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Extended Topological Gauge Theories in Codimension Zero and Higher

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Abstract

Topological field theories (TFT) have been extensively studied by physicists and mathematicians ever since the seminal paper by Edward Witten. Not only can these theories lead to new insights in the world of low-dimensional topology, but they have also been used to better comprehend the mysterious structure of the WZW models and the (fractional) quantum Hall effects, while at the same time alleviating the arduous lives lead by string theorists. The mathematical structure of a topological field theory was completely laid out by Atiyah with his “axioms of a TFT,” and later modernized into the “symmetric, monoidal functor from the category of cobordisms to the category of vector spaces” which we are most familiar. Although a beautiful theory, there are drawbacks to the definition of a TFT set forth by Atiyah. Namely, with the Atiyah-type TFT, one can only talk about the TFT living on manifolds of at most codimension one. Hence, there is no notion of the action of an n-dimensional TFT on a $(n-2)$-submanifold. Most recently, there has been an entire legion of mathematical physicists publishing copious amounts of research towards developing the theory of extended TFT's. These TFT's can live on manifolds of any arbitrary codimension. Extended TFT's have also found their way into the study of quantum gravity; notable via the paper of Morton and Baez. The purpose of this paper is to lead the reader from the usual notions of TFT's all the way to extended TFT's, while covering the relevant mathematical and physical structures arising in between. We begin with a review (or introduction, depending on the readers current knowledge) of all the mathematical concepts required to study such theories; namely, category theory, (co)homological algebra, principal bundles, connections and characteristic classes. Following this review, we then introduce the classical 3-dimensional Chern-Simons theory with compact gauge group. We then restrict to the case of a finite gauge group (also known as Dijkgraaf-Witten theories), which are far simpler to rigorously quantize. Finally, we introduce the extended Dijkgraaf-Witten theory and show how, under quantization, it leads to higher categories. We conclude by explicitly carrying out several calculations of the quantum invariants associated, by the quantum theory, to specific manifolds of varying codimension.

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Contents

Chapter 1

Introduction/Overview

A topological quantum field theory (TQFT) is a metric independent quantum field theory that gives rise to topological invariants of the background manifold. That is, a TQFT is a background-free quantum theory with no local degrees of freedom. TQFT's have been the main source for the interaction between physics and mathematics for the past 25 years. For example mathematicians are interested in topological theories because of the knot invariants they produce, while physicists are interested in topological theories because they are, in a sense, the simplest examples of quantum field theories which are exactly solvable and generally covariant. Although, in general, one can define TQFT's in any arbitrary dimension, most of the research currently being conducted is restricted to the three dimensional Chern-Simons theory. Partly because Witten was able to show that the expectation value of an observable (obtained as the product of the Wilson loops associated with a link) gives the generalized Jones invariant of the link, and partly because of its implications in 3-dimensional gravity.

The purpose of this thesis is to completely work out the details, for the finite structure group case, required to construct an extended TQFT, all the way down to points. This will require the introduction of many concepts from mathematics. In particular, we begin with an overview of category theory, algebraic topology, and the theory of characteristic classes on principal G-bundles. Following this discussion, we introduce the classical Chern-Simons theory on trivial principal bundles over a 3-dimensional manifold, $G \hookrightarrow P \xrightarrow{\pi} M$. Here we take, for the Lagrangian, the pullback of the antiderivative of the Chern-Weil 4-form α associated to a connection ω on P - this form α is also known as the Chern-Simons 3-form - via the section $s : M \to P$. The classical action is then defined by integrating α over the moduli space of connections

$$S = \frac{k}{8\pi^2} \int_{\mathcal{A}/\mathcal{G}} s^*(\alpha(\omega)).$$

Furthermore, we show that this action is gauge-invariant when defined over closed 3-manifolds (up to an integer), while on a compact manifold with boundary, a gauge transformation effects the action by the addition of a WZW term. Finally, we derive the expression for the classical action in the case where the principal bundle is not trivial.

Following the discussion of the classical Chern-Simons theory, we then begin the study of the 3-dimensional Dijkgraaf-Witten theory (Chern-Simons theory with a finite gauge group Γ). Here, rather than integrating the Chern-Simons 3-form, our classical action is given an element in the degree 4 cohomology class $[\alpha] \in H^4(B\Gamma; \mathbb{Z})$

$$S = e^{2\pi i \langle \gamma_M^*([\alpha]), [M] \rangle}.$$

Once the classical action has been defined, we define the quantum theory (i.e., partition function) by summing this action over the moduli space of flat Γ-bundles over manifolds of dimension three and two. We then show that this definition of the path integral obeys the axioms, set forth by Atiyah, defining a TQFT. That is, the path integral defines a symmetric monoidal functor from the category of cobordisms to the category of vector spaces - to closed 2-dimensional manifolds it assigns a vector space, while to compact 3-dimensional manifolds it assigns an element in the vector space(s) associated to its boundary.

Finally, we show how to extend the classical action and the path integral to incorporate manifolds of codimensions higher than one. This will require us to define torsors and gerbes, as well as higher Hilbert spaces and higher categories. We end this section, and the thesis, by performing several explicit calculations, and we show that by assigning the category of vector bundles over Γ to the point gives an extended TQFT down to points, thus fulfilling our beginning objective.

Part I

MATHEMATICAL BACKGROUND

Chapter 2

Category Theory (Abstract Nonsense)

2.1 Categories and Functors

In this section we give several definitions relevant to the sequel. The reader in search of a deeper introduction to category theory is directed to the book by Mac Lane [27].

Definition 2.1.1. A **category** $\mathcal{C}$ consists of a class of **objects**, denoted $\mathrm{Obj}(\mathcal{C})$, along with, for each pair of objects $A, B \in \mathrm{Obj}(\mathcal{C})$, a set, $\mathrm{Hom}_{\mathcal{C}}(A, B)$, of **morphisms** such that:

- for each $A \in \mathrm{Obj}(\mathcal{C})$, an element $id_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$, called the **identity morphism** on the object A;
- for each triple $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C), \tag{2.1}$$

  called **composition**;

such that $id_B \circ f = f = f \circ id_A$ and $f \circ (g \circ h) = (f \circ g) \circ h$, where the equalities hold for all objects and morphisms for which both sides make sense.

N.B. 2.1.2. Note, we will often simplify notation and write $A \in \mathcal{C}$ to mean $A \in \mathrm{Obj}(\mathcal{C})$. Also, if it is clear that we are working in a particular category, we will typically drop the suffix to Hom. Hence, if it is clear that we are looking at morphisms between objects in a category $\mathcal{C}$, then we will usually write $\mathrm{Hom}(A, B)$ rather than $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

Example 2.1.3. Let Set denote the category whose objects are sets X and whose morphisms $f : X \to Y$ are mappings between the sets. In this case, $\mathrm{Hom}(X, Y)$ is the set of all functions from X to Y.

Example 2.1.4. Let G be a group, we can interpret G as a category as follows. Consider the category with only one object, which we denote by $\star$, for which the set of morphisms from this object to itself is just G. The identity and multiplication operations in G provide the identity morphism and composition morphism, respectively.

Example 2.1.5. Let $\mathscr{V}_1$ denote the category whose objects are vector spaces V over some fixed field $\Bbbk$ and whose morphisms are linear maps. Let Grp denote the category whose objects are groups G and whose morphisms are group homomorphisms. Let Ab denote the restriction of Grp to the case where all objects are abelian groups (we still keep group homomorphisms as morphisms). Finally, let Top denote the category whose objects are topological spaces and whose morphisms are continuous functions between topological spaces.

