Representations of the Symmetry Group of Spacetime

Kyle Drake, Michael Feinberg, David Guild, Emma Turetsky

March 11, 2009

Abstract

The Poincaré group consists of the Lorentz isometries combined with Minkowski spacetime translations. Its connected double cover, $\mathrm{SL}(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$, is the symmetry group of spacetime. In 1939 Eugene Wigner discovered a stunning correspondence between the elementary particles and the irreducible representations of this double cover. We will classify these representations and explain their relationship to physical phenomena such as spin.

1 Introduction

Minkowski spacetime is the mathematical model of flat (gravity-less) space and time. The transformations on this space are the Lorentz transformations, known as O(1,3). The identity component of O(1,3) is $SO^+(1,3)$. The component $SO^+(1,3)$ taken with the translations $\mathbb{R}^{1,3}$ is the Poincaré group, which is the symmetry group of Minkowski spacetime. However, physical experiments show that a connected double cover of the Poincaré is more appropriate in creating the symmetry group actual spacetime.

Why the double cover of the Poincaré group? What is so special about this group in particular that it describes the particles? The answer lies in the way the group is defined. Simply, it is the set of all transformations that preserve the Minkowski metric of Minkowski spacetime. In other words, it is the set of all transformations of spacetime that preserve the speed of light.

SL(2,$\mathbb{C}$) is the connected double cover of the identity component of $SO^+(1,3)$. This is the symmetry group of spacetime: the double cover of the Poincaré group, SL(2,$\mathbb{C}$) $\ltimes \mathbb{R}^{1,3}$.

In 1939, Eugene Wigner classified the fundamental particles using the irreducible representations of the double cover of the Poincaré group. Wigner was motivated by the idea that symmetry underlies all physical laws. In particular, a physical experiment should come up with the same results regardless of where, when, or what orientation the experiment is done in. An experiment's results should also be invariant whether the experiment is at rest or moving at a constant velocity. It turns out the symmetries of physics go further than this. It is common to combine several systems together such as when protons and electrons combine to form atoms. When this is done, the overall symmetry of the system should be related to the individual symmetries of its components. These components are the irreducible representations [3].

The double cover of the Poincaré group acts to classify the fundamental particles in physics and explain patterns in their behaviors. In particular, the particles are most easily classified by the irreducible representations of the double cover of the Poincaré group. The representations are determined by the different orbits in the group. These orbits serve to classify types of particles.

The first two types of orbits correspond to the value $m^2 > 0$. The value m^2 is equivalent to the idea of mass in physics. Thus elements in the first two orbits correspond to massive particles which travel slower than the speed of light.

The particles that correspond to the light cones are those with $m^2 = 0$, which are particles that travel at the speed of light. Examples of these particles are photons, from the which the light cone gains its name, and gravitons, which currently only exist in theory.

The particles on the single-sheet hyperboloid with values $m^2 < 0$ are called tachyons. They travel faster than the speed of light and have imaginary mass. As tachyons have never been observed, we will not find the representations corresponding this orbit.

Finally, the orbit in which $\mathrm{m}^2 = 0$ and the particles are not moving through time corresponds to vacuum. Vacuum is devoid of all particles, and thus is uninteresting to the purposes of this paper.

It is somewhat surprising that the correspondences between the orbits of the double cover of the Poincaré group and types of particles line up as well as they do. The spin values of particles predicted by physicists are exactly the values that come as a result of finding the irreducible representations of the double cover of the Poincaré group. This fact is further explained by the fact that the Poincaré group's covering space is connected and has nice properties that emulate those observed in quantum spin.

2 The Poincaré group

In order to describe the Poincaré group, we first need to work through some preliminaries.

The *orthogonal group* $\mathrm{O}(n)$ is the group of $n \times n$ real matrices whose transpose is equal to their inverse. In other words, $A \in \mathrm{O}(n)$ if

$$A^T = A^{-1}. \tag{1}$$

One can define an orthogonal group more generally, as follows. Given a vector space V equipped with a symmetric bilinear form $\langle\ ,\ \rangle$, then the corresponding orthogonal group is the group of all linear transformations of V that leave $\langle\ ,\ \rangle$ invariant. So $A \in \mathrm{O}(V, \langle\ ,\ \rangle)$ if

$$\langle Ax, Ay\rangle = \langle x, y\rangle$$

for all $x, y \in V$. By the Spectral Theorem [1, p. 397], there exists a basis in which we can express $\langle\ ,\ \rangle$ as a diagonal matrix g, where

$$\langle x, y\rangle = x^T gy.$$

Applying this to the definition of the orthogonal group gives us

$$\begin{aligned} (Ax)^T g(Ay) &= x^T gy \\ \Longrightarrow x^T A^T gAy &= x^T gy \\ \Longrightarrow A^T gA &= g. \end{aligned} \tag{2}$$

Recall that a real inner product is symmetric, bilinear, and positive definite. If $V = (\mathbb{R}^n, \cdot)$, the metric tensor g is the $n \times n$ identity matrix and (2) simplifies to (1).

The *signature* of a symmetric bilinear form is the number of positive and negative eigenvalues of its matrix, and is usually written (p, q). It turns out that the signature is enough to define the orthogonal group; this group is usually written $O(p, q)$. This group is well-defined because given any two inner products with identical signature there is an isomorphism between their respective groups. We will simplify things by choosing all eigenvalues to have norm 1. In this notation, the standard orthogonal group in $\mathbb{R}^n$ is $\mathrm{O}(n, 0)$.

2.1 Minkowski Spacetime

Much of our work will be done in Minkowski spacetime, a four dimensional real vector space $\mathbb{R}^{1,3}$. This space is distinguished by its inner product, which is a symmetric nondegenerate bilinear form with signature $(1,3)$ or $(3,1)$. We will be using the former, and will be taking the first dimension to be time. Note that this "inner product" does not match the normal definition of an inner product because it is not positive-definite. This inner product, or *Minkowski metric*, is

$$\langle \vec{v}, \vec{w} \rangle \quad = \quad v_0 w_0 - v_1 w_1 - v_2 w_2 - v_3 w_3,$$

where $\vec{v}$ and $\vec{w}$ are four-dimensional vectors. Similarly, the Minkowski norm is

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = v_0^2 - v_1^2 - v_2^2 - v_3^2.$$

The group $\mathrm{O}(1,3)$ is the set of linear transformations of Minkowski spacetime that preserves the Minkowski metric. We can write the metric tensor

$$g \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Notice that $g^{-1} = g$. From (2), we have $\mathrm{O}(1,3) = \{A \mid A^T g A = g\}$. This is not an easily solvable equation, and we will not do so here. Instead, we will describe this group by its generators.

2.2 $\mathrm{O}(1,3)$

The group $\mathrm{O}(1,3)$ is known as the *Lorentz group*, and elements of this group are *Lorentz transformations*. It has four connected components. Roughly speaking, a Lorentz transformation may or may not preserve the direction of time, and it may or may not preserve the orientation of space; these choices correspond to the four connected components. The two components that preserve time form the subgroup $\mathrm{O}^+(1,3)$. The two components that preserve the orientation of space form the subgroup $\mathrm{SO}(1,3)$ which are the elements with determinant 1. The identity component, which preserves both time and space, is the subgroup $\mathrm{SO}^+(1,3)$. This subgroup is the group of *proper* Lorentz transformations.

There are six types of transformations that generate $\mathrm{SO}^+(1,3)$. Three of the generators are simple spatial rotations, and the other three are time-space operators known as *boosts*. The rotations R_i and boosts B_i are given below:

$$
\begin{aligned}
R_x &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}, \\
R_y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix}, \\
R_z &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
B_x &= \begin{bmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
B_y &= \begin{bmatrix} \cosh\theta & 0 & \sinh\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\theta & 0 & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
B_z &= \begin{bmatrix} \cosh\theta & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \cosh\theta \end{bmatrix}.
\end{aligned}
$$

Special relativity says that two observers moving at constant velocity with respect to one another will perceive a single event at different places and at different times. The change from one observer's reference frame to another's is handled by boosts.

