satisfies

$$\mathbb{E}\|\nabla f(\mathbf{x}_\zeta)\|^2 \leq \frac{2\left(f(\mathbf{x}_0) - f^*\right) + \eta m \alpha_0^2}{2T\phi} + \frac{\eta}{12\phi}\epsilon,$$

where $f^* = \inf_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x})$, and T is the total number of iterations.

Theorem 7 shows that AbaSGD achieves a $\mathcal{O}\left(\frac{1}{T}\right)$ convergence rate for nonconvex optimization by using the adaptive mini-batch size. In the following corollary, we derive the SFO complexity of AbaSGD.

Corollary 5. *Under the setting of Theorem 7, we choose the constant stepsize $\eta = \frac{1}{2L}$. Then, to obtain an ϵ-accurate solution $\mathbf{x}_\zeta$, the total number of iterations required by AbaSGD*

$$T = \frac{16L\left(f(\mathbf{x}_0) - f^*\right) + 4m\alpha_0^2}{\epsilon},$$

and the total number of SFO calls required by AbaSGD is given by

$$\sum_{t=0}^{T} |B_t| = \underbrace{\sum_{t=0}^{T} \min\left\{\frac{2\sigma^2}{\sum_{i=1}^m \|\mathbf{v}_{t-i}\|^2/m}, \frac{24\sigma^2}{\epsilon}, n\right\}}_{\text{complexity of AbaSGD}} \leq \underbrace{T\min\left\{\frac{24\sigma^2}{\epsilon}, n\right\}}_{\text{complexity of vanilla SGD}} = \mathcal{O}\left(\frac{1}{\epsilon^2} \wedge \frac{n}{\epsilon}\right).$$

Corollary 5 shows that the worst-case complexity of AbaSGD is $\mathcal{O}\left(\frac{1}{\epsilon^2} \wedge \frac{n}{\epsilon}\right)$, which is at least as good as those of SGD and GD. More importantly, the actual complexity of AbaSGD can be much lower than those of GD and SGD due to the adaptive batch size.

Technical Proofs

E. Proofs for Results in Section 2

E.1. Proof of Theorem 1

To prove Theorem 1, we first establish the following lemma to upper-bound the estimation variance $\mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s)-v_{t-1}^s\|^2$ for $1\le t\le m$, where $\mathbb{E}_{t,s}(\cdot)$ denotes $\mathbb{E}(\cdot|x_0^1,x_0^2,...,x_2^1,...,x_t^s)$.

Lemma 1. *Let Assumption 1 hold. Then, for $1\le t\le m$, we have*

$$\mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s)-v_{t-1}^s\|^2\le\frac{\eta^2L^2(t-1)}{B}\mathbb{E}_{0,s}\sum_{i=0}^{t-2}\|v_i^s\|^2+\frac{I_{(N_s<n)}}{N_s}\sigma^2 \tag{4}$$

where $I_{(A)}=1$ if the event A occurs and 0 otherwise, and $\sum_{i=0}^{-1}\|v_i^s\|^2=0$.

Proof of Lemma 1. Based on line 10 in Algorithm 1, we have, for $1\le t\le m$,

$$\|v_{t-1}^s-\nabla f(x_{t-1}^s)\|^2=\|\nabla f_{\mathcal{B}}(x_{t-1}^s)-\nabla f_{\mathcal{B}}(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1})+g^s-\nabla f(\tilde{x}^{s-1})\|^2.$$

Taking the expectation $\mathbb{E}_{0,s}(\cdot)$ over the above equality yields

$$\begin{aligned}\mathbb{E}_{0,s}\|v_{t-1}^s-\nabla f(x_{t-1}^s)\|^2=&\,\mathbb{E}_{0,s}\|\nabla f_{\mathcal{B}}(x_{t-1}^s)-\nabla f_{\mathcal{B}}(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1})\|^2\\&+2\underbrace{\mathbb{E}_{0,s}\langle\nabla f_{\mathcal{B}}(x_{t-1}^s)-\nabla f_{\mathcal{B}}(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1}),g^s-\nabla f(\tilde{x}^{s-1})\rangle}_{(*)}\\&+\mathbb{E}_{0,s}\|g^s-\nabla f(\tilde{x}^{s-1})\|^2,\end{aligned} \tag{5}$$

which, in conjunction with the fact that

$$(*)=\mathbb{E}_{x_1^s,...,x_{t-1}^s}\mathbb{E}_{t-1,s}\langle\nabla f_{\mathcal{B}}(x_{t-1}^s)-\nabla f_{\mathcal{B}}(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1}),g^s-\nabla f(\tilde{x}^{s-1})\rangle=0$$

and letting $F_i:=\nabla f_i(x_{t-1}^s)-\nabla f_i(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1})$, implies that

$$\begin{aligned}&\mathbb{E}_{0,s}\|v_{t-1}^s-\nabla f(x_{t-1}^s)\|^2\\&=\mathbb{E}_{0,s}\|\nabla f_{\mathcal{B}}(x_{t-1}^s)-\nabla f_{\mathcal{B}}(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1})\|^2+\mathbb{E}_{0,s}\|g^s-\nabla f(\tilde{x}^{s-1})\|^2\\&=\frac{1}{B^2}\mathbb{E}_{0,s}\sum_{i\in\mathcal{B}}\|\nabla f_i(x_{t-1}^s)-\nabla f_i(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1})\|^2+\mathbb{E}_{0,s}\|g^s-\nabla f(\tilde{x}^{s-1})\|^2\\&\quad+\frac{2}{B^2}\sum_{i<j,\,i,j\in\mathcal{B}}\mathbb{E}_{0,s}\langle F_i,F_j\rangle\\&\overset{(i)}{=}\frac{1}{B}\mathbb{E}_{0,s}\|\nabla f_i(x_{t-1}^s)-\nabla f_i(\tilde{x}^{s-1})-\nabla f(x_{t-1}^s)+\nabla f(\tilde{x}^{s-1})\|^2+\mathbb{E}_{0,s}\|g^s-\nabla f(\tilde{x}^{s-1})\|^2\\&\overset{(ii)}{\le}\frac{1}{B}\mathbb{E}_{0,s}\|\nabla f_i(x_{t-1}^s)-\nabla f_i(\tilde{x}^{s-1})\|^2+\mathbb{E}_{0,s}\|g^s-\nabla f(\tilde{x}^{s-1})\|^2\\&\overset{(iii)}{\le}\frac{1}{B}\mathbb{E}_{0,s}\|\nabla f_i(x_{t-1}^s)-\nabla f_i(\tilde{x}^{s-1})\|^2+\frac{I_{(N_s<n)}}{N_s}\sigma^2\end{aligned} \tag{6}$$

where (i) follows from the fact that

$$\mathbb{E}_{0,s}\langle F_i,F_j\rangle=\mathbb{E}_{x_1^s,...,x_{t-1}^s}\left(\mathbb{E}_{t-1,s}\langle F_i,F_j\rangle\right)=\mathbb{E}_{x_1^s,...,x_{t-1}^s}\left(\langle\mathbb{E}_{t-1,s}(F_i),\mathbb{E}_{t-1,s}(F_j)\rangle\right)=0,$$

(ii) follows from the fact that $\mathbb{E}\|y-\mathbb{E}(y)\|^2\le\mathbb{E}\|y\|^2$ for any $y\in\mathbb{R}^d$, and (iii) follows by combining Lemma B.2 in Lei et al. 2017 and the fact that N_s is fixed given $x_0^1,...,x_0^s$. Then, we obtain from (6) that

$$\mathbb{E}_{0,s}\|v_{t-1}^s-\nabla f(x_{t-1}^s)\|^2\le\frac{L^2}{B}\mathbb{E}_{0,s}\|x_{t-1}^s-\tilde{x}^{s-1}\|+\frac{I_{(N_s<n)}}{N_s}\sigma^2$$

$$
\begin{aligned}
=&\frac{L^2}{B}\mathbb{E}_{0,s}\Big\|\sum_{i=0}^{t-2}(x_{i+1}^s-x_i^s)\Big\|^2+\frac{I_{(N_s<n)}}{N_s}\sigma^2\\
=&\frac{\eta^2L^2}{B}\mathbb{E}_{0,s}\Big\|\sum_{i=0}^{t-2}v_i^s\Big\|^2+\frac{I_{(N_s<n)}}{N_s}\sigma^2\\
\overset{(i)}{\leq}&\frac{\eta^2L^2(t-1)}{B}\mathbb{E}_{0,s}\sum_{i=0}^{t-2}\Big\|v_i^s\Big\|^2+\frac{I_{(N_s<n)}}{N_s}\sigma^2,
\end{aligned}\tag{7}
$$

where (i) follows from the Cauchy–Schwartz inequality that $\|\sum_{i=1}^k a_i\|^2\leq k\sum_{i=1}^k\|a_i\|^2$. □

Proof of Theorem 1. Based on Lemma 1, we next prove Theorem 1.

Since the objective function $f(\cdot)$ has a L-Lipschitz continuous gradient, we obtain that for $1\leq t\leq m$,

$$
\begin{aligned}
f(x_t^s)\leq& f(x_{t-1}^s)+\langle\nabla f(x_{t-1}^s),x_t^s-x_{t-1}^s\rangle+\frac{L\eta^2}{2}\|v_{t-1}^s\|^2\\
=&f(x_{t-1}^s)+\langle\nabla f(x_{t-1}^s)-v_{t-1}^s,-\eta v_{t-1}^s\rangle-\eta\|v_{t-1}^s\|^2+\frac{L\eta^2}{2}\|v_{t-1}^s\|^2\\
\overset{(i)}{\leq}&f(x_{t-1}^s)+\frac{\eta}{2}\|\nabla f(x_{t-1}^s)-v_{t-1}^s\|^2+\frac{\eta}{2}\|v_{t-1}^s\|^2-\Big(\eta-\frac{L\eta^2}{2}\Big)\|v_{t-1}^s\|^2.
\end{aligned}
$$

where (i) follows from the inequality that $\langle a,b\rangle\leq\frac{1}{2}(\|a\|^2+\|b\|^2)$. Then, taking expectation $\mathbb{E}_{0,s}(\cdot)$ over the above inequality yields

$$
\mathbb{E}_{0,s}f(x_t^s)\leq\mathbb{E}_{0,s}f(x_{t-1}^s)+\frac{\eta}{2}\mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s)-v_{t-1}^s\|^2-\Big(\frac{\eta}{2}-\frac{L\eta^2}{2}\Big)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2.\tag{8}
$$

Combining (8) and Lemma 1 yields, for $1\leq t\leq m$

$$
\begin{aligned}
\mathbb{E}_{0,s}f(x_t^s)\leq&\mathbb{E}_{0,s}f(x_{t-1}^s)+\frac{\eta^3L^2(t-1)}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{t-2}\|v_i^s\|^2+\frac{\eta I_{(N_s<n)}}{2N_s}\sigma^2\\
&-\Big(\frac{\eta}{2}-\frac{L\eta^2}{2}\Big)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2.
\end{aligned}
$$

Telescoping the above inequality over t from 1 to m yields

$$
\begin{aligned}
\mathbb{E}_{0,s}f(x_m^s)\leq&\mathbb{E}_{0,s}f(x_0^s)+\sum_{t=1}^m\frac{\eta^3L^2(t-1)}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{t-2}\|v_i^s\|^2+\frac{\eta\sigma^2mI_{(N_s<n)}}{2N_s}\\
&-\Big(\frac{\eta}{2}-\frac{L\eta^2}{2}\Big)\sum_{t=0}^{m-1}\mathbb{E}_{0,s}\|v_t^s\|^2\\
\overset{(i)}{\leq}&\mathbb{E}_{0,s}f(x_0^s)+\frac{\eta^3L^2m^2}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2+\frac{\eta\sigma^2mI_{(N_s<n)}}{2N_s}\\
&-\Big(\frac{\eta}{2}-\frac{L\eta^2}{2}\Big)\sum_{t=0}^{m-1}\mathbb{E}_{0,s}\|v_t^s\|^2,
\end{aligned}\tag{9}
$$

where (i) follows from the fact that $\frac{\eta^3L^2(t-1)}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{t-2}\|v_i^s\|^2\leq\frac{\eta^3L^2m}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2$. Recall that $N_s=\min\{c_\beta\sigma^2\beta_s^{-1},c_\epsilon\sigma^2\epsilon^{-1},n\}$ and $c_\beta,c_\epsilon\geq\alpha$. Then, we have

$$
\frac{I_{(N_s<n)}}{N_s}\leq\frac{1}{\min\{c_\beta\sigma^2\beta_s^{-1},c_\epsilon\sigma^2\epsilon^{-1}\}}=\max\Big\{\frac{\beta_s}{c_\beta\sigma^2},\frac{\epsilon}{c_\epsilon\sigma^2}\Big\}\leq\max\Big\{\frac{\beta_s}{\alpha\sigma^2},\frac{\epsilon}{\alpha\sigma^2}\Big\},\tag{10}
$$

To explain the first inequality in (10), we denote $N_s = \min(n_1, n_2, n)$ for simplicty. If $N_s \geq n$, then the indicator function $I_{(\cdot)} = 0$ and hence $I_{(\cdot)}/N_s = 0 < 1/\min(n_1, n_2)$. If $N_s < n$, then $I_{(\cdot)} = 1$ and $N_s = \min(n_1, n_2)$ and hence $I_{(\cdot)}/N_s = 1/\min(n_1, n_2)$. Combining the above two cases yields the first inequality in (10). Combining (10) and (9) yields

$$\begin{aligned}\mathbb{E}_{0,s} f(x_m^s) \overset{(i)}{\leq} & \mathbb{E}_{0,s} f(x_0^s) + \frac{\eta^3 L^2 m^2}{2B} \mathbb{E}_{0,s} \sum_{i=0}^{m-1} \|v_i^s\|^2 + \eta m \left(\frac{\beta_s}{2\alpha} + \frac{\epsilon}{2\alpha} \right) \\ & - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} \right) \sum_{t=0}^{m-1} \mathbb{E}_{0,s} \|v_t^s\|^2. \end{aligned} \tag{11}$$

where (i) follows from the fact that $\max(a, b) \leq a + b$. Taking the expectation of (11) over $x_0^1, ..., x_0^s$, we obtain

$$\mathbb{E} f(x_m^s) \leq \mathbb{E} f(x_0^s) + \frac{\eta m}{2\alpha} \mathbb{E}\beta_s + \frac{\eta m \epsilon}{2\alpha} - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m^2}{2B} \right) \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2.$$

Recall that $\beta_1 \leq \epsilon S$ and $\beta_s = \frac{1}{m} \sum_{t=1}^{m} \|v_{t-1}^{s-1}\|^2$ for $s = 2, ..., S$. Then, telescoping the above inequality over s from 1 to S and noting that $x_m^s = x_0^{s+1}$, we obtain

$$\begin{aligned}\mathbb{E} f(x_m^S) \leq & \mathbb{E} f(x_0) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m^2}{2B} \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 + \frac{\eta m S \epsilon}{2\alpha} \\ & + \frac{\eta m S \epsilon}{2\alpha} + \sum_{s=2}^{S} \frac{\eta m}{2\alpha} \mathbb{E} \left(\frac{1}{m} \sum_{t=1}^{m} \|v_{t-1}^{s-1}\|^2 \right) \\ \leq & \mathbb{E} f(x_0) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m^2}{2B} - \frac{\eta}{2\alpha} \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 + \frac{\eta m S \epsilon}{\alpha}. \end{aligned} \tag{12}$$