Definition 2.1.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be any two categories. Then, a (covariant) **functor** $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{C}'$ is a map $\mathcal{F} : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}')$, along with, for each pair of objects, $A, B \in \mathrm{Obj}(\mathcal{C})$, a map $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{F}(B))$ such that:

- $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$;
- $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for any compatible morphisms f and g in $\mathcal{C}$.

Remark 2.1.7. If we were to switch the order of composition, $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$, then we still would have a functor, which we call a contravariant functor. Unless otherwise stated, all functors will be covariant.

Example 2.1.8. Consider the two categories Top and Ab. We can construct a functor between them by sending any topological space to some abelian group and any continuous map to a group homomorphism. Hence, we have constructed a functor

$$\mathcal{H} : \mathrm{Top} \longrightarrow \mathrm{Ab}. \tag{2.2}$$

As we will see, this particular functor (with a bit more structure) is the categorical interpretation of homology.

Refining the notion of a category gives a new object, called a *groupoid*.

Definition 2.1.9. A groupoid is a category $\mathcal{G}$ with every morphism an isomorphism. More precisely, A **groupoid** $\mathcal{G}$ consists of a class of objects, denoted $\mathrm{Obj}(\mathcal{G})$, along with, for each pair of objects $A, B \in \mathrm{Obj}(\mathcal{G})$, a set, $\mathrm{Hom}_{\mathcal{G}}(A, B)$, of morphisms such that:

- for each $A \in \mathrm{Obj}(\mathcal{G})$, an element $id_A \in \mathrm{Hom}_{\mathcal{G}}(A, A)$, called the identity morphism on the object A;
- for each triple $A, B, C \in \mathrm{Obj}(\mathcal{G})$, a map

$$\circ : \mathrm{Hom}_{\mathcal{G}}(B, C) \times \mathrm{Hom}_{\mathcal{G}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(A, C) \tag{2.3}$$

  called composition;
- for each morphism $f : A \to B$ there exists an 'inverse' morphism $f^{-1} : B \to A$;

such that $id_B \circ f = f = f \circ id_A$, $f \circ (g \circ h) = (f \circ g) \circ h$, $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$, where the equalities hold for all objects and morphisms for which both sides make sense.

2.2 Natural Transformations

The father of category theory Sanders Mac Lane said: "I did not invent category theory to talk about categories and functors. I invented category theory to talk about natural transformations". In order to understand what Mac Lane implied, let us define natural transformations.

Definition 2.2.1. Given two functors $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{C}'$, a **natural transformation** $\alpha : \mathcal{F} \to \mathcal{F}'$ consists of:

- a function α which assigns to each object A in $\mathcal{C}$ a morphism $\alpha_A : \mathcal{F}(A) \to \mathcal{F}'(A)$ such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, the following diagram commutes

$$
\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\alpha_A"'] & \mathcal{F}(B) \arrow[d, "\alpha_B"] \\
\mathcal{F}'(A) \arrow[r, "\mathcal{F}'(f)"'] & \mathcal{F}'(B)
\end{tikzcd}
$$

Remark 2.2.2. We can compose two natural transformations $\alpha : \mathcal{F} \to \mathcal{F}'$ and $\beta : \mathcal{F}' \to \mathcal{F}''$ to get another natural transformation $\beta \circ \alpha : \mathcal{F} \to \mathcal{F}''$. We can also *identify* natural transformations.

Definition 2.2.3. Given two functors $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{C}'$, a **natural isomorphism** $\alpha : \mathcal{F} \to \mathcal{F}'$ is a natural transformation that has an inverse; i.e., there exists a natural transformation $\beta : \mathcal{F}' \to \mathcal{F}$ such that $\beta \circ \alpha = id_{\mathcal{F}}$ and $\alpha \circ \beta = id_{\mathcal{F}'}$.

Proposition 2.2.4. *$\alpha : \mathcal{F} \to \mathcal{F}'$ is a natural isomorphism iff for every object $A \in Obj(\mathcal{C})$, the morphism $\alpha_A : \mathcal{F}(A) \to \mathcal{F}'(A)$ is invertible.*

Proof. Follows directly from the definitions. □

Definition 2.2.5. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is an **equivalence of categories** if it has a **weak inverse**; that is, if there exists a functor $\mathcal{F}' : \mathcal{C}' \to \mathcal{C}$ along with natural transformations $\alpha : \mathcal{F}' \circ \mathcal{F} \to 1_{\mathcal{C}}$ and $\beta : \mathcal{F} \circ \mathcal{F}' \to 1_{\mathcal{C}'}$.

2.3 Additional Structure: Symmetric Monoidal Categories and Functors

Roughly speaking, a category is *monoidal* if it has 'tensor products' of objects and morphisms which satisfy all of the usual axioms, and an object 1 which plays the role of identity for the tensor product. The same category can often be made into a monoidal category in more than one way. For example the category Set can be made into a monoidal category with the cartesian product or the disjoint union as the 'tensor product'.

Definition 2.3.1. A **monoidal category** $(\mathcal{C}, \otimes)$, is a category $\mathcal{C}$ equipped with:

- a functor
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \tag{2.4}$$
called the **tensor product**;
- an object $1 \in \mathrm{Obj}(\mathcal{C})$, called the **unit object**;
- a natural isomorphism
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C), \tag{2.5}$$
called the **associator**;
- a natural isomorphism
$$\lambda_A : 1 \otimes A \longrightarrow A, \tag{2.6}$$
called the **left unit**;
- a and natural isomorphism
$$\rho_A : A \otimes 1 \longrightarrow A, \tag{2.7}$$
called the **right unit**;

such that, for all $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$, the following diagram, called the **pentagon diagram**, commutes

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes id_D"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "id_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

as well as, for all $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the following diagram, called the **triangle diagram**, commutes

$$\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "\rho_A \otimes id_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "id_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Example 2.3.2. Let $(\mathrm{Set}, \times)$ denote the category Set along with Cartesian products of sets. This gives a monoidal category. Additionally, $(\mathrm{Top}, \sqcup)$, where $\sqcup$ is disjoint union forms a monoidal category. As well as $(\mathrm{Ab}, \oplus)$, where $\oplus$ is the direct sum of abelian groups.

Next we define braided and symmetric monoidal categories. Intuitively speaking, a *braided monoidal category* is a category with a tensor product and an isomorphism called the 'braiding' which lets us 'switch' two objects in a tensor product, $A \otimes B \to B \otimes A$.

Definition 2.3.3. A **braided monoidal category** $(\mathcal{C}, \otimes, \mathcal{B})$ is a monoidal category $(\mathcal{C}, \otimes)$ and a natural isomorphism, called the **braiding**,

$$\mathcal{B}_{A,B} : A \otimes B \longrightarrow B \otimes A, \qquad A, B \in \mathrm{Obj}(\mathcal{C}), \tag{2.8}$$

such that the following two diagrams, known as the hexagonal equations, commute for all $A, B, C \in \mathrm{Obj}(\mathcal{C})$:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\mathcal{B}_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\mathcal{B}_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \mathcal{B}_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha^{-1}_{A,B,C}"] \arrow[d, "\mathcal{B}_{A,B \otimes C}"'] & (A \otimes B) \otimes C \arrow[r, "\mathcal{B}_{A,B} \otimes id_C"] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\
(B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"'] & B \otimes (C \otimes A) \arrow[r, "id_B \otimes \mathcal{B}_{C,A}"'] & B \otimes (A \otimes C)
\end{tikzcd}$$

Definition 2.3.4. A **symmetric monoidal category** $(\mathcal{C}, \otimes, \mathcal{B})$ is a braided monoidal category $(\mathcal{C}, \otimes, \mathcal{B})$ where the braiding $\mathcal{B}$ satisfies the additional constraint

$$\mathcal{B}_{B,A} \circ \mathcal{B}_{A,B} = id_{A \otimes B}, \tag{2.9}$$

for all $A, B \in \mathrm{Obj}(\mathcal{C})$. Equivalently, a symmetric monoidal category is a braided monoidal category such that $\mathcal{B}_{A,B} = \mathcal{B}^{-1}_{B,A}$ for all $A, B \in \mathrm{Obj}(\mathcal{C})$.

