WZW terms in a cohesive ∞-topos

Talk at *Representation Theoretical and Categorical Structures in Quantum Geometry and CFT* 2011

Urs Schreiber

November 1, 2011

With

- Domenico Fiorenza
- Hisham Sati

Details and references at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

Outline

Goal: understand geometry of

- Chern-Simons models

and their

- Wess-Zumino-Witten models.

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and their

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in the spirit of
Carey-Johnson-Murray-Stevenson-Wang (math/0410013),
Waldorf (0804.4835)

Goal: understand geometry of

- higher Chern-Simons models

and their

- higher Wess-Zumino-Witten models.

I
Motivation

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Please!   No need.

The **Holographic Principle** of quantum field theory (QFT):

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such that...

Holographic principle

states of TFT_{n+1}
identify with
correlators of CFT_n

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- $\mathrm{AdS}_{n+1}/\mathrm{CFT}_n$;
  supergravity on asymptotic anti-de-Sitter spacetime

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- $\mathrm{AdS}_{n+1}/\mathrm{CFT}_n$;
  supergravity on asymptotic anti-de-Sitter spacetime
- $\mathrm{CS}_{n+1}/\mathrm{CFT}_n$
  Chern-Simons theory in 3d or higher dim abelian

best understood example:

$\mathrm{ChernSimons}_3/\mathrm{WZW}_2$.

ordinary 3d Chern-Simons / Wess-Zumino-Witten model (WZW)

Witten (hep-th/9812012):
also

- $\mathrm{AdS}_5/\mathrm{CFT}_4$
- $\mathrm{AdS}_7/\mathrm{CFT}_6$

governed by their higher Chern-Simons subsystems

- **Goal**: understand higher CS models and their higher WZW models

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- **Strategy**:
  1. internalize construction in higher topos theory

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- **Strategy**:
  1. internalize construction in higher topos theory
  2. unwind what the machinery spits out

II
Higher WZW bundles

∞-topos theory: pairs

- homotopy theory
- with geometric structure

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Running example:

$$\mathbf{H} := \mathrm{Sh}_\infty(\mathrm{SmthMfd})$$

"smooth ∞-groupoids" /
"smooth ∞-stacks"

∞-topos theory: pairs

- homotopy theory
- with geometric structure

here we have long fiber sequences for smooth higher bundles...

$$G$$

Start with an ∞-*group object*: a grouplike A_∞-space internal to the ∞-topos $\mathbf{H}$.

G

In running example: G is a *smooth ∞-group*.

$\mathbf{B}G$

The moduli stack of G-principal bundles.

$$X \xrightarrow{g} \mathbf{B}G$$

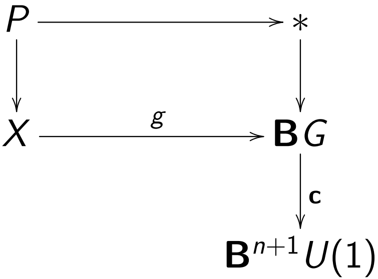
A classifying map.

$$
\begin{array}{ccc}
P & \longrightarrow & * \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & \mathbf{B}G
\end{array}
$$

The corresponding G-principal bundle.

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & \mathbf{B}G \end{array}$$

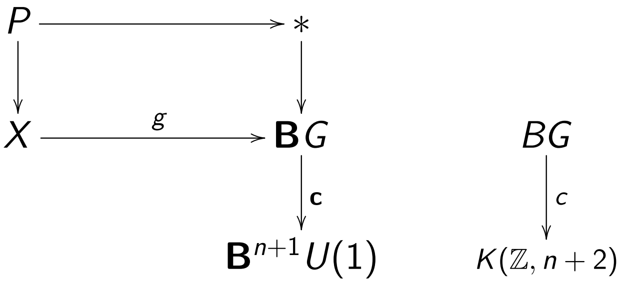
(All squares here and in the following are homotopy pullback squares.)