Dividing the both sides of (12) by $\eta S m$ and rearranging the terms, we obtain

$$\left(\frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta}{2} - \frac{\eta^2 L^2 m^2}{2B} \right) \frac{1}{Sm} \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 \leq \frac{f(x_0) - f^*}{\eta S m} + \frac{\epsilon}{\alpha}, \tag{13}$$

where $f^* = \inf_{x \in \mathbb{R}^d} f(x) > -\infty$. Since the output x_ζ is chosen from $\{x_t^s\}_{t=0,...,m-1, s=1,...,S}$ uniformly at random, we have

$$\begin{aligned} Sm\, \mathbb{E}\|\nabla f(x_\zeta)\|^2 = & \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|\nabla f(x_t^s)\|^2 \\ \leq & 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|\nabla f(x_t^s) - v_t^s\|^2 + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 \\ = & 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}_{x_0^1, ..., x_0^s} \left(\mathbb{E}_{0,s} \|\nabla f(x_t^s) - v_t^s\|^2 \right) + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 \\ \overset{(i)}{\leq} & 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}_{x_0^1, ..., x_0^s} \left(\frac{\eta^2 L^2 m}{B} \mathbb{E}_{0,s} \sum_{i=0}^{m-1} \|v_i^s\|^2 + \frac{\beta_s}{\alpha} + \frac{\epsilon}{\alpha} \right) + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 \\ \leq & 2 \sum_{s=1}^{S} \left(\frac{\eta^2 L^2 m^2}{B} \mathbb{E} \sum_{i=0}^{m-1} \|v_i^s\|^2 + \frac{m\beta_s}{\alpha} + \frac{m\epsilon}{\alpha} \right) + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 \\ \overset{(ii)}{\leq} & \left(\frac{2\eta^2 L^2 m^2}{B} + \frac{2}{\alpha} + 2 \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E}\|v_t^s\|^2 + \frac{4Sm\epsilon}{\alpha} \end{aligned} \tag{14}$$

where (i) follows from Lemma 1 and (10), and (ii) follows from the definition of β_s for $s = 1, ..., S$. Combining (13) and (14) and letting $\phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta}{2} - \frac{\eta^2 L^2 m^2}{2B}$ and $\psi = \frac{2\eta^2 L^2 m^2}{B} + \frac{2}{\alpha} + 2$, we have

$$\mathbb{E}\|\nabla f(x_\zeta)\|^2 \leq \frac{\psi(f(x_0) - f^*)}{\phi \eta S m} + \frac{\psi \epsilon}{\phi \alpha} + \frac{4\epsilon}{\alpha}, \tag{15}$$

which finishes the proof. □

E.2. Proof of Corollary 1

Recall that $\eta = \frac{1}{4L}$, $B = m^2$ and $c_\beta, c_\epsilon \geq 16$. Then, we have $\alpha = 16$, $\phi \geq \frac{5}{16} > \frac{1}{4}$ and $\psi \leq \frac{9}{4}$ in Theorem 1, and thus

$$\mathbb{E}\|\nabla f(x_\zeta)\|^2 \leq \frac{36L(f(x_0) - f^*)}{K} + \frac{13}{16}\epsilon.$$

Thus, to achieve $\mathbb{E}\|\nabla f(x_\zeta)\|^2 < \epsilon$, AbaSVRG requires at most $192L(f(x_0) - f^*)\epsilon^{-1} = \Theta(\epsilon^{-1})$ iterations. Then, the total number of SFO calls is given by

$$\sum_{s=1}^{S} \min\{c_\beta \sigma^2 \beta_s^{-1}, c_\epsilon \sigma^2 \epsilon^{-1}, n\} + KB \leq S(c_\epsilon \sigma^2 \epsilon^{-1} \wedge n) + KB \leq \mathcal{O}\left(\frac{\epsilon^{-1} \wedge n}{\epsilon\sqrt{B}} + \frac{B}{\epsilon}\right).$$

Furthermore, if we choose $B = n^{2/3} \wedge \epsilon^{-2/3}$, then SFO complexity of AbaSVRG becomes

$$\mathcal{O}\left(\frac{\epsilon^{-2/3} \wedge n^{2/3}}{\epsilon}\right) \leq \mathcal{O}\left(\frac{1}{\epsilon}\left(n \wedge \frac{1}{\epsilon}\right)^{2/3}\right). \tag{16}$$

E.3. Complexity under $B = m$

Corollary 6. *Let stepsize $\eta = \frac{1}{4L\sqrt{m}}$, mini-batch size $B = m$ and $c_\beta, c_\epsilon \geq 16$. Then, to obtain an ϵ-accurate solution x_ζ, the total number of SFO calls required by AbaSVRG is given by*

$$\sum_{s=1}^{S} \min\left\{\frac{c_\beta \sigma^2}{\beta_s}, \frac{c_\epsilon \sigma^2}{\epsilon}, n\right\} + KB \leq \mathcal{O}\left(\frac{n \wedge \epsilon^{-1}}{\sqrt{B}\epsilon} + \frac{B^{3/2}}{\epsilon}\right).$$

If we specially choose $B = n^{1/2} \wedge \epsilon^{-1/2}$, then the worst-case complexity is $\mathcal{O}\left(\frac{1}{\epsilon}(n \wedge \frac{1}{\epsilon})^{3/4}\right)$.

Proof. Since $\eta = \frac{1}{4L\sqrt{m}}$, $B = m$ and $c_\beta, c_\epsilon \geq 16$, we obtain $\alpha = 16$, $\phi = \frac{7}{16} - \frac{1}{8\sqrt{m}} \geq \frac{5}{16} > \frac{1}{4}$ and $\psi \leq \frac{9}{4}$ in Theorem 1, and thus

$$\mathbb{E}\|\nabla f(x_\zeta)\|^2 \leq \frac{36L\sqrt{m}(f(x_0) - f^*)}{K} + \frac{13}{16}\epsilon.$$

To achieve $\mathbb{E}\|\nabla f(x_\zeta)\|^2 < \epsilon$, AbaSVRG requires at most $192L\sqrt{m}(f(x_0) - f^*)\epsilon^{-1} = \Theta(\sqrt{m}\epsilon^{-1})$ iterations. Then, the total number of SFO calls is given by

$$\sum_{s=1}^{S} \min\{c_\beta \sigma^2 \beta_s^{-1}, c_\epsilon \sigma^2 \epsilon^{-1}, n\} + KB \leq S(c_\epsilon \sigma^2 \epsilon^{-1} \wedge n) + KB \leq \mathcal{O}\left(\frac{\epsilon^{-1} \wedge n}{\epsilon\sqrt{B}} + \frac{B^{3/2}}{\epsilon}\right).$$

Furthermore, if we choose $B = n^{1/2} \wedge \epsilon^{-1/2}$, then the SFO complexity is $\mathcal{O}\left(\frac{1}{\epsilon}(n \wedge \frac{1}{\epsilon})^{3/4}\right)$. □

E.4. Proof of Theorem 2

In order to prove Theorem 2, we first use the following lemma to provide an upper bound on the estimation variance $\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2$ for $0 \leq t \leq m-1$, where $\mathbb{E}_{t,s}(\cdot)$ denotes $\mathbb{E}(\cdot|x_0^1, x_0^2, ..., x_2^1, ..., x_t^s)$.

Lemma 2 (Adapted from Fang et al. 2018)**.** *Let Assumption 1 hold. Then, for $0 \leq t \leq m-1$,*

$$\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2 \leq \frac{\eta^2 L^2}{B} \sum_{i=0}^{t-1} \mathbb{E}_{0,s}\|v_i^s\|^2 + \frac{I_{(N_s<n)}}{N_s}\sigma^2. \tag{17}$$

where we define the stochastic gradients before the algorithm starts to satisfy $\sum_{i=0}^{-1} \mathbb{E}_{0,s}\|v_i^s\|^2 = 0$ for easy presentation.

Proof of Lemma 2. Combining A.3 and A.4 in Fang et al. 2018 yields, for $1 \leq i \leq m-1$,

$$\begin{aligned}\mathbb{E}_{i,s}\|\nabla f(x_i^s) - v_i^s\|^2 \leq & \frac{L^2}{B}\|x_i^s - x_{i-1}^s\|^2 + \|\nabla f(x_{i-1}^s) - v_{i-1}^s\|^2 \\ = & \frac{\eta^2 L^2}{B}\|v_{i-1}^s\|^2 + \|\nabla f(x_{i-1}^s) - v_{i-1}^s\|^2.\end{aligned}$$

Taking the expectation of the above inequality over $x_1^s, ..., x_i^s$, we have

$$\mathbb{E}_{0,s}\|\nabla f(x_i^s) - v_i^s\|^2 \leq \frac{\eta^2 L^2}{B}\mathbb{E}_{0,s}\|v_{i-1}^s\|^2 + \mathbb{E}_{0,s}\|\nabla f(x_{i-1}^s) - v_{i-1}^s\|^2.$$

Then, telescoping the above inequality over i from 1 to t yields

$$\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2 \leq \frac{\eta^2 L^2}{B}\sum_{i=0}^{t-1}\mathbb{E}_{0,s}\|v_i^s\|^2 + \mathbb{E}_{0,s}\|\nabla f(x_0^s) - v_0^s\|^2. \tag{18}$$

Based on Lemma B.2 in Lei et al. 2017, we have

$$\mathbb{E}_{0,s}\|\nabla f(x_0^s) - v_0^s\|^2 \leq \frac{I_{(N_s<n)}}{N_s}\sigma^2,$$

which, combined with (18), finishes the proof. □

Proof of Theorem 2. Based on Lemma 2, we now prove Theorem 2.

Since the objective function $f(\cdot)$ has a L-Lipschitz continuous gradient, we obtain that for $1 \leq t \leq m$,

$$\begin{aligned}f(x_t^s) \leq & f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s), x_t^s - x_{t-1}^s\rangle + \frac{L\eta^2}{2}\|v_{t-1}^s\|^2 \\ = & f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s) - v_{t-1}^s, -\eta v_{t-1}^s\rangle - \eta\|v_{t-1}^s\|^2 + \frac{L\eta^2}{2}\|v_{t-1}^s\|^2 \\ \overset{(i)}{\leq} & f(x_{t-1}^s) + \frac{\eta}{2}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 + \frac{\eta}{2}\|v_{t-1}^s\|^2 - \eta\|v_{t-1}^s\|^2 + \frac{L\eta^2}{2}\|v_{t-1}^s\|^2 \\ \leq & f(x_{t-1}^s) + \frac{\eta}{2}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\|v_{t-1}^s\|^2,\end{aligned}$$

where (i) follows from the inequality that $\langle a, b\rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$. Then, taking expectation $\mathbb{E}_{0,s}$ over the above inequality and applying Lemma 2, we have, for $1 \leq t \leq m$,

$$\begin{aligned}\mathbb{E}_{0,s}f(x_t^s) \leq & \mathbb{E}_{0,s}f(x_{t-1}^s) + \frac{\eta}{2}\mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2 \\ \leq & \mathbb{E}_{0,s}f(x_{t-1}^s) + \frac{\eta^3 L^2}{2B}\sum_{i=0}^{t-2}\mathbb{E}_{0,s}\|v_i^s\|^2 + \frac{I_{(N_s<n)}}{N_s}\frac{\eta\sigma^2}{2} - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2 \\ \overset{(i)}{\leq} & \mathbb{E}_{0,s}f(x_{t-1}^s) + \frac{\eta^3 L^2}{2B}\sum_{i=0}^{m-1}\mathbb{E}_{0,s}\|v_i^s\|^2 + \max\left\{\frac{\eta\beta_s}{2\alpha}, \frac{\eta\epsilon}{2\alpha}\right\} - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2\end{aligned}$$

where (i) follows from $t - 2 < m - 1$ and (10). Telescoping the above inequality over t from 1 to m and using $\max(a, b) \leq a + b$ yield

$$\mathbb{E}_{0,s}f(x_m^s) \leq \mathbb{E}_{0,s}f(x_0^s) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m}{2B}\right)\sum_{t=0}^{m-1}\mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta m\beta_s}{2\alpha} + \frac{\eta m\epsilon}{2\alpha}.$$

Taking the expectation of the above inequality over $x_0^1, ..., x_0^s$, we obtain

$$\mathbb{E}f(x_m^s) \leq \mathbb{E}f(x_0^s) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m}{2B}\right)\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 + \frac{\eta m}{2\alpha}\mathbb{E}(\beta_s) + \frac{\eta m\epsilon}{2\alpha}.$$

Recall that $\beta_1 \leq \epsilon S$ and $\beta_s = \frac{1}{m}\sum_{t=0}^{m-1}\|v_t^{s-1}\|^2$ for $s = 2, ..., S$. Then, telescoping the above inequality over s from 1 to S and noting that $x_m^s = x_0^{s+1} = \tilde{x}^s$, we have

$$\begin{aligned}\mathbb{E}f(\tilde{x}^S) \leq& \mathbb{E}f(x_0) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m}{2B}\right)\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 + \frac{\eta m S\epsilon}{2\alpha} \\ &+ \frac{\eta}{2\alpha}\sum_{s=1}^{S-1}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \\ \leq& \mathbb{E}f(x_0) - \left(\frac{\eta}{2} - \frac{\eta}{2\alpha} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m}{2B}\right)\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 + \frac{\eta m S\epsilon}{2\alpha}.\end{aligned}$$

Dividing the both sides of the above inequality by $\eta S m$ and rearranging the terms, we obtain

$$\left(\frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta}{2} - \frac{\eta^2 L^2 m}{2B}\right)\frac{1}{Sm}\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \leq \frac{f(x_0) - f^*}{\eta S m} + \frac{\epsilon}{2\alpha}. \tag{19}$$

Since the output x_ζ is chosen from $\{x_t^s\}_{t=0,...,m-1,s=1,...,S}$ uniformly at random, we have

$$\begin{aligned}Sm\,\mathbb{E}\|\nabla f(x_\zeta)\|^2 =& \sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|\nabla f(x_t^s)\|^2 \\ \leq& 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|\nabla f(x_t^s) - v_t^s\|^2 + 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \\ =& 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}_{x_0^1,...,x_0^s}\left(\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2\right) + 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \\ \overset{(i)}{\leq}& 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}_{x_0^1,...,x_0^s}\left(\frac{\eta^2 L^2}{B}\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2 + \frac{\beta_s}{\alpha} + \frac{\epsilon}{\alpha}\right) + 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \\ \leq& 2\sum_{s=1}^{S}\left(\frac{\eta^2 L^2 m}{B}\mathbb{E}\sum_{i=0}^{m-1}\|v_i^s\|^2 + \frac{m\beta_s}{\alpha} + \frac{m\epsilon}{\alpha}\right) + 2\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \\ \overset{(ii)}{\leq}& \left(\frac{2\eta^2 L^2 m}{B} + \frac{2}{\alpha} + 2\right)\sum_{s=1}^{S}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 + \frac{4Sm\epsilon}{\alpha}\end{aligned} \tag{20}$$

where (i) follows from Lemma 2 and (10) and (ii) follows from the definition of β_s for $s = 1, ..., S$. Let $\phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta}{2} - \frac{\eta^2 L^2 m}{2B}$, $\psi = \frac{2\eta^2 L^2 m}{B} + \frac{2}{\alpha} + 2$ and $K = Sm$. Then, combining (20) and (19), we finish the proof.

