Skyrmions and Rational Maps

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Outline of the talk

- Introduction to the Skyrme Model
- The rational map ansatz
- Quantization of Skyrmions
- Homotopy groups and rational maps
- Results

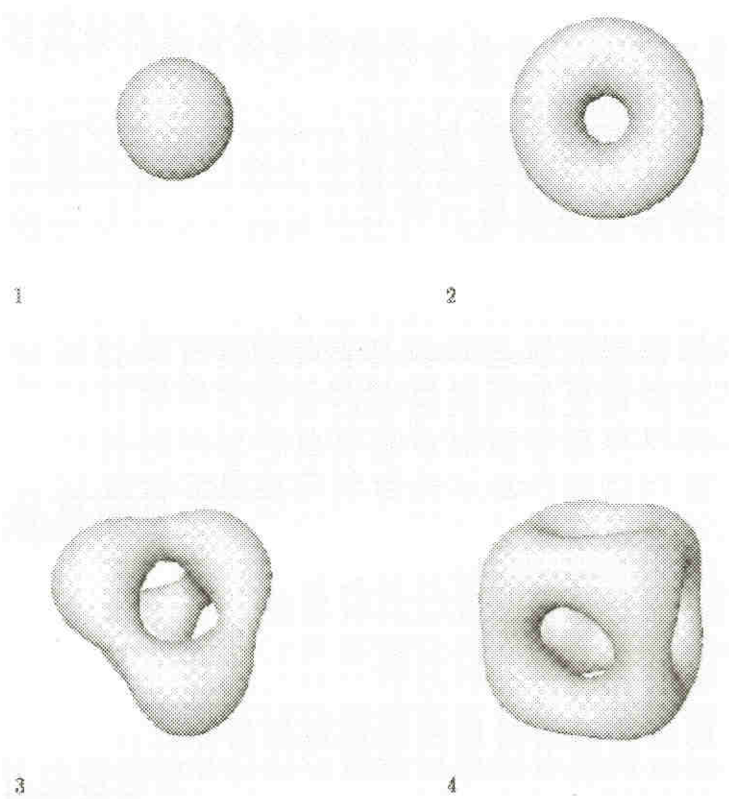
The Skyrme Model

- $U : \mathbb{R}^3 \to SU(2)$ such that $U(\infty) = 1$.

- Topologically: $U : S^3 \to S^3$, $\pi_3(S^3) = \mathbb{Z}$.
  Let $Q_B = \{U : \deg U = B\}$

- Energy: $E = \int |\mathrm{d}U|^2 + |\mathrm{d}U \wedge \mathrm{d}U|^2 \geq B$.

- Symmetry: $SO(3) \times SO(3) \times \mathbb{R}^3$.



