Fiber bundles and non-abelian cohomology

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Abstract

The transition maps of a fiber bundle are often said to satisfy the "cocycle condition." If we take this terminology seriously we are led to consider cohomology with coefficients in a non-abelian group. The resulting long exact sequence makes the first and second Stiefel-Whitney classes (and their interpretations in terms of orientability and spin structures) totally transparent. We will also muse briefly about the Eilenberg–Mac Lane space $K(G, 1)$, the classifying space BG, and the derived functor Ext.

1 Definition and examples

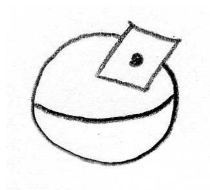
A *smooth fiber bundle* consists of smooth manifolds B, E, and F, called the *base space*, the *total* or *entire space*, and the *fiber*, respectively, and a smooth map $\pi : E \to B$, called the *projection*, that looks locally like the projection $B \times F \to B$ in the following sense: there is an open cover $\{U_i\}$ of B and diffeomorphisms $\varphi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \varphi_i : U_i \times F \to U_i$ is projection onto the first factor. We write

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \pi \\ & & B. \end{array}$$

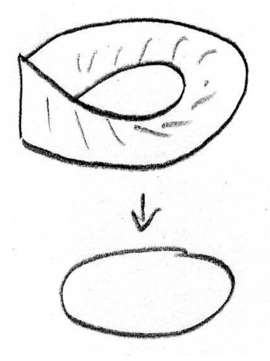
Here are some examples:

- Any vector bundle, e.g. the tangent bundle to the 2-sphere

$$\begin{array}{ccc} \mathbb{R}^2 & \hookrightarrow & TS^2 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$



or the Möbius bundle



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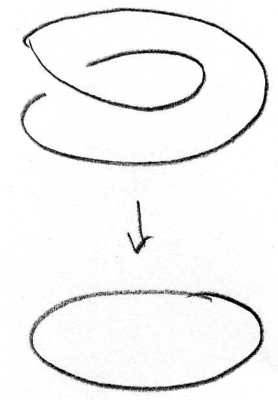
whose fiber is $\mathbb{R}$.

- Any covering space, e.g. the double cover of S^1



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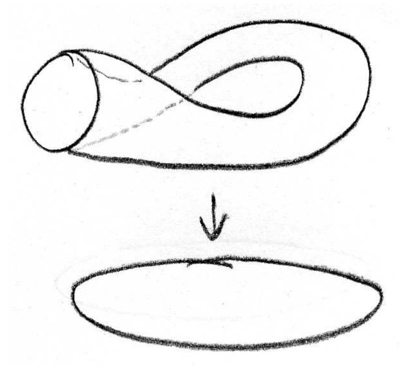
whose fiber is a two-point discrete space.

- The Klein bottle



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whose fiber is S^1.

2 Short exact sequences

According to Steenrod, we should think of a fiber bundle as a product that lost one of its projections. Recall that a product has two projections

$$\begin{array}{ccc} B \times F & \longrightarrow & F \\ \downarrow & & \\ B. & & \end{array}$$

Today, we would like to think of a fiber bundle as a short exact sequence of spaces. Here are some reasons:

- A short exact sequence of abelian groups is like a product that lost one of its projections. The simplest short exact sequence is a product:

  $$0 \longrightarrow A \longrightarrow A \times C \longrightarrow C \longrightarrow 0.$$

  A general short exact sequence

  $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

  has one projection $B \to C$, and if there is a second projection $B \to A$, i.e. a splitting

  $$0 \longrightarrow A \rightleftarrows B \longrightarrow C \longrightarrow 0,$$

  then $B \cong A \times C$.

- A short exact sequence of Lie groups *is* a fiber bundle. If

  $$1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

  is a short exact sequence of Lie groups, then G is a bundle over H with fiber K:

  $$\begin{array}{ccc} K & \hookrightarrow & G \\ & & \downarrow \pi \\ & & H. \end{array}$$

  This is the only place we really need to be working with manifolds—the same statement is not quite true of topological groups.