Definition 2.3.5. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ between two monoidal categories is called a **monoidal functor** if it is equipped with:

- a natural transformation $\Phi_{A,B} : \mathcal{F}(A) \otimes \mathcal{F}(B) \to \mathcal{F}(A \otimes B)$,
- and an isomorphism $\phi : 1_{\mathcal{C}'} \to \mathcal{F}(1_{\mathcal{C}})$,

such that:

- the following diagram commutes for all $A, B, C \in \mathrm{Obj}(\mathcal{C})$

$$\begin{tikzcd}
(\mathcal{F}(A)\otimes\mathcal{F}(B))\otimes\mathcal{F}(C) \arrow[r, "\Phi_{A,B}\otimes id_{\mathcal{F}(C)}"] \arrow[d, "\alpha_{\mathcal{F}(A),\mathcal{F}(B),\mathcal{F}(C)}"'] & \mathcal{F}(A\otimes B)\otimes\mathcal{F}(C) \arrow[r, "\Phi_{A\otimes B,C}"] & \mathcal{F}((A\otimes B)\otimes C) \arrow[d, "\mathcal{F}(\alpha_{A,B,C})"] \\
\mathcal{F}(A)\otimes(\mathcal{F}(B)\otimes\mathcal{F}(C)) \arrow[r, "id_{\mathcal{F}(A)}\otimes\Phi_{B,C}"'] & \mathcal{F}(A)\otimes\mathcal{F}(B\otimes C) \arrow[r, "\Phi_{A,B\otimes C}"'] & \mathcal{F}(A\otimes(B\otimes C))
\end{tikzcd}$$

- and the following diagrams commute for all $A\in\mathrm{Obj}(\mathcal{C})$

$$\begin{tikzcd}
1\otimes\mathcal{F}(A) \arrow[r, "\lambda_{\mathcal{F}(A)}"] \arrow[d, "\phi\otimes id_{\mathcal{F}(A)}"'] & \mathcal{F}(A) \\
\mathcal{F}(1)\otimes\mathcal{F}(A) \arrow[r, "\Phi_{1,A}"'] & \mathcal{F}(1\otimes A) \arrow[u, "\mathcal{F}(\lambda_A)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(A)\otimes 1 \arrow[r, "\rho_{\mathcal{F}(A)}"] \arrow[d, "id_{\mathcal{F}(A)}\otimes\phi"'] & \mathcal{F}(A) \\
\mathcal{F}(A)\otimes\mathcal{F}(1) \arrow[r, "\Phi_{A,1}"'] & \mathcal{F}(A\otimes 1) \arrow[u, "\mathcal{F}(\rho_A)"']
\end{tikzcd}$$

Definition 2.3.6. A functor $\mathcal{F}:\mathcal{C}\to\mathcal{C}'$ between braided monoidal categories is called a **braided monoidal functor** if it is monoidal and it makes the following diagram commute for all $A,B\in\mathrm{Obj}(\mathcal{C})$:

$$\begin{tikzcd}
\mathcal{F}(A)\otimes\mathcal{F}(B) \arrow[r, "\mathcal{B}_{\mathcal{F}(A),\mathcal{F}(B)}"] \arrow[d, "\Phi_{A,B}"'] & \mathcal{F}(B)\otimes\mathcal{F}(A) \arrow[d, "\Phi_{B,A}"] \\
\mathcal{F}(A\otimes B) \arrow[r, "\mathcal{F}(\mathcal{B}_{A,B})"'] & \mathcal{F}(B\otimes A)
\end{tikzcd}$$

Definition 2.3.7. A **symmetric monoidal functor** is a braided monoidal functor that happens to go between symmetric monoidal categories (no extra conditions are needed)

Example 2.3.8. For us, the most important example of a symmetric monoidal functor is the functor between $(\mathrm{Top},\sqcup)$ and $(\mathrm{Ab},\otimes)$, which we call a topological quantum field theory or TQFT[1]. Another example of a symmetric monoidal functor is the homology

[1]This will all be defined later. Also, it is customary to add further restrictions on Top; namely we restrict the objects of Top to be manifolds and the morphisms to be cobordisms between the manifolds. This category is denoted Cob_n, where n specifies the dimension of the cobordisms.

functor $\mathcal{H} : (\mathrm{Top}, \sqcup) \to (\mathrm{Ab}, \oplus)$ (plus some extra conditions on $\mathcal{H}$). Hence, a TQFT is multiplicative while the homology functor is additive.

2.4 Higher Category Theory

Very roughly, an *n-category* is an algebraic structure consisting of a collection of 'objects', a collection of 'morphisms' between objects, a collection of '2-morphisms' between morphisms, and so on up to n, with various reasonable ways of composing these morphisms. For example, a 0-category is a set, while a 1-category is the usual notion of a category. There are two main types of higher categories: the *strict n-categories* and the *weak n-categories*. Strict n-categories are easily defined, however they are usually not sufficient for $n \geq 3$. On the other hand, weak n-categories - which are n-categories where the compositions of morphisms obey the usual associativity, unit, and exchange laws only up to coherent equivalence - are very difficult to define. In fact, there has been numerous proposed definitions of a weak n-category, none of which have been shown to be sufficient. For the reader who is interested in the subtitles in n-categories we direct them to the paper by Baez [12].

For our purposes, defining n-categories is somewhat easier since as far as we will go is $n = 2$. Hence we can use strict n- categories, which, as we have mentioned, are easily defined using induction on n. For instance consider the following:

Construction 2.4.1. (**working definition of a n-category**) First, we define a 0-category to be a set, a 1-category to be a category, a 2-category to be a category whose objects are categories and whose morphisms are functors, and so on. Using induction on n, an **n-category** is a category $\mathscr{C}$ whose objects are $(n-1)$-categories $\mathcal{C}$ and whose morphisms are $(n-1)$-morphisms (or $(n-1)$-functors) $\mathcal{F}$ between the objects.

Remark 2.4.2. The reader should keep in mind the idea of how one goes from a 1-category to a 2-category and then iterate this process when they encounter higher categories later on. For example, a 3-category can be thought of as a category whose objects are 2-categories and whose morphisms are natural transformations.

Although our definition of a higher category is far from precise (which we have done intentionally so as to not bog the reader down in inconsequential details), it encompasses all we will need in what follows.

Chapter 3

(Co)Homological Algebra: Homology and Cohomology

Homological algebra is the study of homology (and cohomology) from a strictly algebraic viewpoint. That is, in homological algebra, there is usually no mention of manifolds, topological spaces, CW complexes, etc; rather, one constructs a chain complex from a collection of abelian groups, from which they later define the homology and cohomology of such complexes. However, there are links between the algebraic version and the topological version, and in this chapter we will exploit these links in order to define concepts which will be used later. We by no means have attempted to give a thorough introduction to the homology and cohomology theories. We are merely trying to develop the necessary tools needed to understand the language which is used throughout the thesis. For a more complete treatment of everything that appears here, the reader is urged to review the books by Bott [13] and by Spanier [40], along with the notes by Moerdijk [34].

