Supplementary Material for Better Depth-Width Trade-offs for Neural Networks through the lens of Dynamical Systems

Vaggos Chatziafratis [1] **Sai Ganesh Nagarajan** [2] **Ioannis Panageas** [2]

A. Proof of Claim 2

Proof. For $p = 3$, the desired equation holds, since the matrix $A^\top$ becomes just

$$A^\top = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

with characteristic polynomial $(\lambda - 1)\lambda - 1 = \lambda^2 - \lambda - 1$. Let I denote the identity matrix of size $(p-1) \times (p-1)$. Assume $p \geq 5$. We consider the matrix:

$$A^\top - \lambda I = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} := \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -\lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\lambda & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -\lambda \end{pmatrix}$, $A_{12} := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$,

$A_{21} := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$, and $A_{22} := \begin{pmatrix} -\lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & -\lambda & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda & 0 \\ 0 & 0 & 0 & \dots & 0 & -\lambda \end{pmatrix}$.

Observe that $\lambda = 0$ is not an eigenvalue of the matrix $A^\top$. Suppose that $A_{11}, A_{12}, A_{21}, A_{22}$ are the four block submatrices of the matrix above. Using Schur's complement, we get that $\det(A^\top - \lambda I) = \det(A_{22}) \times \det(A_{11} - A_{12}A_{22}^{-1}A_{21})$, where $\det(A_{22}) = (-\lambda)^{\frac{p-1}{2}}$ and

$$\lambda^{\frac{p-1}{2}}\det(A_{11} - A_{12}A_{22}^{-1}A_{21}) = \begin{pmatrix} \lambda-\lambda^2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -\lambda^2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\lambda^2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & -\lambda^2+1 \end{pmatrix}.$$

We can multiply the first row by $\frac{1}{\lambda(\lambda-1)}$, the second row by $\frac{1}{\lambda^2} + \frac{1}{\lambda^2\lambda(\lambda-1)}$, the third row by $\frac{1}{\lambda^2} + \frac{1}{\lambda^4} + \frac{1}{\lambda^4\lambda(\lambda-1)}, \dots$, the

[1]Department of Computer Science, Stanford University [2]Singapore University of Technology and Design. Correspondence to: Vaggos Chatziafratis <vaggos@cs.stanford.edu>, Sai Ganesh Nagarajan <sai_nagarajan@mymail.sutd.edu.sg>, Ioannis Panageas <ioannis@sutd.edu.sg>.

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i-th row by $\sum_{j=1}^{i-1}\frac{1}{\lambda^{2j}}+\frac{1}{\lambda^{2(i-1)}\cdot\lambda(\lambda-1)}$ (and so on) and add them to the last row. Let B be the resulting matrix:

$$B=\begin{pmatrix} \lambda-\lambda^2 & 1 & 0 & 0 & 0 & \dots & 0\\ 0 & -\lambda^2 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & -\lambda^2 & 1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 0 & K \end{pmatrix},$$

where $K=-\lambda^2+1+\sum_{j=1}^{\frac{p-5}{2}}\frac{1}{\lambda^{2j}}+\frac{1}{\lambda^{p-5}\cdot\lambda(\lambda-1)}$. It is clear that the equation $\det(B)=0$ has the same roots as $\det(A^\top-\lambda I)=0$. Since B is an upper triangular matrix, it follows that

$$\det(B)=(-1)^{\frac{p-5}{2}}\lambda(\lambda-1)\lambda^{p-5}\cdot\left(-\lambda^2+1+\sum_{j=1}^{\frac{p-5}{2}}\frac{1}{\lambda^{2j}}+\frac{1}{\lambda^{p-5}\cdot\lambda(\lambda-1)}\right).$$

We conclude that the eigenvalues of $A^\top$ (and hence of A) must be roots of

$$(\lambda^{p-3}-\lambda^{p-4})\left(1-\lambda^2+\sum_{j=1}^{\frac{p-5}{2}}\frac{1}{\lambda^{2j}}\right)+1=-\lambda^{p-1}+\lambda^{p-2}+\lambda^{p-3}-\lambda^{p-4}+\sum_{j=1}^{\frac{p-5}{2}}\lambda^{p-3-2j}-\lambda^{p-4-2j}+1$$
$$=-\lambda^{p-1}+\lambda^{p-2}+\sum_{j=0}^{p-3}(-1)^j\lambda^j=\frac{-\lambda^p+\lambda^{p-2}}{\lambda+1}+\frac{1+\lambda^{p-2}}{\lambda+1}=\frac{-\lambda^p+2\lambda^{p-2}+1}{\lambda+1},$$