- From a fiber bundle

  $$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \pi \\ & & B \end{array}$$

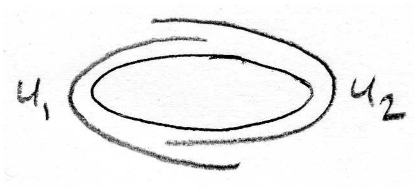
  we get a long exact sequence of homotopy groups

  $$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B)$$
  $$\longrightarrow \pi_{i-1}(F) \longrightarrow \pi_{i-1}(E) \longrightarrow \pi_{i-1}(B) \longrightarrow \cdots,$$

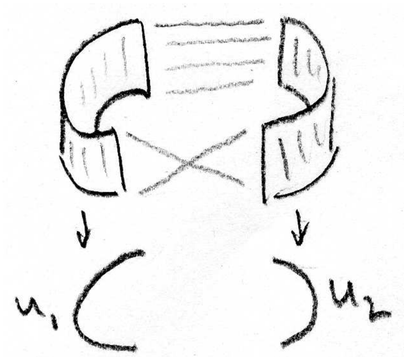
  and long exact sequences often arise from short exact sequences.

3 Transition maps

Recall that we have an open cover $\{U_i\}$ of B and local trivializations $\pi^{-1}(U_i) \cong U_i \times F$. For the Möbius bundle, the open cover is



and the local trivializations look like



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If $p \in U_i \cap U_j$, we have two ways to identify the fiber $\pi^{-1}(p)$ with F: one over U_i and one over U_j. Thus for each p we get an automorphism of F, so there is a map $\psi_{ij} : U_i \cap U_j \to \text{Diffeo}(F)$, called a *transition map*. Transition maps tell us how to glue the $U_i \times F$ together to make E.

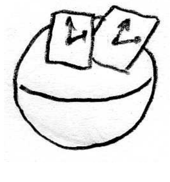
If $p \in U_i \cap U_j \cap U_k$, then

$$\psi_{jk}(p)\psi_{ij}(p) = \psi_{ik}(p).$$

This is called the *cocycle condition*, which we will examine in a moment.

Our transition maps probably don't need to take values in the whole diffeomorphism group of F—if we choose our trivializations carefully, we may be able to make do with a smaller group:

- For a vector bundle, we only need the general linear group $\text{GL}_n(\mathbb{R})$.
- For an oriented vector bundle, we only need $\text{GL}_n^+(\mathbb{R}) = \{A \in \text{GL}_n(\mathbb{R}) : \det A > 0\}$. A local trivialization of a vector bundle is the same as a moving frame

  and if we make all our frames positively oriented, our transition maps will all have positive determinant.

- For a vector bundle with a Riemannian metric, we can make our frames orthonormal using the Gram-Schmidt process, so our transition maps will take values in the orthogonal group $\mathrm{O}(n)$.
- For the Möbius bundle, we only need $\mathbb{Z}/2$.

4 Cohomology

Now we have an open cover $\{U_i\}$ of B and maps $\psi_{ij} : U_i \cap U_j \to G \subset \mathrm{Diffeo}(F)$ which satisfy the "cocycle condition." If there are cocycles, there should be cochain groups, coboundary maps, and a whole cohomology theory hanging around somewhere. Since we are dealing with an open cover, pairwise intersections $U_i \cap U_j$, and triple intersections $U_i \cap U_j \cap U_k$, it should be Čech cohomology. Let's try to define $\check{H}^*(B; G)$ using the usual Čech recipe.

First we define the cochain groups:

- A 0-cochain φ consists of smooth maps $\varphi_i : U_i \to G$ for all i.
- A 1-cochain ψ consists of smooth maps $\psi_{ij} : U_i \cap U_j \to G$ for all i, j.
- A 2-cochain ξ consists of smooth maps $\xi_{ijk} : U_i \cap U_j \cap U_k \to G$ for all i, j, k.

These form groups under pointwise multiplication. Next we define the coboundary maps d:

- If φ is a 0-cochain, $d\varphi$ is the 1-cochain given by $(d\varphi)_{ij}(p) = \varphi_j(p)\varphi_i(p)^{-1}$.
- If ψ is a 1-cochain, $d\psi$ is the 2-cochain given by $(d\psi)_{ijk}(p) = \psi_{ij}(p)\psi_{ik}(p)^{-1}\psi_{jk}(p)$.

But if G is not abelian, these are not group homomorphisms, $\ker d$ and $\operatorname{im} d$ are not subgroups, and if we went on to 2-cochains, we would not be able to get $d^2 = 1$. Let's see what we can do despite these problems.

For $\check{H}^0$, there is no trouble. If $d\varphi = 1$ then for all $p \in U_i \cap U_j$, $\varphi_i(p) = \varphi_j(p)$, so the partial functions $\varphi_i : U_i \to G$ patch together to make a global function $B \to G$. Thus

$$\check{H}^0(B; G) = C^\infty(B, G)$$

which is a group under pointwise multiplication.

