GENERALIZED EQUIVARIANT BUNDLES

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Dedicated to Professor Guy HIRSCH

Let Π and G be compact Lie groups and let $p\colon E \to B$ be a principal Π-bundle, where B is a G-space. Informally, we say that p is an equivariant bundle if the action of G on B lifts appropriately to E. The classical way to make this precise is to require the Π-action on E to extend to a $G \times \Pi$-action such that p is a G-map, and a detailed foundational study of such bundles was given in [2].

However, there are many naturally occurring examples which surely ought to count as equivariant bundles but which don't fit this description. Most simply, if Γ is an extension of G by Π, then one surely wants to consider the quotient homomorphism $q\colon \Gamma \to G$ as an equivariant bundle. Taking this example as a model, we fix such an extension and say that p is a principal $(\Pi;\Gamma)$-bundle if E is a Π-free Γ-space and p is a Γ-map, where G acts through q on B; the classical case is obtained by taking $\Gamma = G \times \Pi$. If F is a Γ-space, we then refer to the induced G-map $E \times_{\Pi} F \to B$ as the associated $(\Pi;\Gamma)$-bundle with fiber F, where $E \times_{\Pi} F$ denotes the orbit G-space $E \times F/\Pi$. This is our preferred general notion of a G-bundle, and it is the purpose of this note to point out that the basic theory of equivariant bundles set out in [2] generalizes without difficulty to this context. More discussion of examples and of the passage back and forth between principal and associated bundles may be found in [3;IV,§1]. Our main result is an analysis of the fixed point spaces of equivariant classifying spaces, and the second author will use this analysis in [4] to prove a new generalization of the Segal conjecture.

Let $\rho\colon Y \to B$ be a G-map which is a Π-bundle with fiber F, where F is a Γ-space on which Π acts effectively. The discussion in [3;IV§1] implies that ρ is the bundle with fiber F associated to a principal $(\Pi;\Gamma)$-bundle $p\colon E \to B$ if and only if each composite

$$F \to \rho^{-1}(b) \to \rho^{-1}(gb) \to F$$

coincides with the action of a (necessarily unique) lift of g to Γ, where the

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first and third arrows are admissible homeomorphisms and the middle arrow is action by $g \in G$. This characterization leads to the following natural examples of equivariant bundles in our sense which are not equivariant bundles in the classical sense.

EXAMPLE 1. — Let M be a smooth oriented n-manifold with a smooth orientation-reversing involution τ. The tangent bundle TM is an $SO(n)$-bundle, but the induced involution $\tau: TM \to TM$ is only an $O(n)$-bundle map. Regarding τ as a generator of $G = Z_2$, we see that $TM \to M$ is an $(SO(n); O(n))$-bundle with fiber $\mathbb{R}^n$.

EXAMPLE 2. — Let $\Pi \subset \Omega$ and let Γ be the normalizer of Π in Ω. If $\mu: \Omega/\Pi \to \Omega/\Gamma$ is the projection, then the pullback $\mu^*T(\Omega/\Gamma)$ is a $(\Pi;\Gamma)$-bundle with fiber the Γ-representation $V = T_{e\Gamma}(\Omega/\Gamma)$. In fact, we may identify $\mu^*T(\Omega/\Gamma)$ with $\Omega \times_\Pi V$ and give it the G-action, $G = \Gamma/\Pi$, specified by $q(\gamma)[\omega, v] = [\omega\gamma^{-1}, \gamma v]$.

We say that two $(\Pi;\Gamma)$-bundles $\rho: Y \to B$ and $\rho': Y' \to B$ with fiber F are equivalent if there is a G-map $f: Y \to Y'$ which is an equivalence of Π-bundles over B. The discussion in [3; V, § 1] implies that f necessarily has the form $\tilde{f} \times_\Pi 1$ for a Γ-map $\tilde{f}: E \to E'$ which is an equivalence of principal Π-bundles over B, where E and E′ are the associated principal $(\Pi;\Gamma)$-bundles of Y and Y′. Therefore equivalence classes of $(\Pi;\Gamma)$-bundles with fiber F over B correspond bijectively to equivalence classes of principal $(\Pi;\Gamma)$-bundles over B. We shall concentrate on principal $(\Pi;\Gamma)$-bundles henceforward.

