Prequantum physics in a cohesive ∞-topos

Talk at *Quantum Physics and Logic* 2011

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- Mike Shulman
- Richard Williamson

Details and references at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

Motivation

Cohesive ∞-toposes

Geometric action functionals

Addendum – Technical details

Quantum Physics and Logic?

Physicist

Logician

Physicist: Logician

I have a theory!

Physicist

Logician:

Nice!

Physicist

Logician:

I am an expert on theories.

Physicist

Logician:

Which one is it?

Physicist: Logician

It has a field ϕ...

Physicist: Logician

It has a field ϕ
with **many** *indices!*

Physicist Logician:

?

Physicist: Logician

And the action functional...

Physicist: Logician

And the action functional...

$S(\phi) = \cdots$

Physicist: Logician

...has a kinetic term... ?

$$S(\phi) = \frac{1}{2}\langle \phi, D\phi \rangle + \cdots$$

Physicist: Logician

...and interaction given by... ?

$$S(\phi) = \frac{1}{2}\langle \phi, D\phi \rangle + \cdots$$

Physicist: Logician

...a cubic term... ??

$$S(\phi) = \frac{1}{2}\langle \phi, D\phi \rangle + \frac{1}{6}\langle \phi, [\phi, \phi] \rangle + \cdots$$

Physicist: Logician

...and a quartic term... ???

$$
\begin{aligned}
S(\phi) &= \frac{1}{2}\langle \phi, D\phi \rangle \\
&\quad + \frac{1}{6}\langle \phi, [\phi, \phi] \rangle \\
&\quad + \frac{1}{24}\langle \phi, [\phi, \phi, \phi] \rangle + \cdots
\end{aligned}
$$

Physicist: Logician

...and a quintic term... ????

$$S(\phi) = \frac{1}{2}\langle \phi, D\phi \rangle + \frac{1}{6}\langle \phi, [\phi, \phi] \rangle + \frac{1}{24}\langle \phi, [\phi, \phi, \phi] \rangle + \frac{1}{120}\langle \phi, [\phi, \phi, \phi, \phi] \rangle + \cdots$$

Physicist: Logician

...and so on. $?^\infty$

$$S(\phi) = \frac{1}{2}\langle \phi, D\phi \rangle + \sum_{k=2}^{\infty} \frac{1}{(k+1)!}\langle \phi, [\phi^k] \rangle$$

Physicist

Logician

Physicist

Logician:

Is that it?

Physicist: Logician

Next I quantize this!

Physicist

Logician:

%#!&

Physicist

Logician:

%#!&

$\boxed{\text{Enough!}}$ $\boxed{\text{Go on!}}$

Physicist

Logician:

%#!&

Physicist

Logician:

Let's see...

Physicist

Logician:

What are the models of your "theory"?

Physicist: Logician

Everything!

Physicist

Logician:

Everything?

Physicist: Logician

Yes, in physics:

Physicist: Logician

Yes, in physics:
electromagnetism,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields, C-fields,

Physicist: Logician

Yes, in physics:

electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields, C-fields, RR-fields,

Physicist: Logician

Yes, in physics:

electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields, C-fields, RR-fields,
Chern-Simons fields,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields, C-fields, RR-fields,
Chern-Simons fields,
Poisson σ-model fields,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields, C-fields, RR-fields,
Chern-Simons fields,
Poisson σ-model fields, Courant σ-model fields,

Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity,
electrons, quarks, gravitinos,
B-fields, C-fields, RR-fields,
Chern-Simons fields,
Poisson σ-model fields, Courant σ-model fields,
string fields,...

Physicist

Logician:

Hold it!

Physicist

Logician:

What's going on here?

Is there a *formal theory* of

1. geometric action functionals;
2. their quantization

that produces the fundamental *physical theories* of interest?

Notice that **quantized** field theory
has been identified with a
universal construction
in *higher category theory*.

Crash course in higher category theory:

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$\boxed{\text{An } (\infty, n)\text{-category is a directed space in which } (k \leq n)\text{-dimensional paths need not be reversible.}}$

Crash course in higher category theory:

An (∞, n)-category
is a directed space
in which $(k \leq n)$-dimensional paths
need not be reversible.

So an $(\infty, 0)$-category
is an ∞-groupoid is a space.

Crash course in higher category theory:

An (∞, n)-category
is a directed space
in which $(k \leq n)$-dimensional paths
need not be reversible.

