SEMICLASSICAL LAGRANGIAN DISTRIBUTIONS

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Abstract. These are (rather sloppy) notes for the talk given in UC Berkeley in March 2012, attempting to explain the global theory of semiclassical Lagrangian distributions and some of its applications.

1. Overview

In this expository note, we will sketch the construction of semiclassical Lagrangian distributions, which are rapidly oscillating functions associated to Lagrangian submanifolds.

Let us first recall several definitions from symplectic geometry. Let M be a smooth manifold of dimension n, then the cotangent bundle T^*M has a naturally ocurring symplectic form, which we denote ω. In fact, we have

$$\omega = d\alpha,$$

where

$$\alpha := \xi\, dx$$

is the canonical 1-form on T^*M. Here x denotes some coordinates on M and ξ the dual coordinates on the fibers of the cotangent bundle; the form α is independent of the choice of coordinates, as one can think of $\alpha_\rho(v)$ for $\rho \in T^*M$ and $v \in T_\rho(T^*M)$ as pairing the cotangent vector ρ with the image of v under the differential of the canonical projection map

$$\pi : T^*M \to M;$$

the latter image is a tangent vector at $\pi(\rho)$.

A *Lagrangian submanifold* is a submanifold $\Lambda \subset T^*M$ of dimension n that is *isotropic*, namely the pullback of ω under the inclusion map $\Lambda \to T^*M$ is zero. In other words, the tangent space of Λ at any point ρ is a Lagrangian subspace of the linear symplectic space $T_\rho(T^*M)$, in the sense that it has dimension n and the symplectic form ω vanishes on this subspace. Note that n is the maximal possible dimension of a Lagrangian subspace on which ω vanishes, because ω is nondegenerate.

If M_1, M_2 are manifolds of the same dimension, then a *canonical transformation* is a symplectomorphism $\kappa : T^*M_1 \to T^*M_2$; in other words, $\kappa^*\omega_2 = \omega_1$. If we consider

the flipped graph

$$\Gamma_\kappa := \{(x,\xi;y,-\eta) \mid \kappa(x,\xi) = (y,\eta)\},$$

then Γ_κ is a Lagrangian submanifold of $T^*(M_1 \times M_2)$. The latter property makes it possible to talk about *canonical relations* (defined as relations whose flipped graphs are Lagrangian submanifolds), which can be considered also when M_1 and M_2 are not necessarily of the same dimension.

The main goal of this note is to associate to each Lagrangian submanifold $\Lambda \subset T^*M$ the space of *Lagrangian distributions* $I(\Lambda)$. These are distributions (for a compact Λ, they will actually be smooth functions) depending on the *semiclassical parameter* $h > 0$, and we will be interested in their asymptotic as $h \to 0$. In particular, distributions that are $\mathcal{O}(h^\infty)$ will be considered negligible (namely, our theory cannot distinguish those from zero). The inverse of the semiclassical parameter corresponds to the frequency at which our distributions oscillate.

The origin of semiclassical analysis (that is, microlocal analysis with the semiclassical parameter h) lies in semiclassical approximation in quantum mechanics, that associates the behaviour of quantum objects (e.g. wave functions of quantum particles, or eigenfunctions of quantum Hamiltonians) to their classical behavior (e.g. Newtonian mechanics, or Hamiltonian flows of the classical Hamiltonians). However, semiclassical techniques have a wide range of uses in the theory linear and nonlinear differential equations which go beyond the original applications to quantum mechanics.

Semiclassical analysis is an extension of *microlocal analysis*; the latter studies the structure of singularities of solutions to linear differential equations. The difference between the two on the classical level is that in microlocal analysis, one studies the growth of the symbols $a(x,\xi)$ as $|\xi| \to \infty$, while in semiclassical analysis, one studies the decay in h of h-dependent symbols $a(x,\xi;h)$. On the quantum (or PDE) level, microlocal analysis studies the singularities of functions (i.e., their failure to be smooth), while semiclassical analysis is more interested in behavior of h-dependent functions as $h \to 0$. Note though that in a full treatment of semiclassical analysis, the two aspects are combined – one studies both decay as $h \to 0$ and asymptotic behavior as $|\xi| \to \infty$; therefore, microlocal analysis could be regarded a subset of semiclassical analysis.

Going back to Lagrangian distributions, we will define a geometrically invariant symbol map

$$\sigma : I(\Lambda) \to C^\infty(\Lambda; \mathcal{M}),$$

where $\mathcal{M}$ is a certain line bundle that will be specified later. The kernel of σ is exactly $hI(\Lambda)$; that is, Lagrangian distributions that decay like h. Therefore, one can write the exact sequence

$$0 \to hI(\Lambda) \to I(\Lambda) \xrightarrow{\sigma} C^\infty(\Lambda;\mathcal{M}) \to 0.$$

The presence of a geometrically invariant symbol map (and in particular the definition of the line bundle $\mathcal{M}$) defines the boundary between *local* and *global* theory of Lagrangian distributions. That is, both theories study the same space of Lagrangian distributions, but while local theory is content with just verifying that something is a Lagrangian distribution, the global theory demands to express it in a geometrically invariant way up to $\mathcal{O}(h)$ errors. The local theory is enough for most applications in analysis (and is used in roughly every second paper on semiclassical analysis). The global theory is needed in much fewer cases, however it is indispensable in writing quantization conditions for quantum integrable systems, an application which we will look at in the present note.