E.5. Proof of Corollary 2

Recall that $1 \leq B \leq n^{1/2} \wedge \epsilon^{-1/2}$, $m = (n \wedge \frac{1}{\epsilon})B^{-1}$, $\eta = \frac{1}{4L}\sqrt{\frac{B}{m}}$ and $c_\beta, c_\epsilon \geq 16$. Then, we have $\alpha = 16$, $m \geq n^{1/2} \wedge \epsilon^{-1/2} \geq B$ and $\eta \leq \frac{1}{4L}$. Thus, we obtain

$$\phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta}{2} - \frac{\eta^2 L^2 m}{2B} \geq \frac{5}{16} > \frac{1}{4} \text{ and } \psi \leq \frac{9}{4},$$

which, in conjunction with Theorem 2, implies that

$$\mathbb{E}\|\nabla f(x_\zeta)\|^2 \leq \frac{36L\sqrt{m}(f(x_0) - f^*)}{\sqrt{B}K} + \frac{17}{32}\epsilon.$$

Thus, to achieve $\mathbb{E}\|\nabla f(x_\zeta)\|^2 < \epsilon$, AbaSPIDER requires at most $\frac{384L\sqrt{m}(f(x_0)-f^*)}{5\sqrt{B}\epsilon} = \Theta\big(\frac{\sqrt{m}}{\sqrt{B}\epsilon}\big)$ iterations. Then, the total number of SFO calls is given by

$$\sum_{s=1}^{S}\min\{c_\beta\sigma^2\beta_s^{-1}, c_\epsilon\sigma^2\epsilon^{-1}, n\} + KB \leq S(c_\epsilon\sigma^2\epsilon^{-1} \wedge n) + KB \leq \mathcal{O}\left(\frac{\epsilon^{-1} \wedge n}{\epsilon\sqrt{mB}} + \frac{\sqrt{mB}}{\epsilon}\right),$$

which, in conjunction with $mB = n \wedge \frac{1}{\epsilon}$, finishes the proof. □

F. Proofs for Results in Section 3

F.1. Useful Lemmas

In this section, we provide some useful lemmas. The following two lemmas follow directly from Assumptions in Subsection 3.4.

Lemma 3 ((Papini et al., 2018)). *Under Assumptions 2 and 3, the following holds:*

(i) ∇J *is* L*- Lipschitz, i.e., for any* $\theta_1, \theta_2 \in \mathbb{R}^d$*:* $\|\nabla J(\theta_1) - \nabla J(\theta_2)\| \leq L \|\theta_1 - \theta_2\|$.

(ii) $g(\tau|\theta)$ *is Lipschitz continuous with Lipschitz constant* L_g*, i.e., for any trajectory* $\tau \in \mathcal{T}$*:*

$$\|g(\tau|\theta_1) - g(\tau|\theta_2)\| \leq L_g \|\theta_1 - \theta_2\| .$$

(iii) $g(\tau|\theta)$ *and* $\nabla \log(p(\tau|\theta))$ *are bounded, i.e., there exist positive constants* $0 \leq \Gamma, M < \infty$ *such that for any* $\tau \in \mathcal{T}$ *and* $\theta \in \Theta$*:*

$$\|\nabla \log(p(\tau|\theta))\|^2 \leq M \quad \text{and} \quad \|g(\tau|\theta)\|^2 \leq \Gamma.$$

Lemma 4 ((Xu et al., 2019b;a) Lemma A.1). *For any* $\theta_1, \theta_2 \in \mathcal{R}^d$*, let* $\omega\big(\tau|\theta_1, \theta_2\big) = p(\tau|\theta_1)/p(\tau|\theta_2)$*. Under Assumptions 3 and 4, it holds that*

$$\mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} \left\| 1 - \frac{p(\tau|\theta_2)}{p(\tau|\theta_1)} \right\|^2 = \mathbb{V}\mathrm{ar}\left(\omega\big(\tau|\theta_1, \theta_2\big)\right) \leq \alpha \|\theta_1 - \theta_2\|_2^2,$$

where α *is a positive constant.*

The following lemma captures an important property for the trajectory gradients, and its proof follows directly from Lemma 4.

Lemma 5. *Under Assumptions 2, 3, and 4 the following inequality holds for any* $\theta_1, \theta_2 \in \mathcal{R}^d$*,*

$$\mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} \|g(\tau|\theta_1) - \omega(\tau|\theta_1, \theta_2) g(\tau|\theta_2)\|^2 \leq Q \|\theta_1 - \theta_2\|^2 ,$$

where the importance sampling function $\omega(\tau|\theta_1, \theta_2) := p(\tau|\theta_2)/p(\tau|\theta_1)$*, and the constant* $Q := 2(L_g^2 + \Gamma\alpha)$ *with constants* L_g, Γ *and* α *given in Lemmas 3 and 4.*

F.2. Proof of Theorem 3

In this section, we provide the convergence analysis for AbaSVRPG. To simplify notations, we use $\mathbb{E}_k[\cdot]$ to denote the expectation operation conditioned on all the randomness before θ_k, i.e., $\mathbb{E}[\cdot|\theta_0, \cdots, \theta_k]$ and $n_k = \lfloor k/m \rfloor \times m$.

To prove the convergence of AbaSVRPG, we first present a general iteration analysis for an algorithm with the update rule taking the form of $\theta_{k+1} = \theta_k + \eta v_k$, for $k = 0, 1, \cdots$. The proof of Lemma 6 can be found in Appendix G.

Lemma 6. *Let* ∇J *be* L*-Lipschitz, and* $\theta_{k+1} = \theta_{k+1} + \eta v_k$*. Then, the following inequality holds:*

$$\mathbb{E}\, J(\theta_{k+1}) - \mathbb{E}\, J(\theta_k) \geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E} \|v_k\|^2 - \frac{\eta}{2} \mathbb{E} \|v_k - \nabla J(\theta_k)\|^2 .$$

Since, we do not specify the exact form of v_k, Lemma 6 is applicable to various algorithms such as AbaSVRPG and AbaSPIDER-PG with the same type of update rules.

We next present the variance bound of AbaSVRPG.

Proposition 1. *Let Assuptions 2, 3, and 4 hold. Then, for* $k = 0, \cdots K$*, the variance of the gradient estimator* v_k *of AbaSVRPG can be bounded as*

$$\mathbb{E} \|v_k - \nabla J(\theta_k)\|^2 \leq (k - n_k) \frac{Q\eta^2}{B} \sum_{i=n_k}^{k} \mathbb{E} \|v_i\|^2 + \mathbb{E} \|v_{n_k} - \nabla J(\theta_{n_k})\|^2 ,$$

where $\|v_i\| = 0$ for $i = -1, \cdots, -m$ for simple notations.

Proof of Proposition 1. To bound the variance $\mathbb{E}\|v_k - \nabla J(\theta_k)\|^2$, it is sufficient to bound $\mathbb{E}_k\|v_k - \nabla J(\theta_k)\|^2$ since by the tower property of expectation we have $\mathbb{E}\|v_k - \nabla J(\theta_k)\|^2 = \mathbb{E}\mathbb{E}_k\|v_k - \nabla J(\theta_k)\|^2$. Thus, we first bound $\mathbb{E}_k\|v_k - \nabla J(\theta_k)\|^2$ for the case with $\mod(k,m) \neq 0$, and then generalize it to the case with $\mod(k,m) = 0$.

$$
\begin{aligned}
&\mathbb{E}_k\|v_k - \nabla J(\theta_k)\|^2 \\
&\overset{\text{(i)}}{=} \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\tilde{\theta})g(\tau_i|\tilde{\theta}) + \tilde{v} - \nabla J(\theta_k)\right\|^2 \\
&= \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\tilde{\theta})g(\tau_i|\tilde{\theta}) + \nabla J(\tilde{\theta}) - \nabla J(\theta_k) + \tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&= \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\tilde{\theta})g(\tau_i|\tilde{\theta}) + \nabla J(\tilde{\theta}) - \nabla J(\theta_k)\right\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\qquad + 2\mathbb{E}_k\left\langle \frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\tilde{\theta})g(\tau_i|\tilde{\theta}) + \nabla J(\tilde{\theta}) - \nabla J(\theta_k), \tilde{v} - \nabla J(\tilde{\theta})\right\rangle \\
&\overset{\text{(ii)}}{=} \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\tilde{\theta})g(\tau_i|\tilde{\theta}) + \nabla J(\tilde{\theta}) - \nabla J(\theta_k)\right\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\overset{\text{(iii)}}{\leq} \mathop{\mathbb{E}_k}_{\tau\sim p(\cdot|\theta_k)}\frac{1}{B}\left\|g(\tau|\theta_k) - \omega(\tau|\theta_k,\tilde{\theta})g(\tau|\tilde{\theta}) + \nabla J(\tilde{\theta}) - \nabla J(\theta_k)\right\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\overset{\text{(iv)}}{\leq} \mathop{\mathbb{E}_k}_{\tau\sim p(\cdot|\theta_k)}\frac{1}{B}\left\|g(\tau|\theta_k) - \omega(\tau|\theta_k,\tilde{\theta})g(\tau|\tilde{\theta})\right\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\overset{\text{(v)}}{\leq} \frac{Q}{B}\left\|\theta_k - \tilde{\theta}\right\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 = \frac{Q}{B}\|\theta_k - \theta_{k-1} + \theta_{k-1}\cdots\theta_{n_k}\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\overset{\text{(vi)}}{\leq} \frac{Q}{B}(k-n_k)\sum_{i=n_k}^{k-1}\|\theta_{i+1} - \theta_i\|^2 + \mathbb{E}_k\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2,
\end{aligned}
$$

where (i) follows from the definition of v_k in Algorithm 3, (ii) follows from the fact that $\mathbb{E}_k\left[\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\tilde{\theta})g(\tau_i|\tilde{\theta}) + \nabla J(\tilde{\theta}) - \nabla J(\theta_k)\right] = 0$, and thus given $\theta_k, \cdots, \tilde{\theta}$, the expectation of the inner product is 0, (iii) follows from Lemma 7, (iv) follows from the fact that $\mathbb{V}\mathrm{ar}(X) \leq \mathbb{E}\|X\|^2$, (v) follows from Lemma 5 we provide in Appendix G, and (vi) follows from the vector inequality that $\|\sum_{i=1}^{m}\theta_i\|^2 \leq m\sum_{i=1}^{m}\|\theta_i\|^2$.

Therefore, we have

$$
\begin{aligned}
\mathbb{E}\|v_k - \nabla J(\theta_k)\|^2 &= \mathbb{E}\mathbb{E}_k\|v_k - \nabla J(\theta_k)\|^2 \\
&\leq \frac{Q}{B}(k-n_k)\sum_{i=n_k}^{k-1}\mathbb{E}\|\theta_{i+1} - \theta_i\|^2 + \mathbb{E}\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\leq \frac{Q}{B}(k-n_k)\sum_{i=n_k}^{k}\mathbb{E}\|\theta_{i+1} - \theta_i\|^2 + \mathbb{E}\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\leq \frac{Q}{B}(k-n_k)\eta^2\sum_{i=n_k}^{k}\mathbb{E}\|v_i\|^2 + \mathbb{E}\left\|\tilde{v} - \nabla J(\tilde{\theta})\right\|^2 \\
&\overset{\text{(i)}}{=} (k-n_k)\frac{Q\eta^2}{B}\sum_{i=n_k}^{k}\mathbb{E}\|v_i\|^2 + \mathbb{E}\|v_{n_k} - \nabla J(\theta_{n_k})\|^2,
\end{aligned}
$$

where (i) follows from the fact that at iteration k, $\tilde{v} = v_{n_k}$ and $\tilde{\theta} = \theta_{n_k}$. It is also straightforward to check that the above inequality holds for any k with $\mod(k, m) = 0$. □

Proof of Theorem 3

Since in Algorithm 3, ∇J is L-Lipschitz, and $\theta_{k+1} = \theta_{k+1} + \eta v_k$, we obtain the following inequality directly from Lemma 6:

$$\mathbb{E}\, J(\theta_{k+1}) - \mathbb{E}\, J(\theta_k) \geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E}\, \|v_k\|^2 - \frac{\eta}{2}\, \mathbb{E}\, \|v_k - \nabla J(\theta_k)\|^2. \tag{21}$$

By Proposition 1, we have following variance bound:

$$\mathbb{E}\, \|v_k - \nabla J(\theta_k)\|^2 \leq (k - n_k)\frac{Q\eta^2}{B} \sum_{i=n_k}^{k} \mathbb{E}\, \|v_i\|^2 + \mathbb{E}\, \|v_{n_k} - \nabla J(\theta_{n_k})\|^2 \tag{22}$$

Moreover, for $\mod(k, m) = 0$, we obtain

$$\begin{aligned}
\mathbb{E}\|v_k - \nabla f(x_k)\|^2 &= \mathbb{E}\left\| \frac{1}{N} \sum_{i=1}^{N} \nabla g(\tau_i|\theta_k) - \nabla J(\theta_k) \right\|^2 \\
&\overset{\text{(i)}}{=} \frac{1}{N} \mathop{\mathbb{E}}_{\tau \sim p(\cdot|\theta_k)} \|\nabla g(\tau|\theta_k) - \nabla J(\theta_k)\|^2 \overset{(ii)}{\leqslant} \frac{\sigma^2}{N} \\
&\overset{(iii)}{\leqslant} \frac{\beta}{\alpha m} \sum_{i=n_k-m}^{n_k-1} \|v_i\|^2 + \frac{\epsilon}{\alpha}.
\end{aligned} \tag{23}$$

where (i) follows from Lemma 7, (ii) follows from Assumption 4, and (iii) follows from the fact that

$$N = \frac{\alpha\sigma^2}{\frac{\beta}{m}\sum_{i=n_k-m}^{n_k-1} \|v_i\|^2 + \epsilon},$$

where $\alpha > 0$ and $\beta \geqslant 0$.