There is a description of principal $(\Pi;\Gamma)$-bundles in terms of the classical kind of equivariant bundles that the reader may find illuminating. If $p: E \to B$ is a principal $(\Pi;\Gamma)$-bundle, then the induced map $E \times_\Pi \Gamma \to B$ is a principal $(\Gamma; G \times \Gamma)$-bundle with a given reduction of its structural group from Γ to Π. Here $E \times \Gamma$ is given the diagonal left action by Γ and $E \times_\Pi \Gamma$ is the orbit G-space. The right action of Γ on itself gives rise to a right action of Γ on $E \times_\Pi \Gamma$ which commutes with the left G-action. The reduction may be viewed as given by the section

$$s: B \to E \times_\Pi (\Gamma/\Pi) \cong B \times G$$

specified by $s(b) = (b, 1)$, and s satisfies $s(gb) = gs(b)g^{-1}$. Note that s is not a $(G \times \Pi)$-reduction since it is not equivariant with respect to the left G-action. Conversely, if $\phi: Z \to B$ is a principal $(\Gamma; G \times \Gamma)$-bundle with a reduction $s: B \to Z/\Pi$ of its structural group to Π such that $s(gb) = gs(b)g^{-1}$, then Z is Γ-homeomorphic over B to $E \times_\Pi \Gamma$ for a principal Π-bundle E over B. The Π-action on E extends to a Γ-action; E can be identified with the space of admissible homeomorphisms

$\psi: \Gamma \to \varphi^{-1}(b)$, and $\gamma\psi$ is the composite

$$\Gamma \xrightarrow{\gamma^{-1}} \Gamma \xrightarrow{\psi} \varphi^{-1}(b) \xrightarrow{q(\gamma)} \varphi^{-1}(q(\gamma)b).$$

We conclude that principal $(\Pi;\Gamma)$-bundles correspond bijectively to principal $(\Gamma; G \times \Gamma)$-bundles together with given reductions of their structural group from Γ to Π which satisfy the specified formula.

A principal $(\Pi;\Gamma)$-bundle is said to be trivial if it is equivalent to one of the form $\Gamma \times_{\Lambda} V \to G \times_{H} V$, where $H \subset G$, $\Lambda \subset \Gamma$, $\Lambda \cap \Pi = e$, Λ maps isomorphically to H under $q: \Gamma \to G$, and V is an H-space regarded as a Λ-space via q. The following observation on the local structure of principal $(\Pi;\Gamma)$-bundles makes clear that this definition is appropriate.

LEMMA 3. — Let $p: E \to B$ be a principal $(\Pi;\Gamma)$-bundle, where E and therefore B is completely regular. Let $b \in B$ and $z \in p^{-1}(b)$ and let G_b and Γ_z be the respective isotropy groups.

(i) $\Gamma_z \cap \Pi = e$ and q maps Γ_z isomorphically onto G_b.

(ii) If W is a slice through b in B, there is a slice V through z in E such that p maps V homeomorphically onto a neighborhood V' of b in W and thus $p \mid \Gamma V$ is trivial.

PROOF. — (i) $\Gamma_z \cap \Pi = e$ since Π acts freely on E; $q(\Gamma_z) = G_b$ since, for $\gamma \in \Gamma$, $q(\gamma)b = b$ if and only if $\gamma z = \nu z$ for some $\nu \in \Pi$, in which case $\nu^{-1}\gamma z = z$ and $q(\nu^{-1}\gamma) = q(\gamma)$.