One can similarly consider *Fourier integral operators* associated to some canonical relation $\kappa \subset T^*M_1 \times T^*M_2$. These are defined as follows: if M_1, M_2 are manifolds of dimension n_1 and n_2, respectively, $A(h) : C^\infty(M_1) \to C^\infty(M_2)$ is an h-dependent family of operators and $K_A(h)$ is its Schwartz kernel:

$$A(h)u(x) = \int K_A(x, y; h)u(y)\, dy,$$

then A is a Fourier integral operator associated to κ (we denote $A \in I(\kappa)$) if $h^{(n_1+n_2)/4}K_A$ is a Lagrangian distribution associated to the flipped graph Γ_κ. (The renormalization $h^{(n_1+n_2)/4}$ occurs because we want the L^2 norm of a Lagrangian distribution to be ~ 1, but the $L^2 \to L^2$ operator norm of a Fourier integral operator to be ~ 1.)

Let us now give several examples of Lagrangian distributions or Fourier integral operators:

(1) Assume that we have an *quantum integrable system*, namely on a compact manifold M of dimension n, we specify n semiclassical pseudodifferential operators $P_1, \dots, P_n$ that commute: $[P_j, P_k] = 0$. Assume also that the principal symbols $p_1, \dots, p_n \in C^\infty(T^*M)$ of these operators are real-valued. Define the map $\mathbf{p} : T^*M \to \mathbb{R}^n$ by $\mathbf{p} = (p_1, \dots, p_n)$ and assume that it has no critical points on the set $\mathbf{p}^{-1}(0)$. Then $\mathbf{p}$ is an integrable Hamiltonian system near $\mathbf{p}^{-1}(0)$, and the latter is a (or perhaps a union of finitely many) Lagrangian torus in T^*M. Then any L^2 normalized joint eigenfunction u:

$$P_j u = \lambda_j u,\ \lambda_j = \mathcal{O}(h),$$

is a Lagrangian distribution associated to $\mathbf{p}^{-1}(0)$. What is also important is that we will be able to approximate the joint spectrum near 0 by a certain quantization condition; to write out the latter, one needs to know the global theory of Lagrangian distributions.

(2) If M_1, M_2 are manifolds (not necessarily of the same dimension) and $\varphi : M_1 \to M_2$ is a smooth map, then the pullback $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ is (after multiplying by a certain power of h) a Fourier integral operator associated to

the following canonical relation:

$$\kappa := \{(x,\xi;y,\eta) \in T^*(M_1 \times M_2) \mid y = \varphi(x),\ \xi = (d\varphi)_x^* \cdot \eta\}.$$

Here $(d\varphi)_x^* : T^*_{\varphi(x)} M_2 \to T^*_x M_1$ is the dual to the map $(d\varphi)_x : T_x M_1 \to T_{\varphi(x)} M_2$. This example in particular includes pullbacks by diffeomorphisms and restriction to submanifolds.

(3) If P is a self-adjoint semiclassical pseudodifferential operator on M, then for each fixed t, the semiclassical Schrödinger propagator

$$\exp(itP/h) : L^2(M) \to L^2(M),$$

is a Fourier integral operator associated to the Hamiltonian flow $\exp(tH_p)$. This in particular includes the Schrödinger propagator $e^{ith\Delta} = e^{it(h^2\Delta)/h}$ in the case when M has a Riemannian metric, which is associated to the (rescaled) geodesic flow on M. Same is true for the wave propagator $e^{it\sqrt{\Delta}}$, which is associated to a different rescaling of the geodesic flow.

Finally, we should remark that the present notes do not contain rigorous proofs or explanations of most facts listed; the reader interested in the presented theory is referred to several treatments of the subject, listed in the last section. We remark however that the main tool is the following *method of stationary phase*: if M is a manifold, Φ is a (real-valued in particular) Morse function on M with critical points $x_1, x_2, \dots$, then there exist differential operators L_{jk} of order $2j$ on M at each critical point x_k such that for each $a \in C_0^\infty(M)$ and each positive integer N,

$$\int_M e^{\frac{i}{h}\Phi(x)} a(x)\,dx = \sum_{0 \le j < N} h^{n/2+j} \sum_k e^{\frac{i}{h}\Phi(x_k)} (L_{jk}a)(x_k) + \mathcal{O}(h^{n/2+N}).$$

Moreover, the following formula for L_0 will prove useful in explaining the structure of the Maslov bundle $\mathcal{M}$:

$$L_{0k} = (2\pi)^{n/2} |\det(d^2\Phi(x_k))|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}(d^2\Phi(x_k))},$$

where $\operatorname{sgn}(d^2\Phi(x_k))$ is the signature of the Hessian $d^2\Phi(x_k)$ (namely, the number of positive eigenvalues minus the number of negative eigenvalues).