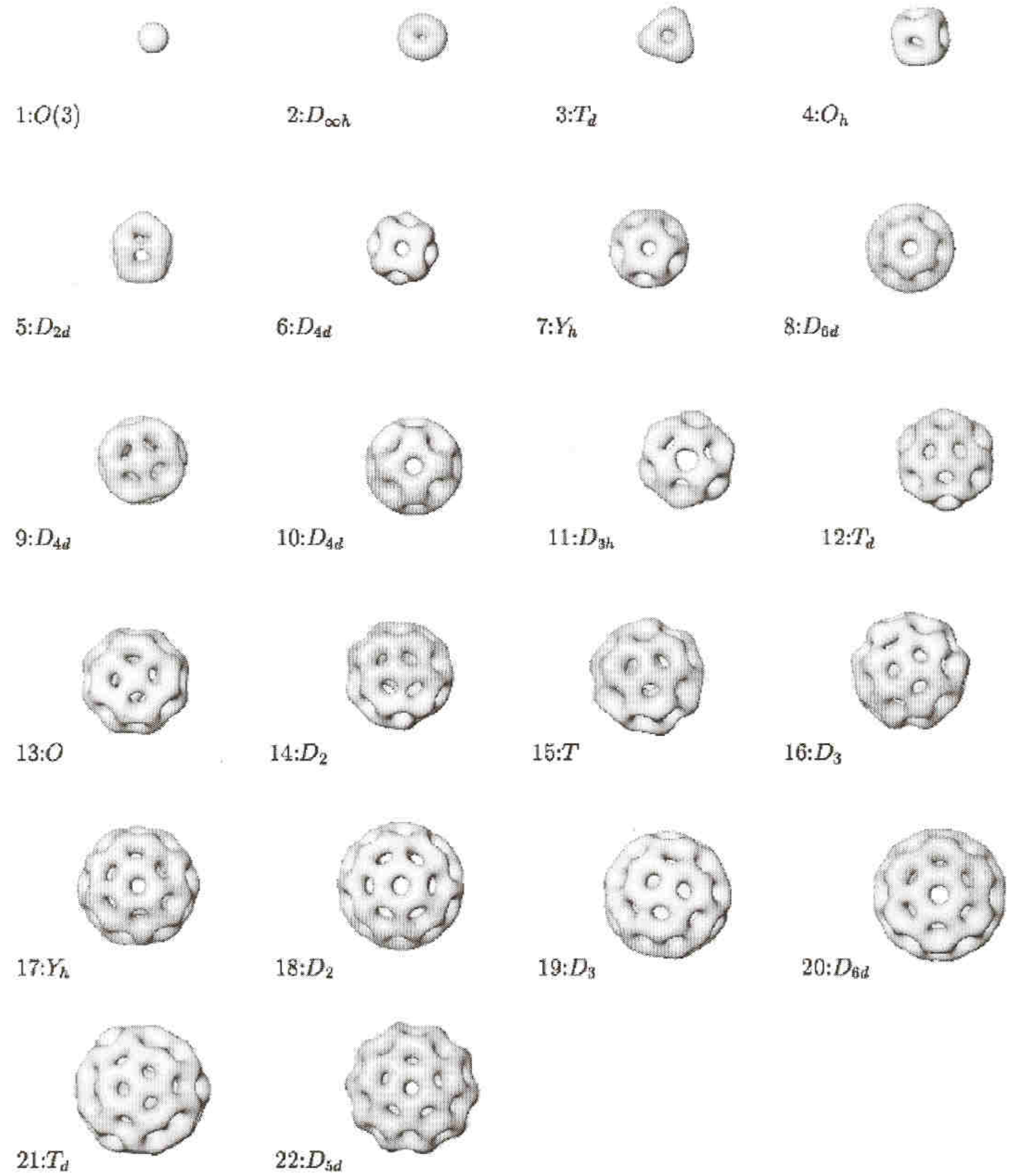
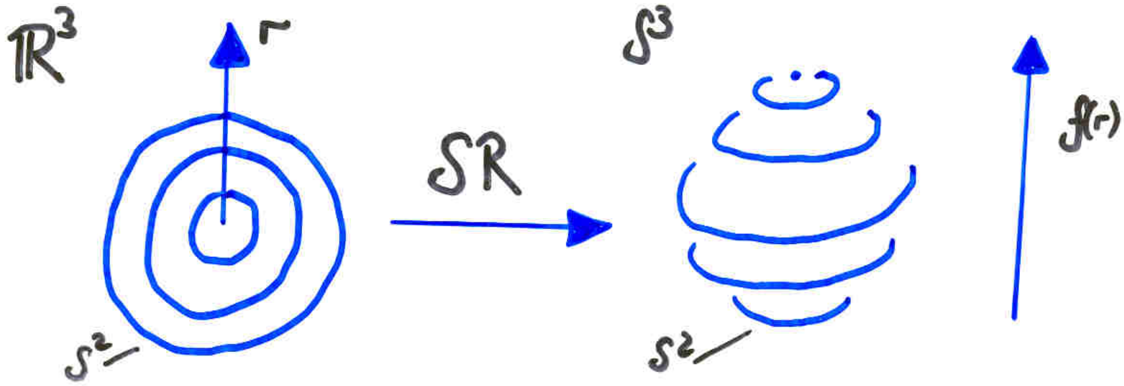


Figure 1: The baryon density isosurfaces of the Skyrmions with $B = 1 - 22$ which are minimum energy configurations (see table 1) within the rational map ansatz. Each corresponds to a value of $\mathcal{B} = 0.035$ and are presented to scale.