So an $(\infty, 0)$-category
is an ∞-groupoid is a space.

End of the crash course.

again:

Notice that **quantized** field theory
has been identified with a
universal construction
in *higher category theory*.

again:

Notice that **quantized** field theory
has recently been identified with a
universal construction
in *higher category theory*.

$\boxed{\text{Sure.}}$ $\boxed{\text{What?}}$

cobordism theorem, roughly:

$$(\infty, n)\mathrm{Cat}$$

All (∞, n)-categories.

cobordism theorem, roughly:

$$\mathrm{SymMon}(\infty, n)\mathrm{Cat} \qquad (\infty, n)\mathrm{Cat}$$

Those with symmetric monoidal structure.

cobordism theorem, roughly:

$$U : \mathrm{SymMon}(\infty, n)\mathrm{Cat} \xrightarrow[\text{forget}]{} (\infty, n)\mathrm{Cat}$$

Forget the structure.

cobordism theorem, roughly:

$$U : \mathrm{SymMon}(\infty, n)\mathrm{Cat} \underset{\text{forget}}{\overset{\text{free}}{\rightleftarrows}} (\infty, n)\mathrm{Cat} \; : F$$

Or generate it freely.

Example:

$$\mathrm{Bord}_n$$

($k \leq n$)-paths are k-dimensional cobordisms, ($k > n$)-paths are diffeomorphisms.

Example:

$$n\mathrm{Vect}$$

Points are higher analogs of vector spaces, paths are higher linear maps.

cobordism theorem, roughly:

$$\mathrm{Bord}_n \xrightarrow{Z} n\mathrm{Vect}$$

A topological quantized field theory is a symmetric monoidal functor.

Lurie (Baez-Dolan):

$$F(*) \xrightarrow{\simeq} \mathrm{Bord}_n \xrightarrow{Z} \quad n\mathrm{Vect}$$

A topological quantized field theory is a symmetric monoidal functor.

cobordism theorem, roughly:

Lurie (Baez-Dolan):

$$\frac{F(*) \xrightarrow{\simeq} \mathrm{Bord}_n \xrightarrow{Z} n\mathrm{Vect}}{* \xrightarrow{Z(*)} U(n\mathrm{Vect})}$$

$Z(*)$ is the *n-space of states*.

So n-dimensional topological QFT
is characterized by its
n-space of states
$Z(*)$.

In nature,
the space of states of a QFT
is not random,
but arises from *quantization*
of **geometric** action functionals,
such as

$$S : \phi \mapsto \frac{1}{2}\langle \phi, D\phi \rangle + \sum_{k=2}^{\infty} \frac{1}{(k+1)!}\langle \phi, [\phi^k] \rangle .$$

Task:

1. formalize differential geometry;
2. formally derive these action functionals;
3. and their quantization.

Task:

1. formalize differential geometry;
2. formally derive these action functionals;
3. and their quantization.

Solution:

By a universal construction
in *higher topos theory*...

II
Cohesive $(\infty, 1)$-toposes

$$\mathrm{Set}$$

The category of sets.

$\mathbf{H}$ $\quad$ Set

A category of geometric structures.

$$\mathbf{H} \xrightarrow{\quad \Gamma \quad} \mathrm{Set}$$

The underlying set.

$$\mathbf{H} \underset{\Gamma}{\overset{\mathrm{Disc}}{\leftrightarrows}} \mathrm{Set}$$

The discrete (free) geometric structure.

$$\mathbf{H} \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \mathrm{Set}$$

Convention throughout:
a morphism on top of another one denotes a *left adjoint*.

$$\mathrm{Disc} \dashv \Gamma$$

$$\begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, "\Gamma"'] & \mathrm{Set} \arrow[l, hook', bend right, "\mathrm{Disc}"'] \arrow[l, hook, bend left, "\color{green}{\mathrm{coDisc}}"] \end{tikzcd}$$

The codiscrete geometric structure.

$$\begin{tikzcd}[column sep=huge] \mathbf{H} \arrow[r, shift left=4.5, "\Pi_0"] \arrow[r, shift right=1.5, "\Gamma" description] & \mathrm{Set} \arrow[l, hook', shift right=1.5, "\mathrm{Disc}" description] \arrow[l, hook', shift left=4.5, "\mathrm{coDisc}"] \end{tikzcd}$$

The set of connected components.