2. Local theory of Lagrangian distributions

We start by considering the function

$$u(x;h) := e^{i\Phi(x)/h} b(x). \tag{2.1}$$

Here $\Phi(x)$ is a real-valued smooth function on M, called the *phase function*, and $b(x)$ is an h-independent smooth function (or more generally, a classical symbol in h in the sense that $b(x;h)$ is smooth up to $h = 0$), called the *amplitude* (or the *symbol*, though we will reserve this term for geometrically invariant objects) of u.

We would like to see where $u(x;h)$ is *microlocalized*, e.g. where it is concentrated in the phase space T^*M. For this, let us quickly introduce semiclassical pseudodifferential operators. If $a(x,\xi) \in C_0^\infty(T^*\mathbb{R}^n)$ (or more generally, if it obeys certain growth conditions as we approach infinity along the fibers of $T^*\mathbb{R}^n$), then we define the operator $\mathrm{Op}_h(a) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$\mathrm{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big) f(y)\, d\xi dy. \tag{2.2}$$

Defining $\mathrm{Op}_h(a)$ on a manifold takes more care, however we are able to do this invariantly modulo $\mathcal{O}(h)$ errors (and, in case of Weyl quantization and half-densities, modulo $\mathcal{O}(h^2)$ errors – this will be needed later in the global theory).

A basic example of a semiclassical pseudodifferential operator is multiplication by a function $f \in C^\infty(M)$; the corresponding symbol is independent of ξ. Another example is the operator $(h/i)X$, where X is any vector field; the symbol is a linear function on the fibers of T^*M, naturally induced by X. In general, any semiclassical differential operator provides an example.

Here are several useful properties of semiclassical quantization:

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h),$$
$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\bar{a}) + \mathcal{O}(h),$$
$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\,\mathrm{Op}_h(\{a,b\}) + \mathcal{O}(h^2).$$

Here $\{a,b\}$ is the Poisson bracket:

$$\{a,b\} := \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b,$$

where H_a is the Hamiltonian vector field:

$$H_a := \partial_\xi a \cdot \partial_x - \partial_x a \cdot \partial_\xi.$$

The commutator identity for pseudodifferential operators motivates the following criterion, known as Beals's theorem [Zw, Theorem 8.3]: an operator $A : L^2(M) \to L^2(M)$ is given by $\mathrm{Op}_h(a)$ for some symbol a, if and only if for any finite set of vector fields $X_1, \dots, X_k$, we have

$$\mathrm{ad}_{hX_1} \dots \mathrm{ad}_{hX_k} A = \mathcal{O}(h^k)_{L^2 \to L^2}.$$

Here we denote $\mathrm{ad}_P A = [P, A]$.

Now, if $v(x;h)$ is any family of smooth functions (which is h-tempered in the sense that all its derivatives are polynomially bounded in h), then we say that a point (x,ξ) does not lie in the wavefront set $\mathrm{WF}_h(v)$, if there exists a neighborhood $U(x,\xi)$ such that for each symbol a supported inside of U, we have

$$\mathrm{Op}_h(a)v(x;h) = \mathcal{O}(h^\infty).$$

This statement provides a definition of the wavefront set $\mathrm{WF}_h(v)$, which is a closed subset of T^*M.

Let us now take u defined by (2.1) and some symbol a, and calculate

$$\mathrm{Op}_h(a)u(x;h) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}((x-y)\cdot\xi+\varphi(y))}a\Big(\frac{x+y}{2},\xi\Big)b(y)\,d\xi dy.$$

By the method of stationary phase, we see that:

(1) If $\operatorname{supp} a$ does not intersect the Lagrangian

$$\Lambda_\Phi := \{(x,\partial_x\Phi(x)) \mid x\in M\}$$

then $\mathrm{Op}_h(a)u(x;h) = \mathcal{O}(h^\infty)$. Therefore, the wavefront set of $u(x;h)$ is contained in Λ_Φ.

(2) $\mathrm{Op}_h(a)u(x;h)$ again has the form (2.1), for some amplitude which is a classical symbol in h.

(3) We have

$$\mathrm{Op}_h(a)u(x;h) = e^{\frac{i}{h}\Phi(x)}a(x,\partial_x\Phi(x))b(x) + \mathcal{O}(h).$$

Therefore, if we consider the amplitude b as a function on Λ_Φ, multiplying by a pseudodifferential operator $\mathrm{Op}_h(a)$ amounts to multiplying the amplitude by $a|_{\Lambda_\Phi}$, up to $\mathcal{O}(h)$ errors.

(4) If $a|_{\Lambda_\Phi} = 0$, then we can find

$$\mathrm{Op}_h(a)u(x;h) = e^{\frac{i}{h}\Phi(x)}hLa + \mathcal{O}(h^2),$$

where L is a first order differential operator given by the Hamiltonian vector field H_a (lifting functions of x to functions on Λ_Φ) plus a zero order term, which we just denote by $(\dots)$.