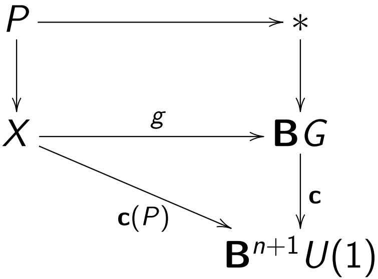
Consider a characteristic map.

$$
\begin{array}{ccc}
P & \longrightarrow & * \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & \mathbf{B}G \\
& & \downarrow{\scriptstyle \mathbf{c}} \\
& & \mathbf{B}^{n+1}U(1)
\end{array}
$$

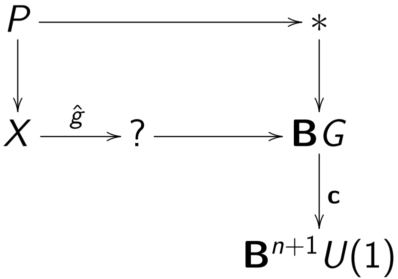
Classifying a *circle* $n+1$-*bundle* / *bundle n-gerbe* on $\mathbf{B}G$.



Lifting a topological cohomology class $[c] \in H^{n+2}(BG, \mathbb{Z})$.



If the obstruction class $[\mathbf{c}(P)]$ vanishes...



... then g lifts...

$$
\begin{array}{ccccc}
P & \longrightarrow & & & * \\
\downarrow & & & & \downarrow \\
X & \xrightarrow{\hat{g}} & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\
 & & \downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1}U(1)
\end{array}
$$

... to the extension $\hat{G} \to G$ classified by $\mathbf{c}$.

$$
\begin{array}{ccccc}
P & \longrightarrow & ? & \longrightarrow & * \\
\downarrow & & & & \downarrow \\
X & \xrightarrow{\hat{g}} & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\
 & & \downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1}U(1)
\end{array}
$$

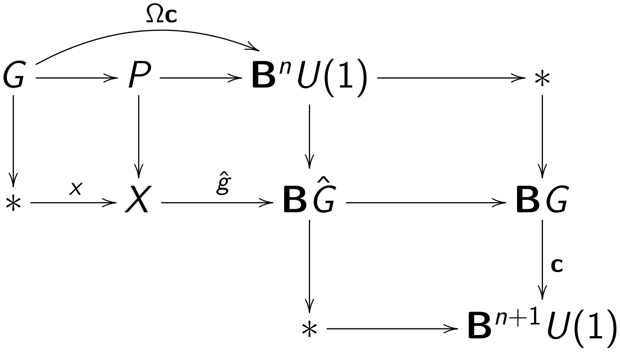
On the total space of P this is,
by the pasting law,...

$$
\begin{array}{ccccc}
P & \longrightarrow & \mathbf{B}^n U(1) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\hat{g}} & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\
 & & \downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1} U(1)
\end{array}
$$

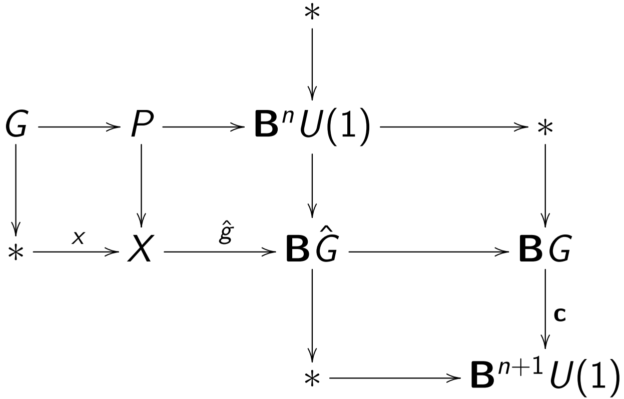
...a circle *n*-bundle...

$$\begin{array}{ccccccc}
G & \longrightarrow & P & \longrightarrow & \mathbf{B}^n U(1) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \xrightarrow{x} & X & \xrightarrow{\hat{g}} & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\
 & & & & \downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\
 & & & & * & \longrightarrow & \mathbf{B}^{n+1}U(1)
\end{array}$$

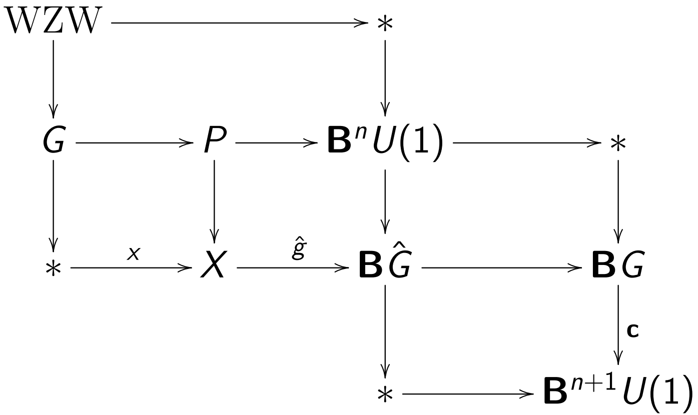
...whose restriction to any fiber...