Plugging (23) into (22), we obtain

$$\mathbb{E}\, \|v_k - \nabla J(\theta_k)\|^2 \leq (k - n_k)\frac{Q\eta^2}{B} \sum_{i=n_k}^{k} \mathbb{E}\, \|v_i\|^2 + \frac{\beta}{\alpha m} \sum_{i=n_k-m}^{n_k-1} \|v_i\|^2 + \frac{\epsilon}{\alpha}. \tag{24}$$

Plugging (24) into (21), we obtain

$$\mathbb{E}\, J(\theta_{k+1}) - \mathbb{E}\, J(\theta_k) \geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E}\, \|v_k\|^2 - \frac{Q\eta^3}{2B}(k - n_k) \sum_{i=n_k}^{k} \mathbb{E}\, \|v_i\|^2 - \frac{\eta\beta}{2\alpha m} \sum_{i=n_k-m}^{n_k-1} \|v_i\|^2 - \frac{\epsilon\eta}{2\alpha}.$$

We note that for a given k, any iteration $n_k \leq i \leq k$ shares the same $\tilde{\theta}$, and all their corresponding n_i satisfies $n_i = n_k$. Thus, take the summation of the above inequality over k from n_k to k, we obtain

$$\begin{aligned}
&\mathbb{E}J(\theta_{k+1}) - \mathbb{E}\, J(\theta_{n_k}) \\
&\geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \sum_{i=n_k}^{k} \mathbb{E}\, \|v_i\|^2 - \frac{Q\eta^3}{2B} \sum_{i=n_k}^{k} (i - n_k) \sum_{j=n_k}^{i} \mathbb{E}\, \|v_j\|^2 - \sum_{i=n_k}^{k} \frac{\eta\beta}{2\alpha m} \sum_{j=n_k-m}^{n_k-1} \|v_j\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha} \\
&\geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \sum_{i=n_k}^{k} \mathbb{E}\, \|v_i\|^2 - \frac{Q\eta^3}{2B} \sum_{i=n_k}^{k} (k - n_k) \sum_{j=n_k}^{k} \mathbb{E}\, \|v_j\|^2 - \sum_{i=n_k}^{k} \frac{\eta\beta}{2\alpha m} \sum_{j=n_k-m}^{n_k-1} \|v_j\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \sum_{i=n_k}^{k} \mathbb{E}\left\|v_i\right\|^2 - \frac{Q\eta^3(k-n_k)(k-n_k+1)}{2B} \sum_{j=n_k}^{k} \mathbb{E}\left\|v_j\right\|^2 - \sum_{i=n_k}^{k} \frac{\eta\beta}{2\alpha m} \sum_{j=n_k-m}^{n_k-1} \left\|v_j\right\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha} \\
&= \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{Q\eta^3(k-n_k)(k-n_k+1)}{2B}\right) \sum_{i=n_k}^{k} \mathbb{E}\left\|v_i\right\|^2 - \sum_{i=n_k}^{k} \frac{\eta\beta}{2\alpha m} \sum_{j=n_k-m}^{n_k-1} \left\|v_j\right\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha} \\
&\overset{\text{(i)}}{\geq} \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{Q\eta^3 m^2}{2B}\right) \sum_{i=n_k}^{k} \mathbb{E}\left\|v_i\right\|^2 - \sum_{i=n_k}^{k} \frac{\eta\beta}{2\alpha m} \sum_{j=n_k-m}^{n_k-1} \left\|v_j\right\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha} \\
&\overset{\text{(ii)}}{=} \phi \sum_{i=n_k}^{k} \mathbb{E}\left\|v_i\right\|^2 - \frac{\eta\beta(k-n_k+1)}{2\alpha m} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha} \\
&\overset{\text{(iii)}}{\geq} \phi \sum_{i=n_k}^{k} \mathbb{E}\left\|v_i\right\|^2 - \frac{\eta\beta}{2\alpha} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha},
\end{aligned} \tag{25}
$$

where (i) follows from the fact that $k - n_k < k - n_k + 1 \leq m$, (ii) follows from the fact that $\phi := \left(\frac{\eta}{2} - \frac{L\eta^2}{2} - \frac{Q\eta^3 m^2}{2B}\right)$, and (iii) follows because $(k - n_k + 1)/m \leqslant 1$.

Now, we are ready to bound $J(\theta_{K+1}) - J(\theta_0)$.

$$
\begin{aligned}
&\mathbb{E}J(\theta_{K+1}) - \mathbb{E}\, J(\theta_0) \\
&= \mathbb{E}\, J(\theta_{K+1}) - \mathbb{E}\, J(\theta_{n_K}) + \mathbb{E}\, J(\theta_{n_K}) \cdots + \mathbb{E}\, J(\theta_m) - \mathbb{E}\, J(\theta_0) \\
&\overset{\text{(i)}}{\geq} \phi \sum_{i=n_K}^{K} \mathbb{E}\left\|v_i\right\|^2 - \frac{\eta\beta}{2\alpha} \sum_{i=n_K-m}^{n_K-1} \left\|v_i\right\|^2 - \sum_{i=n_K}^{K} \frac{\epsilon\eta}{2\alpha} + \cdots + \phi \sum_{i=0}^{m-1} \mathbb{E}\left\|v_i\right\|^2 - \frac{\eta\beta}{2\alpha} \sum_{i=-m}^{-1} \left\|v_i\right\|^2 - \sum_{i=0}^{m} \frac{\epsilon\eta}{2\alpha} \\
&\overset{\text{(ii)}}{\geq} \phi \sum_{i=0}^{K} \mathbb{E}\left\|v_i\right\|^2 - \frac{\eta\beta}{2\alpha} \sum_{i=0}^{n_K-1} \left\|v_i\right\|^2 - \sum_{i=0}^{K} \frac{\epsilon\eta}{2\alpha} \\
&\geqslant \left(\phi - \frac{\eta\beta}{2\alpha}\right) \sum_{i=0}^{K} \mathbb{E}\left\|v_i\right\|^2 - \sum_{i=0}^{K} \frac{\epsilon\eta}{2\alpha},
\end{aligned}
$$

where (i) follows from (25), and (ii) follows from the fact that we define $\|v_{-1}\| = \cdots = \|v_{-m}\| = 0$.

Thus, we obtain

$$
\begin{aligned}
\left(\phi - \frac{\eta\beta}{2\alpha}\right) \sum_{i=0}^{K} \mathbb{E}\left\|v_i\right\|^2 &\leq \mathbb{E}\, J(\theta_{K+1}) - \mathbb{E}\, J(\theta_0) + \sum_{i=0}^{K} \frac{\epsilon\eta}{2\alpha} \\
&\overset{\text{(i)}}{\leq} J(\theta^*) - J(\theta_0) + \sum_{i=0}^{K} \frac{\epsilon\eta}{2\alpha},
\end{aligned}
$$

where (i) follows because $\theta^* := \arg\max_{\theta \in \mathbb{R}^d} J(\theta)$. Here, we assume $\left(\phi - \frac{\eta\beta}{2\alpha}\right) > 0$ to continue our proof. Such an assumption can be satisfied by parameter tuning as shown in (32). Therefore, we obtain

$$
\sum_{i=0}^{K} \mathbb{E}\left\|v_i\right\|^2 \leq \left(\phi - \frac{\eta\beta}{2\alpha}\right)^{-1} \left(J(\theta^*) - J(\theta_0) + \sum_{i=0}^{K} \frac{\epsilon\eta}{2\alpha}\right). \tag{26}
$$

With (26), we next bound the gradient norm, i.e., $\|\nabla J(\theta_\xi)\|$, of the output of AbaSVRPG. Observe that

$$
\mathbb{E}\|\nabla J(\theta_\xi)\|^2 = \mathbb{E}\|\nabla J(\theta_\xi) - v_\xi + v_\xi\|^2 \leq 2\mathbb{E}\|\nabla J(\theta_\xi) - v_\xi\|^2 + 2\mathbb{E}\|v_\xi\|^2. \tag{27}
$$

Therefore, it is sufficient to bound the two terms on the right hand side of the above inequality. First, note that

$$
\mathbb{E}\|v_\xi\|^2 \overset{\text{(i)}}{=} \frac{1}{K+1} \sum_{i=0}^{K} \mathbb{E}\|v_i\|^2 \overset{\text{(ii)}}{\leq} \left(\phi - \frac{\eta\beta}{2\alpha}\right)^{-1} \left(\frac{J(\theta^*) - J(\theta_0)}{K+1} + \frac{\epsilon\eta}{2\alpha}\right), \tag{28}
$$

where (i) follows from the fact that ξ is selected uniformly at random from $\{0,\dots,K\}$, and (ii) follows from (26). On the other hand, we observe that

$$
\begin{aligned}
&\mathbb{E}\|\nabla J(\theta_\xi)-v_\xi\|^2\\
&\overset{\text{(i)}}{=}\frac{1}{K+1}\sum_{k=0}^{K}\mathbb{E}\|\nabla J(\theta_k)-v_k\|^2\\
&\overset{\text{(ii)}}{\leq}\frac{1}{K+1}\sum_{k=0}^{K}\left((k-n_k)\frac{Q\eta^2}{B}\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\right)\\
&\overset{\text{(iii)}}{=}\frac{Q\eta^2 m}{B(K+1)}\sum_{k=0}^{K}\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(iv)}}{=}\frac{Q\eta^2 m}{B(K+1)}\left(\sum_{k=0}^{m-1}\sum_{i=0}^{k}\mathbb{E}\left\|v_i\right\|^2+\cdots+\sum_{k=n_K}^{K}\sum_{i=n_K}^{k}\mathbb{E}\left\|v_i\right\|^2\right)+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\leqslant\frac{Q\eta^2 m}{B(K+1)}\left(\sum_{k=0}^{m-1}\sum_{i=0}^{m-1}\mathbb{E}\left\|v_i\right\|^2+\cdots+\sum_{k=n_K}^{K}\sum_{i=n_K}^{K}\mathbb{E}\left\|v_i\right\|^2\right)+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(v)}}{\leq}\frac{Q\eta^2 m^2}{B(K+1)}\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(vi)}}{\leq}\frac{Q\eta^2 m^2}{B(K+1)}\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m(K+1)}\left(\sum_{k=0}^{m-1}\sum_{i=-m}^{-1}\left\|v_i\right\|^2+\cdots+\sum_{k=n_K}^{K}\sum_{i=n_K-m}^{n_K-1}\left\|v_i\right\|^2\right)+\frac{\epsilon}{\alpha}\\
&\overset{\text{(vii)}}{\leq}\frac{Q\eta^2 m^2}{B(K+1)}\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha(K+1)}\sum_{i=-m}^{n_K-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(viii)}}{\leq}\frac{1}{K+1}\left(\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(viiii)}}{\leq}\frac{1}{K+1}\left(\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\left(J(\theta^*)-J(\theta_0)+\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha}\right)+\frac{\epsilon}{\alpha}\\
&=\left(\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\frac{(J(\theta^*)-J(\theta_0))}{K+1}+\frac{\epsilon}{\alpha}\left(1+\frac{\eta}{2}\left(\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\right)
\end{aligned}
\tag{29}
$$

where (i) follows from the fact that ξ is selected uniformly at random from $\{0,\dots,K\}$, (ii) follows from (24), (iii) follows from the fact that $k-n_k\leq m$, (iv) follows from the fact that for $n_k\leqslant k\leqslant n_k+m-1$, $n_i=n_k$. (v) follows from $\sum_{k=n_k}^{n_k+m-1}\sum_{i=n_k}^{n_k+m-1}\mathbb{E}\left\|v_i\right\|^2=m\sum_{i=n_k}^{n_k+m-1}\mathbb{E}\left\|v_i\right\|^2$, (vi) follows from the same reasoning as in (iv), (vii) follows from $\sum_{k=n_k}^{n_k+m-1}\sum_{i=n_k-m}^{n_k-1}\mathbb{E}\left\|v_i\right\|^2=m\sum_{i=n_k-m}^{n_k-1}\mathbb{E}\left\|v_i\right\|^2$, (viii) follows from $\|v_{-1}\|=\cdots=\|v_{-m}\|=0$, and (viiii) follows from eq. (26).

Substituting (28), (29) into (27), we obtain

$$
\begin{aligned}
\mathbb{E}\|\nabla J(\theta_\xi)\|^2\leq\frac{2}{K+1}&\left(1+\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}(J(\theta^*)-J(\theta_0))\\
&+\frac{2\epsilon}{\alpha}\left(1+\frac{\eta}{2}\left(1+\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\right)
\end{aligned}
\tag{30}
$$

F.3. Proof of Corollary 3

Based on the parameter setting in Theorem 3 that

$$\eta = \frac{1}{2L}, m = \left(\frac{L^2\sigma^2}{Q\epsilon}\right)^{\frac{1}{3}}, B = \left(\frac{Q\sigma^4}{L^2\epsilon^2}\right)^{\frac{1}{3}}, \alpha = 48, \text{ and } \beta = 6, \tag{31}$$

we obtain

$$\phi - \frac{\eta\beta}{2\alpha} = \left(\frac{1}{4L} - \frac{1}{8L} - \frac{1}{16L}\right) - \frac{1}{32L} = \frac{1}{32L} > 0. \tag{32}$$

Plugging (31) and (32) into (30), we obtain

$$\mathbb{E}\left\|\nabla J(\theta_\xi)\right\|^2 \leq \frac{88L}{K+1}\left(J(\theta^*) - J(\theta_0)\right) + \frac{\epsilon}{2}.$$

Hence, AbaSVRPG converges at a rate of $\mathcal{O}(1/K)$. Next, we bound the STO complexity. To acheive ϵ accuracy, we need

$$\frac{88L}{K+1}\left(J(\theta^*) - J(\theta_0)\right) \leq \frac{\epsilon}{2},$$

which gives

$$K = \frac{176L\left(J(\theta^*) - J(\theta_0)\right)}{\epsilon}.$$

We note that for $\mod(k, m) = 0$, the outer loop batch size $N = \frac{\alpha\sigma^2}{\frac{\beta}{m}\sum_{i=n_k-m}^{n_k-1}\|v_i\|^2+\epsilon} \leq \frac{\alpha\sigma^2}{\epsilon}$. Hence, the overall STO complexity is given by

$$\begin{aligned}
K \times 2B + \sum_{k=0}^{n_K} \frac{\alpha\sigma^2}{\frac{\beta}{m}\sum_{i=km-m}^{km-1}\left\|v_i\right\|^2 + \epsilon} &\leqslant K \times 2B + \sum_{k=0}^{n_K} \frac{\alpha\sigma^2}{\epsilon} \leqslant K \times 2B + \left\lceil \frac{K}{m} \right\rceil \times \frac{\alpha\sigma^2}{\epsilon} \\
&\overset{\text{(i)}}{\leq} 2KB + \frac{K}{m}\frac{\alpha\sigma^2}{\epsilon} + \frac{\alpha\sigma^2}{\epsilon} \\
&\overset{\text{(ii)}}{=} \mathcal{O}\left(\left(\frac{L\left(J(\theta^*) - J(\theta_0)\right)}{\epsilon}\right)\left(\left(\frac{Q\sigma^4}{L^2\epsilon^2}\right)^{\frac{1}{3}} + \frac{\sigma^2}{\epsilon}\left(\frac{Q\epsilon}{L^2\sigma^2}\right)^{\frac{1}{3}}\right) + \frac{\sigma^2}{\epsilon}\right) \\
&= \mathcal{O}\left(\left(\frac{L\left(J(\theta^*) - J(\theta_0)\right)}{\epsilon}\right)\left(\frac{Q\sigma^4}{L^2\epsilon^2}\right)^{\frac{1}{3}} + \frac{\sigma^2}{\epsilon}\right) \\
&= \mathcal{O}\left(\epsilon^{-5/3} + \epsilon^{-1}\right),
\end{aligned}$$

where (i) follows from the fact that $\left\lceil \frac{K}{m} \right\rceil \times N \leq \frac{KN}{m} + N$, and (ii) follows from the parameters setting of $K, B,$ and m in (31).