We note that Λ_Φ is always a Lagrangian distribution, as the canonical 1-form α, when restricted to Λ_Φ (which we parametrize by x), is equal to $d\Phi$ and thus $\omega = d\alpha = 0$ when restricted to Λ_Φ.

As an example of an application of functions of the form (2.1), consider the one-dimensional semiclassical Schrödinger operator

$$P(h) = h^2D_x^2 + V(x)$$

Here $D_x = \partial_x/i$ and V is a smooth function such that $V(-1) = V(1) = 0$, $V<0$ on $(-1,1)$, $V>0$ and goes to infinity at infinity outside of $[-1,1]$, and $\partial_xV(\pm1)\neq 0$. The symbol of P is given by

$$p(x,\xi) = \xi^2 + V(x).$$

Consider a phase function Φ solving the following eikonal equation, say, on $(-1/2,1/2)$:

$$(\partial_x\Phi)^2 + V(x) = 0.$$

There are two such functions, we choose for example the one with $\partial_x \Phi > 0$. Then

$$p|_{\Lambda_\Phi} = 0.$$

If u is given by (2.1), then we find

$$P(h)u = e^{\frac{i}{h}\Phi(x)} h(H_p + (\dots))b + \mathcal{O}(h^2).$$

We can solve the *transport equation* $H_p + (\dots)b = 0$ locally and get a u such that $P(h)u = \mathcal{O}(h^2)$ in $(-1/2, 1/2)$. One can repeat this process (by solving the transport equation with a right-hand side) and get a function u such that $P(h)u = \mathcal{O}(h^\infty)$ in $(-1/2, 1/2)$.

However, we are not able to find a function Φ such that $\Lambda_\Phi = \{p = 0\}$ near the turning points ± 1, the reason being that $\{p = 0\}$ does not project nicely onto the x axis at these points. Therefore, we need to consider more general oscillatory expressions, which leads us to the main formula of the present note:

$$u(x; h) = h^{-m/2} \int_{\mathbb{R}^m} e^{\frac{i}{h}\Phi(x,\theta)} b(x, \theta; h)\, d\theta. \tag{2.3}$$

Here $\Phi(m, \theta)$ is a real-valued function on $M \times \mathbb{R}^m$, which is a *nondegenerate phase function* in the sense that on the critical set

$$C_\Phi := \{(x, \theta) \mid \partial_\theta \Phi(x, \theta) = 0\},$$

the differentials $d(\partial_{\theta_1}\Phi), \dots, d(\partial_{\theta_2}\Phi)$ are linearly independent. The dimension m can be anything; if $m = 0$ in particular, we get back (2.1). The normalization $h^{-m/2}$ is introduced to make $\|u\|_{L^2} \sim 1$. As for $b(x, \theta)$, it is simply required to be a classical symbol compactly supported inside $M \times \mathbb{R}^m$.

Using the method of stationary phase (actually, the method of nonstationary phase even) as before, we can see that the wavefront set of $u(x; h)$ is contained in the set

$$\Lambda_\Phi := \{(x, \partial_x \Phi(x, \theta)) \mid (x, \theta) \in C_\Phi\}.$$

This set is a (in principle, immersed, but we will ignore this issue here) Lagrangian submanifold of T^*M.

If $\Lambda \subset T^*M$ is a Lagrangian submanifold and $\Phi(x, \theta)$ is a phase function on some open subset of M, then we say that Φ generates Λ locally if $\Lambda_\Phi \subset \Lambda$. One then calls an h-dependent family of functions $v(x; h)$ a *Lagrangian distribution* associated to Λ, if v can be written as a finite sum of expressions of the form (**??**), with various phase functions locally parametrizing Λ. We denote $v \in I(\Lambda)$. Similarly, an operator $A : C^\infty(M_1) \to C^\infty(M_2)$ is called a *Fourier integral operator* associated to some canonical relation $\kappa \subset T^*M_1 \times T^*M_2$ (we write $A \in I(\kappa)$), if the Schwartz kernel of $h^{(n_1+n_2)/4}A$ is a Lagrangian distribution associated to the flipped graph Γ_κ.

If two phase functions Φ_1 and Φ_2 generate the same Lagrangian, then the formulas (2.3) for these two functions describe the same class of distributions: namely, for each classical symbol b_1, there exists a classical symbol b_2 such that

$$h^{-m_1}\int_{\mathbb{R}^{m_1}} e^{\frac{i}{h}\Phi_1(x,\theta_1)}b_1(x,\theta_1)\,d\theta_1 = h^{-m_2}\int_{\mathbb{R}^{m_2}} e^{\frac{i}{h}\Phi_2(x,\theta_2)}b_2(x,\theta_2)\,d\theta_2 + \mathcal{O}(h^\infty).$$

We will see a particular case of this fact in the next section, when we define the principal symbol of a Lagrangian distribution.