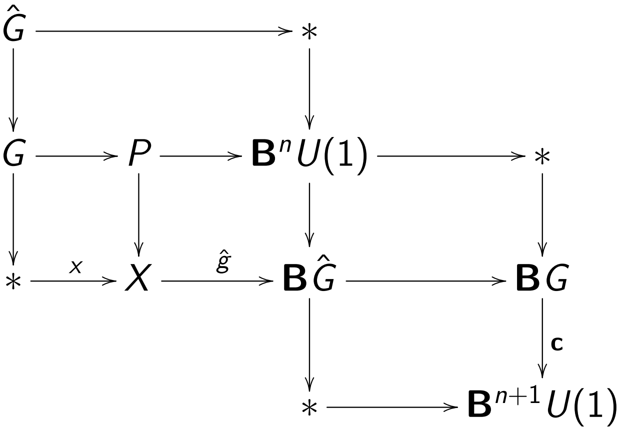
...is the looping of **c**.



This classifies...



the *WZW circle n-bundle* / bundle $(n-1)$-gerbe induced by **c**...



… which is $\hat{G}$ itself (all by the pasting law).

Next: add connections

Next: add connections
same idea of looping
but now with
a differential twist

III
Higher WZW connections

cohesive ∞-topos theory: pairs

- homotopy theory
- with **differential** structure

cohesive ∞-topos theory: pairs

- homotopy theory
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Fact: our example $\mathbf{H} = \mathrm{Sh}_\infty(\mathrm{SmthMfd})$ is cohesive.

cohesive ∞-topos theory: pairs

- homotopy theory
- with **differential** structure

so we have long fiber sequences for smooth higher bundles *with connection*...

$$\flat \mathbf{B} G$$

First, cohesion induces coefficients for *flat* G-connections.

$$X \xrightarrow{\nabla} \flat \mathbf{B} G$$

In that morphisms into it are flat G-principal connections on X.

$$X \xrightarrow{\nabla} \flat \mathbf{B}G \longrightarrow \mathbf{B}G$$

Canonically equipped with a map to the underlying G-bundles.

$$
\begin{array}{ccc}
 & & * \\
 & & \downarrow \\
\flat \mathbf{B}G & \longrightarrow & \mathbf{B}G
\end{array}
$$

The homotopy fiber of this...

$$
\begin{array}{ccc}
\flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & * \\
\downarrow & & \downarrow \\
\flat\mathbf{B}G & \longrightarrow & \mathbf{B}G
\end{array}
$$

...is the coefficient for flat G-valued differential forms.

$$
\begin{array}{ccccc}
X & \xrightarrow{A} & \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & * \\
 & & \downarrow & & \downarrow \\
 & & \flat\mathbf{B}G & \longrightarrow & \mathbf{B}G
\end{array}
$$

In that morphisms into it are flat L_∞-algebra valued forms $A \in \Omega_{\mathrm{flat}}(X, \mathfrak{g})$.

$$\begin{array}{ccccc} & \longrightarrow & \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \flat\mathbf{B}G & \longrightarrow & \mathbf{B}G \end{array}$$

The pasting law gives a universal $\mathfrak{g}$-valued form...

$$\begin{array}{ccccc} G & \xrightarrow{\theta} & \flat_{\mathrm{dR}}\mathbf{B}G & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \flat\mathbf{B}G & \longrightarrow & \mathbf{B}G \end{array}$$

... on G itself. The ∞-*Maurer-Cartan form* θ.

$$
\begin{array}{ccccc}
G & \xrightarrow{\theta} & \flat_{\mathrm{dR}} \mathbf{B} G & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \flat \mathbf{B} G & \longrightarrow & \mathbf{B} G
\end{array}
$$