F.4. Proof of Theorem 4

In this section, we provide the proof of AbaSPIDER-PG. We first bound the variance of AbaSPIDER-PG given in the following proposition.

Proposition 2. *Let Assumptions 2, 3, and 4 hold. For $k = 0, ...K$, gradient estimator v_k of AbaSPIDER-PG satisfies*

$$\mathbb{E}\left\|v_k - \nabla J(\theta_k)\right\|^2 \leq \frac{Q\eta^2}{B}\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2 + \mathbb{E}\left\|v_{n_k} - \nabla J(\theta_{n_k})\right\|^2.$$

Comparing Proposition 2 and Proposition 1, one can clearly see that AbaSPIDER-PG has a much smaller variance bound than AbaSVRPG, particularly as the inner loop iteration goes further (i.e., as k increases). This is because AbaSVRPG uses the initial outer loop batch gradient to construct the gradient estimator in all inner loop iterations, so that the variance in the inner loop accumulates up as the iteration goes further. In contrast, AbaSPIDER-PG avoids such a variance accumulation problem by continuously using the gradient information from the immediate preceding step, and hence has less variance during the inner loop iteration.

Proof of Proposition 2. To bound the variance $\mathbb{E}\left\|v_k - \nabla J(\theta_k)\right\|^2$, it is sufficient to bound $\mathbb{E}_k\left\|v_k - \nabla J(\theta_k)\right\|^2$, and then the tower property of expectation yields the desired result. Thus, we first bound $\mathbb{E}_k\left\|v_k - \nabla J(\theta_k)\right\|^2$ for $\mod(k,m) \neq 0$, and then generalize it to $\mod(k,m) = 0$.

$$
\begin{aligned}
&\mathbb{E}_k\left\|v_k - \nabla J(\theta_k)\right\|^2 \\
&\overset{\text{(i)}}{=} \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\theta_{k-1})g(\tau_i|\theta_{k-1}) + v_{k-1} - \nabla J(\theta_k)\right\|^2 \\
&= \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\theta_{k-1})g(\tau_i|\theta_{k-1}) + \nabla J(\theta_{k-1}) - \nabla J(\theta_k) + v_{k-1} - \nabla J(\theta_{k-1})\right\|^2 \\
&= \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\theta_{k-1})g(\tau_i|\theta_{k-1}) + \nabla J(\theta_{k-1}) - \nabla J(\theta_k)\right\|^2 + \mathbb{E}_k\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2 \\
&\qquad + 2\mathbb{E}_k\left\langle \frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\theta_{k-1})g(\tau_i|\theta_{k-1}) + \nabla J(\theta_{k-1}) - \nabla J(\theta_k), v_{k-1} - \nabla J(\theta_{k-1})\right\rangle \\
&\overset{\text{(ii)}}{=} \mathbb{E}_k\left\|\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\theta_{k-1})g(\tau_i|\theta_{k-1}) + \nabla J(\theta_{k-1}) - \nabla J(\theta_k)\right\|^2 + \mathbb{E}_k\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2 \\
&\overset{\text{(iii)}}{\leq} \mathop{\mathbb{E}_k}_{\tau\sim p(\cdot|\theta_k)} \frac{1}{B}\left\|g(\tau|\theta_k) - \omega(\tau|\theta_k,\theta_{k-1})g(\tau|\theta_{k-1}) + \nabla J(\theta_{k-1}) - \nabla J(\theta_k)\right\|^2 + \mathbb{E}_k\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2 \\
&\overset{\text{(iv)}}{\leq} \mathop{\mathbb{E}_k}_{\tau\sim p(\cdot|\theta_k)} \frac{1}{B}\left\|g(\tau|\theta_k) - \omega(\tau|\theta_k,\theta_{k-1})g(\tau|\theta_{k-1})\right\|^2 + \mathbb{E}_k\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2 \\
&\overset{\text{(v)}}{\leq} \frac{Q}{B}\left\|\theta_k - \theta_{k-1}\right\|^2 + \mathbb{E}_k\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2 \\
&\overset{\text{(vi)}}{\leq} \frac{Q\eta^2}{B}\left\|v_{k-1}\right\|^2 + \mathbb{E}_k\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2
\end{aligned} \tag{33}
$$

where (i) follows from the definition of v_k in Algorithm 3, (ii) follows from the fact that

$$
\mathop{\mathbb{E}}_{k}\left[\frac{1}{B}\sum_{i=1}^{B} g(\tau_i|\theta_k) - \frac{1}{B}\sum_{i=1}^{B}\omega(\tau_i|\theta_k,\theta_{k-1})g(\tau_i|\theta_{k-1}) + \nabla J(\theta_{k-1}) - \nabla J(\theta_k)\right] = 0,
$$

thus given $\theta_k, \cdots, \theta_0$, the expectation of the inner product equals 0, (iii) follows from Lemma 7, (iv) follows from the fact that $\mathbb{V}\mathrm{ar}(X) \leq \mathbb{E}\left\|X\right\|^2$, (v) follows from Lemma 5, and (vi) follows because $\theta_k = \theta_{k-1} + \eta v_{k-1}$.

Therefore, we have

$$
\begin{aligned}
\mathbb{E}\left\|v_k - \nabla J(\theta_k)\right\|^2 &\overset{\text{(i)}}{=} \mathbb{E}\,\mathbb{E}_k\left\|v_k - \nabla J(\theta_k)\right\|^2 \\
&\overset{\text{(ii)}}{\leq} \frac{Q\eta^2}{B}\,\mathbb{E}\left\|v_{k-1}\right\|^2 + \mathbb{E}\left\|v_{k-1} - \nabla J(\theta_{k-1})\right\|^2,
\end{aligned} \tag{34}
$$

where (i) follows from the tower property of expectation, and (ii) follows from (33).

Telescoping (34) over k from $n_k + 1$ to k, we obtain

$$
\mathbb{E}\left\|v_k - \nabla J(\theta_k)\right\|^2 = \sum_{i=n_k+1}^{k} \frac{Q\eta^2}{B}\,\mathbb{E}\left\|v_{i-1}\right\|^2 + \mathbb{E}\left\|v_{n_k} - \nabla J(\theta_{n_k})\right\|^2
$$

$$\leq \sum_{i=n_k}^{k} \frac{Q\eta^2}{B} \mathbb{E} \left\|v_i\right\|^2 + \mathbb{E} \left\|v_{n_k} - \nabla J(\theta_{n_k})\right\|^2.$$

It is straightforward to check that the above inequality also holds for any k with $\mod(k, m) = 0$. □

Proof of Theorem 4

Since in Algorithm 4, ∇J is L-Lipschitz, and $\theta_{k+1} = \theta_{k+1} + \eta v_k$, we obtain the following inequality directly from Lemma 6:

$$\mathbb{E}\, J(\theta_{k+1}) - \mathbb{E}\, J(\theta_k) \geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E} \left\|v_k\right\|^2 - \frac{\eta}{2} \mathbb{E} \left\|v_k - \nabla J(\theta_k)\right\|^2. \tag{35}$$

By Proposition 2, we obtain

$$\mathbb{E} \left\|v_k - \nabla J(\theta_k)\right\|^2 \leq \sum_{i=n_k}^{k} \frac{Q\eta^2}{B} \mathbb{E} \left\|v_i\right\|^2 + \mathbb{E} \left\|v_{n_k} - \nabla J(\theta_{n_k})\right\|^2 \tag{36}$$

Moreover, for $\mod(k, m) = 0$, we obtain

$$\begin{aligned}
\mathbb{E}\|v_k - \nabla f(x_k)\|^2 &= \mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^{N} \nabla g(\tau_i|\theta_k) - \nabla J(\theta_k) \right\|^2 \\
&\overset{\text{(i)}}{=} \frac{1}{N} \mathop{\mathbb{E}}_{\tau \sim p(\cdot|\theta_k)} \left\|\nabla g(\tau|\theta_k) - \nabla J(\theta_k)\right\|^2 \overset{(ii)}{\leqslant} \frac{\sigma^2}{N} \\
&\overset{(iii)}{\leqslant} \frac{\beta}{\alpha m} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 + \frac{\epsilon}{\alpha},
\end{aligned} \tag{37}$$

where (i) follows from Lemma 7, (ii) follows from Assumption 4, and (iii) follows from the fact that

$$N = \frac{\alpha\sigma^2}{\frac{\beta}{m} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 + \epsilon},$$

where $\alpha > 0$ and $\beta \geqslant 0$.

Plugging (37) into (36), we obtain

$$\mathbb{E} \left\|v_k - \nabla J(\theta_k)\right\|^2 \leq \frac{Q\eta^2}{B} \sum_{i=n_k}^{k} \mathbb{E} \left\|v_i\right\|^2 + \frac{\beta}{\alpha m} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 + \frac{\epsilon}{\alpha}. \tag{38}$$

Plugging (38) into (35), we obtain

$$\mathbb{E}\, J(\theta_{k+1}) - \mathbb{E}\, J(\theta_k) \geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E} \left\|v_k\right\|^2 - \frac{Q\eta^3}{2B} \sum_{i=n_k}^{k} \mathbb{E} \left\|v_i\right\|^2 - \frac{\eta\beta}{2\alpha m} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 - \frac{\epsilon\eta}{2\alpha}.$$

We note that for a given k, any iteration $n_k \leq i \leq k$, all their corresponding n_i satisfies $n_i = n_k$. Thus, telescoping the above inequality over k from n_k to k, we obtain

$$\begin{aligned}
&\mathbb{E}J(\theta_{k+1}) - \mathbb{E}\, J(\theta_{n_k}) \\
&\geq \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \sum_{i=n_k}^{k} \mathbb{E} \left\|v_i\right\|^2 - \frac{Q\eta^3}{2B} \sum_{i=n_k}^{k} \sum_{j=n_k}^{i} \mathbb{E} \left\|v_j\right\|^2 - \sum_{i=n_k}^{k} \frac{\eta\beta}{2\alpha m} \sum_{i=n_k-m}^{n_k-1} \left\|v_i\right\|^2 - \sum_{i=n_k}^{k} \frac{\epsilon\eta}{2\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{\eta}{2}-\frac{L\eta^2}{2}\right)\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2-\frac{Q\eta^3}{2B}\sum_{i=n_k}^{k}\sum_{j=n_k}^{k}\mathbb{E}\left\|v_j\right\|^2-\sum_{i=n_k}^{k}\frac{\eta\beta}{2\alpha m}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2-\sum_{i=n_k}^{k}\frac{\epsilon\eta}{2\alpha}\\
&=\left(\frac{\eta}{2}-\frac{L\eta^2}{2}\right)\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2-\frac{Q\eta^3(k-n_k+1)}{2B}\sum_{j=n_k}^{k}\mathbb{E}\left\|v_j\right\|^2-\sum_{i=n_k}^{k}\frac{\eta\beta}{2\alpha m}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2-\sum_{i=n_k}^{k}\frac{\epsilon\eta}{2\alpha}\\
&=\left(\frac{\eta}{2}-\frac{L\eta^2}{2}-\frac{Q\eta^3(k-n_k+1)}{2B}\right)\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2-\sum_{i=n_k}^{k}\frac{\eta\beta}{2\alpha m}\sum_{j=n_k-m}^{n_k-1}\left\|v_j\right\|^2-\sum_{i=n_k}^{k}\frac{\epsilon\eta}{2\alpha}\\
&\overset{\text{(i)}}{\geq}\left(\frac{\eta}{2}-\frac{L\eta^2}{2}-\frac{Q\eta^3 m}{2B}\right)\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2-\sum_{i=n_k}^{k}\frac{\eta\beta}{2\alpha m}\sum_{j=n_k-m}^{n_k-1}\left\|v_j\right\|^2-\sum_{i=n_k}^{k}\frac{\epsilon\eta}{2\alpha}\\
&\overset{\text{(ii)}}{=}\phi\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2-\frac{\eta\beta(k-n_k+1)}{2\alpha m}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2-\sum_{i=n_k}^{k}\frac{\epsilon\eta}{2\alpha}\\
&\overset{\text{(iii)}}{\geq}\phi\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2-\frac{\eta\beta}{2\alpha}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2-\sum_{i=n_k}^{k}\frac{\epsilon\eta}{2\alpha},
\end{aligned}
\tag{39}
$$

where (i) follows from the fact that $k-n_k+1\leq m$, (ii) follows from the fact that $\phi:=\left(\frac{\eta}{2}-\frac{L\eta^2}{2}-\frac{Q\eta^3 m}{2B}\right)$, and (iii) follows because $(k-n_k+1)/m\leqslant 1$.