and the claim follows. □

B. Proof of Corollary 3.5

Proof. We first need to relate the spectral radius with the number of oscillations. We follow the idea from (Chatziafratis et al., 2020) which concludes that $\delta_0^t\geq\|A^t\|_\infty\geq\mathrm{spec}(A^t)=\mathrm{spec}(A)^t=\rho_p^t$ (where $\mathrm{spec}(A)$ denotes the spectral radius), that is the growth rate of the number of oscillations of compositions of f is at least ρ_p.

Assume $1<p$ be an odd number. It suffices to show that $\rho_{p+2}<\rho_p$ (and then use induction). Observe that $\lambda^{p+2}-2\lambda^p-1=\lambda^2(\lambda^p-2\lambda^{p-2}-1)+\lambda^2-1$. Therefore

$$0=q_{p+2}(\rho_{p+2})=\rho_{p+2}^2q_p(\rho_{p+2})+\rho_{p+2}^2-1,$$

hence since $\rho_{p+2}>1$ we conclude that $q_p(\rho_{p+2})<0$. Since $\lim_{\lambda\to\infty}q_p(\lambda)=+\infty$, by Bolzano's theorem it follows that q_p has a root in the interval $(\rho_{p+2},+\infty)$. Thus $\rho_p>\rho_{p+2}$. One can also see that $\sqrt{2}^p-2\sqrt{2}^{p-2}-1=-1<0$ and $2^p-2\cdot2^{p-2}-1>0$, thus from Bolzano's again, it follows that $\rho_p>\sqrt{2}$ for all p. □

C. Proof of Lemma 3.6

Proof. It suffices to show that f has period p (the Lipschitz constant is trivially ρ_p). We start from $z_0=0$ and we get $z_t=f(z_{t-1})=\rho_p|z_{t-1}|-1$ for $1\leq t\leq p$. Observe that $z_1=-1$, $z_2=\rho_p-1>0$. Set $q_i(\lambda)=\frac{\lambda^i-2\lambda^{i-2}-1}{\lambda+1}$. First, we shall show that for $t\in\{3,\dots,p-1\}$, we have $z_t\leq0$ and that $z_t=q_t(\rho_p)$, whereas for t even, we have $z_t=-q_{t-1}(\rho_p)\rho_p-1$ in the interval above.

For $t=3$ we get that $z_3=\rho_p^2-\rho_p-1=q_3(\rho_p)\leq0$ because we showed ρ_p is decreasing in p and moreover holds $q_3(\rho_3)=0$. Since $z_3\leq0$ we get that $z_4=-\rho_pz_3-1=q_3(\rho_p)\rho_p-1$. Let us show that $z_4\leq0$. Observe that $z_4=-\rho_p^3+\rho_p^2+\rho_p-1=(\rho_p-1)(1-\rho_p^2)<0$ (since $\rho_p>\sqrt{2}$).

We will use induction. Assume now, that we have the result for some t even, we need to show that $z_{t+1}=q_{t+1}(\rho_p)$, $z_{t+2}=-q_{t+1}(\rho_p)\rho_p-1$ and moreover $z_{t+1},z_{t+2}\leq0$.

By induction, we have that $z_{t-1},z_t\leq0$ and $z_t=-q_{t-1}(\rho_p)\rho_p-1$, hence $z_{t+1}=-\rho_p(-q_{t-1}(\rho_p)\rho_p-1)-1=\frac{\rho_p^{t+1}-2\rho_p^t-\rho_p^2}{\rho_p+1}+\rho_p-1=q_{t+1}(\rho_p)$. Since ρ_p is decreasing in p and $q_{t+1}(\rho_{t+1})=0$, we conclude that $z_{t+1}\leq0$. Since

$z_{t+1} \leq 0$, we get that $z_{t+2} = -\rho_p z_{t+1} - 1 = -\rho_p q_{t+1}(\rho_p) - 1$. To finish the claim, it suffices to show that $z_{t+2} \leq 0$. Observe that

$$\begin{aligned} -\rho_p q_{t+1}(\rho_p) - 1 &= -\rho_p \left(\rho_p^t - \rho_p^{t-1} - \sum_{j=0}^{t-2} (-\rho_p)^j \right) - 1 \\ &= -\rho_p^{t+1} + \rho_p^t - \sum_{j=1}^{t-1} (-\rho_p)^j - 1 \\ &= -2\rho_p^{t+1} + 2\rho_p^t + \frac{q_{t+1}(\rho_p)}{\rho_p + 1}. \end{aligned}$$