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=3, "\Pi_0"] & \mathrm{Set} \arrow[l, hook', shift left=1, "\mathrm{Disc}"'] \arrow[l, hook', shift right=3, "\mathrm{coDisc}"] \\ \end{tikzcd}$$

$$\mathbf{H} \xrightarrow{\Gamma} \mathrm{Set}$$

If $\mathbf{H}$ a topos and $\Pi_0(*) \simeq *$: **cohesive topos** (Lawvere).

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=3, "\Pi_0"] \arrow[r, phantom, shift left=1, ""] & \mathrm{Set} \arrow[l, hook', shift right=1, "\mathrm{Disc}" description] \arrow[l, hook', shift left=3, "\mathrm{coDisc}"'] \end{tikzcd}$$

with $\Gamma : \mathbf{H} \to \mathrm{Set}$

For instance $\mathbf{H}$ = sheaves on smooth $\mathbb{R}^n$s: smooth manifolds and diffeological spaces.

We may consider this also in *higher topos theory*.

$$\infty\mathrm{Grpd}$$

The $(\infty, 1)$-category of ∞-groupoids ($\simeq$ spaces).

$\mathbf{H}$ $\qquad\qquad$ $\infty\mathrm{Grpd}$

An $(\infty, 1)$-category of geometric structures.

$$\mathbf{H} \xrightarrow{\quad\quad\Gamma\quad\quad} \infty\mathrm{Grpd}$$

The underlying ∞-groupoid.

$$\mathbf{H} \underset{\Gamma}{\overset{\mathrm{Disc}}{\leftrightarrows}} \infty\mathrm{Grpd}$$

The discrete (free) geometric structure.

$$\mathbf{H} \underset{\color{green}{\mathrm{coDisc}}}{\overset{\mathrm{Disc}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\Gamma}}}}} \infty\mathrm{Grpd}$$

The codiscrete geometric structure.

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H} \arrow[r, shift left=6, "\Pi"] \arrow[r, shift right=2, "\Gamma" description] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=2, "\mathrm{Disc}" description] \arrow[l, hook', shift left=6, "\mathrm{coDisc}"]
\end{tikzcd}
$$

The...

$$\begin{tikzcd}[row sep=small]
\mathbf{H} \arrow[r, shift left=6, "\Pi"] \arrow[r, shift right=2, "\Gamma"] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=2, "\mathrm{Disc}"'] \arrow[l, hook', shift left=6, "\mathrm{coDisc}"]
\end{tikzcd}$$

The ∞-groupoid of paths.

$$\begin{tikzcd} \mathbf{H} \arrow[r, shift left=3, "\Pi"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, shift right=1, "\Gamma"'] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1, "\mathrm{Disc}"'] \arrow[l, hook', shift left=3, "\mathrm{coDisc}"] \end{tikzcd}$$

The ∞-groupoid of **paths**!

$$
\begin{tikzcd}
\mathbf{H} \arrow[r, shift left=3, "\Pi"] \arrow[r, shift right=1, "\Gamma"'] & \infty\mathrm{Grpd} \arrow[l, hook', shift right=1, "\mathrm{Disc}"'] \arrow[l, hook', shift left=3, "\mathrm{coDisc}"]
\end{tikzcd}
$$

If $\mathbf{H}$ an $(\infty,1)$-topos and $\Pi(*) \simeq *$: **cohesive $(\infty,1)$-topos**.

$$\mathbf{H} \underoverset{\underset{\mathrm{coDisc}}{\hookleftarrow}}{\overset{\overset{\Pi}{\longrightarrow}}{\underset{\Gamma}{\overset{\mathrm{Disc}}{\hookleftarrow}\longrightarrow}}}{} \infty\mathrm{Grpd}$$

For instance $\mathbf{H} = \infty$-stacks on smooth $\mathbb{R}^n$s: smooth ∞-groupoids.

The intrinsic existence of paths gives rise to an intrinsic *dynamics*...