Assume that the two observers agree on the origin, and that their x, y, and z axes are respectively parallel. Without loss of generality we will consider their relative velocity to be β in the x direction. In order to avoid conversion factors between units of measurement, we will express time in seconds, distance in light-seconds, and velocity in light-seconds per second (equivalently, as a fraction of the speed of light). Let

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

The transformation from one observer's reference frame (t, x) to the second observer's reference frame (t', x') is

$$\begin{aligned} t' &= \gamma(t + \beta x) \\ x' &= \gamma(x + \beta t) \\ y' &= y \\ z' &= z. \end{aligned}$$

Then

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$$

This is a symmetric matrix of one variable. Introducing the angle

$$\theta = \ln \frac{1 + \beta}{\sqrt{1 - \beta^2}}$$

allows us to rewrite this matrix as B_x. Notice how boosts are similar to rotations, but involve the hyperbolic functions rather than the circular functions. This is due to the difference in sign between space and time.

So far, we have not proven that these are generators of $SO^+(1, 3)$. We will do so by using the Lie algebra.

2.3 The Lie Algebra $\mathfrak{so}(1, 3)$

Recall that a Lie group, such as $O(1, 3)$, is a smooth manifold as well as a group. The Lie algebra of a Lie group is defined in terms of matrix exponentiation, which we will cover later. For now we will use the fact that the Lie algebra is also the tangent space at the identity equipped with a binary operator known as the Lie bracket. For matrix groups, the bracket is the familiar matrix commutator $[A, B] = AB - BA$.

We will proceed to find the Lie algebra corresponding to $O(1, 3)$. Since we are only considering the local properties of the identity element, the structure of the algebra will only reflect the identity component of the group. Thus the Lie group $O(1, 3)$ and the subgroups discussed earlier all have the same Lie algebra, which we denote as $\mathfrak{so}(1, 3)$.

Let $A : \mathbb{R} \to O(1, 3)$ be a continuous function with $A(0) = I$. Recalling the definition of $O(1, 3)$ from (2), we see that

$$\begin{aligned} A(t)^T g A(t) &= g \\ \implies A(t)^T g &= g A(t)^{-1}. \end{aligned}$$

Any element of the tangent space is the derivative of such a curve A at $t = 0$. Finding the characteristic equation of the algebra is now simply a calculus problem:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(A(t)^T g\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(gA(t)^{-1}\big) \\
\implies \left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)^T g &= -g\left(\frac{\mathrm{d}}{\mathrm{d}t}dA(t)\right)A(t)^{-2} \\
\implies \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A(t)\right)^T g &= -g\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A(t)\right)A(0)^{-2} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A(t) &= -g\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} A(t)\right)^T g.
\end{aligned}
$$

Let $B = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} A(t)$. Then $B = -gB^T g$, which can be worked out as

$$
\begin{aligned}
\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} b_{11} & b_{21} & b_{31} & b_{41} \\ b_{12} & b_{22} & b_{32} & b_{42} \\ b_{13} & b_{23} & b_{33} & b_{43} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -b_{11} & b_{21} & b_{31} & b_{41} \\ b_{12} & -b_{22} & -b_{32} & -b_{42} \\ b_{13} & -b_{23} & -b_{33} & -b_{43} \\ b_{14} & -b_{24} & -b_{34} & -b_{44} \end{bmatrix}.
\end{aligned}
$$

Therefore B is of the form

$$
B = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{bmatrix}. \tag{3}
$$

Let V_a be the matrix of this form where $a = 1$ and $b = c = \cdots = 0$, and define V_b, V_c, V_d, V_e, and V_f likewise. These six matrices are a basis for $\mathfrak{so}(1,3)$ as a vector space. From this we know that $\mathfrak{so}(1,3)$ is 6-dimensional. Since a manifold has the same number of dimensions as its tangent space, $\mathrm{SO}^+(1,3)$ must be 6-dimensional as well. We claimed earlier that the six R_i and B_i matrices were generators of $\mathrm{SO}^+(1,3)$. Now we will show how the basis elements V_i of $\mathfrak{so}(1,3)$ correspond to the generators of $\mathrm{SO}^+(1,3)$.

The exponential map

The exponentiation operator exp is defined as

$$
\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.
$$

The famous irrational e can be defined as exp 1. However, we are not limited to exponentiating scalars; the exp operator works equally well on matrices. For example,

$$\begin{aligned}
\exp \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} &= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}^k \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 0 & a^3 \\ a^3 & 0 \end{bmatrix} + \frac{1}{24} \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix} + \dots \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{(2k)!} \begin{bmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{bmatrix} + \frac{1}{(2k+1)!} \begin{bmatrix} 0 & a^{2k+1} \\ a^{2k+1} & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{bmatrix}.
\end{aligned}$$

This example shows how to compute exp V_a. If we were to exponentiate the full 4×4 matrix, which merely adds rows and columns filled with zeroes, we would get $e^0 = 1$ on the diagonals. In other words,

$$\exp \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is simply B_x.

Similar calculations show that the infinitesimal generators V_b and V_c exponentiate to B_y and B_z, respectively. Furthermore, the exponentiation of the generators V_d, V_e, and V_f yield R_z, R_y, and R_x.

This leads to an alternative definition of the Lie algebra $\mathfrak{g}$ corresponding to a matrix Lie group G:

$$\mathfrak{g} = \{A \mid \exp A \in G\}.$$

Since $\mathrm{SO}^+(1,3)$ is connected, the image of $\mathfrak{so}(1,3)$ under the exponential map is the entire group [2, p. 116], and this definition matches the earlier one.

The symmetry group of Minkowsky spacetime

After exploring the Lorentz group, we turn to the full group of symmetries of Minkowski spacetime: the Poincaré group. Like the Lorentz group, elements of the Poincaré group preserve the Minkowsi metric; however, they need not be linear transformations. We further restrict the Poincaré group to those transformations that preserve both the direction of time and the orientation of space. The *Poincaré group* is $\mathrm{SO}^+(1,3) \ltimes \mathbb{R}^{1,3}$, the semidirect product of the proper Lorentz transformations with the translations of Minkowski spacetime.

Now that we have expressed the Poincaré group, it would seem that we are in a position to answer any questions about the symmetries of Minkowski spacetime. Unfortunately, the universe is not completely defined by this 4-dimensional vector space, and we will have to slightly expand our view.

3 The Double Cover of the Poincaré Group

In 1922, Otto Stern and Walther Gerlach performed an experiment which showed the existence of a physical phenomenon called quantum spin. The experiment involved shooting silver atoms through a magnetic field and measuring where they hit a screen on the other end. Under the classical model of physics, it would be expected that the atoms would land somewhere on a spectrum of points on the screen. However, the atoms struck only two discrete points. This is due to a feature called spin. The two paths the atoms take correspond to the two spin states of electrons [6].

An electron can either have spin $+\frac{1}{2}$ or spin $-\frac{1}{2}$. This spin factor essentially tacks on a fourth piece of information to the three velocity coordinates that already describe the electron's motion. The existence of this spin information implies that $\mathrm{SO}^+(1,3)$ is not a perfect model for classifying the fundamental particles. Instead, we are interested in the covering space of the proper Lorentz group, namely SL(2, $\mathbb{C}$).

In order to account for spin, we must show that the special linear group SL(2, $\mathbb{C}$) is the double cover of the proper Lorentz group. SL(2,$\mathbb{C}$) is the set of all 2×2 matrices A with entries in $\mathbb{C}$ such that $\det A = 1$. For any matrix $A \in$ SL(2,$\mathbb{C}$), since A has complex entries, A has a Schur decomposition [1, p. 430]. This means that there exists a unitary matrix B such that

$$A \quad = \quad B \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} B^{-1}.$$

We can now set $a(t)$ to be a curve of non-zero complex numbers such that $a(0) = 1$ and $a(1) = a$ and similarly have $b(t)$ be a curve with $b(0) = 0$ and $b(1) = b$. This means that

$$A = B \begin{bmatrix} a(t) & b(t) \\ 0 & a(t)^{-1} \end{bmatrix} B^{-1}$$

is a curve of matrices from the identity to A in SL(2,$\mathbb{C}$). So SL(2,$\mathbb{C}$) is path-connected.