Suspensions and the Rational Map Ansatz

- *Suspension*: Given an interval $I = [0,1]$, a manifold M define $\mathcal{S}M = I \times M/(\{0\} \times M \cup \{1\} \times M)$.
- Note: $\mathcal{S}S^n = S^{n+1}$.
- Let $R : S^2 \to S^2$, holomorphic, let $Rat_B = \{R : \deg R = B\}$. (This implies $R(z) = p(z)/q(z)$).
- Define $U = \mathcal{S}R : S^3 \to S^3 : (r, z) \mapsto (f(r), R(z))$.



- The rational map ansatz is a good approximation to the numerical solutions. (Energy, symmetries.)
- This construction has nice properties with respect to homotopy groups.

More Details about Rational Map Ansatz

- Insert ansatz into E:

$$E = 4\pi \int \left(r^2 f'^2 + 2B(f'^2 + 1)\sin^2 f + \mathcal{I}\, \frac{\sin^4 f}{r^2} \right) \mathrm{d}r,$$

where

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{\mathrm{d}R}{\mathrm{d}z} \right| \right)^4 \frac{2i\, \mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}.$$

- Minimize $\mathcal{I}$ on the space of rational maps. Then calculate $f(r)$, subject to $f(0) = \pi$ and $f(\infty) = 0$.
- Good approximations for symmetries and energies of Skyrmions.
- The degree B:

$$B = \frac{1}{4\pi} \int \left| \frac{\mathrm{d}R}{\mathrm{d}z} \right|^2 \frac{2i\, \mathrm{d}z\mathrm{d}\bar{z}}{(1+|R|^2)^2}.$$

- Note:

$$\frac{\mathrm{d}R}{\mathrm{d}z} \propto p'q - pq'$$

which is a polynomial of degree $2B - 2$. Zeros correspond to faces where the nucleon density vanishes and the energy density is very low. (There are also polynomials whose zeros correspond to edges and vertices.)

Examples of Rational Maps

- $B = 1$ (The hedgehog)

$$R(z) = z$$

has spherical symmetry.

- $B = 2$ (The torus)

$$R(z) = z^2$$

has the symmetries

$$R\left(e^{i\alpha}z\right) = e^{2i\alpha}R(z) \quad \text{and} \quad R\left(\frac{1}{z}\right) = \frac{1}{R(z)}.$$

- $B = 3$ (The tetrahedron)

$$R(z) = \frac{\sqrt{3}iz^2 - 1}{z^3 - \sqrt{3}iz}$$

has symmetries

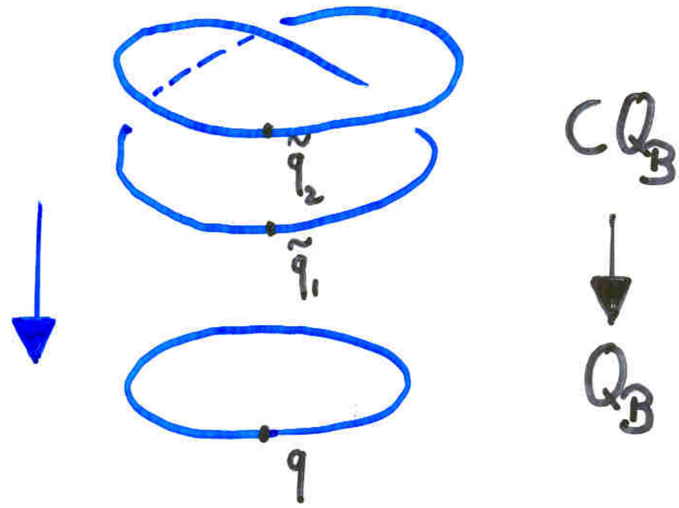
$$R\left(\frac{1}{z}\right) = \frac{1}{R(z)} \quad \text{and} \quad R\left(\frac{iz+1}{-iz+1}\right) = \frac{iR(z)+1}{-iR(z)+1}.$$

- Use group theory to construct "very symmetric" Skyrmions.