For $\check{H}^1$, neither the 1-cocycles nor the 1-coboundaries form a group, but the group of 0-cochains acts on the set 1-cocycles by

$$(\varphi \cdot \psi)_{ij}(p) = \varphi_j(p)\psi_{ij}(p)\varphi_i(p)^{-1}.$$

To get $\check{H}^1(B; G)$, take the quotient of this set by this action (this should be viewed as taking cocycles modulo coboundaries) and take the direct limit over all open covers. While $\check{H}^1(B; G)$ is not a group, it does have a distinguished element, represented by the trivial cocycle $\psi_{ij} \equiv 1$.

From a fiber bundle with structure group G, we get transition maps, hence a class in $\check{H}^1(B; G)$. But several bundles may have the same transition maps, as for example the Möbius bundle, the double cover of the circle, and the Klein bottle do. We can eliminate this ambiguity by choosing the principal G-bundle, where the fiber F is the group G, and G acts on itself by left multiplication. Of the three bundles just mentioned, the double cover is the principal $\mathbb{Z}/2$-bundle. Conversely, a cocycle tells us how to construct a principal G-bundle over B, and if two cocycles represent the same class in $\check{H}^1(B; G)$ then the resulting bundles are isomorphic. Thus

$$\check{H}^1(B; G) = \{ \text{ isomorphism classes of principal } G\text{-bundles over } B \ \}.$$

We cannot define $\check{H}^2$ and higher unless G is abelian. This is reminiscent of the situation with relative homotopy groups, where $\pi_3(X, A)$ and higher are abelian, π_2 is a group but not necessarily abelian, π_1 is a pointed set, and π_0 is not defined.

How does our cohomology theory relate to singular cohomology? If G is abelian then singular cohomology $H^*_{\text{sing}}(B; G)$ agrees with Čech cohomology $\check{H}^*(B; \mathcal{G})$, where $\mathcal{G}$ is the sheaf of locally constant functions $B \to G$. We are working with smooth functions, not locally constant functions, but if G is discrete (e.g. $\mathbb{Z}$ or $\mathbb{Z}/2$) then these are same, so our $\check{H}^*(B; G)$ agrees with singular cohomology. But note that our $\check{H}^1(B; \mathbb{R}) = 0$ for all B, unless we give $\mathbb{R}$ the discrete topology.

5 Aside: representable functors

We have defined $\check{H}^1(B; G)$ following the Čech recipe and found that it classifies principal G-bundles over B. This should not come as a great surprise. If G is abelian,

$$H^1_{\text{sing}}(B; G) = \{ \text{ homotopy classes of maps } B \to K(G, 1) \ \}$$

where $K(G, 1)$ is an Eilenberg–Mac Lane space, i.e. its fundamental group is G and all its higher homotopy groups vanish. On the other hand, for any G,

$$\{ \text{ principal } G\text{-bundles over } B \ \} = \{ \text{ homotopy classes of maps } B \to BG \ \}$$

where BG is the classifying space of G. But if you're like me, you can never keep $K(G, 1)$ and BG straight, because you build them the same way: you start with a basepoint, add a loop for every $g \in G$, glue in a triangle with edges g, h, and k whenever $gh = k$, etc. This is because BG is a $K(G, 1)$ if G is discrete.

6 Aside: derived functors

We are working with Čech cohomology, which in its natural habitat is a sheaf cohomology theory. Grothendieck teaches us that sheaf cohomology can also be defined as the derived functors of global sections $\Gamma = H^0$. For our sheaf, $H^0(B; G)$ is $C^\infty(B, G)$, which is $\text{Hom}(B, G)$ in the category of smooth manifolds. In the category of abelian groups, the derived functor of Hom is Ext, and $\text{Ext}(C, A)$ classifies 1-step extensions of C by A, that is, short exact sequences

$$0 \to A \to B \to C \to 0.$$

For us, $H^1(B; G)$ classifies principal G-bundles over B, but earlier we said that we should consider

$$\begin{array}{ccc} G & \hookrightarrow & E \\ & & \big\downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

as a short exact sequence of spaces.

7 Induced maps

If $f : B' \to B$ is a smooth map and $\alpha : G \to H$ is a group homomorphism, we get induced maps $f^* : \check{H}^*(B; G) \to \check{H}^*(B'; G)$ and $\alpha_* : \check{H}^*(B; G) \to \check{H}^*(B; H)$ in the usual way: if a class $\psi \in \check{H}^1(B; G)$ is represented over an open cover $\{U_i\}$ of B by maps $\psi_{ij} : U_i \cap U_j \to G$, then $\{f^{-1}(U_i)\}$ is an open cover of B' and we let $(f^*\psi)_{ij} = \psi_{ij} \circ f$ and $(\alpha_*\psi)_{ij} = \alpha \circ \psi_{ij}$.