Now, we are ready to bound $J(\theta_{K+1})-J(\theta_0)$.

$$
\begin{aligned}
&\mathbb{E}J(\theta_{K+1})-\mathbb{E}\,J(\theta_0)\\
&=\mathbb{E}\,J(\theta_{K+1})-\mathbb{E}\,J(\theta_{n_K})+\mathbb{E}\,J(\theta_{n_K})\cdots+\mathbb{E}\,J(\theta_m)-\mathbb{E}\,J(\theta_0)\\
&\overset{\text{(i)}}{\geq}\phi\sum_{i=n_K}^{K}\mathbb{E}\left\|v_i\right\|^2-\frac{\eta\beta}{2\alpha}\sum_{i=n_K-m}^{n_K-1}\left\|v_i\right\|^2-\sum_{i=n_K}^{K}\frac{\epsilon\eta}{2\alpha}+\cdots+\phi\sum_{i=0}^{m-1}\mathbb{E}\left\|v_i\right\|^2-\frac{\eta\beta}{2\alpha}\sum_{i=-m}^{-1}\left\|v_i\right\|^2-\sum_{i=0}^{m}\frac{\epsilon\eta}{2\alpha}\\
&\overset{\text{(ii)}}{\geq}\phi\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2-\frac{\eta\beta}{2\alpha}\sum_{i=0}^{n_K-1}\left\|v_i\right\|^2-\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha}\\
&\geqslant\left(\phi-\frac{\eta\beta}{2\alpha}\right)\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2-\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha},
\end{aligned}
$$

where (i) follows from (39), and (ii) follows from the fact that we define $\|v_{-1}\|=\cdots=\|v_{-m}\|=0$. Thus, we obtain

$$
\begin{aligned}
\left(\phi-\frac{\eta\beta}{2\alpha}\right)\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2&\leq\mathbb{E}\,J(\theta_{K+1})-\mathbb{E}\,J(\theta_0)+\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha}\\
&\overset{\text{(i)}}{\leq}J(\theta^*)-J(\theta_0)+\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha},
\end{aligned}
$$

where (i) follows because $\theta^*:=\arg\max_{\theta\in\mathbb{R}^d}J(\theta)$. Here, we assume $\left(\phi-\frac{\eta\beta}{2\alpha}\right)>0$ to continue our proof. Such an assumption will be satisfied by parameter tuning as shown in (32). Therefore, we obtain

$$
\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2\leq\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\left(J(\theta^*)-J(\theta_0)+\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha}\right).
\tag{40}
$$

With (40), we are now able to bound the gradient norm, i.e., $\|\nabla J(\theta_\xi)\|$, of the output of AbaSPIDER-PG. Observe that

$$
\mathbb{E}\|\nabla J(\theta_\xi)\|^2=\mathbb{E}\|\nabla J(\theta_\xi)-v_\xi+v_\xi\|^2\leq 2\mathbb{E}\|\nabla J(\theta_\xi)-v_\xi\|^2+2\mathbb{E}\|v_\xi\|^2.
\tag{41}
$$

Therefore, it is sufficient to bound the two terms on the right hand side of the above inequality. First, note that

$$\mathbb{E}\|v_\xi\|^2 \overset{\text{(i)}}{=} \frac{1}{K+1}\sum_{i=0}^{K}\mathbb{E}\|v_i\|^2 \overset{\text{(ii)}}{\leq} \left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\left(\frac{J(\theta^*)-J(\theta_0)}{K+1}+\frac{\epsilon\eta}{2\alpha}\right), \tag{42}$$

where (i) follows from the fact that ξ is selected uniformly at random from $\{0,\dots,K\}$, and (ii) follows from (40). On the other hand, we observe that

$$\begin{aligned}
&\mathbb{E}\|\nabla J(\theta_\xi)-v_\xi\|^2\\
&\overset{\text{(i)}}{=} \frac{1}{K+1}\sum_{k=0}^{K}\mathbb{E}\|\nabla J(\theta_k)-v_k\|^2\\
&\overset{\text{(ii)}}{\leq} \frac{1}{K+1}\sum_{k=0}^{K}\left(\frac{Q\eta^2}{B}\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\right)\\
&\overset{\text{(iii)}}{=} \frac{Q\eta^2}{B(K+1)}\sum_{k=0}^{K}\sum_{i=n_k}^{k}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(iv)}}{=} \frac{Q\eta^2}{B(K+1)}\left(\sum_{k=0}^{m-1}\sum_{i=0}^{k}\mathbb{E}\left\|v_i\right\|^2+\cdots+\sum_{k=n_K}^{K}\sum_{i=n_K}^{k}\mathbb{E}\left\|v_i\right\|^2\right)+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\leqslant \frac{Q\eta^2}{B(K+1)}\left(\sum_{k=0}^{m-1}\sum_{i=0}^{m-1}\mathbb{E}\left\|v_i\right\|^2+\cdots+\sum_{k=n_K}^{K}\sum_{i=n_K}^{K}\mathbb{E}\left\|v_i\right\|^2\right)+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(v)}}{\leq} \frac{Q\eta^2 m}{B(K+1)}\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m(K+1)}\sum_{k=0}^{K}\sum_{i=n_k-m}^{n_k-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(vi)}}{\leq} \frac{Q\eta^2 m}{B(K+1)}\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha m(K+1)}\left(\sum_{k=0}^{m-1}\sum_{i=-m}^{-1}\left\|v_i\right\|^2+\cdots+\sum_{k=n_K}^{K}\sum_{i=n_K-m}^{n_K-1}\left\|v_i\right\|^2\right)+\frac{\epsilon}{\alpha}\\
&\overset{\text{(vii)}}{\leq} \frac{Q\eta^2 m}{B(K+1)}\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\beta}{\alpha(K+1)}\sum_{i=-m}^{n_K-1}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(viii)}}{\leq} \frac{1}{K+1}\left(\frac{Q\eta^2 m}{B}+\frac{\beta}{\alpha}\right)\sum_{i=0}^{K}\mathbb{E}\left\|v_i\right\|^2+\frac{\epsilon}{\alpha}\\
&\overset{\text{(viiii)}}{\leq} \frac{1}{K+1}\left(\frac{Q\eta^2 m}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\left(J(\theta^*)-J(\theta_0)+\sum_{i=0}^{K}\frac{\epsilon\eta}{2\alpha}\right)+\frac{\epsilon}{\alpha}\\
&= \left(\frac{Q\eta^2 m}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\frac{(J(\theta^*)-J(\theta_0))}{K+1}+\frac{\epsilon}{\alpha}\left(1+\frac{\eta}{2}\left(\frac{Q\eta^2 m^2}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\right)
\end{aligned} \tag{43}$$

where (i) follows from the fact that ξ is selected uniformly at random from $\{0,\dots,K\}$, (ii) follows from (38), (iii) follows from the fact that $k-n_k\leq m$, (iv) follows from the fact that for $n_k\leqslant k\leqslant n_k+m-1$, $n_i=n_k$. (v) follows from $\sum_{k=n_k}^{n_k+m-1}\sum_{i=n_k}^{n_k+m-1}\mathbb{E}\left\|v_i\right\|^2=m\sum_{i=n_k}^{n_k+m-1}\mathbb{E}\left\|v_i\right\|^2$, (vi) follows from the same reasoning as in (iv), (vii) follows from $\sum_{k=n_k}^{n_k+m-1}\sum_{i=n_k-m}^{n_k-1}\mathbb{E}\left\|v_i\right\|^2=m\sum_{i=n_k-m}^{n_k-1}\mathbb{E}\left\|v_i\right\|^2$, (viii) follows from $\|v_{-1}\|=\cdots=\|v_{-m}\|=0$, and (viiii) follows from eq. (40).

Substituting (42), (43) into (41), we obtain

$$\begin{aligned}
\mathbb{E}\|\nabla J(\theta_\xi)\|^2 \leq{}& \frac{2}{K+1}\left(1+\frac{Q\eta^2 m}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}(J(\theta^*)-J(\theta_0))\\
&+\frac{2\epsilon}{\alpha}\left(1+\frac{\eta}{2}\left(1+\frac{Q\eta^2 m}{B}+\frac{\beta}{\alpha}\right)\left(\phi-\frac{\eta\beta}{2\alpha}\right)^{-1}\right)
\end{aligned} \tag{44}$$

F.5. Proof of Corollary 4

Based on the parameter setting in Theorem 4 that

$$\eta = \frac{1}{2L}, m = \frac{L\sigma}{\sqrt{Q\epsilon}}, B = \frac{\sigma\sqrt{Q}}{L\sqrt{\epsilon}}, \alpha = 48 \text{ and } \beta = 16, \tag{45}$$

we obtain

$$\phi - \frac{\eta\beta}{2\alpha} = \left(\frac{1}{4L} - \frac{1}{8L} - \frac{1}{16L}\right) - \frac{1}{32L} = \frac{1}{32L} > 0. \tag{46}$$

Plugging (45) and (46) into (44), we obtain

$$\mathbb{E}\left\|\nabla J(\theta_\xi)\right\|^2 \leq \frac{88L}{K+1}\left(J(\theta^*) - J(\theta_0)\right) + \frac{\epsilon}{2}$$

To obtain ϵ accuracy, we need

$$\frac{88L}{K+1}\left(J(\theta^*) - J(\theta_0)\right) \leq \frac{\epsilon}{2},$$

which gives

$$K = \frac{176L\left(J(\theta^*) - J(\theta_0)\right)}{\epsilon}.$$

We note that for $\mod(k, m) = 0$, the outer loop batch size $N = \frac{\alpha\sigma^2}{\frac{\beta}{m}\sum_{i=n_k-m}^{n_k-1}\|v_i\|^2+\epsilon} \leq \frac{\alpha\sigma^2}{\epsilon}$. Hence, the overall STO complexity is given by

$$\begin{aligned}
K \times 2B + \sum_{k=0}^{n_K} \frac{\alpha\sigma^2}{\frac{\beta}{m}\sum_{i=km-m}^{km-1}\|v_i\|^2 + \epsilon} &\leqslant K \times 2B + \sum_{k=0}^{n_K} \frac{\alpha\sigma^2}{\epsilon} \leqslant K \times 2B + \left\lceil \frac{K}{m} \right\rceil \times \frac{\alpha\sigma^2}{\epsilon} \\
&\overset{\text{(i)}}{\leq} 2KB + \frac{K}{m}\frac{\alpha\sigma^2}{\epsilon} + \frac{\alpha\sigma^2}{\epsilon} \\
&\overset{\text{(ii)}}{=} \mathcal{O}\left(\left(\frac{L\left(J(\theta^*) - J(\theta_0)\right)}{\epsilon}\right)\left(\frac{\sigma\sqrt{Q}}{L\sqrt{\epsilon}} + \frac{\sigma^2}{\epsilon}\frac{\sqrt{Q\epsilon}}{L\sigma}\right) + \frac{\sigma^2}{\epsilon}\right) \\
&= \mathcal{O}\left(\left(\frac{L\left(J(\theta^*) - J(\theta_0)\right)}{\epsilon}\right)\left(\frac{\sigma\sqrt{Q}}{L\sqrt{\epsilon}}\right) + \frac{\sigma^2}{\epsilon}\right) \\
&= \mathcal{O}\left(\epsilon^{-3/2} + \epsilon^{-1}\right).
\end{aligned}$$

where (i) follows from the fact that $\left\lceil \frac{K}{m} \right\rceil \times N \leq \frac{KN}{m} + N$, and (ii) follows from the parameters setting of $K, B,$ and m in (45).

G. Proof of Technical Lemmas

G.1. Proof of Lemma 3

(i), (ii), (iii) follow from Lemma B.2, Lemma B.3 and Lemma B.4 in Papini et al. 2018, respectively.

G.2. Proof of Lemma 5

Note that

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} \left\|g(\tau|\theta_1) - \omega(\tau|\theta_1,\theta_2)g(\tau|\theta_2)\right\|^2 \\
&\quad = \mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} \left\|g(\tau|\theta_1) - g(\tau|\theta_2) + g(\tau|\theta_2) - \omega(\tau|\theta_1,\theta_2)g(\tau|\theta_2)\right\|^2
\end{aligned}$$

$$\overset{\text{(i)}}{=} \mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} 2\left\|g(\tau|\theta_1) - g(\tau|\theta_2)\right\|^2 + \mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} 2\left\|g(\tau|\theta_2) - \omega(\tau|\theta_1,\theta_2)g(\tau|\theta_2)\right\|^2$$
$$\overset{\text{(ii)}}{\leq} 2L_g^2\left\|\theta_1-\theta_2\right\|^2 + \mathop{\mathbb{E}}_{\tau\sim p(\cdot|\theta_1)} 2\left\|g(\tau|\theta_2)\right\|^2\left\|1-\omega(\tau|\theta_1,\theta_2)\right\|^2$$
$$\overset{\text{(iii)}}{\leq} 2L_g^2\left\|\theta_1-\theta_2\right\|^2 + 2\Gamma\alpha\left\|\theta_1-\theta_2\right\|^2 = 2(L_g^2+\Gamma\alpha)\left\|\theta_1-\theta_2\right\|^2 \overset{\text{(iv)}}{=} Q\left\|\theta_1-\theta_2\right\|^2,$$

where (i) follows from the fact that $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$, (ii) follows from item (ii) in Lemma 3, and (iii) follows from item (iii) in Lemma 3 and Lemma 4. Then, the proof is complete.

G.3. Proof of Lemma 6

We derive the following lower bound

$$\begin{aligned}
J(\theta_{k+1}) - J(\theta_k) &\overset{\text{(i)}}{\geq} \langle \nabla J(\theta_k), \theta_{k+1}-\theta_k\rangle - \frac{L}{2}\left\|\theta_{k+1}-\theta_k\right\|^2 \\
&\overset{\text{(ii)}}{=} \eta\langle \nabla J(\theta_k), v_k\rangle - \frac{L\eta^2}{2}\left\|v_k\right\|^2 \\
&= \eta\langle \nabla J(\theta_k) - v_k + v_k, v_k\rangle - \frac{L\eta^2}{2}\left\|v_k\right\|^2 \\
&= \eta\left\|v_k\right\|^2 + \eta\langle \nabla J(\theta_k) - v_k, v_k\rangle - \frac{L\eta^2}{2}\left\|v_k\right\|^2 \\
&\overset{\text{(iii)}}{\geq} \eta\left\|v_k\right\|^2 - \eta\frac{\left\|v_k-\nabla J(\theta_k)\right\|^2 + \left\|v_k\right\|^2}{2} - \frac{L\eta^2}{2}\left\|v_k\right\|^2 \\
&= \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\left\|v_k\right\|^2 - \frac{\eta}{2}\left\|\nabla J(v_k) - \theta_k\right\|^2,
\end{aligned}$$

where (i) follows from the fact that ∇J is L-Lipschitz, (ii) follows from the update rule $x_{k+1} = x_k + \eta v_k$, and (iii) follows from Young's inequality. Taking the expectation over the entire random process on both sides, we obtain the desired result.

G.4. Proof of Lemma 7

Lemma 7. *Let $X, X_1, \cdots, X_n$ be independent and identically distributed (i.i.d.) random variables with mean $\mathbb{E}[X]$, then, the following equation holds:*

$$\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}\,X\right\|^2 = \frac{\mathbb{E}\left\|X-\mathbb{E}\,X\right\|^2}{n}$$

Proof. Standard calculation yields

$$\begin{aligned}
\mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}\,X\right\|^2 &= \mathbb{E}\left\|\frac{1}{n}\sum_{i=1}^n (X_i - \mathbb{E}\,X)\right\|^2 = \frac{1}{n^2}\mathbb{E}\left\|\sum_{i=1}^n (X_i - \mathbb{E}\,X)\right\|^2 \\
&= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\langle X_i - \mathbb{E}\,X, X_j - \mathbb{E}\,X\rangle \\
&\overset{\text{(i)}}{=} \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}\langle X_i - \mathbb{E}\,X, X_i - \mathbb{E}\,X\rangle \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}\left\|X_i - \mathbb{E}\,X\right\|^2 \\
&\overset{\text{(ii)}}{=} \frac{\mathbb{E}\left\|X-\mathbb{E}\,X\right\|^2}{n},
\end{aligned}$$

where (i) follows from the fact that $X_1, \cdots, X_n$ are i.i.d. random variables such that if $i \neq j$, $\mathbb{E}\langle X_i - \mathbb{E}X, X_j - \mathbb{E}X\rangle = 0$, and (ii) follows from the fact that for i.i.d. random variables $\mathbb{E}\|X - \mathbb{E}X\|^2 = \mathbb{E}\|X_1 - \mathbb{E}X\|^2 \cdots = \mathbb{E}\|X_n - \mathbb{E}X\|^2$. □

H. Proofs for Results in Appendix C

H.1. Proof for Theorem 5

To simplify notations, we let $c_\beta = c_\epsilon = \alpha = \left(2\tau + \frac{2\tau}{1-\exp(\frac{-4}{c_\eta(c_\eta-2)})}\right) \vee \frac{16c_\eta L\tau}{m}$.