3.1 Chain Complexes and Homology

We begin with a definition.

Definition 3.1.1. By a **sequence of abelian groups** $\{C_\bullet, \partial_\bullet\}$ we mean a sequence

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \longrightarrow \cdots \tag{3.1}$$

of abelian groups C_n together with homomorphisms $\partial_n : C_n \to C_{n-1}$ for all $n \in \mathbb{Z}$.

Remark 3.1.2. These sequences form a category where objects are sequences $\{C_\bullet, \partial_\bullet\}$ and morphisms are homomorphisms $f_n : C_n \to D_n$ (here $\{D_\bullet, \tilde{\partial}_\bullet\}$ is another sequence) such that

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & \cdots \\
\cdots \arrow[r, "\tilde{\partial}_3"] & D_2 \arrow[r, "\tilde{\partial}_2"] & D_1 \arrow[r, "\tilde{\partial}_1"] & D_0 \arrow[r, "\tilde{\partial}_0"] & \cdots
\end{tikzcd}$$

commutes everywhere. We call such maps $f_\bullet$ **chain maps**.

Definition 3.1.3. A **chain complex** is a sequence of abelian groups and homomorphisms $\{C_\bullet, \partial_\bullet\}$

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \longrightarrow \cdots \tag{3.2}$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.

N.B. 3.1.4. We abuse notation and write the chain complex as $\{C_\bullet, \partial_\bullet\}$ or even $(C_\bullet, \partial_\bullet)$.

Remark 3.1.5. Observe that the condition to be a chain complex, namely $\partial_n \circ \partial_{n+1} = 0$, is equivalent to

$$\operatorname{im}(\partial_{n+1}) \subseteq \ker(\partial_n).$$

We call the chain complex $\{C_\bullet, \partial_\bullet\}$ **exact at position** n if

$$\operatorname{im}(\partial_{n+1}) = \ker(\partial_n). \tag{3.3}$$

Furthermore, it is obvious that the set of chain complexes forms a category, which we denote by Ch, where the objects are chain complexes and the morphisms are given by chain maps f.

We are now in a position to define the homology of a chain complex. Roughly speaking, homology measures the obstruction from a chain complex being exact. To be more precise, consider the following definition:

Definition 3.1.6 (Homology of a Chain Complex)**.** Define the subgroups $Z_n(C_\bullet) := \ker(\partial_n) \subseteq C_n$ and $B_n(C_\bullet) := \operatorname{im}(\partial_{n+1}) \subseteq C_n$. Then, the n**-th homology group** $H_n(C_\bullet)$ of the chain complex $\{C_n, \partial_n\}$ is defined as the quotient group

$$H_n(C_\bullet) := \frac{\ker \partial_n}{\operatorname{im}(\partial_{n+1})} = \frac{Z_n(C_\bullet)}{B_n(C_\bullet)}. \tag{3.4}$$

Remark 3.1.7. Note, $H_n(C_\bullet) = 0$ iff $\{C_\bullet, \partial_\bullet\}$ is exact at position n. Additionally, Z_n, B_n and H_n are all functors from Ch to the category of abelian groups Ab.

We call the subgroup $Z_n(C_\bullet)$ the group of n**-cycles** and the subgroup $B_n(C_\bullet)$ the group of n**- boundaries**, while elements of $Z_n(C_\bullet)$ are called n-cycles and the elements of $B_n(C_\bullet)$ are called n-boundaries. Elements $[c]$ in $H_n(C_\bullet)$ are equivalence classes of n-cycles; where two n-cycles c, c' are said to be equivalent (or **homologous**) iff their difference is a n-boundary, $c - c' = b$ (where we can write b as $d = \partial a$, with $a \in C_{n+1}$ some $(n+1)$-chain).

Example 3.1.8. Consider the chain complex

$$\cdots \xrightarrow{0} \mathbb{Z}_{\{2\}} \xrightarrow{n} \mathbb{Z}_{\{1\}} \xrightarrow{0} \mathbb{Z}_{\{0\}} \xrightarrow{n} \mathbb{Z}_{\{-1\}} \xrightarrow{0} \mathbb{Z}_{\{-2\}} \xrightarrow{n} \cdots,$$

where by 0 and n we mean multiplication by either 0 or $n \in \mathbb{Z}$ and the subscripts on the $\mathbb{Z}_{\{i\}}$ are there to help with positions; i.e., by $\mathbb{Z}_{\{2\}}$ we mean the abelian group $\mathbb{Z}$ in the position 2, $C_2 \equiv \mathbb{Z}_{\{2\}}$. This sequence is clearly exact since $0 \cdot n = n \cdot 0 = 0$. Now, consider the $\mathbb{Z}$ at position 2, (i.e., $\mathbb{Z}_{\{2\}}$), here we have that $Z_2 = \ker(\partial_2) = \ker(n) = 0$, while $B_2 = \operatorname{im}(\partial_3) = \operatorname{im}(0) = 0$. Thus, $H_2 = 0$. Next, consider $\mathbb{Z}_{\{1\}}$. Here we have that $Z_1 = \ker(0) = \mathbb{Z}$ and $B_1 = \operatorname{im}(n) = n\mathbb{Z}$. Thus, $H_1 = \mathbb{Z}/n\mathbb{Z}$. And so, in general, we have that

$$H_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \mathbb{Z}/n\mathbb{Z}, & \text{if } n \text{ is odd.} \end{cases}$$

3.1.1 Exact Sequences

Recall, a sequence

$$\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \longrightarrow \cdots \tag{3.5}$$

is exact is exact at position n, if $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$. We call such a sequence $(C_\bullet, \partial_\bullet)$ **exact** iff it is exact at all positions.

Definition 3.1.9. Let A, B, and C be abelian groups and let $f : A \to B$ and $g : B \to C$ be homomorphisms. Then, an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0, \tag{3.6}$$

where by 0 we mean the trivial group, is called a **short exact sequence**.

Remark 3.1.10. Exactness of the above sequence implies that f is injective and that g is surjective, which we see as follows. First, note that the map $0 \to A$ is the zero map. Hence, by exactness, $\ker(f) = \mathrm{im}(0) = 0$ which implies that f is injective. Next, since $C \to 0$ is also trivial, we have that $\mathrm{im}(g) = \ker(0) = C$, or that $g : B \to C$ is surjective. Moreover, via the isomorphism theorems (for groups), we see that the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

implies $C \cong A/B$. For example, the following sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} C \longrightarrow 0,$$

is exact iff $C \cong \mathbb{R}/\mathbb{Z}$. Thus, we have our first meaningful, see proposition 7.5.4, short exact sequence, namely

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0. \tag{3.7}$$

We now list some basic properties of certain "special" (short) exact sequences.

Proposition 3.1.11.