If $u \in I(\Lambda)$ and $A = \mathrm{Op}_h(a)$ is a pseudodifferential operator, then Au again lies in $I(\Lambda)$. Moreover, if $a|_\Lambda = 0$, then $Au \in hI(\Lambda)$. A more precise formula is possible (and essential for applications for spectral problems); to have it, one needs to define invariantly the principal symbol of a Lagrangian distribution, which is the purpose of the global theory, described in the next section. Some other properties of the local theory are listed below:

(1) By (2.2), we see that a pseudodifferential operator is a Fourier integral operator associated to the identity map on T^*M.
(2) As mentioned in the introduction, the pullback map is a Fourier integral operator; we see this by writing by the Fourier inversion formula

$$\varphi^* u(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(x)-y)\cdot\xi}u(y)\,dyd\xi.$$

(The integration here is understood in terms of oscillatory integrals, see for example [GrSj, Chapter 1].)
(3) If $u \in I(\Lambda)$ is a Lagrangian distribution and $A \in I(\kappa)$ is a Fourier integral operator, and a certain cleanness condition holds (it is always true if κ is a canonical transformation, rather than just a relation), then $Au \in I(\kappa(\Lambda))$.
(4) If $A \in I(\kappa_A), B \in I(\kappa_B)$ are two Fourier integral operators, and a certain cleanness condition holds (always true if at least one of κ_A, κ_B is a canonical transformation), then $AB \in I(\kappa_A \circ \kappa_B)$.

Finally, one can characterize Lagrangian distributions in the following invariant way: $u \in I(\lambda)$ if and only if for each finite set of pseudodifferential operators $A_1, \dots, A_k$, with $A_k = \mathrm{Op}_h(a_k)$, and such that $a_j|_\Lambda = 0$ for each j, we have

$$A_1 \dots A_k u = \mathcal{O}(h^k)_{L^2}.$$

3. Global theory of Lagrangian distributions

In this section, we will explain how to define invariantly the principal symbol of a Lagrangian distribution $u \in I(\Lambda)$. The problem here is that there are many phase functions locally parametrizing Λ, and the expressions (2.3) for the same distribution u with different phase functions will have different amplitudes. One then needs to

understand how the principal part of the amplitude changes when we switch to a different phase function.

Rather than studying two arbitrary phase functions, we pass from an arbitrary phase function to a specific coordinate-dependent phase function. One can (almost, but the exceptions do not cause essential problems) write a Lagrangian Λ locally in some appropriately chosen coordinate system x on M as

$$\Lambda = \{(\partial_\xi F(\xi), \xi) \mid \xi \in \mathbb{R}^n\}$$

Here (x, ξ) are the coordinates on T^*M induced by x and $F(\xi)$ is some smooth real-valued function. Such parametrization is possible once Λ projects nicely onto the ξ variables, as then we can write $\Lambda = \{(G(\xi), \xi)\}$ for some function $G : \mathbb{R}^n \to \mathbb{R}^n$; since Λ is Lagrangian, the restriction $G(\xi)\, d\xi$ of the 1-form $x\, d\xi$ to Λ is closed (since $d(x\, d\xi) = -\omega$), thus it is exact and we have $G(\xi) = \partial_\xi F(\xi)$ for some function F.

The phase function

$$\Phi_F(x, \xi) := x \cdot \xi - F(\xi)$$

parametrizes Λ locally. Note that F is uniquely determined up to a constant; this freedom will cause phase shift to appear later.

Take some other phase function $\Phi(x, \theta)$ parametrizing Λ locally, with $\theta \in \mathbb{R}^m$, and some symbol $b_\Phi(x, \theta; h)$; we then want to find a symbol $b_F(\xi; h)$ such that

$$h^{-m/2} \int_{\mathbb{R}^m} e^{\frac{i}{h}\Phi(x,\theta)} b_\Phi(x, \theta; h)\, d\theta = h^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x\cdot\xi - F(\xi))} b_F(\xi; h)\, d\xi + \mathcal{O}(h^\infty).$$

For this, we use the Fourier inversion formula, writing (henceforth we omit all factors of 2π)

$$b_F(\xi; h) := h^{-(m+n)/2} \int_{\mathbb{R}^m \times \mathbb{R}^n} e^{\frac{i}{h}(\Phi(x,\theta) - x\cdot\xi + F(\xi))} b_\Phi(x, \theta; h)\, d\theta dx.$$

We now apply the method of stationary phase. The critical points (depending on ξ) are given by

$$\widetilde{C} := \{(x, \theta, \xi) \mid \partial_\theta \Phi(x, \theta) = 0,\ \partial_x \Phi(x, \theta) = \xi\}.$$

In other words, $(x, \theta) \in C_\Phi$ and the corresponding point on Λ_Φ is $(\partial_\xi F(\xi), \xi)$. Note that $\widetilde{C}$ is also parametrized by $\xi \in \mathbb{R}^n$, or by $(x, \xi) \in \Lambda$; the latter makes it possible to identify $\widetilde{C}$ with Λ.