Since the objective function $f(\cdot)$ has a L-Lipschitz continuous gradient, we obtain that for $1 \leq t \leq m$,

$$\begin{aligned} f(x_t^s) \leq & f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s), x_t^s - x_{t-1}^s\rangle + \frac{L\eta^2}{2}\|v_{t-1}^s\|^2 \\ = & f(x_{t-1}^s) + \frac{\eta}{2}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 - \frac{\eta}{2}\|\nabla f(x_{t-1}^s)\|^2 - \frac{\eta}{2}\|v_{t-1}^s\|^2 + \frac{L\eta^2}{2}\|v_{t-1}^s\|^2, \end{aligned}$$

which, in conjunction with the PL condition that $\|\nabla f(x_{t-1}^s)\|^2 \geq \frac{1}{\tau}(f(x_{t-1}^s) - f(x^*))$, implies that

$$f(x_t^s) - f(x^*) \leq \left(1 - \frac{\eta}{2\tau}\right)(f(x_{t-1}^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\|v_{t-1}^s\|^2 + \frac{\eta}{2}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2.$$

Recall that $\mathbb{E}_{t,s}(\cdot)$ denotes $\mathbb{E}(\cdot|x_0^1, x_0^2, ..., x_2^1, ..., x_t^s)$. Then, taking expectation $\mathbb{E}_{0,s}(\cdot)$ over the above inequality yields, for $1 \leq t \leq m$,

$$\begin{aligned} \mathbb{E}_{0,s}(f(x_t^s) - f(x^*)) \leq & \left(1 - \frac{\eta}{2\tau}\right)\mathbb{E}_{0,s}(f(x_{t-1}^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2 \\ & + \frac{\eta}{2}\mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2, \end{aligned} \tag{47}$$

which, in conjunction with Lemma 1, implies that

$$\begin{aligned} \mathbb{E}_{0,s}(f(x_t^s) - f(x^*)) \leq & \left(1 - \frac{\eta}{2\tau}\right)\mathbb{E}_{0,s}(f(x_{t-1}^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2 \\ & + \frac{\eta^3 L^2(t-1)}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{t-2}\|v_i^s\|^2 + \frac{\eta I_{(N_s<n)}}{2N_s}\sigma^2. \end{aligned}$$

Let $\gamma := 1 - \frac{\eta}{2\tau}$. Then, telescoping the above inequality over t from 1 to m and using the fact that $t - 1 < m$, we have

$$\begin{aligned} \mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq & \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2 \\ & + \frac{\eta^3 L^2 m}{2B}\sum_{t=0}^{m-2}\gamma^{m-2-t}\mathbb{E}_{0,s}\sum_{i=0}^{t}\|v_i^s\|^2 + \left(\sum_{t=0}^{m-1}\gamma^t\right)\frac{\eta I_{(N_s<n)}}{2N_s}\sigma^2. \end{aligned} \tag{48}$$

Note that $\gamma^{m-1-t} \geq \gamma^m$ for $0 \leq t \leq m-1$ and $\sum_{t=0}^{m-1}\gamma^t = \frac{1-\gamma^m}{1-\gamma} \leq \frac{1}{1-\gamma} = \frac{2\tau}{\eta}$. Then, we obtain from (48) that

$$\begin{aligned} \mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq & \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\frac{\eta}{4} - \frac{L\eta^2}{2}\right)\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2 \\ & - \frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta^3 L^2 m}{2B}\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2\left(\sum_{t=0}^{m-2}\gamma^{m-2-t}\right) \\ & + \frac{\tau I_{(N_s<n)}}{N_s}\sigma^2 \\ \leq & \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\frac{\eta}{4} - \frac{L\eta^2}{2}\right)\gamma^m\sum_{t=0}^{m-1}\mathbb{E}_{0,s}\|v_t^s\|^2 \end{aligned}$$

$$
\begin{aligned}
&-\frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2+\frac{\eta^2L^2m\tau}{B}\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2+\frac{\tau I_{(N_s<n)}}{N_s}\sigma^2\\
\leq&\gamma^m\mathbb{E}_{0,s}(f(x_0^s)-f(x^*))-\left(\left(\frac{\eta}{4}-\frac{L\eta^2}{2}\right)\gamma^m-\frac{\eta^2L^2m\tau}{B}\right)\sum_{t=0}^{m-1}\mathbb{E}_{0,s}\|v_t^s\|^2\\
&+\frac{\tau I_{(N_s<n)}}{N_s}\sigma^2-\frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2. \qquad (49)
\end{aligned}
$$

Recall $\eta=\frac{1}{c_\eta L}$ ($c_\eta>4$), $\frac{8L\tau}{c_\eta-2}\leq m<4L\tau$ and $B=m^2$. Then, we have

$$
\begin{aligned}
\left(\frac{\eta}{4}-\frac{L\eta^2}{2}\right)\gamma^m&=\eta\left(\frac{1}{4}-\frac{1}{2c_\eta}\right)\left(1-\frac{1}{2c_\eta\tau L}\right)^m>\eta\left(\frac{1}{4}-\frac{1}{2c_\eta}\right)\left(1-\frac{1}{2m}\right)^m\\
&\overset{(i)}{\geq}\frac{\eta}{2}\left(\frac{1}{4}-\frac{1}{2c_\eta}\right)\geq\frac{\eta^2L^2\tau}{m}=\frac{\eta^2L^2m\tau}{B}, \qquad (50)
\end{aligned}
$$

where (i) follows from the fact that $\left(1-\frac{1}{2m}\right)^m\geq\frac{1}{2}$ for $m\geq1$. Recall $c_\beta=c_\epsilon=\alpha$ and $N_s=\min\{c_\beta\sigma^2\beta_s^{-1},c_\epsilon\sigma^2\epsilon^{-1},n\}$. Then, combining (10), (49) and (50) yields

$$
\begin{aligned}
\mathbb{E}_{0,s}(f(x_m^s)-f(x^*))\leq&\gamma^m\mathbb{E}_{0,s}(f(x_0^s)-f(x^*))+\frac{\tau I_{(N_s<n)}}{N_s}\sigma^2-\frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2.\\
\leq&\gamma^m\mathbb{E}_{0,s}(f(x_0^s)-f(x^*))+\tau\left(\frac{\beta_s}{\alpha}+\frac{\epsilon}{\alpha}\right)-\frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2.
\end{aligned}
$$

Further taking expectation of the above inequality over $x_0^1,...,x_0^s$, we obtain

$$
\mathbb{E}(f(x_m^s)-f(x^*))\leq\gamma^m\mathbb{E}(f(x_0^s)-f(x^*))+\frac{\tau}{\alpha}\mathbb{E}\beta_s+\frac{\tau\epsilon}{\alpha}-\frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}\|v_t^s\|^2
$$

Recall that $\beta_1\leq\epsilon\left(\frac{1}{\gamma}\right)^{m(S-1)}$ and $\beta_s=\frac{1}{m}\sum_{t=1}^m\|v_{t-1}^{s-1}\|^2$ for $s\geq2$. Then, telescoping the above inequality over s from 1 to S yields

$$
\begin{aligned}
\mathbb{E}(f(x_m^S)-f(x^*))\leq&\gamma^{Sm}\mathbb{E}(f(x_0)-f(x^*))+\sum_{s=1}^{S-1}\gamma^{m(S-1-s)}\frac{\tau}{\alpha m}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2\\
&+\gamma^{m(S-1)}\frac{\tau\beta_1}{\alpha}+\sum_{s=1}^{S}\gamma^{m(S-s)}\frac{\tau\epsilon}{\alpha}-\frac{\eta}{4}\sum_{s=1}^{S}\gamma^{m(S-s)}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}\|v_t^s\|^2\\
\overset{(i)}{\leq}&\gamma^K\mathbb{E}(f(x_0)-f(x^*))-\left(\frac{\eta}{4}\gamma^{2m}-\frac{\tau}{\alpha m}\right)\sum_{s=1}^{S-1}\gamma^{m(S-1-s)}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2\\
&+\left(1+\frac{1}{1-\exp(-\frac{4}{c_\eta(c_\eta-2)})}\right)\frac{\tau\epsilon}{\alpha}, \qquad (51)
\end{aligned}
$$

where (i) follows from the fact that $\gamma^{m-1-t}\geq\gamma^m$ for $0\leq t\leq m-1$, $\gamma^{m(S-1)}\leq1$, $\sum_{s=1}^S\gamma^{m(S-s)}\leq\frac{1}{1-\gamma^m}$ and

$$
\gamma^m=\left(1-\frac{1}{2c_\eta\tau L}\right)^m\leq\left(1-\frac{4}{c_\eta(c_\eta-2)m}\right)^m\leq\exp\left(-\frac{4}{c_\eta(c_\eta-2)}\right).
$$

Since $\alpha=\left(2\tau+\frac{2\tau}{1-\exp(-\frac{4}{c_\eta(c_\eta-2)})}\right)\vee\frac{16c_\eta L\tau}{m}$, we have

$$
\left(1+\frac{1}{1-\exp(-\frac{4}{c_\eta(c_\eta-2)})}\right)\frac{\tau\epsilon}{\alpha}\leq\frac{1}{2},\quad\frac{\eta}{4}\gamma^{2m}\overset{(i)}{>}\frac{1}{16c_\eta L}\geq\frac{\tau}{\alpha m} \qquad (52)
$$

where (i) follows from (50) that $\gamma^m \geq \left(1-\frac{1}{2m}\right)^m \geq \frac{1}{2}$. Note that $x_m^S = \tilde{x}^S$. Then, combining (52) and (51) yields

$$\mathbb{E}(f(\tilde{x}^S) - f(x^*)) \leq \gamma^K (f(x_0) - f(x^*)) + \frac{\epsilon}{2}. \tag{53}$$

Let $K = (2c_\eta \tau L - 1)\log\left(\frac{2(f(x_0)-f(x^*))}{\epsilon}\right)$. Then, we have

$$\gamma^K (f(x_0) - f(x^*)) = \exp\left[(2c_\eta\tau L - 1)\log\frac{1}{\gamma}\log\left(\frac{\epsilon}{2(f(x_0)-f(x^*))}\right)\right](f(x_0)-f(x^*)) \overset{(i)}{\leq} \frac{\epsilon}{2},$$

where (i) follows from the fact that $\log\frac{1}{\gamma} = \log\left(1+\frac{1}{2c_\eta\tau L-1}\right) \leq \frac{1}{2c_\eta\tau L-1}$. Thus, the total number of SFO calls is

$$\begin{aligned}\sum_{s=1}^{S}\min\left\{\frac{c_\beta}{\beta_s}, \frac{c_\epsilon}{\epsilon}, n\right\} + KB \leq& \mathcal{O}\left(\left(\frac{c_\epsilon}{\epsilon}\wedge n\right)\frac{\tau}{m}\log\frac{1}{\epsilon} + B\tau\log\frac{1}{\epsilon}\right)\\ \overset{(i)}{\leq}& \mathcal{O}\left(\left(\frac{\tau}{\epsilon}\wedge n\right)\log\frac{1}{\epsilon} + \tau^3\log\frac{1}{\epsilon}\right),\end{aligned}$$

where (i) follows from the fact that $m = \Theta(\tau)$ and $c_\epsilon = \Theta(\tau)$.

H.2. Proof of Theorem 6

To simplify notations, we let $c_\beta = c_\epsilon = \alpha = \left(2\tau + \frac{2\tau}{1-\exp(\frac{-4}{c_\eta(c_\eta-2)})}\right) \vee \frac{16c_\eta L\tau}{m}$.

Using an approach similar to (47), we have, for $1 \leq t \leq m$

$$\begin{aligned}\mathbb{E}_{0,s}(f(x_t^s) - f(x^*)) \leq& \left(1-\frac{\eta}{2\tau}\right)\mathbb{E}_{0,s}(f(x_{t-1}^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2\\ &+ \frac{\eta}{2}\mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2,\end{aligned}$$

which, in conjunction with Lemma 2, implies that

$$\begin{aligned}\mathbb{E}_{0,s}(f(x_t^s) - f(x^*)) \leq& \left(1-\frac{\eta}{2\tau}\right)\mathbb{E}_{0,s}(f(x_{t-1}^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\mathbb{E}_{0,s}\|v_{t-1}^s\|^2\\ &+ \frac{\eta^3L^2}{2B}\sum_{i=0}^{t-2}\mathbb{E}_{0,s}\|v_i^s\|^2 + \frac{\eta I_{(N_s<n)}}{2N_s}\sigma^2.\end{aligned}$$

Let $\gamma := 1 - \frac{\eta}{2\tau}$. Then, telescoping the above inequality over t from 1 to m yields

$$\begin{aligned}\mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq& \gamma^m\mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right)\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2\\ &+ \frac{\eta^3L^2}{2B}\sum_{t=0}^{m-2}\gamma^{m-2-t}\sum_{i=0}^{t}\mathbb{E}_{0,s}\|v_i^s\|^2 + \left(\sum_{t=0}^{m-1}\gamma^t\right)\frac{\eta I_{(N_s<n)}}{2N_s}\sigma^2,\end{aligned}$$

which, in conjunction with $\sum_{t=0}^{m-1}\gamma^t = \frac{1-\gamma^m}{1-\gamma} \leq \frac{1}{1-\gamma} = \frac{2\tau}{\eta}$ and $\gamma^{m-1-t} \geq \gamma^m$ for $0 \leq t \leq m-1$, implies that

$$\begin{aligned}\mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq& \gamma^m\mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\frac{\eta}{4} - \frac{L\eta^2}{2}\right)\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2\\ &- \frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta^3L^2}{2B}\left(\sum_{t=0}^{m-2}\gamma^{m-2-t}\right)\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2\\ &+ \frac{\tau I_{(N_s<n)}}{N_s}\sigma^2\end{aligned}$$

$$
\begin{aligned}
\leq & \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\frac{\eta}{4} - \frac{L\eta^2}{2}\right)\gamma^m \sum_{t=0}^{m-1} \mathbb{E}_{0,s}\|v_t^s\|^2 \\
& - \frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta^2 L^2 \tau}{B}\mathbb{E}_{0,s}\sum_{i=0}^{m-1}\|v_i^s\|^2 + \frac{\tau I_{(N_s<n)}}{N_s}\sigma^2 \\
\leq & \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left(\left(\frac{\eta}{4} - \frac{L\eta^2}{2}\right)\gamma^m - \frac{\eta^2 L^2 \tau}{B}\right)\sum_{t=0}^{m-1}\mathbb{E}_{0,s}\|v_t^s\|^2 \\
& + \frac{\tau I_{(N_s<n)}}{N_s}\sigma^2 - \frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2. \qquad (54)
\end{aligned}
$$