(*a*) *A sequence* $0 \longrightarrow A \xrightarrow{f} B$ *is exact iff* f *is injective.*

(*b*) *A sequence* $A \xrightarrow{g} B \longrightarrow 0$ *is exact iff* g *is surjective.*

(*c*) *A sequence* $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$ *is exact iff* $h : A \longrightarrow B$ *is an isomorphism.*

Proof. We have already shown that (*a*) and (*b*) hold, while (*c*) follows immediately from the properties of exactness. □

Proposition 3.1.12 (Splicing Sequences)**.**

(*a*) *If the following short sequences*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$
$$0 \longrightarrow C \xrightarrow{\alpha} D \xrightarrow{\beta} E \longrightarrow 0$$

are exact, then they can be 'spliced' together to give a longer exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\alpha\circ\psi} D \xrightarrow{\beta} E \longrightarrow 0. \tag{3.8}$$

(*b*) *Conversely, any long sequence* (*such as the one in* (3.5)) *is exact at n iff the following sequence is short exact*

$$0 \longrightarrow im(\partial_{n+1}) \longrightarrow A_n \longrightarrow \ker(\partial_n) \longrightarrow 0. \tag{3.9}$$

Proof. See [34]. □

Using these basic properties we can show that any compact G with finite fundamental group is built from a connected, simply connected group and some finite groups - the two extreme cases that we consider. Indeed, any Lie group G appears in the short exact sequence (we are using 1 to denote the trivial group, rather than 0, since we are not assuming abelian groups for this sequence)

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \Gamma \longrightarrow 1, \tag{3.10}$$

where G_0 is the component of the identity of G and Γ is the group of components. Note, for G compact, Γ is finite. Additionally, G also appears in the short exact sequence

$$1 \longrightarrow \pi_1(G) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1, \tag{3.11}$$

where $\tilde{G}$ is the simply connected universal cover of G. Now, combining these two exact sequences using the previous propositions, we see that any compact group G with finite fundamental group is built from a connected, simply connected group and some finite groups.

We now show the relation between short exact sequences and (long) exact sequences of homology groups.

Proposition 3.1.13. *Let A, B and C be chain complexes* (*ignoring writing boundary homomorphisms for simplicity*) *and assume that there exists chain maps $f : A \to B$ and $g : B \to C$ such that for each n, the following sequence*

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0, \tag{3.12}$$

is short exact. Then:

(*a*) *there exists a homomorphism, called the* ***connecting homomorphism***, $\partial_\star : H_n(C) \to H_{n-1}(A)$ *defined by* (*for all n*)

$$\partial_\star([c]) := [a], \tag{3.13}$$

where $[c] \in H_n(C)$ and $[a] \in H_{n-1}(A)$.

(*b*) *there exists a long exact sequence*

$$\cdots \xrightarrow{g_\star} H_{n+1}(C) \xrightarrow{\partial_\star} H_n(A) \xrightarrow{f_\star} H_n(B) \xrightarrow{g_\star} H_n(C) \xrightarrow{\partial_\star} H_{n-1}(A) \xrightarrow{f_\star} H_{n-1}(B) \xrightarrow{g_\star} H_{n-1}(C) \xrightarrow{\partial_\star} H_{n-2}(A) \xrightarrow{f_\star} \cdots \tag{3.14}$$

between the homology groups of $H_\bullet(A)$, $H_\bullet(B)$, *and* $H_\bullet(C)$. *Here* $f_\star$ *and* $g_\star$ *are the maps in homology induced by the chain maps* $f : A \to B$ *and* $g : B \to C$ (*remember,* H_n *is a functor*).

Proof. In order to show that the connecting homomorphism is (well-)defined, one usually performs a diagram chase through the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial"] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial"] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "g_{n+1}"] & B_n \arrow[r, "\partial"] \arrow[d, "g_n"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & ,
\end{tikzcd}$$

which can be found in almost any book on homological algebra. Proving exactness of the long sequence in homology groups follows from a diagram chase as well. □

We now switch to the topological side of homology theory.

3.1.2 Singular Homology

Definition 3.1.14. The **standard n-simplex** $\triangle^n \in \mathbb{R}^{n+1}$ is defined as the convex hull of the standard basis vectors $e_0, ..., e_n \in \mathbb{R}^{n+1}$; i.e., $\triangle^n$ is defined as the set

$$\triangle^n := \Big\{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \forall i,\ t_i \geq 0 \text{ and } \textstyle\sum_i t_i = 1\Big\}. \tag{3.15}$$

See figure 3.1.

Definition 3.1.15. Let T be any topological space. A **singular n-simplex** is a continuous map $\sigma_n : \triangle^n \to T$.

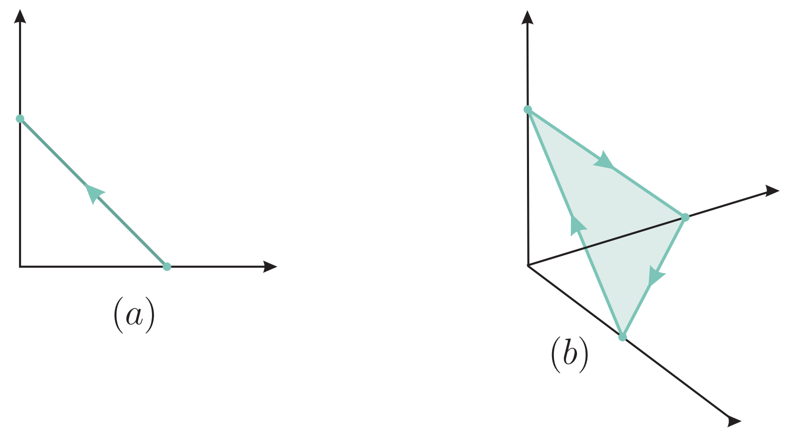


Figure 3.1: (a) The standard (oriented) 1-simplex in $\mathbb{R}^2$, and (b) the standard (oriented) 2-simplex in $\mathbb{R}^3$.

Remark 3.1.16. Let $C_n(T;\mathbb{F})$ denote the free abelian group, with coefficients in some abelian group $\mathbb{F}$, on all singular n-simplicies, where we set $C_n(T;\mathbb{F}) = 0$ for all $n \leq 0$. A basis for $C_n(T;\mathbb{F})$ is given by all continuous maps $\sigma_n : \triangle^n \to T$ (which is uncountable). Hence, an element $s \in C_n(T;\mathbb{F})$ looks like

$$s = \sum_{\sigma_n:\triangle^n \to T} z_{\sigma_n}\, \sigma_n \qquad (z_{\sigma_n} \in \mathbb{F}), \tag{3.16}$$

such that only finitely many z_{σ_n} are nonzero. We call $C_n(T;\mathbb{F})$ the group of **singular** n**-chains** on T with coefficients in $\mathbb{F}$. Also, $C_n : \mathrm{Top} \to \mathrm{Ab}$ is a functor from the category of topological spaces to the category of abelian groups.

N.B. 3.1.17. We take $\mathbb{F} = \mathbb{Z}$, unless otherwise noted, and sometimes denote $C_n(T;\mathbb{Z})$ simply as $C_n(T)$. However, don't fret, if we are ever using any coefficient group other than $\mathbb{Z}$, it will ALWAYS appear in the notation.

We now describe the equivalent boundary operators in the topological version. To begin, note that there are natural embeddings

$$\begin{aligned} d_i^n : \triangle^{n-1} &\longrightarrow \triangle^n \\ (t_0, t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_{q-1}) &\longmapsto (t_0, t_1, ..., t_{i-1}, 0, t_i, t_{i+1}, ..., t_{q-1}), \end{aligned}$$

of $\triangle^{n-1}$ as the facet opposite the vertex $e_i \in \triangle^n$.

Definition 3.1.18. For a singular n-chain σ we define the **boundary operator** $\partial_n : C_n(T) \to C_{n-1}(T)$ (which is linear) as

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n \circ d_i^n \in C_{n-1}(T). \tag{3.17}$$

Example 3.1.19. Let us denote the n-simplex σ_n as the ordered tuple $[p_0, ..., p_n] = \sigma([e_0, ..., e_n])$, where each p_i corresponds to a vertex of the n-simplex embedded in T, see figure 3.2.