III
Geometric action functionals

Reflect paths back to $\mathbf{H}$:

$$(\mathbf{\Pi} \dashv \flat) : \mathbf{H} \underset{\Gamma}{\overset{\mathrm{Disc}}{\rightleftarrows}} \infty\mathrm{Grpd} \underset{\mathrm{Disc}}{\overset{\mathbf{\Pi}}{\rightleftarrows}} \mathbf{H}$$

Reflect paths back to $\mathbf{H}$:

$$(\mathbf{\Pi} \dashv \flat) : \mathbf{H} \underset{\Gamma}{\overset{\mathrm{Disc}}{\leftrightarrows}} \infty\mathrm{Grpd} \underset{\mathrm{Disc}}{\overset{\mathbf{\Pi}}{\leftrightarrows}} \mathbf{H}$$

"$\flat$" is pronounced "flat"

A morphism

$$\phi : \frac{\mathbf{\Pi}(X) \to A}{X \to \flat A}$$

is *flat parallel transport* with values in A.

$$
\begin{tikzcd}
A \\
\mathbf{\Pi}(X) \arrow[u, "\phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \phi(x_2) \arrow[dr, "\phi(\gamma_2)"] \arrow[d, phantom, "\Downarrow \phi(\Sigma)"] & \\
\phi(x_1) \arrow[ur, "\phi(\gamma_1)"] \arrow[rr, "\phi(\gamma_3)"'] & {} & \phi(x_3)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & x_2 \arrow[dr, "\gamma_2"] \arrow[d, phantom, "\Downarrow \Sigma"] & \\
x_1 \arrow[ur, "\gamma_1"] \arrow[rr, "\gamma_3"'] & {} & x_3
\end{tikzcd}
$$

Alternative perspective:

$$\phi : \frac{\mathbf{\Pi}(X) \to A}{X \to \flat A}$$

is *A-valued field*

with vanishing field strength.

For instance we could have

$$A = U(1)$$

the circle group.

For instance we could have

$$A = \mathbf{B}U(1)$$

the one-object groupoid with $\mathrm{End}(*) = U(1)$.

For instance we could have

$$A = \mathbf{B}^2 U(1)$$

the one-object 2-groupoid with $\mathrm{End}(*) = \mathbf{B}U(1)$.

For instance we could have

$$A = \mathbf{B}^{n+1} U(1)$$

the one-object $(n+1)$-groupoid with $\mathrm{End}(*) = \mathbf{B}^n U(1)$.

A morphism

$$X \to \flat \mathbf{B}^{n+1} U(1)$$

encodes locally a closed $(n+1)$-form on X

(and globally a bit more).

X an $(n+1)$-dim compact manifold:

volume holonomy functional

$$[X, \flat \mathbf{B}^{n+1} U(1)] \xrightarrow{\int_X} U(1)$$

field space

Therefore every morphism

$$\mathbf{c} : A \to \mathbf{B}^{n+1} U(1)$$

induces a functional

$$\exp(i S_{\mathrm{CS}_{\mathbf{c}}}) : [\Sigma, \flat A] \xrightarrow{\int_\Sigma \flat \mathbf{c}} U(1)$$

on flat A-valued fields.

After generalization
to *non-flat* fields
this is the
geometric action functional
that we are after
("higher Chern-Weil theory").

Refinement to *non-flat* fields; by canonical factorization:

$$\flat \mathbf{B}^{n+1}U(1) \xrightarrow{\text{counit}} \mathbf{B}^{n+1}U(1)$$

flat fields

underlying cocycles

Refinement to *non-flat* fields; by canonical factorization:

$$\flat \mathbf{B}^{n+1} U(1) \longrightarrow \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^{n+1} U(1)$$

flat fields — general fields — underlying cocycles

Refinement to *non-flat* fields; by canonical factorization:

$$\flat \mathbf{B}^{n+1} U(1) \longrightarrow \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \longrightarrow \mathbf{B}^{n+1} U(1)$$

flat fields | general fields | underlying cocycles

explained in $\boxed{\text{Addendum}}$.

By naturality, $\mathbf{c}$ gives

$$\begin{tikzcd} \flat A \arrow[r, "\flat \mathbf{c}"] \arrow[d] & \flat \mathbf{B}^{n+1} U(1) \arrow[d] \\ A \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1} U(1) \end{tikzcd}$$

with factorization

$$
\begin{tikzcd}
\flat A \arrow[r, "\flat \mathbf{c}"] \arrow[dd] & \flat \mathbf{B}^{n+1} U(1) \arrow[d] \\
& \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[d] \\
A \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1} U(1)
\end{tikzcd}
$$

then consider

$$\begin{tikzcd}
\flat A \arrow[r, "\flat \mathbf{c}"] \arrow[d] & \flat \mathbf{B}^{n+1} U(1) \arrow[d] \\
A_{\mathrm{conn}} \arrow[r, "\hat{\mathbf{c}}"] \arrow[d] & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[d] \\
A \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1} U(1)
\end{tikzcd}$$

A morphism

$$X \to A_{\mathrm{conn}}$$

encodes A-valued fields

(and globally a bit more).