The Finkelstein–Rubinstein Constraints

- How do you define a fermionic quantization only with scalar fields?
- Note: $\pi_1(Q_B) = \mathbb{Z}_2$.
- Define wave functions ψ on covering space CQ_B:

$$\psi : CQ_B \to \mathbb{C}.$$



- Impose $\psi(\tilde{q}_1) = -\psi(\tilde{q}_2)$.
- Action of symmetries $SO(3) \times SO(3)$ on ψ:

$$\exp\left(-i\alpha\mathbf{n}\cdot\mathbf{J}\right)\exp\left(-i\beta\mathbf{N}\cdot\mathbf{I}\right)\psi(\tilde{q}) = \chi_{FR}\,\psi(\tilde{q}).$$

$$\text{where } \chi_{FR} = \begin{cases} 1 & \text{if the induced loop is contractible,} \\ -1 & \text{otherwise.} \end{cases}$$

- Calculate $\chi_{FR} \in \pi_1(Q_B)$?

Whitehead's Theorem

- **Theorem (Whitehead):** *Let $F^p(X, x_0)$ be the space of based maps $f : S^p \to X$ such that $f(1) = x_0$. Then the connected components of $F^p(X, x_0)$ are homotopy equivalent.*

- It follows that the homotopy groups $\pi_k(Q_B)$ are independent of B.

- (Physical motivation: Physics should be the same if we add a particle or antiparticle, provided it is far enough away.)

- Let $Rat_B^* = \{R \in Rat_B : R(\infty) = 1\}$.

- Let M_B the space of continuous maps $S^2 \to S^2$ of degree B and M_B^* the space of based maps in M_B.

- **Theorem (Segal):** *M_B^* and Rat_B^* are homotopy equivalent up to B.*

Relationship between $\pi_1(Rat_B^*)$ and $\pi_1(Q_B^*)$

- **Theorem (Freudenthal):** *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.*

- $\pi_2(S^2) \to \pi_3(S^3)$ is an isomorphism:
- It follows that rational maps of degree B give rise to Skyrme configurations of degree B.
- $\pi_3(S^2) \to \pi_4(S^3)$ is surjective:
- Note:

$$\pi_1(M_0^*) \cong \pi_3(S^2) \cong \mathbb{Z}$$

and

$$\pi_1(Q_0^*) \cong \pi_4(S^3) \cong \mathbb{Z}_2.$$

- **Theorem (S.K.):** *The rational map ansatz induces a surjective homomorphism $\pi_1(Rat_B^*) \to \pi_1(Q_B^*)$.*

Topology of Rational Maps

- **Theorem (Segal):** *$\pi_1(Rat_B^*) \cong \mathbb{Z}$ and it is generated by moving one zero once around one pole.*

- ($\pi_1(Rat_B) = \mathbb{Z}_{2B}$.)

- $R \in Rat_B^*$ can be written as

$$R(z) = \frac{z^B + a_{B-1}z^{B-1} + \cdots + a_0}{z^B + b_{B-1}z^{B-1} + \cdots + b_0} = \prod_{i,j=1}^{B} \frac{z - z_i}{z - p_j}.$$

- Given a loop L that moves zeros z_i and poles p_j around the complex plane as a function of $\phi \in [0, \Phi]$, let

$$N(L) = \frac{i}{2\pi} \sum_{i,j=1}^{B} \int_0^{\Phi} \frac{(z_i'(\phi) - p_j'(\phi))\mathrm{d}\phi}{(z_i(\phi) - p_j(\phi))}.$$

- **Lemma:** *$N(L)$ is a homotopy invariant and counts the number of times zeros move around poles. Therefore, $N(L)$ provides an isomorphism $\pi_1(Rat_B^*) \to \mathbb{Z}$.*

A Formula for $N(L)$

- Consider the axially symmetric map

$$R(z) = \frac{z^B - b}{z^B + b}$$

for $b \neq 0$.