The Hessian of the phase is just the Hessian $\partial^2\Phi$ of Φ. The value

$$c_{\Phi,F} := \Phi(x, \theta) - x \cdot \xi + F(\xi),\ (x, \theta, \xi) \in \widetilde{C},$$

is locally constant (the reason being that Φ and Φ_F parametrize the same Lagrangian). We then write by the method of stationary phase,

$$b_F(\xi; h) = e^{\frac{i}{h} c_{\Phi,F}} |\det \partial^2\Phi(x, \theta)|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sgn}(\partial^2\Phi(x,\theta))} b_\Phi(x, \theta; h) + \mathcal{O}(h),\ (x, \theta, \xi) \in \widetilde{C}.$$

We see then that the function on Λ

$$\sigma_0(x,\xi) := e^{\frac{i}{h}c_{\Phi,F}}|\det \partial^2\Phi(x,\theta)|^{-1/2}e^{\frac{i\pi}{4}\operatorname{sgn}(\partial^2\Phi(x,\theta))}b_\Phi(x,\theta;h),\ (x,\theta,\xi)\in\widetilde{C},$$

is independent of the choice of the phase function Φ, but it depends on the choice of local coordinates on M. On the other hand, the function $b_\Phi(x,\theta;h)$ does not depend on the choice of local coordinates, but it does depend on the choice of the phase function. We thus need to understand how the prefactors in the definition of $\sigma_0(x,\xi)$ change when we pass to different coordinates, which we denote $\tilde{x}$, but keep the phase function fixed; this change will be independent of the choice of Φ. Once we do this, we can consider the principal symbol as an element of a certain line bundle; we will not specify the line bundle explicity (except for the half density part), rather stating how its transition functions change when multiplied along some curve.

We start with the $e^{\frac{i}{h}c_{\Phi,F}}$ term. Assumethat $\tilde{x}$ is a different set of local coordinates and $\widetilde{F}$ is the corresponding function parametrizing Λ in these coordinates (recall that it is defined uniquely up to a constant). Assume that (x,ξ) and $(\tilde{x},\tilde{\xi})$ are the coordinates of the same point of Λ in the corresponding systems; then

$$e^{\frac{i}{h}c_{\Phi,\widetilde{F}}} = e^{\frac{i}{h}c_{F,\widetilde{F}}}e^{\frac{i}{h}c_{\Phi,F}},$$

where

$$c_{F,\widetilde{F}} := \widetilde{F}(\widetilde{\xi}) - \tilde{x}\cdot\tilde{\xi} - F(\xi) + x\cdot\xi$$

is (locally) constant. Now, assume that $\gamma:[0,1]\to\Lambda$ is a closed curve on Λ and we consider some coordinate systems $x^{(1)},\dots,x^{(k)}$ such that the domain of $x^{(j)}$ contains, say, $\gamma([j/k-2/3,j/k+2/3])$. Choose also the corresponding functions $F^{(1)},\dots,F^{(k)}$. Put $\rho_j=\gamma(j/k+1/2)\in\Lambda$ and and denote by $(\hat{x}^{(j)},\hat{\xi}^{(j)})$ the coordinates of this point in with respect to $(x^{(j)},\xi^{(j)})$ and by $(\check{x}^{(j)},\check{\xi}^{(j)})$ its coordinates with respect to $(x^{(j+1)},\xi^{(j+1)})$. Then we see that

$$\begin{aligned}\sum_{j=1}^k c_{F^{(j)},F^{(j+1)}} &= \sum_{j=1}^k F^{(j+1)}(\check{\xi}^{(j)}) - \check{x}_j\cdot\check{\xi}_j - F^{(j)}(\hat{\xi}_j) + \hat{x}_j\cdot\hat{\xi}_j\\ &= \sum_{j=1}^k (F^{(j)}(\check{\xi}^{(j-1)}) - F^{(j)}(\hat{\xi}_j)) - (\check{x}_{j-1}\cdot\check{\xi}_{j-1} - \hat{x}_j\cdot\hat{\xi}_j)\end{aligned}$$

Each term in the sum on the right-hand side is taken in the coordinate system $(x^{(j)},\xi^{(j)})$, with $(\check{x}^{(j-1)},\check{\xi}^{(j-1)})$ corresponding to ρ_{j-1} and $(\hat{x}^{(j)},\hat{\xi}^{(j)})$ corresponding to ρ_j. We can then write each term in the sum as an integral over $\gamma[\rho_{j-1},\rho_j]$ of the form (recalling that we are on Λ and denoting $(x,\xi)=(x^{(j)},\xi^{(j)})$)

$$d(x\cdot\xi - F(\xi)) = x\,d\xi + \xi\,dx - \partial_\xi F(\xi)\,d\xi = \alpha,$$

where $\alpha = \xi\, dx$ is the canonical 1-form. Therefore, we get

$$\sum_{j=1}^{k} c_{F^{(j)},F^{(j+1)}} = \int_\gamma \alpha. \tag{3.1}$$

We now go to the determinant term. Consider the map $j_\Phi : C_\Phi \to \Lambda$ defined by