(ii) Let D be a bi-invariant normal disc to $\Pi\Gamma_z$ through e in Γ. Then q maps D homeomorphically onto a bi-invariant normal disc D' to G_b through e in G, and $q(\gamma d \gamma^{-1}) = g q(d) g^{-1}$ if $\gamma \in \Gamma$ and $g = q(\gamma)$. The action of G on B maps $D' \times W$ homeomorphically onto a G_b-invariant neighborhood of b. Choose a slice $\overline{V}$ through z sufficiently small that $p(\overline{V}) \subset D' \times W$ and let $\lambda: \overline{V} \to D$ be the composite Γ_z-map $q^{-1}\pi_1 p$, where $\pi_1: D' \times W \to D'$ is the projection. It is easily checked that $V = \{\lambda(x)^{-1}x \mid x \in \overline{V}\}$ is then a slice through z which maps homeomorphically into W under p.

A principal $(\Pi;\Gamma)$-bundle $p: E \to B$ is said to be locally trivial if there is an open cover $\{GV_\alpha\}$ of B such that V_α is an H_α-slice and $p/p^{-1}(GV_\alpha)$ is trivial; p is numerable if, in addition, the cover can be chosen to have a G-equivariant partition of unity $\{\lambda_\alpha: GV_\alpha \to I\}$. With these definitions, we have the following generalizations of [2, 1.5] and [2, 1.13].

PROPOSITION 4. — A principal $(\Pi;\Gamma)$-bundle with completely regular total space is locally trivial.

PROPOSITION 5. — A locally trivial principal $(\Pi;\Gamma)$-bundle over a paracompact base space is numerable.

Using Lemma 3, the arguments of [2, §2] generalize to give the following analogs of [2, 2.10-2.12 and 2.14].

THEOREM 6. — A numerable principal $(\Pi;\Gamma)$-bundle E over $B \times I$ is equivalent to $(E|B \times \{0\}) \times I$.

COROLLARY 7. — The pullbacks of a numerable principal $(\Pi;\Gamma)$-bundle along homotopic G-maps into its base space are equivalent.

COROLLARY 8. — A numerable principal $(\Pi;\Gamma)$-bundle satisfies the equivariant bundle covering homotopy property.

THEOREM 9. — A numerable principal $(\Pi;\Gamma)$-bundle $p: E \to B$ is universal if and only if E^Λ is contractible for all (closed) subgroups Λ of Γ such that $\Lambda \cap \Pi = e$.

We let $B(\Pi;\Gamma)$ denote the base space of a universal principal $(\Pi;\Gamma)$-bundle; it is uniquely determined up to G-homotopy type. Our main result, which generalizes [2, 2.17], gives the homotopy types of the fixed point spaces of such a classifying G-space.

THEOREM 10. — For $H \subset G$,

$$B(\Pi;\Gamma)^H = \amalg B(\Pi \cap N_\Gamma\Lambda; W_\Gamma\Lambda),$$

where the union runs over the Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$. In particular, $B(\Pi;\Gamma)^H$ is empty if there is no such Λ.

Here $N_\Gamma\Lambda$ is the normalizer of Λ in Γ and $W_\Gamma\Lambda = N_\Gamma\Lambda/\Lambda$. When $\Lambda \cap \Pi = e$, $\Pi \cap N_\Gamma\Lambda$ is a normal subgroup of $W_\Gamma\Lambda$ since Π is a normal subgroup of Γ. Actually, since the theorem is only a statement about nonequivariant spaces, the Λ^{th} summand on the right may be viewed simply as the ordinary classifying space $B(\Pi \cap N_\Gamma\Lambda)$, with $W_\Gamma\Lambda$ ignored. However, it is crucial to the application to the Segal conjecture in [4] to know the structure of $B(\Pi;\Gamma)^H$ as a W_GH-space, or, more generally, as a W_KH-space for any K containing H, and the theorem admits the following elaboration.