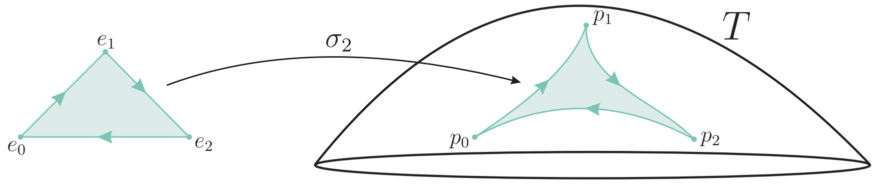


Figure 3.2: Embedding of a 2-simplex into the topological space T, $\sigma_2 : \triangle^2 \to T$.

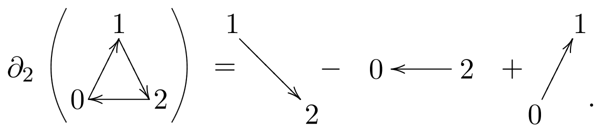
Then,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n}(-1)^i[p_0, ..., p_{i-1}, p_{i+1}, ..., p_n].$$

For example, for the 2-simplex $\sigma_2 = [p_0, p_1, p_2]$ we have that

$$\partial_2(\sigma_2) = [p_1, p_2] - [p_0, p_2] + [p_0, p_1],$$

or, more figuratively speaking,



We now show that there does indeed exists a chain complex for singular chains. To begin, we have the following theorem:

Theorem 3.1.20. *$\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}_+$.*

Proof. The proof is irrelevant for what follows and can be found in any book on algebraic topology. □

Thus, we obtain the singular chain complex $(C_\bullet(T), \partial_\bullet)$ of the topological space T. As before, we define the n**-th singular homology** of T to be

$$H_n(T) := \frac{\ker \partial_n}{\mathrm{im}(\partial_{n+1})} = \frac{Z_n(C_\bullet(T))}{B_n(C_\bullet(T))}. \tag{3.18}$$

Example 3.1.21 (Homology of a point). Before we end this section, let us first calculate the singular homology groups for $T = pt$. Since only constant maps are allowed for $\triangle^n \to pt.$, we only have one distinct n-simplex $\sigma_n : \triangle^n \to pt.$. From (3.16), we see that, for all n (since each free abelian group has exactly one basis element),

$$C_n(pt.) \cong \mathbb{Z}.$$

Now, for the boundary operators, we have

$$\partial_n(\sigma_n) = \sum_{i=1}^{n}(-1)^n \sigma_n \circ d_i^n,$$

but $\sigma_n \circ d_i^n : \triangle^{n-1} \to pt.$ and so must be a constant map. Hence, as before, there is only one distinct map - which we denote by σ_{n-1}. Therefore, we have

$$\partial_n(\sigma_n) = \left(\sum_{i=1}^{n}(-1)^n\right)\sigma_{n-1},$$

or

$$\partial_n(\sigma_n) = \begin{cases} \sigma_{n-1}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Collecting everything, we see that the general chain complex,

$$\cdots \xrightarrow{\partial_5} C_4(pt.) \xrightarrow{\partial_4} C_3(pt.) \xrightarrow{\partial_3} C_2(pt.) \xrightarrow{\partial_2} C_1(pt.) \xrightarrow{\partial_1} C_0(pt.) \xrightarrow{0} 0,$$

becomes

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \underset{\substack{\big\uparrow \text{0th position} \\ \text{i.e., } C_0(pt.)}}{\mathbb{Z}} \xrightarrow{0} 0, \tag{3.19}$$

where we denote the 0-th position for convenience. So, unless $n = 0$ we only get trivial homology. Indeed, for n odd we have that $Z_n(pt.) = \ker(0) = \mathbb{Z}$, while $B_n(pt.) = \mathrm{im}(id) = \mathbb{Z}$, and so, we conclude $H_n(pt.) = \mathbb{Z}/\mathbb{Z} = 0$ for n odd. Next, suppose n is even. Then, $Z_n(pt.) = \ker(id) = 0$ and $B_n(pt.) = \mathrm{im}(0) = 0$, giving $H_n(pt.) = 0$ for n even. Finally, for $n = 0$, we get that $Z_0(pt.) = \ker(0) = \mathbb{Z}$ and $B_0(pt.) = \mathrm{im}(0) = 0$. Hence,

$$H_n(pt.) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

3.1.3 Fundamental Class

Let M be a closed n-dimensional manifold. Then, one can show that the top homology group, $H_n(M;\mathbb{Z})$, is either trivial, $H_n(M;\mathbb{Z}) = 0$, or infinite cyclic, $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

Definition 3.1.22. An n-dimensional closed manifold M is said to be **orientable** iff

$$H_n(M;\mathbb{Z}) \cong \mathbb{Z}. \tag{3.20}$$

Furthermore, an **orientation** of M is a choice of a particular (out of two) isomorphism

$$\zeta : \mathbb{Z} \longrightarrow H_n(M;\mathbb{Z}); \tag{3.21}$$

i.e., a choice of one of the two possible generators of $\mathbb{Z}$, either 1 or -1.

Remark 3.1.23. For any oriented manifold (M, ζ), we call $\zeta(1)$ the **fundamental class** of M and denote it by $[M]$. Hence, choosing the opposite orientation $\zeta(-1)$, namely picking the opposite generator of $\mathbb{Z}$, amounts to $-[M]$. Note, if M is disconnected (but still orientable), a fundamental class is a fundamental class for each connected component (corresponding to an orientation for each component).

Intuitively speaking, the fundamental class represents an "integration over M". In fact, for the de Rham cohomology, this is exactly the case; namely, for any n-manifold M and any n-form ω, one can form the pairing

$$\langle \omega, [M] \rangle = \int_M \omega, \tag{3.22}$$

where the pairing $\langle \cdot, \cdot \rangle$ is defined in section 3.2. The reader who does not follow this last analogy should not fray, it was only mentioned to remind those readers who have taken a course in algebraic topology.

The astute reader will note that in the above definitions we only considered closed manifolds. There is a reason for this. The above was a warm-up for the more general case of compact manifolds. In fact, the top dimensional homology of a non-closed manifold is always trivial and so we must take a different approach to defining orientation and fundamental classes for compact manifolds. We do this by using relative homology.

Relative Homology

Let X be a topological space and let $A \subseteq X$ be a subspace of X together with an inclusion map $\iota : A \hookrightarrow X$. Then, the inclusion map induces a homomorphism $C_\bullet(A) \to C_\bullet(X)$, giving $C_\bullet(A)$ a subgroup of $C_\bullet(X)$. Indeed, note that (by functoriality) a continuous map $f : X \to Y$ induces a chain map

$$\begin{aligned} C_\bullet(f) : C_\bullet(X) &\longrightarrow C_\bullet(Y) \\ \sigma &\longmapsto f \circ \sigma, \end{aligned}$$

which, in turn, induces a homomorphism

$$H_\bullet(X) \longrightarrow H_\bullet(Y).$$

For more details the reader should consult any book on homological algebra. Now, since the boundary map on $C_\bullet(X)$, $\partial_n : C_n(X) \to C_{n-1}(X)$, restricts to a boundary map on $C_\bullet(A)$, $\partial_n|_A : C_n(A) \to C_{n-1}(A)$, we are allowed to form the quotient group

$$C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A). \tag{3.23}$$

Definition 3.1.24. The homology groups, $H_\bullet(X, A)$, of the complex formed by abelian groups $C_\bullet(X, A)$ and homomorphisms $\partial|_A$ are called the **relative homology groups** of the pair (X, A).