We will now describe a map from SL(2,$\mathbb{C}$) onto SO^+(1,3). Recall that we are working in the Minkowski space M and that Lorentz transformations can be applied to its elements. For each $x \in M$, we identify it with a 2×2 complex matrix via

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix}.$$

We can see that

$$\begin{bmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{bmatrix} = \begin{bmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{bmatrix}^*$$

so the matrix representing x is self-adjoint. Also, the determinant is the Minkowski norm of x. This transformation allows us to take the product of elements of M with a 2×2 matrix.

Let A be a 2×2 matrix. Define a map

$$\phi_A : M \to M$$

by

$$\phi_A(x) = A \begin{bmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{bmatrix} A^*.$$

Let $A \in \mathrm{SL}(2,\mathbb{C})$. Then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{C}$ and $\det A = 1$. So

$$\begin{aligned} \phi_A(x) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} \\ &= \begin{bmatrix} y_0+y_3 & y_1-iy_2 \\ y_1+iy_2 & y_0-y_3 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} 2y_0 &= x_0(||a||^2+||b||^2+||c||^2+||d||^2) + x_1(\bar{a}b + a\bar{b} + \bar{c}d + c\bar{d}) \\ &\quad + ix_2(\bar{a}b - a\bar{b} + \bar{c}d - c\bar{d}) + x_3(||a||^2-||b||^2+||c||^2-||d||^2), \\ 2y_1 &= x_0(a\bar{c} + \bar{a}c + b\bar{d} + \bar{b}d) + x_1(a\bar{d} + \bar{a}d + b\bar{c} + \bar{b}c) \\ &\quad + ix_2(\bar{a}d + b\bar{c} - a\bar{d} - \bar{b}c) + x_3(a\bar{c} + \bar{a}c - b\bar{d} - \bar{b}d), \\ 2y_2 &= -ix_0(\bar{b}d + \bar{a}c - a\bar{c} - b\bar{d}) - ix_1(a\bar{d} + \bar{b}c - \bar{a}d - b\bar{c}) \\ &\quad + x_2(a\bar{d} + \bar{a}d - b\bar{c} - \bar{b}c) - ix_3(\bar{a}c + b\bar{d} - a\bar{c} - \bar{b}d), \\ 2y_3 &= x_0(||a||^2+||b||^2-||c||^2-||d||^2) + x_1(a\bar{b} + \bar{a}b - c\bar{d} - \bar{c}d) \\ &\quad + ix_2(\bar{a}b + c\bar{d} - a\bar{b} - \bar{c}d) + x_3(||a||^2-||b||^2-||c||^2+||d||^2). \end{aligned}$$

This means that $\phi_A x = A'x$ where

$$A' = \frac{1}{2} \begin{bmatrix} ||a||^2+||b||^2+||c||^2+||d||^2 & \bar{a}b + a\bar{b} + \bar{c}d + c\bar{d} & \cdots & ||a||^2-||b||^2+||c||^2-||d||^2 \\ a\bar{c} + \bar{a}c + b\bar{d} + \bar{b}d & & \ddots & \vdots \\ \vdots & & & \\ ||a||^2+||b||^2-||c||^2-||d||^2 & & \cdots & |a||^2-||b||^2-||c||^2+||d||^2 \end{bmatrix}.$$

One can check that $A' \in \mathrm{SO}^+(1,3)$. So we have a map $\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ defined by $\phi(A) = \phi_A$. To see that ϕ is a homomorphism, let $A, B \in \mathrm{SL}(2,\mathbb{C})$ and $x \in M$. Then

$$\begin{aligned}
\phi(AB)x &= \phi_{AB}(x) \\
&= ABx(AB)^* \\
&= ABxB^*A^* \\
&= A(BxB^*)A^* \\
&= \phi_A(BxB^*) \\
&= \phi_A\phi_B x \\
&= \phi(A)\phi(B)x.
\end{aligned}$$

Lemma 3.1 *The homomorphism ϕ is a surjective $2 \to 1$ map.*

Because $\mathrm{SL}(2,\mathbb{C})$ is connected and ϕ is continuous, its image is connected. This, however, does not prove that its image is all of $\mathrm{SO}^+(1,3)$. For that, it suffices to show that there are elements in $\mathrm{SL}(2,\mathbb{C})$ which map to the six generators of $\mathrm{SO}^+(1,3)$ under ϕ. Here they are:

$$\begin{aligned}
\phi\left(\begin{bmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{bmatrix}\right) &= \begin{bmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\phi\left(\begin{bmatrix} \cosh\frac{\theta}{2} & i\sinh\frac{\theta}{2} \\ -i\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{bmatrix}\right) &= \begin{bmatrix} \cosh\theta & 0 & \sinh\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\theta & 0 & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\phi\left(\begin{bmatrix} e^{\frac{\theta}{2}} & 0 \\ 0 & e^{-\frac{\theta}{2}} \end{bmatrix}\right) &= \begin{bmatrix} \cosh\theta & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \cosh\theta \end{bmatrix}, \\
\phi\left(\begin{bmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{-i\theta}{2}} \end{bmatrix}\right) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

$$\phi\left(\begin{bmatrix}\cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2})\\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2})\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & 0 & \sin\theta\\ 0 & 0 & 1 & 0\\ 0 & -\sin\theta & 0 & \cos\theta\end{bmatrix},$$

$$\phi\left(\begin{bmatrix}\cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2})\\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2})\end{bmatrix}\right) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\theta & -\sin\theta\\ 0 & 0 & \sin\theta & \cos\theta\end{bmatrix}.$$

Therefore the image of SL(2,$\mathbb{C}$) under ϕ is all of SO$^+$(1,3). Notice how the angles in the elements in SL(2,$\mathbb{C}$) are doubled under the map to SO$^+$(1,3).

Since ϕ is a homomorphism, the identity of SL(2,$\mathbb{C}$) maps to the identity of SO$^+$(1,3) under it. It is also the case that

$$\phi(-I) = \frac{1}{2}\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{bmatrix} = I.$$

To show that only I and $-I$ in SL(2,$\mathbb{C}$) map to the identity, we set the diagonal entries in $A' = \phi_A$ to 1. This yields the equations

$$\begin{aligned}
||a||^2 + ||b||^2 + ||c||^2 + ||d||^2 &= 2,\\
a\bar{d} + \bar{a}d + b\bar{c} + \bar{b}c &= 2,\\
a\bar{d} + \bar{a}d - b\bar{c} - \bar{b}c &= 2,\\
||a||^2 - ||b||^2 - ||c||^2 + ||d||^2 &= 2.
\end{aligned}$$

This system has only two solutions, $a = 1$, $b = 0$, $c = 0$, $d = 1$ and $a = -1$, $b = 0$, $c = 0$, $d = -1$. These are then the only elements which map to I.

Now, if $A, B \in$ SL(2,$\mathbb{C}$) and $\phi_A = \phi_B$, then

$$\phi_{AB^{-1}} = \phi_A\phi_{B^{-1}} = \phi_A\phi_B^{-1} = I$$

so

$$AB^{-1} = \pm I$$

and therefore $A = \pm B$. So every element in SO$^+$(1,3) has exactly two preimages, which are negatives of each other. Therefore SL(2,$\mathbb{C}$) double covers SO$^+(1,3)$under ϕ. $\square$

As SL(2,$\mathbb{C}$) is the connected double cover of SO$^+(1,3)$, we can find the double cover of the Poincaré group, SL(2,$\mathbb{C}$) $\ltimes \mathbb{R}^{1,3}$.