- Rotation by α around the x_3-axis

$$z \to e^{i\alpha} z.$$

- Isorotation by β around the X_1-axis

$$R \to \frac{\cos(\beta/2)R - i\sin(\beta/2)}{-i\sin(\beta/2)R + \cos(\beta/2)}.$$

- Calculate N:

$$N = \frac{B}{2\pi}\left(B\alpha - \beta\right).$$

- **Lemma:** *The formula holds for general $R \in Rat_B$.*

$R = z^2 \quad n_R =$

$z = e^{i\varphi} \tan\frac{\theta}{2}$

$$n = \frac{1}{1+|z|^2}\begin{pmatrix} z+\bar{z} \\ \frac{1}{i}(z-\bar{z}) \\ 1-|z|^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$$

Calculation of Groundstates

- Recall $\pi_1(Q_B) = \mathbb{Z}_2$. Note: $\pi_1(Rat_B) = \mathbb{Z}$.

- **Theorem** (S.K., 2002):
  The homomorphism $\pi_1(Rat_B) \to \pi_1(Q_B)$ is surjective.

- Recall: $\exp\left(-i\alpha\mathbf{n}\cdot\mathbf{J}\right)\exp\left(-i\beta\mathbf{N}\cdot\mathbf{I}\right)\psi(\tilde{q}) = \chi_{FR}\,\psi(\tilde{q})$.

- $\chi_{FR} = (-1)^N$, where $N = \frac{B}{2\pi}(B\alpha - \beta)$.

- $SO(3)\times SO(3)$ action commutes with energy E.
  Invariant subspaces are labelled by J and I. $|J\,J_3 L_3\rangle|I\,I_3 K_3\rangle$

- Approximation: The ground state for given B occurs for the lowest values of I and J which are compatible with the FR constraints.

- Count number of the irreducible representation χ_{FR} using representation theory.

Results

- Calculate FR constraints from the rational map ansatz.
- Assume wave function has the same symmetry as the classical minimal energy configuration.
- The isospin I_3 measures the difference between the numbers of protons and neutrons. ($\#p = B/2 + I_3$ and $\#n = B/2 - I_3$)
- Calculate groundstates $|J, I\rangle$ for $B = 1, \ldots, 22$.
- $B = 1$: $|\frac{1}{2}, \frac{1}{2}\rangle$ corresponds to ${}^1_1$H.
- $B = 2$: $|1, 0\rangle$ corresponds to the deuteron ${}^2_1$H.
- $B = 3$: $|\frac{1}{2}, \frac{1}{2}\rangle$ corresponds to ${}^3_2$He.
- $B = 4$: $|0, 0\rangle$ corresponds to the alpha particle ${}^4_2$He.
- For $B = 8, 12, 16, 20$ the ground state is $|0, 0\rangle$ in agreement with experiment. These states are particularly stable.
- For $B = 5$ my calculations disagree with experiments. (However, there is no stable nucleus with $B = 5$.)
- For odd B my calculations also disagree with experiment for $B = 7, 9, 11, 13, 17$ and 21.
- For even B my calculations disagree with experiment for $B = 10, 18$ and 22.

Conclusion

- Found a way to calculate χ_{FR} directly.
- My calculations of the ground states show whether the symmetries of the classical configurations are compatible with the ground states.
- Consider the energy of a state:

$$E \approx M_{classical} + \frac{\hbar^2}{2\Theta_J} J(J+1) + \frac{\hbar^2}{2\Theta_I} I(I+1)$$

- Works well for even B, but not so well for odd B (take local minima into account)
- Note: when the pion mass is taken into account then the solutions and their symmetries might change.