Therefore every morphism

$$\hat{\mathbf{c}} : A_{\mathrm{conn}} \to \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}$$

induces a functional

$$\exp(iS_{\mathrm{CS}_{\hat{\mathbf{c}}}}) : [X, A_{\mathrm{conn}}] \xrightarrow{\int_X \hat{\mathbf{c}}} U(1)$$

on A-valued fields.

Example.

- $\mathfrak{g}$ an L_∞-algebra
- invariant bilinear $\langle -, - \rangle$
- $A := \exp(\mathfrak{g})$
- $\mathbf{c} := \exp(\langle -, - \rangle)$

$$S_{\mathrm{CS}_{\hat{\mathbf{c}}}}(\phi) = \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \langle \phi, [\phi^k] \rangle \, .$$

But much more:

$$\hat{\mathbf{c}} : A_{\mathrm{conn}} \to \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}$$

is the full

"prequantum

circle $(n+1)$-bundle

with connection

on the moduli ∞-stack

of fields"

Therefore next:
apply
higher geometric quantization
to $\hat{\mathbf{c}}$
to obtain $Z(*)$.

Therefore next:
apply
higher geometric quantization
to $\hat{\mathbf{c}}$
to obtain $Z(*)$.

But not today.

End.

view $\boxed{\text{Addendum}}$

Addendum

Some technical details.

On the derivation of
geometric action functionals.

G

A group object in **H**.

G

A group object: “cohesive ∞-group”.

G

A group object: "grouplike cohesive A_∞-space".

$\mathbf{B}G$

Equivalently its *delooping*: $\mathrm{Hom}_{\mathbf{B}G}(*,*) \simeq G$.

$\mathbf{B}G$

Equivalently its *delooping*: $\infty\mathrm{Grp}(\mathbf{H}) \underset{\mathbf{B}}{\overset{\mathbf{\Omega}}{\underset{\simeq}{\rightleftarrows}}} \mathbf{H}^{*/}_{\geq 0}$

$\mathbf{B}G$ $\mathbf{B}U(1)$

For instance for $U(1) := \mathbb{R}/\mathbb{Z}$.

$\mathbf{B}G$ $\qquad\qquad$ $\mathbf{B}U(1)$

Since $U(1)$ is abelian, the delooping $\mathbf{B}U(1)$ is a group object itself.

$\mathbf{B}G$ $\mathbf{B}^2 U(1)$

Hence there is a second delooping.

$\mathbf{B}G$ $\mathbf{B}^3 U(1)$

And a third.

$\mathbf{B}G \qquad\qquad \mathbf{B}^{n+1}U(1)$

And so on.

$$\begin{array}{ccc} \mathbf{B}G & \qquad & \mathbf{B}^{n+1}U(1) \\ \Big\downarrow{\scriptstyle \Pi} & & \Big\downarrow{\scriptstyle \Pi} \\ BG & & K(\mathbb{Z}, n+2) \end{array}$$

Under geometric realization...

$$
\begin{tikzcd}
\mathbf{B}G \arrow[d, mapsto, "\Pi"] & \mathbf{B}^{n+1}U(1) \arrow[d, mapsto, "\Pi"] \\
BG \arrow[r, "c"] & K(\mathbb{Z}, n+2)
\end{tikzcd}
$$

... this classifies integral cohomology $[c] \in H^{n+2}(BG, \mathbb{Z})$.

$$\mathbf{B}G \qquad\qquad \mathbf{B}^{n+1}U(1)$$

$$BG \xrightarrow{c} K(\mathbb{Z}, n+2)$$

Such $[c]$ is a *universal characteristic class*.

$\mathbf{B}G \qquad\qquad \mathbf{B}^{n+1}U(1)$

$$
\begin{tikzcd}
BG \arrow[r, "c"] & K(\mathbb{Z}, n+2) \\
X \arrow[u, "P"] \arrow[ur, "c(P)"'] &
\end{tikzcd}
$$

It takes equivalence classes $[P]$ of G-bundles to integral cohomology.