Remark 3.1.25. For "nice" inclusions $\iota : A \hookrightarrow X$, we have that

$$H_\bullet(X, A) \cong H_\bullet(X/A). \tag{3.24}$$

With relative homology groups at our disposal, the problem of defining an orientation (equivalently, a fundamental class) on some compact manifold (possibly with boundary) is a walk in the park. This is because if M is a compact orientable manifold with boundary, then the top relative homology group is again infinite cyclic, $H_n(M, \partial M) \cong \mathbb{Z}$. Consequently, as with closed manifolds, we define the fundamental class of M to be the choice of isomorphism $\mathbb{Z} \to H_n(M, \partial M)$.

We now look at the other side of the coin: the cohomology theories.

3.2 Cochain Complexes and Their Cohomology

3.2.1 Baby Steps: The de Rham Cohomology

We assume the reader has a working knowledge of differential forms and of taking exterior derivatives of them. However, we briefly remind the reader of these concepts, before delving into the de Rham cohomology of differential forms. To begin, M will denote a smooth n-dimensional manifold and, for each $k \in \mathbb{Z}$, $\Omega^k(M;\mathbb{R})$ will denote its vector space of (real-valued) differential k-forms. Hence, an element $\omega \in \Omega^k(M;\mathbb{R})$ can be thought of as a skew-symmetric multilinear mapping

$$\omega : \underbrace{T_xM \times \cdots \times T_xM}_{k} \longrightarrow \mathbb{R},$$

where T_xM represents the tangent space to M at a point $x \in M$. It is customary to take $\Omega^0(M;\mathbb{R}) \cong C^\infty(M;\mathbb{R})$ (the space of infinitely differentiable (smooth) functions $f : M \to \mathbb{R}$), while setting $\Omega^k(M;\mathbb{R}) = 0$ for all $k \leq 0$ - customs which we will adhere to. Next, recall that if $f \in \Omega^0(M;\mathbb{R})$ the exterior derivative of f is then the unique 1-form $d^0 f \in \Omega^1(M;\mathbb{R})$ (here the 0 is a superscript, not a power - $d^0 f$ is usually written as df) defined by $d^0 f(V) = V(f)$ for $V \in T_xM$. In local coordinates we have the much more familiar formula

$$d^0 f = \sum_{i=1}^{n} \frac{\partial f(x^1, ..., x^n)}{\partial x^i}\ dx^i. \tag{3.25}$$

In general, we define the exterior derivative as follows:

Definition 3.2.1. The **exterior derivative** of an arbitrary k-form is defined to be the unique set of maps

$$\Big\{ d^k : \Omega^k(M;\mathbb{R}) \longrightarrow \Omega^{k+1}(M;\mathbb{R}) \mid k \in \mathbb{Z} \Big\}, \tag{3.26}$$

such that

(a) $d^0 : \Omega^0(M;\mathbb{R}) \to \Omega^1(M;\mathbb{R})$ is defined as before (see (3.25)),

(b) for all $\omega_1, \omega_2 \in \Omega^k(M;\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$

$$d^k(c_1\omega_1 + c_2\omega_2) = c_1 d^k\omega_1 + c_2 d^k\omega_2,$$

(c) for $\omega_1 \in \Omega^k(M;\mathbb{R})$ and $\omega_2 \in \Omega^l(M;\mathbb{R})$

$$d^{k+l}(\omega_1 \wedge \omega_2) = d^k\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d^l\omega_2,$$

(d) $d^k(d^k\omega) = 0$ for any $\omega \in \Omega^k(M;\mathbb{R})$, and finally that

(e) d^k is trivial, $d^k = 0$, for all $k < 0$.

Thus, via the exterior derivative (maps), we have the following *cochain complex* (see definition 3.2.6) of vector spaces $(\Omega^\bullet(M;\mathbb{R}), d^\bullet)$

$$0 \longrightarrow \Omega^0(M;\mathbb{R}) \xrightarrow{d^0} \Omega^1(M;\mathbb{R}) \xrightarrow{d^1} \Omega^2(M;\mathbb{R}) \xrightarrow{d^2} \Omega^3(M;\mathbb{R}) \xrightarrow{d^3} \cdots. \tag{3.27}$$

Furthermore, since $d^k \circ d^{k-1} = 0$, we have that $\mathrm{im}(d^{k-1})$ is a subspace of $\ker(d^k)$, $\mathrm{im}(d^{k-1}) \subseteq \ker(d^k)$. And so, we can form the quotient vector space $\ker(d^k)/\mathrm{im}(d^{k-1})$. We call elements of $\ker(d^k)$ **closed** k-forms (or **de Rham** k**-cocycles**) and elements of $\mathrm{im}(d^{k-1})$ **exact** k-forms (or **de Rham** k**-coboundaries**).

Definition 3.2.2 (de Rham Cohomology of a manifold)**.** Define the subspaces $Z^k_{dR}(M;\mathbb{R}) := \ker(d^k) \subseteq \Omega^k(M;\mathbb{R})$ and $B^k_{dR}(M;\mathbb{R}) := \mathrm{im}(d^{k-1}) \subseteq \Omega^k(M;\mathbb{R})$. Then the k**-th de Rham cohomology group** of M is the quotient vector space

$$H^k_{dR}(M;\mathbb{R}) := \frac{\ker(d^k)}{\mathrm{im}(d^{k-1})} = \frac{Z^k_{dR}(M;\mathbb{R})}{B^k_{dR}(M;\mathbb{R})}. \tag{3.28}$$

Remark 3.2.3. Note, $H^k_{dR}(M;\mathbb{R})$ is not merely a group, it is a vector space. However, in all of the literature it is called a group, so why fight the system?

Elements in $H^k_{dR}(M;\mathbb{R})$, which we denote $[\omega]$, are not k-forms. They are equivalence classes of k-forms, where two k-forms $\omega, \omega' \in \Omega^k(M;\mathbb{R})$ are said to be *equivalent* (or *cohomologous*) iff there exists some $(k-1)$-form η such that $\omega' - \omega = d\eta$.

Example 3.2.4 (0-th de Rham Group of M). As an example, let's calculate the 0-th de Rham cohomology group for a manifold M. It has been shown that, in terms of local coordinates, $d^0 f$ (for some $f \in \Omega^0(M;\mathbb{R})$) can be written as

$$d^0 f = \sum_{i=1}^{n} \frac{\partial f(x^1, ..., x^n)}{\partial x^i}\, dx^i.$$

Thus, we have that $f \in Z^0_{dR}(M;\mathbb{R}) \equiv \ker(d^0)$ if and only if it is locally constant on M - i.e., if M has m connected components then on the i-th component f takes the value c_i for some constant c_i - since then it vanishes under d^0 and hence belongs to $\ker(d^0)$. Now, clearly the set $B^0_{dR}(M;\mathbb{R}) \equiv \mathrm{im}(d^{-1})$ is empty, since by definition $d^k = 0$ for all $k < 0$ and $\mathrm{im}(0) = 0$ for linear maps. Consequently, the 0-th de Rham cohomology group is given by

$$H^0_{dR}(M;\mathbb{R}) = Z^0_{dR}(M;\mathbb{R})/\{0\} \cong Z^0_{dR}(M;\mathbb{R}),$$

which, according to the above, is just the space of locally constant functions on M. Consequently, we conclude that the dimension of $H^0_{dR}(M;\mathbb{R})$ is equal to the number of connected components of M and that $H^0_{dR}(M;\mathbb{R}) \cong \mathbb{R}$ if and only if M is connected.