COROLLARY 11. — If $H \subset K \subset G$, then, as a W_KH-space,

$$B(\Pi;\Gamma)^H = \amalg W_kH \times_{V(\Lambda;\Omega)} B(\Pi \cap N_\Omega\Lambda; W_\Omega\Lambda),$$

where $\Omega = q^{-1}(K)$, the union runs over the $q^{-1}(N_KH)$-conjugacy classes of subgroups Λ of Ω such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$, and $V(\Lambda;\Omega) = W_\Omega\Lambda/\Pi \cap N_\Omega\Lambda$ is the image of $W_\Omega\Lambda$ in W_KH.

To prove Theorem 10 and Corollary 11, we shall analyze the fixed point structure of general principal $(\Pi;\Gamma)$-bundles. For notational simplicity, we agree to write $\Pi^\Lambda = \Pi \cap N_\Gamma\Lambda$ when $\Lambda \subset \Gamma$ and $\Lambda \cap \Pi = e$. It is perhaps worth observing that Π^Λ is also equal to $\Pi \cap Z_\Gamma\Lambda$, where $Z_\Gamma\Lambda$ is the centralizer of Λ in Γ.

THEOREM 12. — Let $p: E \to B$ be a principal $(\Pi;\Gamma)$-bundle, where E is a completely regular Γ-space. Let $H \subset G$ and $\Lambda \subset \Gamma$, with $\Lambda \cap \Pi = e$.

(i) E^Λ is a principal $(\Pi^\Lambda; W_\Gamma\Lambda)$-bundle and $E^\Lambda/\Pi^\Lambda = p(E^\Lambda)$.

(ii) $p^{-1}(p(E^\Lambda)) = \amalg E^\theta$, where the union runs over the distinct Π-conjugates θ of Λ.

(iii) $B^H = \amalg p(E^\Lambda)$, where the union runs over the Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$.

(iv) As a W_GH-space, $B^H = \amalg W_GH \times_{V(\Lambda)} p(E^\Lambda)$, where the union runs over the $q^{-1}(N_GH)$-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$ and where $V(\Lambda) = W_\Gamma\Lambda/\Pi^\Lambda$ is the image of $W_\Gamma\Lambda$ in W_GH.

PROOF. — (i) Obviously Π^Λ acts freely on E^Λ. If $z \in E^\Lambda$, $\nu \in \Pi$, and $\nu z \in E^\Lambda$, then $\nu \in \Pi^\Lambda$ since, for any $\lambda \in \Lambda$, $\nu z = \lambda\nu z = \lambda\nu\lambda^{-1}\lambda z = \lambda\nu\lambda^{-1}z$, hence $\nu = \lambda\nu\lambda^{-1}$. Therefore the natural map $E^\Lambda/\Pi^\Lambda \to p(E^\Lambda)$ is a homeomorphism.

(ii) Since $\gamma E^\Lambda = E^{\gamma\Lambda\gamma^{-1}}$ for $\gamma \in \Gamma$, $p(z) \in p(E^\Lambda)$ if and only if $z \in E^{\nu\Lambda\nu^{-1}}$ for some $\nu \in \Pi$. If $z \in E^\Lambda \cap E^{\nu\Lambda\nu^{-1}}$, $\nu \in \Pi$, then $\lambda z = \nu\lambda\nu^{-1}z$ for all $\lambda \in \Lambda$, hence $\lambda^{-1}\nu\lambda = \nu$ and $\Lambda = \nu\Lambda\nu^{-1}$.