4 Representation Theory

A *representation* [2, p. 3] of a group G in a vector space V is a homomorphism $\rho : G \to GL(V)$, where $GL(V)$ is the group of invertible linear transformations $V \to V$ with composition as the group operator. For a given group G, there may be many different choices for V and ρ. The *regular representation* for finite groups assigns each element of G its own basis element $\vec{e}_h$ in V, so $\rho : G \to \mathbb{R}^n$, where $n = |G|$. Define the representation by $\rho(g)(\vec{e}_h) = \vec{e}_{gh}$.

Recall that the characteristic of a field is the smallest integer m such that $m \cdot 1 = 0$ if it exists. If there is no such x, the characteristic of the field is 0. For both the complex numbers and the real numbers, the characteristic is 0. A subspace W is called *invariant* under ρ if $\rho(G)(W) \subseteq W$.

Any nontrivial vector spaces V has $\{\vec{0}\}$ and V as invariant subspaces. It is easy to check that these subspaces are invariant. A representation $\rho : G \to GL(V)$ is *irreducible* if the only invariant subspaces of V are $\{\vec{0}\}$ and V. A useful tool for finding representations is Maschke's Theorem[7, p. 320].

Theorem 4.1 (Maschke's Theorem) *Let G be a finite group and let $\rho : G \to GL(V)$ where V is finite dimensional, $dim(V) > 0$, and V is over a field F such that char $F = 0$ or char $F = p$ where p divides $|G|$. Then ρ can be decomposed into the sum of one or more irreducible representations.*

4.1 Characters

A reference for this material is *Representation Theory: A First Course*[2].

Often, rather than looking at the representation itself, we instead look at the trace of the representation. The trace of a matrix is the sum of the elements on the main diagonal. We then define the *character* of ρ to be $\chi_\rho : G \to \mathbb{C}$,

$$\chi_\rho(g) = Tr\left(\rho(g)\right). \tag{4}$$

Character have useful properties that makes working with them worthwhile.

1. $\chi_\rho(g) = \chi_\rho(aga^{-1})$
   (a) χ_ρ is constant on each conjugacy class.
   (b) χ_ρ is invariant under change of basis.
2. $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.
3. $\chi_{\rho_1 \oplus \rho 2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$.

For finite G, there is an inner product from $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g) \overline{\chi_{\rho_2}(g)}. \tag{5}$$

If ρ is an irreducible representation then $\langle \chi_\rho, \chi_\rho \rangle = 1$, while if ρ_1 and ρ_2 are different irreducible representations then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 0$. Thus the characters of the irreducible representations form an orthonormal basis of some function space. Using the properties of character, we can show that the function space is the space of functions that map conjugates to the same values. These functions are called the conjugacy class functions of G [2, p. 22].

5 Semidirect Products

Given two groups N and H such that $N \cap H = \{e\}$, the *semidirect product* of N and H is a group G. The result is that each $g \in G$

$$g = nh,$$

where $n \in N$ and $h \in H$. Under the semidirect product N is normal in G and H is a subgroup of G. The notation is $G = H \ltimes N$. Our focus is on the group created by $N = \mathbb{R}^{1,3}$ and $H = SL(2, \mathbb{C})$.

5.1 Irreducible Representations of Semidirect Products

Let $G = H \ltimes N$. Recall that this means every element g of G can be uniquely written as $g = nh$, where $n \in N$ and $h \in H$. Now we will build the irreducible representations of G from the irreducible representations of its normal subgroup, N. Since N is Abelian, each conjugacy class has only one element [4, p. 68], and each irreducible representation can be shown to be one dimensional in N.

Define N^* as the space of all one dimensional characters of N. Also define N_j^* as the orbit of G acting on N^* containing χ_j. Now follow these steps [9, p. 138]:

1. Break N^* into orbits under G and pick a representative character χ_j from each orbit.
2. For each χ_j, find the *isotropy group* L_j of χ_j; meaning the subgroup of H that fixes χ_j.
3. Choose an irreducible representation ρ_j of L_j and extend it to $G_j = L_j \ltimes N$ by $\rho_j(hn)v = \chi_j(n)\rho(h)v$.
4. Create the vector bundle E, and let G act on $\Gamma(E)$.
5. Have V_j be identified with the δ-section concentrated on χ_j such that $V_j = (\Gamma(E))_{\chi_j}$.

Once we have constructed the irreducible representations of G from the irreducible representations of N. While some of the steps we haven't totally explained, the method will become clear when we apply it to the double cover of the Poincaré group.

6 Representations of the Poincaré Group

6.1 Poincaré Group as Semi-Sirect Product

From here, we can describe the orbits of the double cover of the Poincaré Group. Note that every element in the double cover of the Poincaré group acting on $\mathbb{R}^{1,3}$ preserves m^2. Therefore each orbit is contained in a level set of m^2. It turns out that there are six families of orbits. They have the following values for m^2 and x_0. Note that there is not a single orbit for all vectors with $m^2 > 0$, $x_0 > 0$ or $m^2 > 0$, $x_0 < 0$, but an infinite family. In fact, there is a unique orbit for each value of m^2, given either $x < 0$ or $x > 0$. The six types listed above correspond to the six families [9, p. 32].

The first two orbit types are those corresponding to the values $m^2 > 0$, $x_0 > 0$ and $m^2 > 0$, $x_0 < 0$. To find the representative point of an orbit, we simply consider the general form of a point in that orbit. In the case of $m^2 > 0$, $x_0 > 0$ or $m^2 > 0$, $x_0 < 0$, these orbits have representative points of

$$\pm \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \leftrightarrow \begin{bmatrix} \pm m \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with $m \neq 0$, because as can clearly be seen, the time coordinate is nonzero and real, so $m^2 > 0$. The orbits are represented by a two-sheet three-dimensional hyperboloid in $\mathbb{R}^{1,3}$.

It is not at all difficult to see that a linear transformation that fixes the representative point would look like

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \pm m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pm m \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the matrix that fixes the representative point of our first two orbits is a member of the rotation subgroup of the connected double cover of the proper Lorentz group. This double cover is SU(2).

Now we consider the isotropy groups of the third orbit. This orbit corresponds to the values $m^2 = 0$, $x_0 > 0$. This is the forward looking light cone (which is three-dimensional) in $\mathbb{R}^{1,3}$ respectively.

The representative points of this orbit will be $\begin{bmatrix}\omega & 0 & \omega & 0\end{bmatrix}^T$ where $\omega \neq 0$. We use this point because the value m^2 for this vector in $\mathbb{R}^{1,3}$ is 0, and because the time coordinate of this vector, namely ω is not 0.

In the case of these two orbits, however, there are two types of linear transformations that keep this representative point fixed. Rotations about the y-axis will fix the point:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ \omega \\ 0 \end{bmatrix} = \begin{bmatrix} \omega \\ 0 \\ \omega \\ 0 \end{bmatrix}.$$

By similar argument, we see the fourth orbit is also fixed by transformations of this form.

Now consider just the third orbit's representative point $\begin{bmatrix}\omega & 0 & \omega & 0\end{bmatrix}^T$, then the transformation

$$T(u,v) = \begin{bmatrix} 1+(u^2+v^2)/2 & v & -(u^2+v^2)/2 & u \\ v & 1 & -v & 0 \\ (u^2+v^2)/2 & v & 1-(u^2+v^2)/2 & u \\ u & 0 & -u & 1 \end{bmatrix} \in SL(2,\mathbb{C})$$

will also leave the point fixed as shown by the following calculations.

$$\begin{aligned} & \begin{bmatrix} 1+(u^2+v^2)/2 & v & -(u^2+v^2)/2 & u \\ v & 1 & -v & 0 \\ (u^2+v^2)/2 & v & 1-(u^2+v^2)/2 & u \\ u & 0 & -u & 1 \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ \omega \\ 0 \end{bmatrix} \\ = & \begin{bmatrix} \omega + \omega\cdot(u^2+v^2)/2 - \omega\cdot(u^2+v^2)/2 \\ v\cdot\omega - v\cdot\omega \\ \omega + \omega\cdot(u^2+v^2)/2 - \omega\cdot(u^2+v^2)/2 \\ u\cdot\omega - u\cdot\omega \end{bmatrix} \\ = & \begin{bmatrix} \omega \\ 0 \\ \omega \\ 0 \end{bmatrix}. \end{aligned}$$

Thus $T(u,v)$ also leaves the fourth orbit's representative point invariant [5, p. 58].