$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr, "\mathbf{c}"] \arrow[d, mapsto, "\Pi"] & & \mathbf{B}^{n+1}U(1) \arrow[d, mapsto, "\Pi"] \\
BG \arrow[rr, "c"] & & K(\mathbb{Z}, n+2)
\end{tikzcd}
$$

Let **c** be a *cohesive refinement*.

$$\begin{tikzcd} X \arrow[d, "P"] & \\ \mathbf{B}G \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1}U(1) \end{tikzcd}$$

This takes G-bundles P...

$$\begin{tikzcd} X \arrow[d, "P"] \arrow[drr, "\mathbf{c}(P)"] & & \\ \mathbf{B}G \arrow[rr, "\mathbf{c}"] & & \mathbf{B}^{n+1}U(1) \end{tikzcd}$$

... to $\mathbf{B}^{n+1}U(1)$-bundles $\mathbf{c}(P)$.

$$
\begin{tikzcd}
\mathbf{B}U(n) \arrow[d, mapsto, "\Pi"] & & \mathbf{B}^1U(1) \arrow[d, mapsto, "\Pi"] \\
BU(n) \arrow[rr, "c_1"] & & K(\mathbb{Z}, 2)
\end{tikzcd}
$$

Example: first Chern class of unitary bundles.

$$
\begin{tikzcd}
\mathbf{B}U(n) \arrow[r, "\mathbf{B}\,\mathbf{det}"] \arrow[d, maps to, "\Pi"] & \mathbf{B}^1U(1) \arrow[d, maps to, "\Pi"] \\
BU(n) \arrow[r, "c_1"] & K(\mathbb{Z}, 2)
\end{tikzcd}
$$

Lifted by determinant function.

$$\begin{tikzcd} X \arrow[d, "P"] & \\ \mathbf{B}U(n) \arrow[r, "\mathbf{B}\,\mathbf{det}"] & \mathbf{B}U(1) \end{tikzcd}$$

This takes a unitary bundle P...

$$\begin{tikzcd}
X \arrow[d, "P"] \arrow[drr, "\mathrm{det}(P)"] & & \\
\mathbf{B}U(n) \arrow[rr, "\mathbf{B}\,\mathbf{det}"] & & \mathbf{B}U(1)
\end{tikzcd}$$

... to its *derminant line bundle* $\mathrm{det}(P) \otimes_{U(1)} \mathbb{C}$.

$$\mathbf{B}G \xrightarrow{\quad \mathbf{c} \quad} \mathbf{B}^{n+1}U(1)$$

$$BG \xrightarrow{\quad c \quad} K(\mathbb{Z}, n+2)$$

Generally, for any cohesive characteristic map...

$$\xrightarrow{\qquad \mathbf{c}_{\mathrm{conn}} \qquad}$$

$$\mathbf{B}G \xrightarrow{\qquad \mathbf{c} \qquad} \mathbf{B}^{n+1}U(1)$$

$$BG \xrightarrow{\qquad c \qquad} K(\mathbb{Z}, n+2)$$

... we may ask for a further *differential* refinement...

$$\mathbf{B}G_{\mathrm{conn}} \xrightarrow{\mathbf{c}_{\mathrm{conn}}} \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}$$

$$\mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^{n+1}U(1)$$

$$BG \xrightarrow{c} K(\mathbb{Z}, n+2)$$

... which takes *G-connections* to $\mathbf{B}^n U(1)$*-connections*.

$$\mathbf{B}^{n+1}U(1)$$

To construct this, first differentially refine the coefficient object...

$$\mathbf{B}^{n+1}U(1)$$

$$\mathbf{B}^{n+2}U(1)$$

To that end, first consider one more delooping...

$$
\begin{tikzcd}
\mathbf{B}^{n+1}U(1) & & \\
& \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

..and the universal map it receives from the flat coefficients.

$$\mathbf{B}^{n+1}U(1) \qquad \begin{tikzcd} \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1) \end{tikzcd}$$

The homotopy fiber of this...

$$\begin{tikzcd}
X \arrow[drr, bend right, "\nabla"'] & & & \\
& & \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
& & \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}$$

...classifies flat $\mathbf{B}^{n+1}U(1)$-connections

$$
\begin{tikzcd}
X \arrow[drr, bend left, ""] \arrow[ddr, bend right, "\nabla"'] & & \\
& \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
& \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

...whose underlying $\mathbf{B}^{n+1}U(1)$-bundle is trivial.