(iii) Certainly $p(E^\Lambda) \subset B^H$ if $q(\Lambda) = H$. If $b \in B^H$ and $z \in p^{-1}(b)$, then $q(\Gamma_z) = G_b$ and there is a subgroup Λ of Γ_z such that $q(\Lambda) = H$. If $z \in E^\Lambda \cap E^\theta$, where $\Lambda \cap \Pi = e = \theta \cap \Pi$, and $q(\Lambda) = H = q(\theta)$, then $z \in E^{\Lambda\theta}$, hence $(\Lambda\theta) \cap \Pi = e$, and $q(\Lambda\theta) = H$. Therefore $\Lambda = \Lambda\theta = \theta$. With (ii), this implies that $B^H = \amalg p(E^\Lambda)$ as sets, where Λ runs over the appropriate Π-conjugacy classes. It is to ensure that this decomposition is a homeomorphism that we require E to be completely regular. Certainly $p(E^\Lambda)$ is a closed subspace of B^H. We must show that it is also open. Let $b = p(z)$, where $z \in E^\Lambda$. Let $\Sigma = \Gamma_z$ and $K = q(\Sigma) = G_b$. Then z has a slice neighborhood $\Gamma V \cong \Gamma \times_\Sigma V$ whose image under p is a slice neighborhood $GV' \cong G \times_K V'$ of b, where p maps V Σ-homeomorphically onto V' regarded as a Σ-space via $q: \Sigma \cong K$. Let D and D' be as in the proof of (ii) of Lemma 3. $D' \times V'$ is a K-invariant neighborhood of b, and $p^{-1}(D' \times V') = D\Pi \times V$. Since p maps $D \times V$ Σ-homeomorphically onto $D' \times V'$, it maps the open neighborhood $(D \times V)^\Lambda$ of z homeomorphically onto the open neighborhood $(D' \times V')^H$ of b in B^H.

(iv) For each subgroup Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$, $p(E^\Lambda)$ is fixed by $V(\Lambda)$, hence the action of W_GH on B^H induces a W_GH-map

$W_GH \times_{V(\Lambda)} p(E^\Lambda) \to B^H$. If $g = q(\gamma) \varepsilon N_GH$, then $q(\gamma\Lambda\gamma^{-1}) = H$, $\gamma\Lambda\gamma^{-1} \cap \Pi = e$, and $gp(E^\Lambda) = p(\gamma E^\Lambda) = p(E^{\gamma\Lambda\gamma^{-1}})$. It is easily checked that, as the images in W_GH of such elements g run through a set of $V(\Lambda)$-coset representatives, the groups $\gamma\Lambda\gamma^{-1}$ run through one group in each Π-conjugacy class of subgroups Σ of Γ such that $\Sigma \cap \Pi = e$, $q(\Sigma) = H$, and Σ is $q^{-1}(N_GH)$-conjugate to Λ. Thus, as Λ runs over $q^{-1}(N_GH)$-conjugacy classes, the images of the cited W_GH-maps account for all of B^H.

The following observation makes clear that parts (iii) and (iv) are consistent with the topology.

LEMMA 13. — For each Λ, $V(\Lambda)$ has finite index in W_GH.

PROOF. — It is standard that the group $N_\Gamma\Lambda/(Z_\Gamma\Lambda)\Lambda$ is finite since, via conjugation, it is a compact subgroup of the discrete group of automorphisms modulo inner automorphisms of Λ. Applying this with Λ replaced by $\Lambda\Pi$ and observing that $Z_\Gamma(\Lambda\Pi) \subset Z_\Gamma\Lambda \subset N_\Gamma\Lambda$, we see that $(N_\Gamma\Lambda)\Pi$ has finite index in $N_\Gamma(\Lambda\Pi)$. Since $q^{-1}(H) = \Lambda\Pi$, $q^{-1}(N_GH) = N_\Gamma(\Lambda\Pi)$. Therefore $q(N_\Gamma\Lambda)$ has finite index in N_GH.