Let $D(u,v,\phi) = T(u,v)R(\phi)$. Then any transformation $D(u,v,\phi)$ will fix the representative point of the light cones as well. Note, then, that any such translation can be written as a composition of a translation by some 2-vector with a rotation of some angle. In other words, the transformation $D(u,v,\phi)$ is an arbitrary transformation from the isometry group of $\mathbb{R}^2$, namely E(2). This is the isotropy group for the third and fourth orbits of the Poincaré group.

As a result of these previous transformations we do not end up with the Euclidean group E(2), but its connected double cover. This double cover is $\tilde{E}(2)$. It is isomorphic to E(2), except that rotations by θ in E(2) correspond to rotations of 2θ in $\tilde{E}(2)$.

The other two orbits of SU(2) correspond to the values of $m^2 < 0$ and $m^2 = p_0 = 0$. Their isotropy groups are SL(2, $\mathbb{R}$) and SL(2, $\mathbb{C}$) respectively, but we will not prove these facts. These two orbits are not as relevant to the study of physics as the first four are, and as such will not be studied as much in this paper. This is because the $m^2 < 0$ orbit corresponds to tachyons, which are faster than light particles. These have never been observed and only recently have been theorized. The $m^2 = p_0 = 0$ orbit corresponds to vacuum, and as such is entirely uninteresting to study [9].

6.2 The Representations of the Isotropy Groups

Let us consider the isotropy group of a representative point in one of the light cone orbits. Recall that it is the double cover, $\tilde{E}(2)$, of the set of all Euclidean transformations of the plane. It is clear to see from the way we constructed this isotropy group that $\tilde{E}(2)$ is, itself, a semi-direct product of the connected double cover of SO(2) and the set of translations $\mathbb{R}^2$. So we can find the group's representations using the fact that it is a semi-direct product.

The orbits of SO(2) acting on $\mathbb{R}^2$ are circles, and the one-point orbit consisting of the origin. It turns out that representations corresponding to the circle (or its connected double cover) orbit do not occur in physics [9, p. 147], so we will restrict our focus to the origin. The isotropy group of $\{0\}$ is exactly SO(2), the set of rotations of the plane.

Theorem 6.1 *The irreducible representations of the double cover of SO(2) are classified by*

$$s = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots, \text{ where } n \in \mathbb{Z}$$

The irreducible representations of SO(2)'s double cover are all one-dimensional, as the group is abelian. The representations are given by

$$\theta \to e^{in\theta}, n \in \mathbb{Z}$$

These representations correspond to rotations of $\mathbb{S}^1$. It must be the case that $n \in \mathbb{Z}$ as, in order for the representation, which can be thought of as a map $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ to be continuous, the function needs to map the circle to itself some integer number of times.

However, in this representation, θ corresponds to a rotation by 2θ as given by the double covering nature of $\tilde{E}(2)$. Thus, it is conventional in physics to let

$$s = \frac{1}{2}n$$

so that the irreducible representations are classified by

$$s = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots. \quad \square$$

Unfortunately, finding the irreducible representations of SU(2) is more difficult.

7 Haar measure

In order to complete some of the work that follows, we will need to integrate a function f over a Lie group G. To that end, we will define an *invariant integral*.

A function is compactly supported if it is identically zero outside some compact set. Let $\mathcal{F}_0(G)$ be the space of compactly supported continuous functions on G. Then integration da is *left invariant* if

$$\int_G (r(b)f)(a)\mathrm{d}a \quad = \quad \int_G f(a)\mathrm{d}a \tag{6}$$

for all $f \in \mathcal{F}_0(G)$ and $b \in G$. Here r is the regular representation of G on $\mathcal{F}_0(G)$, namely

$$r(b)f(a) = f(b^{-1}a).$$

If G is a compact group, any function on a compact group is compactly supported. Furthermore, the invariant integral is unique up to scalar multiples and the total volume of the group is finite [9, p. 174]. Then we can fix an integral by simply declaring

$$\int_G 1 \;\mathrm{d}a \quad = \quad 1.$$

This integral is known as the *Haar measure*.

It remains to find an invariant integral on SU(2) and normalize it.

7.1 Differential forms

A *differential k-form* on a Lie group G is an alternating k-tensor, which can be used in a k-dimensional integral [8, p. 88]. A 0-form on G is the same as a function $f : G \to \mathbb{K}$. We will not go into the details of differential forms in this paper.

Differential forms are combined using the alternating wedge operator

$$(k\text{-form}) \wedge (l\text{-form}) \to (k+l)\text{-form}$$
$$\alpha \wedge \beta = -\beta \wedge \alpha.$$

We will occasionally drop the $\wedge$ symbol when its use is clearly implied.

If $\phi : R \to S$ is differentiable and ω is a differential form on S, then the pullback $\phi^*\omega$ is a differential form on R. Furthermore, given certain assumptions we have

$$\int_R \phi^*\omega \quad = \quad \int_S \omega.$$

(The necessary assumptions are that ϕ is a diffeomorphism, ω is an n-form where $n = \dim S$, R and S are oriented, and ϕ is orientation-preserving. For the specific uses that follow, all of these are true.)

If ω does not have compact support, this integral might not be defined. However, we can multiply it by a compactly supported function f to get

$$\int_R \phi^* f \; \phi^* \omega \quad = \quad \int_S f \; \omega. \tag{7}$$

In order to find a Haar measure, we are going to set $R = S = G$. Now the pullback ϕ^* is not moving the differential form from one group to another, but it might still move it around within our group G. Choose some $b \in G$ and define $\phi_b : G \to G$ to be left multiplication by b^{-1}. For the 0-form f, we find that $\phi^* f = r(b) f$ [9, p. 174]. Now we can simplify (7) to

$$\int_G r(b) f \; \phi_b^* \omega \quad = \quad \int_G f \; \omega. \tag{8}$$

Our task is to find an ω such that $\phi_b^* \omega = \omega$ for all $b \in G$. If we can do that, then (8) simplifies to (6), aside from changes in notation. Suppose we have a representation A of G on $\mathbb{C}^k$:

$$A : G \to GL(k, \mathbb{C}).$$

We can consider A as k^2 component functions $A_{ij} : G \to \mathbb{C}$, which are also 0-forms on G. Then define $\mathrm{d}A$ component-wise to get a $k \times k$ matrix of 1-forms.

Theorem 7.1 *Each entry of $A^{-1}\mathrm{d}A$ is a left invariant differential form.*

Because A is a homomorphism,

$$\begin{aligned} \phi_b^* A(g) &= A(\phi_b^* g) \\ &= A(b^{-1} g) \\ &= A(b)^{-1} A(g) \end{aligned}$$

and

$$\phi_b^* A \quad = \quad B^{-1} A$$

where $B = A(b)$. Furthermore, because B is constant,

$$\begin{aligned} \phi_b^* (\mathrm{d}A) &= \mathrm{d}(\phi_b^* A) \\ &= \mathrm{d}(B^{-1} A) \\ &= B^{-1} \mathrm{d}A. \end{aligned}$$

Now it is a simple computation to show that $A^{-1}\mathrm{d}A$ is left invariant:

$$\begin{aligned}
\phi_b^*(A^{-1}\mathrm{d}A) &= \phi_b^*(A^{-1})\phi_b^*(\mathrm{d}A)\\
&= \phi_b^*(A^{-1})B^{-1}\mathrm{d}A\\
&= \phi_b^*(A)^{-1}B^{-1}\mathrm{d}A\\
&= (B^{-1}A)^{-1}B^{-1}\mathrm{d}A\\
&= A^{-1}BB^{-1}\mathrm{d}A\\
&= A^{-1}\mathrm{d}A. \quad \Box
\end{aligned}$$

The n-form ω that we want is a wedge product of n of these 1-forms. Since any such n-form is an invariant integral, and there is only one such integral up to scalar multiples, any n choices will give us something that can be normalized to the Haar measure.