The term $-2(\rho_p^{t+1} - \rho_p^t) < 0$ (since $\rho_p > 1$) and moreover $\frac{q_{t+1}(\rho_p)}{\rho_p+1} \leq 0$ because ρ_p is decreasing in p and $t + 1 \leq p - 1$. Hence $z_{t+2} \leq 0$ and the induction is complete.

From the above, we conclude that $z_p = -\rho_p z_{p-1} - 1 = q_p(\rho_p) = 0$, thus $z_0, ..., z_{p-1}$ form a cycle. If we show that $z_0, ..., z_{p-1}$ are distinct, the proof of the lemma follows.

First observe that $q_t(\lambda) = \frac{\lambda^t - 2\lambda^{t-2} - 1}{\lambda+1}$ is strictly increasing in t as long as $\lambda > \sqrt{2}$ (by computing the derivative). Therefore it holds that $z_3 < z_5 < \ldots < z_p = 0$ (for all the odd indices) and also $z_1 < z_3$. Furthermore, $-\lambda q_t(\lambda) - 1$ is decreasing in t for $\lambda > \sqrt{2}$, therefore we conclude $z_4 > \ldots > z_{p-1}$ (and also $z_2 > 0 \geq z_4$).

We will show that $z_3 > z_4$ and finally $z_{p-1} > -1 = z_1$ and the lemma will follow. Recall $z_3 = \rho_p^2 - \rho_p - 1$ and $z_4 = -\rho_p^3 + \rho_p^2 + \rho_p - 1$. Equivalently, we need to show that $\rho_p^2 - \rho_p - 1 > -\rho_p^3 + \rho_p^2 + \rho_p - 1$ or $\rho_p^3 - 2\rho_p > 0$ which holds because $\rho_p > \sqrt{2}$. Finally $z_{p-1} = -\rho_p z_{p-2} - 1 > -1$ since $z_{p-2} < z_p = 0$.

□

D. Sensitivity to Lipschitzness and separation examples based on periods

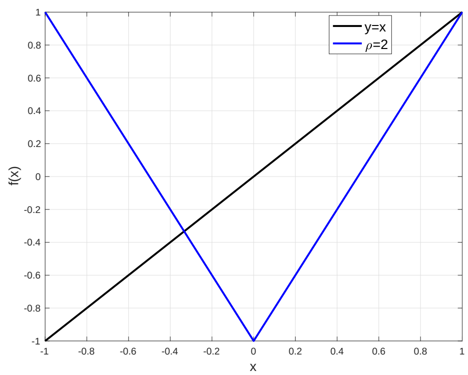
We consider three regimes. The first regime corresponds to the functions that appear in Lemma 3.2, where $L = \rho_p$ and $\rho_p \in [\sqrt{2}, \phi]$, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. The second regime corresponds to the case when $L > \phi$ and the third regime corresponds to the case when $L < \sqrt{2}$. We can see in Figure 1 that the function $f(x) := 2|x| - 1$ has period 3 and a Lipschitz constant of $L = 2$, while in Figure 2, we can see that the function $f(x) := 1.2|x| - 1$, does not have any odd period and $L = 1.2$.

Figure 1 and Figure 2 correspond to cases where the Lipschitz constant of the function does not match ρ_p.

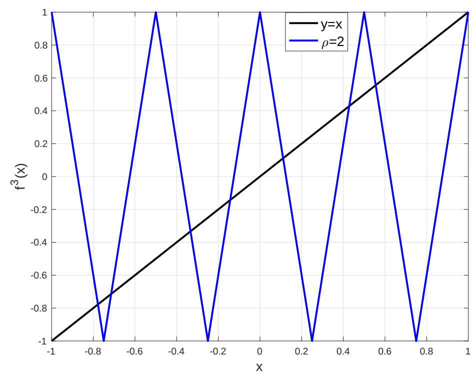
- When $\sqrt{2} \leq L \leq \phi$, we see from Figure 3, how small differences in the values of the slope can lead to the existence of different (prime) periods, which consequently lead to different depth-width trade-offs.
- When $L > \phi$, we can see from Figure 1 that $L = 2$ and also the growth rate of oscillations is 2. This means that $L = \rho$ and that L^1 separation is achievable. Note that period 3 is present in the tent map, so $\rho_3 = \phi$ for this case.
- When $L < \sqrt{2}$, we can see from Figure 2 that the oscillations do not grow exponentially with compositions and that the existing ones are of small magnitude, which means that the L^1 error can be made arbitrarily small. Observe here that no odd period is present in the function (as this would imply that $L \geq \rho \geq \sqrt{2}$).