Finally, we turn to the proofs of Theorem 10 and Corollary 11. It is convenient to think of the universal principal $(\Pi;\Gamma)$-bundle in terms of families. A family $\mathscr{F}$ of subgroups of Γ is a set of subgroups closed under subconjugacy. (Subgroups are understood to be closed.) A Γ-space X is said to be an $\mathscr{F}$-space if all of its isotropy groups are in $\mathscr{F}$ or, equivalently, If X^Λ is empty for Λ not in $\mathscr{F}$. An $\mathscr{F}$-space $E\mathscr{F}$ is said to be universal if any $\mathscr{F}$-space maps into it, uniquely up to Γ-homotopy. (We restrict attention to Γ-CW homotopy types to avoid technical problems.) It is equivalent to require $E\mathscr{F}^\Lambda$ to be contractible for Λ in $\mathscr{F}$. A pleasant conceptual construction of universal $\mathscr{F}$-spaces $E\mathscr{F}$ has been given by Elmendorf [1]. Let $\mathscr{F}(\Pi;\Gamma)$ be the family of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and let $E(\Pi;\Gamma) = E\mathscr{F}(\Pi;\Gamma)$. Either quoting Theorem 9 or using $E(\Pi;\Gamma)$ to prove it, we see that $E(\Pi;\Gamma)$ is a universal principal $(\Pi;\Gamma)$-bundle, so that $B(\Pi;\Gamma) = E(\Pi;\Gamma)/\Pi$. We insert a few general observations about universal $\mathscr{F}$-spaces. The proofs are easy.

LEMMA 14. — Let $\mathscr{F}$ be a family in Γ and let Λ be a subgroup of Γ.
(i) $E\mathscr{F}$ regarded as a Λ-space is $E(\mathscr{F}|\Lambda)$, where

$$\mathscr{F}|\Lambda = \{\theta \mid \theta \subset \Lambda \text{ and } \theta \in \mathscr{F}\}.$$

(ii) If $\Lambda \in \mathscr{F}$, then $(E\mathscr{F})^\Lambda$ regarded as a $W_\Gamma\Lambda$-space is $E(\mathscr{F}^\Lambda)$, where $\mathscr{F}^\Lambda$ is the family in $W_\Gamma\Lambda$ specified by

$$\mathscr{F}^\Lambda = \{\theta \mid \theta = \Sigma/\Lambda, \text{ where } \Lambda \subset \Sigma \subset N_\Gamma\Lambda \text{ and } \Sigma \in \mathscr{F}\}.$$

Specializing to $\mathscr{F}(\Pi;\Gamma)$, we obtain the following conclusions.

LEMMA 15. — Let Π be a normal subgroup of Γ and Λ be any subgroup.

(i) $\mathscr{F}(\Pi;\Gamma)|\Lambda = \mathscr{F}(\Pi \cap \Lambda;\Lambda)$, hence $E(\Pi;\Gamma) = E(\Pi \cap \Lambda;\Lambda)$ as a Λ-space and

$$E(\Pi;\Gamma)/\Pi \cap \Lambda = B(\Pi \cap \Lambda;\Lambda)$$

as a $(\Lambda/\Pi \cap \Lambda)$-space.

(ii) If $\Lambda \in \mathscr{F}(\Pi;\Gamma)$, then $\mathscr{F}(\Pi;\Gamma)^{\Lambda} = \mathscr{F}(\Pi^{\Lambda};W_{\Gamma}\Lambda)$, hence $E(\Pi;\Gamma)^{\Lambda} = E(\Pi^{\Lambda};W_{\Gamma}\Lambda)$ as a $W_{\Gamma}\Lambda$-space and

$$E(\Pi;\Gamma)^{\Lambda}/\Pi^{\Lambda} = B(\Pi^{\Lambda};W_{\Gamma}\Lambda)$$

as a $V(\Lambda)$-space.

PROOF OF THEOREM 10. — This is now immediate from (ii) of Lemma 15 and (i) and (iii) of Theorem 12.

PROOF OF COROLLARY 11. — By (i) of Lemma 15 (with Λ replaced by Ω of Corollary 11), it suffices to consider the case $K = G$ and thus $\Omega = \Gamma$. Here the conclusion is immediate from (iv) of Theorem 12.

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