7.2 Haar measure of SU(2)

Any element of SU(2) is of the form

$$A = \begin{bmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{bmatrix},$$

where $|\alpha|^2 + |\beta|^2 = 1$. So

$$\begin{aligned}
A^{-1}\mathrm{d}A &= \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}\begin{bmatrix} \mathrm{d}\bar{\alpha} & -\mathrm{d}\beta \\ \mathrm{d}\bar{\beta} & \mathrm{d}\alpha \end{bmatrix}\\
&= \begin{bmatrix} \alpha\mathrm{d}\bar{\alpha} + \beta\mathrm{d}\bar{\beta} & -\alpha\mathrm{d}\beta + \beta\mathrm{d}\alpha \\ -\bar{\beta}\mathrm{d}\bar{\alpha} + \bar{\alpha}\mathrm{d}\bar{\beta} & \bar{\beta}\mathrm{d}\beta + \bar{\alpha}\mathrm{d}\alpha \end{bmatrix}.
\end{aligned}$$

By theorem 7.1, each entry is a left invariant form. From here, we choose to take the wedge product of the second, third and fourth entries:

$$\begin{aligned}
& (\bar{\alpha}\mathrm{d}\alpha + \bar{\beta}\mathrm{d}\beta) \wedge (-\alpha\mathrm{d}\beta + \beta\mathrm{d}\alpha) \wedge (-\bar{\beta}\mathrm{d}\bar{\alpha} + \bar{\alpha}\mathrm{d}\bar{\beta})\\
=\; & (-\alpha\bar{\alpha} - \beta\bar{\beta})\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge (-\bar{\beta}\mathrm{d}\bar{\alpha} + \bar{\alpha}\mathrm{d}\bar{\beta})\\
=\; & \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge (\bar{\beta}\mathrm{d}\bar{\alpha} - \bar{\alpha}\mathrm{d}\bar{\beta}).
\end{aligned}$$

In order to simplify this, we can express $\mathrm{d}\bar{\beta}$ in terms of the other forms by differentiating:

$$\begin{aligned}
\alpha\bar{\alpha} + \beta\bar{\beta} &= 1\\
\implies \alpha\mathrm{d}\bar{\alpha} + \bar{\alpha}\mathrm{d}\alpha + \beta\mathrm{d}\bar{\beta} + \bar{\beta}\mathrm{d}\beta &= 0\\
\implies \mathrm{d}\bar{\beta} &= -\beta^{-1}(\alpha\mathrm{d}\bar{\alpha} + \bar{\alpha}\mathrm{d}\alpha + \bar{\beta}\mathrm{d}\beta).
\end{aligned}$$

Remember that $\mathrm{d}x \wedge \mathrm{d}x = 0$, so two of the terms in $\mathrm{d}\bar{\beta}$ will cancel out. The invariant 3-form is now

$$\begin{aligned}
\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge (\bar{\beta}\mathrm{d}\bar{\alpha} - \bar{\alpha}\mathrm{d}\bar{\beta}) &= \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge (\bar{\beta}\mathrm{d}\bar{\alpha} - \bar{\alpha}(-\beta^{-1}(\alpha\mathrm{d}\bar{\alpha} + \bar{\alpha}\mathrm{d}\alpha + \bar{\beta}\mathrm{d}\beta))) \\
&= \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge (\bar{\beta}\mathrm{d}\bar{\alpha} + \bar{\alpha}\alpha\beta^{-1}\mathrm{d}\bar{\alpha}) \\
&= \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge ((\bar{\beta}\beta + \bar{\alpha}\alpha)\beta^{-1}\mathrm{d}\bar{\alpha}) \\
&= \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge (\beta^{-1}\mathrm{d}\bar{\alpha}) \\
&= \beta^{-1}\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\bar{\alpha}.
\end{aligned}$$

Thus $\frac{1}{\beta}\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\bar{\alpha}$ is a left invariant form.

It will be useful later to express this in spherical coordinates. Let

$$\begin{aligned}
\alpha &= \cos\theta + i\sin\theta\cos\psi, \\
\beta &= \sin\theta\sin\psi e^{i\phi},
\end{aligned}$$

with $0 \leq \theta \leq \pi$, $0 \leq \psi \leq \pi$, and $0 \leq \phi \leq 2\pi$. This parameterizes SU(2); we will use these coordinates later to simplify the Haar measure to a single variable.

Let u and v be real functions. Then

$$\mathrm{d}(u + iv) \wedge \mathrm{d}(u - iv) = -2i\ \mathrm{d}u \wedge \mathrm{d}v.$$

Therefore

$$\begin{aligned}
\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} &= -2i\ \mathrm{d}(\cos\theta) \wedge \mathrm{d}(\sin\theta\cos\psi) \\
&= 2i\sin\theta\ \mathrm{d}\theta \wedge (\sin\theta\ \mathrm{d}(\cos\psi) + \cos\psi\cos\theta\ \mathrm{d}\theta) \\
&= 2i\sin\theta\ \mathrm{d}\theta \wedge \sin\theta\ \mathrm{d}(\cos\psi) \\
&= 2i\sin\theta\ \mathrm{d}\theta \wedge (-\sin\theta\sin\psi\ \mathrm{d}\psi) \\
&= -2i\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi.
\end{aligned}$$

When expanding $\mathrm{d}\beta$ we can safely ignore any $\mathrm{d}\theta$ and $\mathrm{d}\psi$ terms, since they will drop out when wedged with $\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}$. Then

$$\begin{aligned}
\frac{1}{\beta}\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\bar{\alpha} &= -\frac{1}{\beta}\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} \wedge \mathrm{d}\beta \\
&= \frac{1}{\beta}2i\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{d}\beta \\
&= \frac{1}{\beta}2i\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{d}(\sin\theta\sin\psi e^{i\phi}) \\
&= \frac{1}{\beta}2i\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge (i\sin\theta\sin\psi e^{i\phi})\mathrm{d}\phi \\
&= \frac{1}{\beta}2i\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge i\beta\mathrm{d}\phi \\
&= \frac{1}{\beta}2i(i\beta)\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{d}\phi \\
&= -2\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{d}\phi.
\end{aligned}$$

So $\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{d}\phi$ is a left invariant 3-form on SU(2).

We want the total volume of the group to be 1. We can integrate in the normal way:

$$\begin{aligned}
\int\int\int_{\mathrm{SU}(2)} \sin^2\theta\sin\psi\ \mathrm{d}\theta\ \mathrm{d}\psi\ \mathrm{d}\phi &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin^2\theta\sin\psi\ \mathrm{d}\theta\ \mathrm{d}\psi\ \mathrm{d}\phi \\
&= 2\pi\int_0^{\pi}\int_0^{\pi} \sin^2\theta\sin\psi\ \mathrm{d}\theta\ \mathrm{d}\psi \\
&= 4\pi\int_0^{\pi} \sin^2\theta\ \mathrm{d}\theta \\
&= 2\pi^2.
\end{aligned}$$

Since we can multiply this by any constant, we end up with the Haar measure

$$\Omega = \frac{1}{2\pi^2}\sin^2\theta\sin\psi\ \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{d}\phi.$$

8 The irreducible representations of SU(2)

In order to find the representations of SU(2), we will show that each finite-dimensional irreducible representation of a topological group occurs as a subrepresentation of the regular representation of the group on the space of continuous functions of that group.