Proceeding by similar constructions...

$$\begin{array}{ccc} G & \xrightarrow{\theta} & \flat_{\mathrm{dR}}\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \flat\mathbf{B}G \end{array}$$

$$\mathbf{B}G_{\mathrm{conn}}$$

...one finds coeffiecients for non-flat G-connections.

$$
\begin{array}{ccc}
G & \xrightarrow{\theta} & \flat_{\mathrm{dR}}\mathbf{B}G \\
\downarrow & & \downarrow \\
* & \longrightarrow & \flat\mathbf{B}G \\
 & & \downarrow \\
 & & \mathbf{B}G_{\mathrm{conn}}
\end{array}
$$

The flat connections sit inside.

$$\begin{array}{ccc} G & \xrightarrow{\theta} & \flat_{\mathrm{dR}}\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \flat\mathbf{B}G \\ & & \downarrow \\ & & \mathbf{B}G_{\mathrm{conn}} \end{array}$$

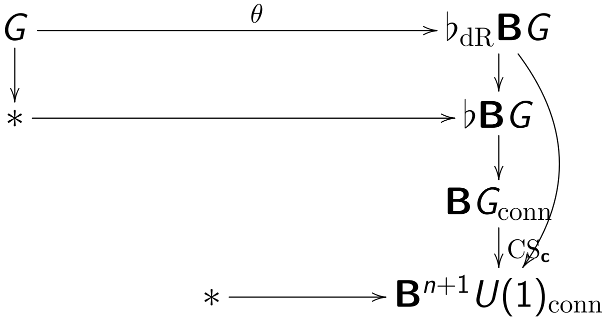
$$\mathbf{B}^{n+1}U(1)_{\mathrm{conn}}$$

For $G = \mathbf{B}^n U(1)$, this object classifies circle

$(n+1)$-connections / n-gerbes with connection.

$$\begin{array}{ccc} G & \xrightarrow{\theta} & \flat_{\mathrm{dR}}\mathbf{B}G \\ \downarrow & & \downarrow \\ * & \longrightarrow & \flat\mathbf{B}G \\ & & \downarrow \\ & & \mathbf{B}G_{\mathrm{conn}} \\ & & \downarrow{\scriptstyle \mathrm{CS}_{\mathbf{c}}} \\ & & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}} \end{array}$$

Characteristic maps $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+1}U(1)$ may lift to *differential* characteristics $\mathrm{CS}_{\mathbf{c}}$.



The homotopy fiber of the total map...

$$
\begin{array}{ccccc}
G & & \mathbf{B}^n U(1)^{\theta}_{\mathrm{conn}} & \longrightarrow & \flat_{\mathrm{dR}} \mathbf{B} G \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \cdot & \longrightarrow & \flat \mathbf{B} G \\
 & & \downarrow & & \downarrow \\
 & & & & \mathbf{B} G_{\mathrm{conn}} \\
 & & \downarrow & & \downarrow {\scriptstyle \mathrm{CS}_{\mathbf{c}}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}}
\end{array}
$$

... is the coefficient for circle n-connections with

curvature given by $\mathbf{c}(\theta)$.

$$
\begin{array}{ccccc}
G & \longrightarrow & \mathbf{B}^n U(1)^{\theta}_{\mathrm{conn}} & \longrightarrow & \flat_{\mathrm{dR}} \mathbf{B} G \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \cdot & \longrightarrow & \flat \mathbf{B} G \\
 & & \downarrow & & \downarrow \\
 & & & & \mathbf{B} G_{\mathrm{conn}} \\
 & & \downarrow & & \downarrow {\scriptstyle \mathrm{CS}_{\mathbf{c}}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}}
\end{array}
$$

By universality, θ factors through this...

$$
\begin{array}{ccccc}
G & \xrightarrow[\mathrm{WZW}_{\mathbf{c}}]{} & \mathbf{B}^n U(1)^{\theta}_{\mathrm{conn}} & \longrightarrow & \flat_{\mathrm{dR}} \mathbf{B} G \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \cdot & \longrightarrow & \flat \mathbf{B} G \\
 & & \downarrow & & \downarrow \\
 & & & & \mathbf{B} G_{\mathrm{conn}} \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{CS}_{\mathbf{c}}} \\
 & & * & \longrightarrow & \mathbf{B}^{n+1} U(1)_{\mathrm{conn}}
\end{array}
$$

... and this defines the WZW circle n-connection on G induced by $\mathrm{CS}_{\mathbf{c}}$.

