Unramified morphisms

Definition 1 *Let $f: X \to Y$ be a morphism of schemes and let $T \to T'$ be an nth order thickening of Y-schemes. Let X/Y_T denote the sheaf on T which assigns to each open set U in T the set of Y-morphisms $U \to X$, and let $X/Y_{T'}$ the sheaf which assigns to each open subset U' of T' the set of Y-morphisms $T' \to X$. In fact since $T \to T'$ is a homeomorphism, we can view $X/Y_{T'}$ as a sheaf on T.*

1. *X/Y is formally unramified if for all $T \to T'$, the map $X/Y_{T'} \to X/Y_T$ is injective.*
2. *X/Y is formally smooth if for all $T \to T'$, the map $X/Y_{T'} \to X/Y_T$ is surjective.*
3. *X/Y is formally étale if for all $T \to T'$, the map $X/Y_{T'} \to X/Y_T$ is bijective.*

Note that if X/Y is formally unramified (resp. étale), then the map on global sections $X/Y(T') \to X/Y(T)$ is injective, resp. bijective. If X/Y is formally smooth, we cannot conclude that $X/Y(T') \to X/Y(T)$ is surjective in general. However, if T is affine, X/Y is locally of finite presentation, and $h: T \to X$ is quasi-compact and quasi-separated, then h can be lifted to T'. Indeed, by induction it is enough to check this for first order thickenings, and it is enough to check that in this case, $Def_h(T')$ is not empty. The smoothness hypothesis implies that this is so locally on T, but not globally. Then $Def_h(T')$ is a torsor under the abelian sheaf $Der_{X/Y}(h_*I) \cong Hom(\Omega_{X/Y}, h_*I)$, which is quasi-coherent, and we know that every such torsor is trivial when T is affine.

It is clear that the family of formally smooth (resp unramified or étale) maps is closed under composition and base change.

A morphism X/Y is said to be smooth (resp. étale) if it is locally of finite presentation and formally smooth (resp. étale) A morphism X/Y is said to be unramified if it is locally of finite type and formally unramified.

Proposition 2 *A morphism X/Y is formally unramified if and only if $\Omega_{X/Y} = 0$.*

Proof: Indeed, the vanishing of $\Omega_{X/Y}$ implies that there is at most one deformation of any first order thickening, and hence of any nth order thickening by induction. Conversely, if X/Y is unramified, then the two deformation p_1 and p_2 from $P^2_{X/Y} \to X$ of the identify map must be equal, and this implies that $p_1^* a = p_2^*(a) \in \Omega_{X/Y}$ for all a, hence $\Omega = 0$. □

Proposition 3 *Let $X \to Y$ be locally of finite type.*

1. *$X \to Y$ is unramified if and only the diagonal morphism is an open immersion.*

2. *If x is a point of X and the fiber $\Omega_{X/Y}(x)$ of $\Omega_{X/Y}$ at x vanishes, then $\Omega_{X/Y}$ vanishes in a neighborhood of x.*

3. *X/Y is unramified if and only if for ever point y of Y, the fiber X_y is unramified over* $\operatorname{Spec} k(y)$.

4. *If k is a field and $\overline{k}$ is an algebraic closure of k, then a k-scheme X/k is unramified if and only if $\overline{X}/\overline{k}$ is unramified.*

5. *Let k be an algebraically closed field and X a k-scheme of finite type. Then X/k is unramified if and only if X is a finite disjoint union of copies of k.*

6. *A finite field extension is unramified if and only if it is separable.*

Proof: If $X \to Y$ is of finite type, then the ideal $I_{X/Y}$ of the diagonal is finitely generated, and a finitely generated ideal I of a local ring with $I = I^2$ must either be the zero ideal or the unit ideal, by Nakayama's lemma. (1) follows.

(2) is from the semicontinuity of the dimension of $\Omega(x)$. For (3): if X/Y is unramified, so are the fibers. Say all the fibers are unramified. Then for each $x \in X$, let y be its image. We claim that $\Omega_{X/Y}(x) = 0$, since this is true for all x and $\Omega_{X/Y}$ is finitely generated, it follows that $\Omega_{X/Y} = 0$. Since the fibers are unramified, each $\Omega_{X_y/y} = 0$, and we have a diagram

$$\begin{array}{ccccc} x & \longrightarrow & X_y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ y & \longrightarrow & y & \longrightarrow & Y \end{array}$$

Since the pullback of $\Omega_{X/Y}$ to X_y is $\Omega_{X_y/y}$, it vanishes, hence so does its fiber at x. (4) is easy. For (5): Suppose without loss of generality that X is affine, say $X = \operatorname{Spec} A$. Then if m is any maximal ideal of A, $m = m^2$, and hence in the localization A_m, $mA_m = 0$. Since m is finitely generated, there exists an $a \in A \setminus m$ such that $am = 0$, and then $mA_a = 0$. This means that A_a is a field, isomorphic to k, and the point corresponding to m is both open and closed. Furthermore, the open subset D_a of X is just *speck*, scheme theoretically.n This shows that every closed point of X is also open, and by quasi-compactness X is just a disjoint union of a finite set of closed points. □

Example 4 The map $k[t] \to k[s]$ sending t to s^2 is ramified, but uramified away from $s = 0$ if 2 is invertible. Indeed, Ω is the free $k[s]$ module generated by ds with relation $2sds = 0$.