Let G be a topological group and V be a finite dimensional irreducible Hilbert space. We say (ρ, V) is a *subrepresentation* of (σ, W) if there exists an injective map $\phi : V \to W$ such that for all b in G, $\sigma(b)\phi(x) = \phi(\rho(b)x)$.

Let $\rho : G \to \mathrm{Hom}(V, V)$ be any representation. For any V, representation ρ, $\ell \in V^*$, $x \in V$, let $a \in G$, let $\phi_x : G \to \mathbb{C}$ be

$$\phi_x(a) = \ell(\rho(a)^{-1}x).$$

For any V, we will define the topology on $\mathrm{Hom}(V, V)$ to be the weakest one such that for all representations ρ, for all $\ell \in V^*$, for all $x \in V$, the map $\phi_x : G \to \mathbb{C}$ is continuous.

Recall that $\mathcal{F}(G)$ is the space of continuous functions on G and there exists a regular representation of G on $\mathcal{F}(G)$. Then we have the map $\phi : V \to \mathcal{F}(G)$ where for $x \in V$,

$$\phi(x) = \phi_x.$$

The regular representation acts on $\phi_x \in \mathcal{F}(G)$ by

$$r(b)\phi(x)(a) = \phi(x)(b^{-1}a).$$

It is simple to show that r is well defined and a homomorphism. As $\phi(x) \in \mathcal{F}(G)$, a space of continuous functions on G, it is continuous on G. Since G is a topological group, multiplication by an element in G is continuous. This means that $r(b)\phi(x)$ is the composition of continuous functions and is therefore continuous, so it is in $\mathcal{F}(G)$.

Also, we can see that $r(b) : \mathcal{F}(G) \to \mathcal{F}(G)$ is a linear map for any $b \in G$. Let $\ell_1, \ell_2 \in V^*$ and $x \in V$. Define $\phi_1(x) = \ell_1(\rho(a)^{-1}x)$ and $\phi_2(x) = \ell_2(\rho(a)^{-1}x)$. Then

$$\begin{aligned}
r(b)(\phi_1(x) + \phi_2(x))(a) &= (\phi_1(x) + \phi_2(x))(b^{-1}a) \\
&= r(b)(\phi_1(x) + \phi_2(x))(a) \\
&= \phi_1(x)(b^{-1}a) + \phi_2(x)(b^{-1}a) \\
&= r(b)\phi_1(x)(a) + r(b)\phi_2(x)(a).
\end{aligned}$$

For some constant k,

$$r(b)(k\phi(x)) = k\phi(x) \circ b^{-1} = k(r(b)\phi(x))$$

so r is well-defined.

We can also see that r is a homomorphism:

$$\begin{aligned}
r(ab)\phi(x) &= \phi(x) \circ (ab)^{-1} \\
&= \phi(x) \circ b^{-1}a^{-1} \\
&= r(a)(\phi(x) \circ b^{-1}) \\
&= r(a)(r(b)\phi(x)).
\end{aligned}$$

Theorem 8.1 *Any finite-dimensional irreducible representation of G occurs as a subrepresentation of r. In other words, let (ρ, V) be a finite dimensional irreducible representation of G. Then there exists a subspace $W \subseteq \mathcal{F}(G)$ with an injective linear transformation $\phi : V \to W$ such that for all $b \in G$, $r(b)\phi(x) = \phi(\rho(b)x)$.*

First we will show ϕ is a G-morphism, meaning $r(b)\phi(x) = \phi(\rho(b)x)$. Let $x \in V$ and $a, b \in G$. Then

$$\phi(\rho(b)x)(a) = \ell(\rho(a)^{-1}\rho(b)x)$$

by the definition of ϕ. Also

$$\begin{aligned} r(b)\phi(x)(a) &= \phi(a)(b^{-1}a) \\ &= \ell(\rho(b^{-1}a)^{-1}x) \\ &= \ell((\rho(b^{-1})\rho(a))^{-1}x) \\ &= \ell(\rho(a)^{-1}\rho(b)x). \end{aligned}$$

So $\phi(\rho(b)x)(a) = r(b)\phi(x)(a)$.

Now we will show that ϕ is injective. We say V is *topologically irreducible* if the only closed invariant subspaces of V are V itself and $\{0\}$. We will take for granted that since V is finite dimensional and irreducible, it is also topologically irreducible [9, p. 178]. We can show that $\ker\phi$ is a closed invariant subspace of V. It is fixed by the action of G on V, namely ρ. Let $x \in \ker\phi$, then

$$\phi(\rho(b)(x)) = r(b)\phi(x) = r(b)0 = 0.$$

So $\rho(b)x$ is also in $\ker\phi$.

To show that ϕ is injective, let $\ell \in V^*\backslash\{0\}$. Then there exists $x \in V$ such that $\ell(x) \neq 0$. This means that

$$\phi(x)I = l(\rho(I)^{-1}x) = l(x) \neq 0.$$

Therefore if $\ell \neq 0$, then $\phi \neq 0 \in \mathrm{Hom}(V, \mathcal{F}(G))$. So then $\ker\phi = \{0\}$ as it cannot be V.

We now have a finite dimensional irreducible representation of G on V, ρ, with an injective map $\phi : V \to W \subseteq \mathcal{F}(G)$ and for all $b \in G$, $r(b)\phi(x) = \phi(\rho(b)x)$. So every finite dimensional irreducible representation of G occurs as a subrepresentation of r on $\mathcal{F}(G)$. □

If we define the scalar product on $\mathcal{F}(G)$ by

$$(f_1, f_2)_G = \int_G (f_1(a)\overline{f_2(a)}\Omega$$

then we make $\mathcal{F}(G)$ into a pre-Hilbert space. Notice that this uses the Haar measure discussed in the previous section. This allows us to obtain a Hilbert space called $L^2(G)$. The regular representation is now a unitary representation of G on $L^2(G)$ [9, p. 179].

Having now created $L^2(G)$, take $G = \mathrm{SU}(2)$, which is a compact group. The Peter-Weyl theorem says the space $L^2(G)$ decomposes into a Hilbert space direct sum of irreducible representations of G, each of which is finite dimensional. This result is similar to Maschke's Theorem, as any representation in $L^2(G)$ can be decomposed into a direct sum of irreducible representations. An important consequence of this theorem is that the irreducible characters form an orthonormal basis of the Hilbert space of square integrable central functions [9, p. 179]. This is exactly the same statement we had for finite G, only then we were considering conjugacy class functions. But this requires an inner product, which is the inner product on $\mathcal{F}(G)$ derived from the Haar measure. To find these representations, we use the Peter-Weyl theorem and the fact that irreducible characters have unit length under the Haar inner product. From here, our task is to determine the form of the possibilities for the characters of SU(2).

Let (ρ, V) be an irreducible representation of SU(2). Then

$$\rho \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

is a unitary matrix. This is because a representation of SU(2) must preserve the operation of SU(2). Thus any element to which the representation ρ maps decomposes to be a direct sum of eigenspaces. Using this fact, we can then define a basis $v_1, \ldots, v_n$ of V such that for $j = 1, \ldots, n$

$$\rho \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} v_j = \lambda_j(\theta) v_j$$

for some function λ_j.