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right, "\nabla"'] & & \\
& \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
& \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}$$

(By the universal property of homotopy pullbacks.)

$$
\begin{tikzcd}
X \arrow[dr, "\omega"] & & \\
& \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
& \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

But this are closed differential $(n+2)$-forms $\omega \in \Omega^{n+2}_{\mathrm{cl}}(X)$!

$$
\begin{tikzcd}
 & \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

Canonical example: pull back further along point inclusion...

$$
\begin{tikzcd}
\mathbf{B}^{n+1}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

... to recover $\mathbf{B}^{n=1}U(1) \simeq \Omega\mathbf{B}^{n+2}U(1)$...

$$
\begin{tikzcd}
\mathbf{B}^{n+1}U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

...equipped with universal form **curv**.

$$
\begin{tikzcd}
\mathbf{B}^{n+1}U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

These are the first steps in constructing a long fiber sequence...

$$
\begin{tikzcd}
 & * \arrow[d] & \\
\mathbf{B}^{n+1}U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}^{n+2}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat\mathbf{B}^{n+2}U(1) \arrow[r] & \mathbf{B}^{n+2}U(1)
\end{tikzcd}
$$

… the next step…

$$
\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] & \\
\mathbf{B}^{n+1} U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}} \mathbf{B}^{n+2} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat \mathbf{B}^{n+2} U(1) \arrow[r] & \mathbf{B}^{n+2} U(1)
\end{tikzcd}
$$

...recovers the flat coefficients of $\mathbf{B}^{n+1} U(1)$.

$$
\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "0"] \arrow[dl, Rightarrow, "\simeq"] & \\
\mathbf{B}^{n+1} U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}} \mathbf{B}^{n+2} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat \mathbf{B}^{n+2} U(1) \arrow[r] & \mathbf{B}^{n+2} U(1)
\end{tikzcd}
$$

We learn: “flat” means $\mathrm{curv} \simeq 0$.

$$
\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[r] \arrow[dd] \arrow[ddr, phantom, very near start, "\lrcorner"] & * \arrow[d, "0"] & \\
& \Omega^{n+2}_{\mathrm{cl}}(-) \arrow[d] & \\
\mathbf{B}^{n+1} U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \flat_{\mathrm{dR}} \mathbf{B}^{n+2} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d] \\
* \arrow[r] & \flat \mathbf{B}^{n+2} U(1) \arrow[r] & \mathbf{B}^{n+2} U(1)
\end{tikzcd}
$$

Refine this by considering all curvature forms in $\Omega^{n+2}_{\mathrm{cl}}(-)$.

$$
\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "0"] & \\
\mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^{n+2}_{\mathrm{cl}}(-) \arrow[d] & \\
\mathbf{B}^{n+1} U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}} \mathbf{B}^{n+2} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & \flat \mathbf{B}^{n+2} U(1) \arrow[r] & \mathbf{B}^{n+2} U(1)
\end{tikzcd}
$$

This finally gives the coefficients for $\mathbf{B}^n U(1)$-connections...

$$
\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d, "0"] & \\
\mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[r, "F"] \arrow[d, "\eta"] \arrow[dr, phantom, very near start, "\lrcorner"] & \Omega^{n+2}_{\mathrm{cl}}(-) \arrow[d] & \\
\mathbf{B}^{n+1} U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \flat_{\mathrm{dR}} \mathbf{B}^{n+2} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d] \\
* \arrow[r] & \flat \mathbf{B}^{n+2} U(1) \arrow[r] & \mathbf{B}^{n+2} U(1)
\end{tikzcd}
$$

With underlying bundle η and curvature form F.

$$\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[d, "\text{inclusion}"] & \text{flat connections} \\
\mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[d, "\text{forget connection}"] & \text{connections} \\
\mathbf{B}^{n+1} U(1) & \text{bundles}
\end{tikzcd}$$

In summary, so far, we have *abelian* differential cohomology.

$$\begin{tikzcd}
\flat \mathbf{B}^{n+1} U(1) \arrow[d] \arrow[dd, bend left] \\
\mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}^{n+1} U(1)
\end{tikzcd}$$

Recall that the total map is still the canonical counit.