$$j_\Phi(x,\theta) = (x, \partial_x \Phi(x,\theta)),\ (x,\theta) \in C_\Phi.$$

Consider the following differential forms on $M \times \mathbb{R}^m$:

$$\beta_\theta := d\theta_1 \wedge \cdots \wedge d\theta_m,$$
$$\beta_\Phi := d(\partial_{\theta_1}\Phi) \wedge \cdots \wedge d(\partial_{\theta_m}\Phi);$$

note that they do not depend on the choice of the coordinate system. Consider next the following differential forms on T^*M:

$$\beta_x := dx_1 \wedge \cdots \wedge dx_n,$$
$$\beta_\xi := d\xi_1 \wedge \cdots \wedge d\xi_n;$$

these forms depend on the choice of the coordinate sytem. We then have

$$j_\Phi^* \beta_\xi \wedge \beta_\Phi = \det(\partial^2 \Phi) \beta_x \wedge \beta_\theta. \tag{3.2}$$

Define now the following volume form on Λ:

$$\alpha_\Lambda := \alpha_\xi|_\Lambda.$$

It depends on the choice of coordinates. We can now write for two different coordinate systems x and $\tilde{x}$,

$$\frac{|\det \partial^2 \Phi|^{-1/2} \cdot |\alpha_\Lambda|^{1/2}}{|\det \tilde{\partial}^2 \Phi|^{-1/2} \cdot |\tilde{\alpha}_\Lambda|^{1/2}} = \frac{|dx|^{1/2}}{|d\tilde{x}|^{1/2}}.$$

Here $|\alpha_\Lambda|^{1/2}$ and $|dx|^{1/2}$ are half-densities on Λ and M, respectively. We then consider Lagrangian half-densities, rather than functions, on M. If x is some local coordinate system on M and in this system, our Lagrangian half-density is given by $u(x)|dx|^{1/2}$, where u is a Lagrangian function of the form (2.3) for some phase function Φ, then we define the principal symbol

$$\sigma(u(x)|dx|^{1/2})_{(x,\xi)} := e^{\frac{i}{h} c_{\Phi,F}} |\det \partial^2 \Phi(x,\theta)|^{-1/2} e^{\frac{i\pi}{4} \operatorname{sgn}(\partial^2 \Phi(x,\theta))} b_\Phi(x,\theta;h) |\alpha_\Lambda|^{1/2},$$
$$(x,\theta) \in C_\Phi,\ j_\Phi(x,\theta) = (x,\xi).$$

Therefore $\sigma(u(x)|dx|^{1/2})$ is a half-density on Λ.

It remains to handle the term $e^{\frac{i\pi}{4} \operatorname{sgn}(\partial^2 \Phi(x,\theta))}$. For that, define the *Hörmander–Kashiwara index* of any three Lagrangian subspaces $\Lambda_1, \Lambda_2, \Lambda_3$ of some finite-dimensional vector space V with some symplectic form ω as follows: consider the quadratic form S on $\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$ given by

$$Q(v_1,v_2,v_3) = \omega(v_1,v_2) + \omega(v_2,v_3) + \omega(v_3,v_1),\ v_j \in \Lambda_j,$$

and let

$$s(\Lambda_1, \Lambda_2, \Lambda_3) = \operatorname{sgn} Q$$

be the signature of this form. This index satisfies the following cocycle identity:

$$s(\Lambda_2, \Lambda_3, \Lambda_4) - s(\Lambda_1, \Lambda_3, \Lambda_4) = s(\Lambda_3, \Lambda_1, \Lambda_2) - s(\Lambda_4, \Lambda_1, \Lambda_2).$$

This immediately gives a definition of the index of a triple of Lagrangian submanifolds of T^*M intersecting at some point.

Now, we need to define the intersection number. Let $\gamma : [a, b] \to T^*M$ be a path in T^*M and $\Lambda_1(t), \Lambda_2(t)$ a family of Lagrangian subspaces of $T_{\gamma(t)}(T^*M)$. If $\Lambda_3(t)$ is another Lagrangian transverse to both $\Lambda_1(t)$ and $\Lambda_2(t)$ at every point, we define the intersection number

$$[\Lambda_1(t) : \Lambda_2(t)]_\gamma := s(\Lambda_1(a), \Lambda_2(a), \Lambda_3(a)) - s(\Lambda_1(b), \Lambda_2(b), \Lambda_3(b)).$$

One can show that this intersection number is independent of the choice of Λ_3. If no such Λ_3 exists, we can break γ into smaller segments and define the intersection number as the sum of the intersection numbers over different segments. This definition does not depend on the choice of the segments, so it gives an invariant way of defining $[\Lambda_1(t) : \Lambda_2(t)]_\gamma$ for any path γ. (See for instance [Dy, Proposition 9] for details.) Moreover, if γ is a loop homotopic to some other loop γ', and we have a homotopy of the corresponding Lagrangian subspaces $\Lambda_j(t)$ to $\Lambda_j'(t)$, then

$$[\Lambda_1(t) : \Lambda_2(t)]_\gamma = [\Lambda_1'(t) : \Lambda_2'(t)]_{\gamma'}.$$