References

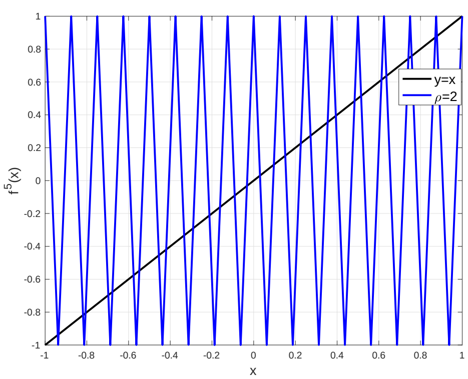
Chatziafratis, V., Nagarajan, S. G., Panageas, I., and Wang, X. Depth-width trade-offs for relu networks via sharkovsky's theorem. *International Conference on Learning Representations, Addis Ababa, Africa*, 2020.



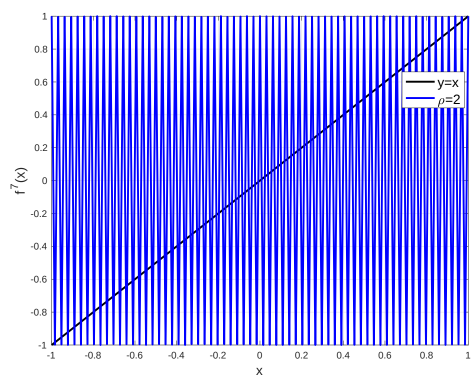
(a) Graph of $f(x)$ intersected with $y = x$, to identify period 1 points.



(b) Graph of $f^3(x)$ intersected with $y = x$, to identify period 3 points.

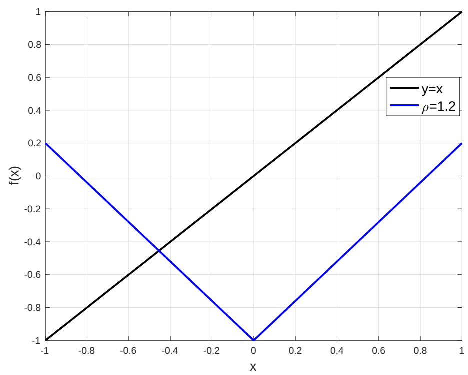


(c) Graph of $f^5(x)$ intersected with $y = x$, to identify period 5 points.

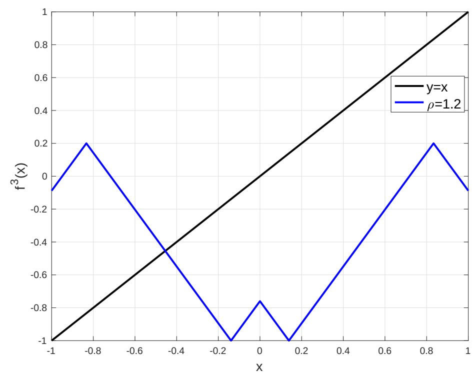


(d) Graph of $f^7(x)$ intersected with $y = x$, to identify period 7 points.

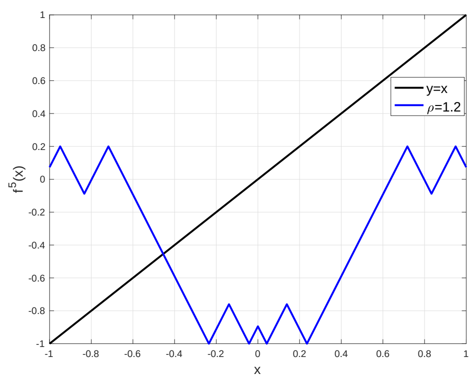
Figure 1. Here $L = 2$, and this function has period 3. However, the growth rate of oscillations is exactly 2 and since we have equality $L = \rho$ we get L^1 separations even though the largest root $\rho_3 = \phi < 2$.