From these maps $\mathrm{CS}_{\mathbf{c}}$ and $\mathrm{WZW}_{\mathbf{c}}$ on moduli stacks one obtains

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- **higher CS functional**

$$\exp(iS_{\mathrm{CS}_{\mathbf{c}}}(-)) :$$

$$[\Sigma_{n+1}, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathrm{CS}_{\mathbf{c}}} [\Sigma_{n+1}, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}] \xrightarrow{\int_{\Sigma}} U(1)$$

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$$G\text{-connections} \xrightarrow[\text{CS Lagrangian}]{} \xrightarrow[\text{volume holonomy}]{}$$

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- **higher WZW functional**:

$$\exp(iS_{\mathrm{WZW}_{\mathbf{c}}}(-)) :$$

$$[\Sigma_n, G] \xrightarrow{\mathrm{WZW}_{\mathbf{c}}} [\Sigma_n, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

From these maps $\mathrm{CS}_{\mathbf{c}}$ and $\mathrm{WZW}_{\mathbf{c}}$ on moduli stacks one obtains

- **higher CS functional**:

$$\exp(iS_{\mathrm{CS}_{\mathbf{c}}}(-)) :$$

$$[\Sigma_{n+1}, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{\mathrm{CS}_{\mathbf{c}}} [\Sigma_{n+1}, \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

- **higher WZW functional**:

$$\exp(iS_{\mathrm{WZW}_{\mathbf{c}}}(-)) :$$

$$[\Sigma_n, G] \xrightarrow{\mathrm{WZW}_{\mathbf{c}}} [\Sigma_n, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\int_\Sigma} U(1)$$

$$\text{maps to } G \xrightarrow[\text{WZW Lagrangian}]{} \xrightarrow[\text{surface holonomy}]{}$$

IV
Example

Theorem. Let G be a compact, simply connected Lie group. Then...

Theorem. Let G be a compact, simply connected Lie group. Then

- the canonical topological class

  $$c : BG \to K(\mathbb{Z}, 4)$$

  has unique smooth lift

  $$\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3 U(1)$$

  to smooth moduli ∞-stacks
- the "Chern-Simons 2-gerbe"

Theorem. Let G be a compact, simply connected Lie group. Then

- the extension it classifies is the *smooth* $\mathrm{String}(G)$*-2-group*

$$\hat{G} \simeq \mathrm{String}(G)\,.$$

Theorem. Let G be a compact, simply connected Lie group. Then

- $\mathbf{B}G_{\mathrm{conn}}$ is moduli stack of G-connections;
- there is a differential refinement

$$\mathrm{CS}_{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}} \, .$$

Theorem. Let G be a compact, simply connected Lie group. Then

- $\exp(iS_{\mathrm{CS}_{\mathbf{c}}})$ is ordinary CS-functional;
- $\exp(iS_{\mathrm{WZW}_{\mathbf{c}}})$ is ordinary WZW functional (topological term).

Next:
consider the same for
higher groups
and higher differential classes.

Next:
consider the same for
higher groups
and higher differential classes.

But not today.

End.

view Addendum

Addendum

Pasting law and looping

Consider two squares in an ∞-category:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

Consider a pasting diagram of two squares in an ∞-category:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

Pasting law A): If both squares are homotopy pullbacks, then so is the total rectangle.

Appl.: **long fiber sequence**
Define loop space objects ΩA of pointed objects A:

$$\begin{array}{ccc} \Omega A & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & A \end{array} .$$

Appl.: **long fiber sequence**

for any $f : A \to B$ we get

$$\begin{CD}
\Omega A @>>> * \\
@V{\Omega f}VV @VVV \\
\Omega B @>>> \mathrm{hofib}(f) @>>> * \\
@VVV @VVV @VVV \\
* @>>> A @>f>> B
\end{CD}$$

Back

to first occurence of pasting.