What are these induced maps in terms of bundles? If a class $\psi \in \check{H}^1(B; G)$ corresponds to a principal G-bundle $E \to B$ then $f^*\psi$ corresponds to the pullback bundle $f^*E \to B'$. The class $\alpha_*\psi$ corresponds to the associated bundle $E \times_G H$: we swap out the fibers G with H, letting the transition maps act on H by $g \cdot h = \alpha(g)h$, and because α is a group homomorphism, this is a principal H-bundle.

Let us emphasize that we are not defining f^* and α_* in terms of bundles—we just do what Čech cohomology obliges us to do, and the bundle interpretation comes for free.

From a short exact sequence of coefficient groups

$$1 \to K \xrightarrow{i} G \xrightarrow{\pi} H \to 1,$$

we get a long exact sequence in cohomology

$$\begin{aligned} 1 \to \check{H}^0(B;K) &\xrightarrow{i_*} \check{H}^0(B;G) \xrightarrow{\pi_*} \check{H}^0(B;H) \\ \xrightarrow{\delta} \check{H}^1(B;K) &\xrightarrow{i_*} \check{H}^1(B;G) \xrightarrow{\pi_*} \check{H}^1(B;H). \end{aligned}$$

Even though $\check{H}^1$ is not a group, it is a pointed set, so it still makes sense to talk about kernels, and thus exactness. The usual Čech proof goes through with no trouble, but we can also give a fun and easy proof by interpreting $\check{H}^0(B; G)$ as smooth maps and $\check{H}^1(B; G)$ as bundles. To give the flavor, let us construct the connecting homomorphism $\delta : \check{H}^0(B; H) \to \check{H}^1(B; K)$, which in most cohomology theories is a chore. Since $1 \to K \to G \to H \to 1$ is a short exact sequence of Lie groups, G is a principal K-bundle over H, hence determines a class $\eta \in \check{H}^1(H; K)$. If $f \in \check{H}^0(B; H)$ then f is a smooth map $B \to H$, hence induces a map $f^* : \check{H}^1(H; K) \to \check{H}^1(B; K)$. We define $\delta f = f^*\eta$.

If K is abelian then $\check{H}^2(B; K)$ is defined, and if $i(K)$ is contained in the center of G then we can continue the exact sequence

$$\begin{aligned} 1 \to \check{H}^0(B;K) &\xrightarrow{i_*} \check{H}^0(B;G) \xrightarrow{\pi_*} \check{H}^0(B;H) \\ \xrightarrow{\delta} \check{H}^1(B;K) &\xrightarrow{i_*} \check{H}^1(B;G) \xrightarrow{\pi_*} \check{H}^1(B;H) \\ \xrightarrow{\delta} \check{H}^2(B;K). & \end{aligned}$$

Unfortunately there is no slick construction of this second connecting homomorphism.

If we wanted to go on to $\check{H}^2(B; G)$ we would need G to be abelian, so H would also be abelian and the long exact sequence would go on forever, but that story is well-known.

8 Stiefel-Whitney classes

We conclude with an application. If E is a vector bundle over B, Milnor defines the Stiefel-Whitney classes $w_i(E) \in H^i_{\text{sing}}(B; \mathbb{Z}/2) = \check{H}^i(B; \mathbb{Z}/2)$ by the following axioms:

- Dimension: $w_i(E) = 0$ for $i > \operatorname{rank} E$.

- Naturality: If $f : B' \to B$ then $w_i(f^*E) = f^*w_i(E)$.
- Direct Sum: If E' is another vector bundle over B then

$$w(E \oplus E') = w(E) \smile w(E')$$

  where $w(E) = 1 + w_1(E) + w_2(E) + w_3(E) + \cdots$.
- Normalization: If $B = S^1$ and E is the Möbius bundle then $w_1(E)$ is the non-trivial element of $\check{H}^1(S^1; \mathbb{Z}/2) = \mathbb{Z}/2$.

This definition is, of course, completely opaque; Milnor only proves that such classes exist fifty pages later. Let's use our cohomology theory to get a more concrete understanding of w_1 and w_2.