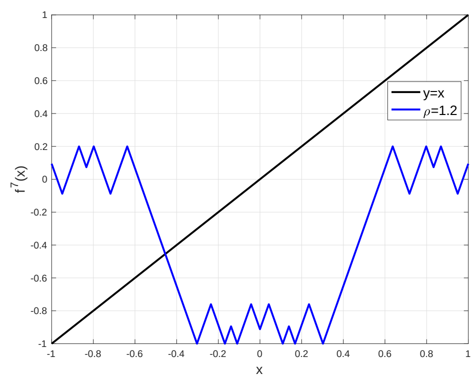
(a) Graph of $f(x)$ intersected with $y = x$, to identify period 1 points.



(b) Graph of $f^3(x)$ intersected with $y = x$, to identify period 3 points.

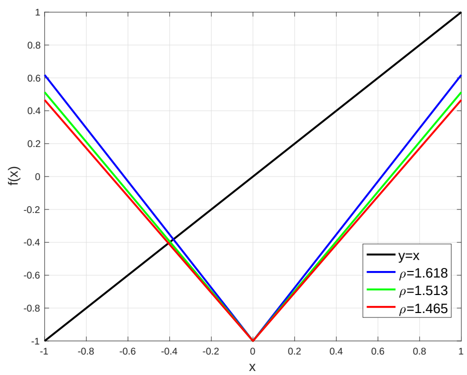


(c) Graph of $f^5(x)$ intersected with $y = x$, to identify period 5 points.

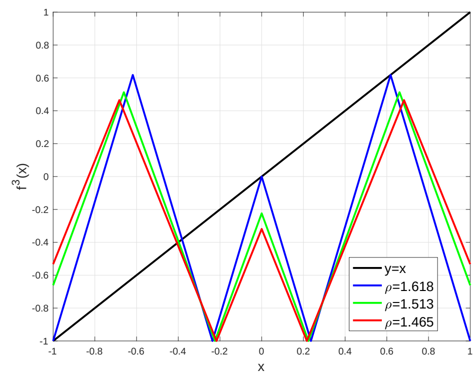


(d) Graph of $f^7(x)$ intersected with $y = x$, to identify period 7 points.

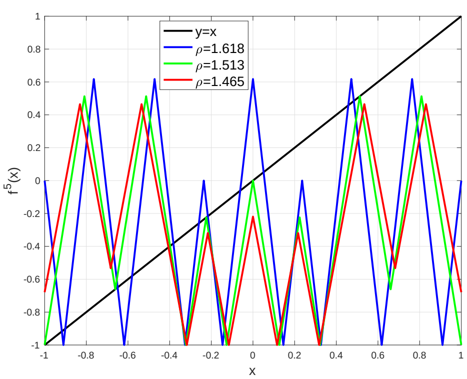
Figure 2. Here $L = 1.2$ that corresponds to the regime where $L < \sqrt{2}$. It follows that this function cannot have any odd period (because then $L \geq \rho \geq \sqrt{2}$). Observe that the oscillations do not grow exponentially fast and they shrink in area, hence no L^1 separation is achievable.



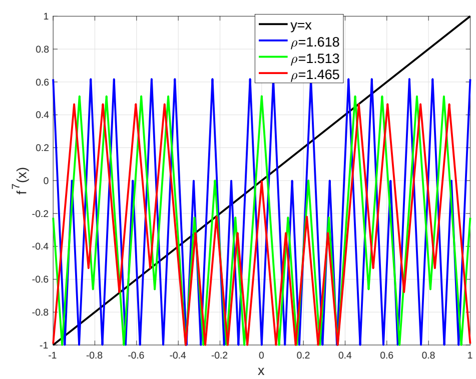
(a) Graph of $f(x)$ is shown. The regime $\sqrt{2} \leq L \leq \phi$ with small slope variations.



(b) Graph of $f^3(x)$. When $L = \phi$, period 3 is present (trade-offs with base ϕ).



(c) Graph of $f^5(x)$. When $L = 1.513$, period 5 is present (trade-offs with base 1.513).



(d) Graph of $f^7(x)$. When $L = 1.465$, period 7 is present (trade-offs with base 1.465).

Figure 3. Functions parameterized by ρ_p for $L = \rho_p$ and $\rho = 1.618, 1.513, 1.465$ with periods 3, 5 and 7 respectively (see intersection with $y = x$). Slight changes lead to different trade-offs.