$$\begin{tikzcd}
 & \flat \mathbf{B}^{n+1} U(1) \arrow[d] \arrow[d, bend left] \\
 & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B} G \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1} U(1)
\end{tikzcd}$$

Therefore for a cohesive characteristic map $\mathbf{c}$...

$$\begin{tikzcd}
\flat \mathbf{B}G \arrow[r, "\flat \mathbf{c}"] \arrow[dd] & \flat \mathbf{B}^{n+1}U(1) \arrow[d] \arrow[dd, bend left] \\
 & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}G \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1}U(1)
\end{tikzcd}$$

...we have a canonical refinement to *flat* differential cohomology.

$$
\begin{tikzcd}
\flat \mathbf{B}G \arrow[r, "\flat \mathbf{c}"] \arrow[d] & \flat \mathbf{B}^{n+1}U(1) \arrow[d] \\
{} \arrow[r, "\mathbf{c}_{\mathrm{conn}}"] \arrow[d] & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}G \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1}U(1)
\end{tikzcd}
$$

Hence a differential refinement of **c** should fit into...

$$
\begin{tikzcd}
\flat \mathbf{B} G \arrow[r, "\flat \mathbf{c}"] \arrow[d] & \flat \mathbf{B}^{n+1} U(1) \arrow[d] \\
\mathbf{B} G_{\mathrm{conn}} \arrow[r, "\mathbf{c}_{\mathrm{conn}}"] \arrow[d] & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B} G \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1} U(1)
\end{tikzcd}
$$

... a diagram of this form.

$$
\begin{tikzcd}
\flat \mathbf{B}G \arrow[r, "\flat \mathbf{c}"] \arrow[d] & \flat \mathbf{B}^n U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "0"] & \\
\mathbf{B}G_{\mathrm{conn}} \arrow[r, "\mathbf{c}_{\mathrm{conn}}"] \arrow[d] & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^{n+2}_{\mathrm{cl}}(-) \arrow[d] & \\
\mathbf{B}G \arrow[r, "\mathbf{c}"] & \mathbf{B}^{n+1} U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}} \mathbf{B}^{n+2} U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
 & * \arrow[r] & \flat \mathbf{B}^{n+2} U(1) \arrow[r] & \mathbf{B}^{n+2} U(1)
\end{tikzcd}
$$

(In total we looked at this situation in the cohesive ∞-topos.)

$$\mathbf{B}G_{\mathrm{conn}} \xrightarrow{\mathbf{c}_{\mathrm{conn}}} \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}$$

So given the differential characteristic map...

$$
\begin{tikzcd}
X \arrow[d, "\nabla"] & \\
\mathbf{B}G_{\mathrm{conn}} \arrow[r, "\mathbf{c}_{\mathrm{conn}}"] & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}
\end{tikzcd}
$$

...we canonically send G-connections ∇...

$$
\begin{tikzcd}
X \arrow[d, "\nabla"] \arrow[drr, "\mathbf{c}_{\mathrm{conn}}(\nabla)"] & & \\
\mathbf{B}G_{\mathrm{conn}} \arrow[rr, "\mathbf{c}_{\mathrm{conn}}"] & & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}
\end{tikzcd}
$$

...to $\mathbf{B}^n U(1)$-connections $\mathbf{c}_{\mathrm{conn}}(\nabla)$.

$$[\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathbf{c}_{\mathrm{conn}}} [\Sigma, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}]$$

This statement refines to the full moduli stack $[X, \mathbf{B}G_{\mathrm{conn}}]$ of G-connections.

$$[\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathbf{c}_{\mathrm{conn}}} [\Sigma, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}] \longrightarrow \mathrm{conc}\tau_0[\Sigma, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}]$$

Consider the projection to the concretefied 0-truncation.

$$[\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathbf{c}_{\mathrm{conn}}} [\Sigma, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}] \longrightarrow U(1)$$

If $\dim \Sigma = n+1$, then this is $\simeq U(1)$...

$$[\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathbf{c}_{\mathrm{conn}}} [\Sigma, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

… and the map computes the *holonomy* $\int_\Sigma \mathbf{c}_{\mathrm{conn}}(\nabla)$.

$$\exp(iS_{\mathrm{CS}_{\mathbf{c}}}) : [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathbf{c}_{\mathrm{conn}}} [\Sigma, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

... we call the ∞-*Chern-Simons functional* induced by **c**.