Recall that $\eta = \frac{1}{c_\eta L}$ with $c_\eta > 4$ and $B = m$ with $\frac{8L\tau}{c_\eta - 2} \leq m < 4L\tau$. Then, we have

$$
\begin{aligned}
\left(\frac{\eta}{4} - \frac{L\eta^2}{2}\right)\gamma^m &= \eta\left(\frac{1}{4} - \frac{1}{2c_\eta}\right)\left(1 - \frac{1}{2c_\eta \tau L}\right)^m > \eta\left(\frac{1}{4} - \frac{1}{2c_\eta}\right)\left(1 - \frac{1}{2m}\right)^m \\
&\geq \frac{\eta}{2}\left(\frac{1}{4} - \frac{1}{2c_\eta}\right) \geq \frac{\eta^2 L^2 \tau}{m} = \frac{\eta^2 L^2 \tau}{B}, \qquad (55)
\end{aligned}
$$

which, combined with (54) and (10) , implies that

$$
\mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) + \tau\left(\frac{\beta_s}{\alpha} + \frac{\epsilon}{\alpha}\right) - \frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}_{0,s}\|v_t^s\|^2.
$$

Taking the expectation of the above inequality $x_0^1, ..., x_0^s$, we obtain

$$
\mathbb{E}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}(f(x_0^s) - f(x^*)) + \frac{\tau}{\alpha}\mathbb{E}\beta_s + \frac{\tau\epsilon}{\alpha} - \frac{\eta}{4}\sum_{t=0}^{m-1}\gamma^{m-1-t}\mathbb{E}\|v_t^s\|^2.
$$

Recall $\beta_1 \leq \epsilon\left(\frac{1}{\gamma}\right)^{m(S-1)}$ and $\beta_s = \frac{1}{m}\sum_{t=0}^{m-1}\|v_t^{s-1}\|^2$ for $s = 2, ..., S$. Then, telescoping the above inequality over s from 1 to S and using an approach similar to (51), we have

$$
\begin{aligned}
\mathbb{E}(f(x_m^S) - f(x^*)) \leq & \gamma^K \mathbb{E}(f(x_0) - f(x^*)) - \left(\frac{\eta}{4}\gamma^{2m} - \frac{\tau}{\alpha m}\right)\sum_{s=1}^{S-1}\gamma^{m(S-1-s)}\sum_{t=0}^{m-1}\mathbb{E}\|v_t^s\|^2 \\
& + \left(1 + \frac{1}{1 - \exp(-\frac{4}{c_\eta(c_\eta - 2)})}\right)\frac{\tau\epsilon}{\alpha},
\end{aligned}
$$

which, in conjunction with (52), yields

$$
\mathbb{E}(f(x_m^S) - f(x^*)) \leq \gamma^K \mathbb{E}(f(x_0) - f(x^*)) + \frac{\epsilon}{2}. \qquad (56)
$$

Let $K = (2c_\eta \tau L - 1)\log\left(\frac{2(f(x_0) - f(x^*))}{\epsilon}\right)$. Then, we have $\gamma^K(f(x_0) - f(x^*)) \leq \frac{\epsilon}{2}$. Thus, the total number of SFO calls is given by

$$
\begin{aligned}
\sum_{s=1}^{S}\min\left\{\frac{c_\beta}{\beta_s}, \frac{c_\epsilon}{\epsilon}, n\right\} + KB \leq & \mathcal{O}\left(\left(\frac{c_\epsilon}{\epsilon} \wedge n\right)\frac{\tau}{m}\log\frac{1}{\epsilon} + B\tau\log\frac{1}{\epsilon}\right) \\
\overset{(i)}{\leq} & \mathcal{O}\left(\left(\frac{\tau}{\epsilon} \wedge n\right)\log\frac{1}{\epsilon} + \tau^2\log\frac{1}{\epsilon}\right),
\end{aligned}
$$

where (i) follows from the fact that $B = m = \Theta(\tau)$ and $c_\epsilon = \Theta(\tau)$.

I. Proofs for Results in Appendix D

I.1. Proof of Theorem 7

Since the gradient ∇f is L-Lipschitz, we obtain that, for $t \geq 0$,

$$\begin{aligned}
f(x_{t+1}) \leq & f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2}\|x_{t+1} - x_t\|^2 \\
\overset{(i)}{=} & f(x_t) - \eta \langle \nabla f(x_t), v_t \rangle + \frac{L\eta^2}{2}\|v_t\|^2 \\
= & f(x_t) - \eta \langle \nabla f(x_t) - v_t + v_t, v_t \rangle + \frac{L\eta^2}{2}\|v_t\|^2 \\
= & f(x_t) - \eta \langle \nabla f(x_t) - v_t, v_t \rangle - \eta \|v_t\|^2 + \frac{L\eta^2}{2}\|v_t\|^2 \\
= & f(x_t) - \eta \langle \nabla f(x_t) - v_t, v_t - \nabla f(x_t) + \nabla f(x_t) \rangle - \Big(\eta - \frac{L\eta^2}{2}\Big)\|v_t\|^2 \\
= & f(x_t) - \eta \langle \nabla f(x_t) - v_t, v_t - \nabla f(x_t) \rangle - \eta \langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle - \Big(\eta - \frac{L\eta^2}{2}\Big)\|v_t\|^2 \\
= & f(x_t) + \eta \|\nabla f(x_t) - v_t\|^2 - \eta \langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle - \Big(\eta - \frac{L\eta^2}{2}\Big)\|v_t\|^2
\end{aligned}$$

where (i) follows from the fact that $x_{t+1} = x_t - \eta v_t$. Then, taking expectation $\mathbb{E}(\cdot)$ over the above inequality yields

$$\begin{aligned}
\mathbb{E}f(x_{t+1}) \leq & \mathbb{E}f(x_t) + \eta \mathbb{E}\|\nabla f(x_t) - v_t\|^2 - \eta \mathbb{E}\langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle - \Big(\eta - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 \\
\overset{(i)}{=} & \mathbb{E}f(x_t) + \eta \mathbb{E}\|\nabla f(x_t) - v_t\|^2 - \Big(\eta - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 \\
= & \mathbb{E}f(x_t) - \Big(\eta - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 + \eta \mathbb{E}\|\nabla f(x_t) - v_t\|^2,
\end{aligned} \tag{57}$$

where (i) follows from $\mathbb{E}\langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle = \mathbb{E}_{x_0,\ldots,x_t}\left(\mathbb{E}_t \langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle\right) = 0$.

Next, we upper-bound $\mathbb{E}\|\nabla f(x_t) - v_t\|^2$. For the case when $|B_t| < n$, we have

$$\begin{aligned}
\mathbb{E}\|\nabla f(x_t) - v_t\|^2 = & \mathbb{E}\left\|\nabla f(x_t) - \frac{1}{|B_t|}\sum_{i \in B_t} \nabla f_i(x_t)\right\|^2 = \mathbb{E}\left\|\frac{1}{|B_t|}\sum_{i \in B_t} (\nabla f(x_t) - \nabla f_i(x_t))\right\|^2 \\
= & \mathbb{E}\frac{1}{|B_t|^2}\left\|\sum_{i \in B_t} (\nabla f(x_t) - \nabla f_i(x_t))\right\|^2 \\
= & \mathbb{E}\frac{1}{|B_t|^2}\sum_{i \in B_t}\sum_{j \in B_t} \langle \nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t) \rangle \\
= & \mathbb{E}_{x_0,\ldots,x_t}\left(\mathbb{E}_t \frac{1}{|B_t|^2}\sum_{i \in B_t}\sum_{j \in B_t} \langle \nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t) \rangle\right) \\
= & \mathbb{E}_{x_0,\ldots,x_t}\frac{1}{|B_t|^2}\sum_{i \in B_t}\sum_{j \in B_t} \mathbb{E}_t \langle \nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t) \rangle \\
\overset{(i)}{=} & \mathbb{E}_{x_0,\ldots,x_t}\frac{1}{|B_t|^2}\sum_{i \in B_t} \mathbb{E}_t \|\nabla f(x_t) - \nabla f_i(x_t)\|^2 \overset{(ii)}{\leq} \mathbb{E}\frac{\sigma^2}{|B_t|},
\end{aligned}$$

where (i) follows from $\mathbb{E}_t \nabla f_i(x_t) = \nabla f(x_t)$, and $\mathbb{E}_t \langle \nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t) \rangle = 0$ for $i \neq j$, and (ii) follows from item (3) in Assumption 1. For the case when $|B_t| = n$, we have $v_t = \nabla f(x_t)$, and thus $\mathbb{E}\|\nabla f(x_t) - v_t\|^2 = 0$. Combining the above two cases, we have

$$\mathbb{E}\|\nabla f(x_t) - v_t\|^2 \leq \mathbb{E}\left(\frac{I_{(|B_t|<n)}}{|B_t|}\sigma^2\right). \tag{58}$$

Plugging (58) into (57), we obtain

$$\Big(\eta - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 \leq \mathbb{E}f(x_t) - \mathbb{E}f(x_{t+1}) + \mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|}\eta\sigma^2.$$

Telescoping the above inequality over t from 0 to T yields

$$\sum_{t=0}^{T}\Big(\eta - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 \leq \mathbb{E}f(x_0) - \mathbb{E}f(x_{T+1}) + \sum_{t=0}^{T}\mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|}\eta\sigma^2. \tag{59}$$

Next, we upper-bound $\sum_{t=0}^{T}\mathbb{E}\big(\frac{I_{(|B_t|<n)}}{|B_t|}\eta\sigma^2\big)$ in the above inequality through the following steps.

$$\begin{aligned}
\sum_{t=0}^{T}\mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|}\eta\sigma^2 &\overset{(i)}{\leq} \sum_{t=0}^{T}\mathbb{E}\left(\frac{\sum_{i=1}^{m}\|v_{t-i}\|^2}{2m\sigma^2} + \frac{\epsilon}{24\sigma^2}\right)\eta\sigma^2 \\
&= \frac{\eta}{2m}\sum_{t=0}^{T}\sum_{i=1}^{m}\mathbb{E}\|v_{t-i}\|^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24} \\
&= \frac{\eta}{2m}\sum_{t=1}^{T}\sum_{i=1}^{\min\{m,t\}}\mathbb{E}\|v_{t-i}\|^2 + \frac{\eta}{2m}\sum_{t=0}^{\min\{m-1,T\}}\sum_{i=t+1}^{m}\mathbb{E}\|v_{t-i}\|^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24} \\
&\overset{(ii)}{\leq} \frac{\eta}{2m}\sum_{t=1}^{T}\sum_{i=1}^{\min\{m,t\}}\mathbb{E}\|v_{t-i}\|^2 + \frac{\eta}{2m}\sum_{t=0}^{m-1}\sum_{i=t+1}^{m}\mathbb{E}\|v_{-1}\|^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24} \\
&= \frac{\eta}{2m}\sum_{t=1}^{T}\sum_{i=1}^{\min\{m,t\}}\mathbb{E}\|v_{t-i}\|^2 + \frac{\eta m}{2}\|v_{-1}\|^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24} \\
&= \frac{\eta}{2m}\sum_{i=0}^{T-1}\mathbb{E}\|v_i\|^2\sum_{t=i+1}^{\min\{i+m,T\}}1 + \frac{\eta m}{2}\|v_{-1}\|^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24} \\
&\leq \frac{\eta}{2}\sum_{i=0}^{T}\mathbb{E}\|v_i\|^2 + \frac{\eta m}{2}\alpha_0^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24}
\end{aligned} \tag{60}$$

where (i) follows from the definition of $|B_t|$, (ii) follows from the fact that $\|v_{-1}\| = \|v_{-2}\| = \cdots = \|v_{-m}\| = \alpha_0$.

Plugging (60) into (59), we obtain

$$\sum_{t=0}^{T}\Big(\eta - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 \leq \mathbb{E}f(x_0) - \mathbb{E}f(x_{T+1}) + \frac{\eta}{2}\sum_{i=0}^{T}\mathbb{E}\|v_i\|^2 + \frac{\eta m}{2}\alpha_0^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24},$$

which further yields

$$\begin{aligned}
\sum_{t=0}^{T}\Big(\frac{\eta}{2} - \frac{L\eta^2}{2}\Big)\mathbb{E}\|v_t\|^2 &\leq \mathbb{E}f(x_0) - \mathbb{E}f(x_{T+1}) + \frac{\eta m}{2}\alpha_0^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24} \\
&\leq f(x_0) - f^* + \frac{\eta m}{2}\alpha_0^2 + \sum_{t=0}^{T}\frac{\eta\epsilon}{24}.
\end{aligned} \tag{61}$$

Recall that $\phi := \Big(\eta - \frac{L\eta^2}{2}\Big) > 0$. Then, we obtain from (61) that

$$\sum_{t=0}^{T}\mathbb{E}\|v_t\|^2 \leq \frac{2\left(f(x_0) - f^*\right) + \eta m\alpha_0^2}{2\phi} + \frac{(T+1)\eta\epsilon}{24\phi}. \tag{62}$$

Recall that the output x_ζ is chosen from $\{x_t\}_{t=0,\dots,T}$ uniformly at random. Then, based on (62), we have

$$\begin{aligned}\mathbb{E}\left\|\nabla f(x_\zeta)\right\|^2 &= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left\|\nabla f(x_t)\right\|^2 = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left\|\mathbb{E}_t v_t\right\|^2 \overset{(i)}{\leq} \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big(\mathbb{E}_t\left\|v_t\right\|^2\big)\\ &= \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left\|v_t\right\|^2 \overset{(ii)}{\leq} \frac{2\left(f(x_0)-f^*\right)+\eta m\alpha_0^2}{2T\phi} + \frac{(T+1)\eta\epsilon}{24T\phi}\\ &\leq \frac{2\left(f(x_0)-f^*\right)+\eta m\alpha_0^2}{2T\phi} + \frac{\eta\epsilon}{12\phi},\end{aligned}$$

where (i) follows from the Jensen's inequality, and (ii) follows from (62).

I.2. Proof of Corollary 5

Since $\eta = \frac{1}{2L}$, have

$$\phi = \left(\eta - \frac{L\eta^2}{2}\right) = \frac{1}{8L} > 0.$$

Then, plugging $\eta = \frac{1}{2L}, \phi = \frac{1}{8L}$ and $T = \left(16L\left(f(x_0)-f^*\right)+4m\alpha_0^2\right)\epsilon^{-1}$ in Theorem 7, we have

$$\mathbb{E}\left\|\nabla f(x_\zeta)\right\|^2 \leq \frac{8L\left(f(x_0)-f^*\right)+2m\alpha_0^2}{T} + \frac{\epsilon}{3} \leq \frac{5}{6}\epsilon \leq \epsilon.$$

Thus, the total SFO calls required by AbaSGD is given by

$$\sum_{t=0}^{T}|B_t| = \sum_{t=0}^{T}\min\left\{\frac{2\sigma^2}{\sum_{i=1}^{m}\left\|v_{t-i}\right\|^2/m}, \frac{24\sigma^2}{\epsilon}, n\right\} \leq (T+1)\left(\frac{24\sigma^2}{\epsilon}\wedge n\right) = \mathcal{O}\left(\frac{1}{\epsilon^2}\wedge\frac{n}{\epsilon}\right).$$