Because a representation is a homomorphism, we have

$$\lambda_j(\theta_1 + \theta_2) = \lambda_j(\theta_1)\lambda_j(\theta_2)$$

and so

$$\lambda_j = e^{ic_j\theta}$$

for some c_j. What this means is that if we consider the representation's character χ, we have

$$\chi(\theta) = \sum e^{ic_j\theta}.$$

Now our problem is reduced to finding what the possibilities for c_j are. By definition, $\chi(\theta)$ is a continuous map. In order for this to be the case, c_j must be an integer. Two more facts will help us determine the possibilities for c_j. First off, we know that, since χ is an even function of θ, if c_j occurs, then $-c_j$ must occur as well. Secondly, we know that, as was stated earlier, $\langle \chi, \chi \rangle = 1$. So

$$1 = \langle \chi, \chi \rangle = \frac{2}{\pi} \int_0^{\pi} \chi(\theta)\overline{\chi}(\theta) \sin^2(\theta) d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \chi(\theta)\overline{\chi(\theta)} \sin^2(\theta) d\theta.$$

Using the identity

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)) = \frac{1}{4}(2 - e^{2i\theta} - e^{-2i\theta})$$

we have

$$1 = \langle \chi, \chi \rangle = \frac{1}{4} \cdot \frac{1}{\pi} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_0^{2\pi} e^{ic_j\theta} \cdot e^{-ic_j\theta}(2 - e^{2i\theta} - e^{-2i\theta}) d\theta. \tag{9}$$

Notice that

$$\int_0^{2\pi} e^{mi\theta} d\theta = 0$$

for integers $m \neq 0$; if $m = 0$, the integral is equal to 2π. So in solving for (9), we mulitply out and keep only the terms that multiply to 1 while discarding the ones that multiply to zero. With this logic in mind, we expand (9) to get

$$\langle \chi, \chi \rangle = \left(\frac{1}{4\pi}\right) \sum_{j=1}^{n} \sum_{k=1}^{n} \left(2 \int_0^{2\pi} e^{(c_j - c_k)i\theta} d\theta + \int_0^{2\pi} e^{(c_j - c_k + 2)i\theta} d\theta + \int_0^{2\pi} e^{(c_j - c_k - 2)i\theta} d\theta\right). \tag{10}$$

So we have n terms in which $j = k$, each of which yields a term of the form $\frac{1}{4} \cdot \frac{1}{\pi} \cdot 2\pi \cdot 2 = 1$. Therefore the integral (9) equals 1 on at least n occasions. Furthermore, we have $2a$ terms where $j \neq k$ where $c_j = c_k$ yielding $2a$ occasions where (9) equals 1 for some unknown quantity a. There are also $2b$ terms in which $c_j = c_k + 2$ for some unknown number b, which yields as a solution for (9) $\frac{1}{4} \cdot \frac{1}{\pi} \cdot 2\pi \cdot (-1)$ $2b$ times. Note $\frac{1}{4} \cdot \frac{1}{\pi} \cdot 2\pi \cdot (-1) \cdot 2b = -b$.

So $\langle \chi, \chi \rangle = n + 2a - b$. It is the case that b can at most be $n - 1$. Therefore, $n - b + 2a \geqslant 1 + 2a$. However, since $\langle \chi, \chi \rangle = 1$, it must be the case that $a = 0$ and $b = n - 1$.

Thus we have found all of the possibilities for the c_js. For each n, $\{c_j\}$ has the properties that the c_js are symmetric about the origin and that if $-c_j \in \{c_j\}$, then $c_j \in \{c_j\}$.

So, for a given n, $\{c_j\} = \{(-n+1, -n+3, \ldots, n-3, n-1\}$. It is the convention to label the characters and representations, which we will prove exist shortly, by $s = (n-1)/2$.

Theorem 8.2 *The irreducible characters of SU(2) are the following*

$$\begin{aligned}
\chi_0 &= 1, \\
\chi_{1/2} &= e^{-i\theta} + e^{i\theta}, \\
\chi_1 &= e^{-2i\theta} + 1 + e^{2i\theta}, \\
\chi_{3/2} &= e^{-3i\theta} + e^{-i\theta} + e^{i\theta} + e^{3i\theta}, \\
&\vdots \\
\chi_s &= e^{-2si\theta} + e^{(2-2s)i\theta} + \cdots + e^{(2s-2)i\theta} + e^{2si\theta}, \ldots.
\end{aligned}$$

It is not difficult to find the representations explicitly. The group SU(2) acts on $\mathbb{C}^2$. Thus the group has representations in the space of functions on $\mathbb{C}^2$. Let V_s be the space of all homogeneous polynomials of degree $2s$. A basis for V_s is

$$\left\{z_1^{2s}, z_1^{2s-1} z_2, z_1^{2s-2} z_2^2, \ldots, z_2^{2s}\right\}.$$

To develop a method for computing these irreducible representations, we will first find the trivial representation ρ_0.

Let $A \in \mathrm{SU}(2)$. Thus A is of the form

$$A = \begin{bmatrix} \overline{a} & -b \\ \overline{b} & a \end{bmatrix},$$

and

$$A^{-1} = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$$

where $|a|^2 + |b|^2 = 1$. So we can treat $|a|$ as $\cos(\theta)$ and $|b|$ as $\sin(\theta)$ for some θ.

Let $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2$. Consider

$$A^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a z_1 + b z_2 \\ -\overline{b} z_1 + \overline{a} z_2 \end{bmatrix}.$$

Note that the basis for V_0 is $\{1\}$, which can be thought of as $\{z_1^0\}$. To find the matrix representation, we simply plug $a z_1 + b z_2$ into z_1^0, which is the first element of V_0's basis. For the $s = 0$ case, we end up with the matrix

$$\rho_0 = [1].$$

For another example, consider s $= \frac{1}{2}$. The basis for V_1 is $\{z_1,\, z_2\}$. Replacing the basis elements z_1 and z_2 with $az_1 + bz_2$ and $\overline{b}z_1 + \overline{a}z_2$ yields the polynomials

$$\begin{gathered} az_1 + bz_2 \\ \overline{b}z_1 + \overline{a}z_2. \end{gathered}$$

Note that these two polynomials are linear transformations of the basis elements z_1 and z_2. The coordinates of the polynomials are the rows of the representation matrix. So the matrix for this representation is exactly

$$\rho_{1/2} = \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix}.$$

These first two cases seemed a bit basic. So let us now compute ρ_1. The basis for V_2 is $\{z_1^2, z_1z_2, z_2^2\}$. Using the same algorithm used in computing the previous representations, we find that

$$\rho_1 = \begin{bmatrix} a^2 & -\overline{b}a & \overline{b}^2 \\ 2ab & a\overline{a} - b\overline{b} & -2\overline{a}\overline{b} \\ b^2 & \overline{a}b & \overline{a}^2 \end{bmatrix}.$$

Note that the trace of this matrix, after the a and b values are replaced with their respective $\cos(\theta)$ and $\sin(\theta)$ values, is

$$e^{-2i\theta} + 1 + e^{2i\theta} = \chi_1.$$

Using this method and solving for the traces of the representations we get does, in fact, confirm that there exists a representation of trace

$$\chi_s \quad = \quad e^{-2si\theta} + e^{(2-2s)i\theta} + \cdots + e^{(2s-2)i\theta} + e^{2si\theta}$$

for any value of s.

Thus, we have found the irreducible representations of SU(2). $\square$

Having found these representations, we now have found the irreducible representations of the double cover of the Poincaré group which correspond to particles and. Here are a few examples of fundamental particles that with spin values that correspond to spin values found using the method developed in this paper.

Spin	$m^2 > 0$	$m^2 = 0$
s = 0	Higgs boson	
s = 1/2	neutrino, electron, quark	
s = 1	W boson, Z boson	photon, gluon
s = 3/2	Ω baryon, gravitino	
s = 2		graviton

References

[1] J. Fraleigh and R. Beauregard, *Linear Algebra, Second Edition*, Addison-Wesley Publishing Company, 1990.

[2] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer-Verlag, 1991.

[3] David J. Gross Physics Today **46 - 50**, (1995).

[4] I.N Hernstein, *Abstract Algebra, Second Edition*, Macmillan Publishing Company, 1990.

[5] Y.S. Kim and M.E. Noz, *Theory and Applications of the Poincaré Group*, D. Reidel Publishing Company, 1986.

[6] M. Reid and B. Szendrői, *Geometry and Topology*, Cambridge University Press, 2005.

[7] W.R. Scott, *Group Theory* Prentice-Hall Inc., 1964.

[8] Michael Spivak, *Calculus on Manifolds*, Westview Press, 1998.

[9] S. Sternberg, *Group theory and physics*, Cambridge University Press, 1994.