Returning to the signature term in the formula for the principal symbol, we can compute that

$$\operatorname{sgn} \tilde\partial^2 \Phi - \operatorname{sgn} \partial^2 \Phi = s(\Lambda, \Lambda_x, \Lambda_V) - s(\Lambda, \Lambda_{\tilde x}, \Lambda_V), \tag{3.3}$$

where Λ is the original Lagrangian, Λ_x and $\Lambda_{\tilde x}$ are the horizontal Lagrangians $\{\xi = \text{const}\}$ and $\{\tilde\xi = \text{const}\}$, respectively, for the coordinate systems x and $\tilde x$, and Λ_V is the fiber of T^*M. The details of the computation are not trivial; they can be found in the discussion preceding [HöIV, Theorem 25.1.9], which relies in turn on [HöIII, Section 21.6].

We can then find that if we multiply the transition functions for $e^{\frac{i\pi}{4} \operatorname{sgn} \partial^2 \Phi}$ over some loop γ like we did for the $e^{\frac{i}{h} c_{\Phi, F}}$ terms above, we get

$$e^{\frac{i\pi}{4} [\Lambda : \Lambda_V]_\gamma}.$$

We now describe how the principal symbol changes under multiplication by pseudodifferential operators. Let $A = \operatorname{Op}_h(a)$; if we require A to act on half-densities instead of functions and use the Weyl quantization of (2.2), then the symbol a is defined (as a function on T^*M) invariantly modulo $\mathcal{O}(h^2)$, not just $\mathcal{O}(h)$. Now, if $u \in I(\Lambda)$, then

$$\sigma(Au) = a|_\Lambda \sigma(u) + \mathcal{O}(h).$$

If moreover $a|_\Lambda = \mathcal{O}(h)$, then, letting H_a be the Hamiltonian flow of a and noting that it is tangent to Λ, we get

$$\sigma(Au) = \frac{h}{i}\mathcal{L}_{H_a}\sigma(u) + au + \mathcal{O}(h^2).$$

Here $\mathcal{L}$ is the Lie derivative, which is defined naturally on the half-density bundle.

We finally give an application to the joint spectrum of operators $P_1, \dots, P_n$ that form a quantum completely integrable system (see above). We attempt to write an $\mathcal{O}(h^2)$ approximate solution (in fact, an iterative procedure yielding an $\mathcal{O}(h^\infty)$ approximate solution is possible) to the equation

$$P_j u = \lambda_j u,\ \lambda_1, \dots, \lambda_n = \mathcal{O}(h).$$

Let $\Lambda = \mathbf{p}^{-1}(\lambda_1, \dots, \lambda_n)$ be a Liouville torus of the corresponding classical integrable system; we try to find $u \in I(\Lambda)$. We have $P_j u = \mathcal{O}(h)$, as $p_j = \mathcal{O}(h)$ on Λ; therefore, it suffices to solve the transport equations

$$\frac{h}{i}\mathcal{L}_{H_{p_j}}(\sigma(u)) = 0.$$

These are however ordinary differential equations on the torus, and we need a certain periodicity condition to make them solvable. Given what we have gathered so far about the principal symbol, this periodicity condition, also known as the *Bohr–Sommerfeld quantization condition*, can be expressed as follows. Let $\alpha = \xi\, dx$ be the canonical 1-form on T^*M. Let $\mu \in H^1(\Lambda; \mathbb{Z})$ be the Maslov class, defined as follows: if γ is a loop, then the pairing of μ and γ is the intersection index $[\Lambda : \Lambda_V]_\gamma$. Then we need for each loop γ that

$$\exp\left(\frac{i}{h}\int_\gamma \alpha + \frac{i\pi}{4}\mu(\gamma)\right) = 1,$$

or simply,

$$\alpha + \frac{\pi}{4}h\mu \in 2\pi h H^1(\Lambda; \mathbb{Z}).$$

This is a condition on λ_j, as the homology class of α on Λ depends on them.

4. A guide to bibliography

A general introduction to the field of semiclassical analysis is given in [Zw], though it does not feature the global theory of Lagrangian distributions outlined here.

Lagrangian distributions and Fourier integral operators in the microlocal case (without the semiclassical parameter h) are described [HöIV, Chapter 25]. In the semiclassical case, they can be found in [GuiSt] or the lecture notes [GuiSt2]. The author himself studied the theory (in the microlocal case) using a much shorter [GrSj, Chapter 11].

Global quantization conditions for quantum integrable systems, including quantum monodromy, have been studied in [VũNg], which also serves as a great introduction to this particular application of the global theory of Lagrangian distributions.

A complete treatment of the Maslov index can be found in [HöIII, Section 21.6]. Some details about the Hörmander–Kashiwara index and the intersection number can also be found in the expository notes [Dy], though the connection to Fourier integral operators is not explained there.

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