The first Stiefel-Whitney class $w_1(E)$ lives in $\check{H}^1(B; \mathbb{Z}/2)$, and vector bundles of rank n live in $\check{H}^1(B; \mathrm{GL}_n(\mathbb{R}))$, so w_1 is a map $\check{H}^1(B; \mathrm{GL}_n(\mathbb{R})) \to \check{H}^1(B; \mathbb{Z}/2)$. We would like it to be induced by a group homomorphism $\mathrm{GL}_n(\mathbb{R}) \to \mathbb{Z}/2$. The first such map that comes to mind is the sign of the determinant. In fact, one can check that $(\mathrm{sign} \circ \det)_* : \check{H}^1(B; \mathrm{GL}_n(\mathbb{R})) \to \check{H}^1(B; \mathbb{Z}/2)$ satisfies Milnor's axioms.

The kernel of $\mathrm{sign} \circ \det$ is $\mathrm{GL}_n^+(\mathbb{R})$, so we have a short exact sequence

$$1 \to \mathrm{GL}_n^+(\mathbb{R}) \xrightarrow{i} \mathrm{GL}_n(\mathbb{R}) \xrightarrow{\mathrm{sign} \circ \det} \mathbb{Z}/2 \to 1.$$

A piece of the resulting long exact sequence is

$$\check{H}^1(B; \mathrm{GL}_n^+(\mathbb{R})) \xrightarrow{i_*} \check{H}^1(B; \mathrm{GL}_n(\mathbb{R})) \xrightarrow{w_1} \check{H}^1(B; \mathbb{Z}/2),$$

that is,

$$\{\text{oriented vector bundles over } B\} \xrightarrow{i_*} \{\text{vector bundles over } B\} \xrightarrow{w_1} \check{H}^1(B; \mathbb{Z}/2).$$

The first map i_* just forgets the orientation. The image of i_* is the kernel of w_1, which is to say that $w_1(E) = 0$ if and only if E is orientable.

Elements of $\check{H}^1(B; \mathbb{Z}/2)$ correspond to principal $\mathbb{Z}/2$-bundles over B, i.e. double covers of B, so we can also interpret w_1 as a map

$$\{\text{vector bundles over } B\} \xrightarrow{w_1} \{\text{double covers of } B\}.$$

If E is tangent bundle of B then $w_1(E)$ is the orientation cover of B. An orientation of B is a section of this cover, and a principal bundle has a global section if and only if it is trivial.

Another piece of the long exact sequence is

$$\check{H}^0(B; \mathbb{Z}/2) \xrightarrow{\delta} \check{H}^1(B; \mathrm{GL}_n^+(\mathbb{R})) \xrightarrow{i_*} \check{H}^1(B; \mathrm{GL}_n(\mathbb{R})),$$

that is,

$$\{\text{smooth maps } B \to \mathbb{Z}/2\} \xrightarrow{\delta} \{\text{oriented vector bundles over } B\} \xrightarrow{i_*} \{\text{vector bundles over } B\}.$$

Exactness means that given an orientable vector bundle, i.e. an element of $\operatorname{im} i_*$, the possible orientations on it are parametrized by the smooth maps $B \to \mathbb{Z}/2$. A smooth map $B \to \mathbb{Z}/2$ is just a choice of ± 1 for each connected component of B.

To interpret the second Stiefel-Whitney class, let's assume that our vector bundle $E \to B$ is oriented and has a Riemannian metric, so $w_2 : \check{H}^1(B; \mathrm{SO}(n)) \to \check{H}^2(B; \mathbb{Z}/2)$. If we are to understand this in terms of the long exact sequence, w_2 will have to be the connecting homomorphism of a central extension

$$1 \to \mathbb{Z}/2 \to\ ?\ \to \mathrm{SO}(n) \to 1.$$

If $n > 2$ then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$, and its universal covering group is called $\mathrm{Spin}(n)$. Now from the short exact sequence of coefficent groups

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1$$

we get a long exact sequence in cohomology

$$\check{H}^1(B; \mathbb{Z}/2) \to \check{H}^1(B; \mathrm{Spin}(n)) \to \check{H}^1(B; \mathrm{SO}(n)) \xrightarrow{w_2} \check{H}^2(B; \mathbb{Z}/2)$$

and one checks that the connecting homomorphism satisfies Milnor's axioms—the hard part is the direct sum axiom. Thus $w_2(E) = 0$ if and only if E admits a spin structure, and the inequivalent spin structures on E are parametrized by $\check{H}^1(B; \mathbb{Z}/2)$. Spin structures are important in physics; they are what you need to write the Dirac equation.

We didn't really need to assume that E was oriented—we could have used the double cover $\mathrm{Pin}(n)$ of $\mathrm{O}(n)$, or if we didn't want to choose a Riemannian metric, the double cover of $\mathrm{GL}_n(\mathbb{R})$.