Induced Homomorphisms

Let M and N be smooth manifolds and let $f : M \to N$ be a smooth map between them. Then, f induces a pullback map $f^* : \Omega^k(N;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$. Furthermore, one can show (by commutativity of f^* with d) that closed k-forms on M are pulled back to closed

k-forms on N via f^*. Indeed, let $\omega \in \Omega^k(M;\mathbb{R})$ be a closed k-form. Then, $d\omega = 0$ implies that $d(f^*(\omega)) = f^*(d\omega) = f^*(0) = 0$, or $f^*(\omega) \in \Omega^k(N;\mathbb{R})$ is closed. Likewise, f^* maps exact forms to exact forms, since $\omega = d\eta$ implies that $f^*\omega = f^*(d\eta) = d(f^*(\eta))$. Hence, f^* maps $Z^k_{dR}(N;\mathbb{R})$ to $Z^k_{dR}(M;\mathbb{R})$ and $B^k_{dR}(N;\mathbb{R})$ to $B^k_{dR}(M;\mathbb{R})$. Consequently, f^* factors through to a map, which we denote either by $f^{\#}$ or $f^{\star}$, on the quotient space

$$f^{\star} : H^k_{dR}(N;\mathbb{R}) \longrightarrow H^k_{dR}(M;\mathbb{R}), \tag{3.29}$$

defined by $f^{\star}([\omega]) = [f^*(\omega)]$. Additionally, if $f : M \to N$ and $g : N \to P$ are smooth maps, then their composition induces a map between de Rham cohomology groups

$$(g \circ f)^{\star} = f^{\star} \circ g^{\star} : H^k_{dR}(P;\mathbb{R}) \longrightarrow H^k_{dR}(M;\mathbb{R}).$$

In particular, if $f : M \to N$ and $g : N \to M$ are smooth maps, then

$$f^{\star} : H^k_{dR}(N;\mathbb{R}) \longrightarrow H^k_{dR}(M;\mathbb{R}) \quad \text{and} \quad g^{\star} : H^k_{dR}(M;\mathbb{R}) \longrightarrow H^k_{dR}(N;\mathbb{R}),$$

are inverse isomorphisms (for all k). Therefore, the de Rham cohomology groups are diffeomorphism invariant. We can proceed further to show that de Rham cohomology groups are actually homotopy invariant (a trait obeyed by all cohomology theories). Indeed, it is known that any two manifolds of the same smooth homotopic type have the same (isomorphic) de Rham cohomology groups.

We end our discussion (for now) of de Rham cohomology groups with a theorem that will prove beneficial in what follows.

Theorem 3.2.5. *Any smooth vector bundle and its base manifold have the same de Rham cohomology groups.*

Proof. The proof goes beyond the instructive, so we refer the curious reader to the book by Bott [13]. □

3.2.2 Exact Sequences and Cohomology

The de Rham cohomology groups play a very substantial role in physics. However, for our purposes we need to introduce cohomology theories which are more general than those of de Rham[1]. This is the goal of this section - namely, we want to define cohomology theories in general. We will proceed along the same lines as before, where we defined the homology theories of a complex. Let us start by defining a few fundamental objects of a cohomology theory.

Definition 3.2.6. A **cochain complex** is a sequence of abelian groups and homomorphisms $\{C^\bullet, \delta^\bullet\}$

$$\cdots \longrightarrow C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \longrightarrow \cdots \tag{3.30}$$

such that $\delta^k \circ \delta^{k-1} = 0$ for all $k \in \mathbb{Z}$.

[1] In particular, the de Rham cohomology groups do not have a way to deal with, nor detect, torsion elements (see section 3.3); a subject which is vital to the study of topological field theories.

Remark 3.2.7. We call $\delta^k : C^k \to C^{k+1}$ the k-th **coboundary operator** and elements of C^k are called k**-cochains**. The set $\mathrm{im}(\delta^{k-1})$ is a subgroup of C^k and its elements are called k**-coboundaries**, while $\ker(\delta^k)$ is also a subgroup of C^k with its elements called k**-cocycles**.

Recall, from before, that $\delta^k \circ \delta^{k-1} = 0$ implies that $\mathrm{im}(\delta^{k-1}) \subseteq \ker(\delta^k)$. Hence, $\mathrm{im}(\delta^{k-1})$ is a subgroup of $\ker(\delta^k)$ and thus, their quotient $\ker(\delta^k)/\mathrm{im}(\delta^{k-1})$ is defined.

Definition 3.2.8 (Cohomology of a Chain Complex)**.** Let $Z^k(C^\bullet) := \ker(\delta^k)$ and $B^k(C^\bullet) := \mathrm{im}(\delta^{k-1})$. Then, we define the k**-th cohomology group** of the cochain complex $(C^\bullet, \delta^\bullet)$ to be

$$H^k(C^\bullet) := \frac{Z^k(C^\bullet)}{B^k(C^\bullet)} = \frac{\ker(\delta^k)}{\mathrm{im}(\delta^{k-1})}. \tag{3.31}$$

The elements of $H^k(C^\bullet)$, $[c]$, are equivalence classes (called **cohomology classes**), where c is a k-cocycle and the equivalence relation is defined as follows: If $c, c' \in Z^k(C^\bullet)$ are two k-cocycles, then $c \sim c'$ are equivalent (or **cohomologous**) iff $c - c' = y$, where $y \in B^k(C^\bullet)$ is a k-coboundary (i.e., there exists a $(k-1)$-chain x such that $\delta^{k-1}(x) = y$.).

Definition 3.2.9. If $(C^\bullet, \delta^\bullet)$ and $(A^\bullet, \tilde{\delta}^\bullet)$ are two cochain complexes of abelian groups, then a **cochain map**

$$\alpha : (C^\bullet, \delta^\bullet) \longrightarrow (A^\bullet, \tilde{\delta}^\bullet), \tag{3.32}$$

is a sequence of maps $\alpha^k : C^k \to A^k$ such that

$$\begin{tikzcd}
\cdots \arrow[r, "\delta^{k-3}"] & C^{k-2} \arrow[r, "\delta^{k-2}"] \arrow[d, "\alpha^{k-2}"] & C^{k-1} \arrow[r, "\delta^{k-1}"] \arrow[d, "\alpha^{k-1}"] & C^k \arrow[r, "\delta^k"] \arrow[d, "\alpha^k"] & C^{k+1} \arrow[r, "\delta^{k+1}"] \arrow[d, "\alpha^{k+1}"] & C^{k+2} \arrow[r, "\delta^{k+2}"] \arrow[d, "\alpha^{k+2}"] & \cdots \\
\cdots \arrow[r, "\tilde{\delta}^{k-3}"] & A^{k-2} \arrow[r, "\tilde{\delta}^{k-2}"] & A^{k-1} \arrow[r, "\tilde{\delta}^{k-1}"] & A^k \arrow[r, "\tilde{\delta}^k"] & A^{k+1} \arrow[r, "\tilde{\delta}^{k+1}"] & A^{k+2} \arrow[r, "\tilde{\delta}^{k+2}"] & \cdots
\end{tikzcd}$$

commutes everywhere.

Remark 3.2.10. It can be shown that $\alpha^k(\ker(\delta^k)) \subseteq \ker(\tilde{\delta}^k)$ and that $\alpha^k(\mathrm{im}(\delta^{k-1})) \subseteq \mathrm{im}(\tilde{\delta}^{-1}))$ for each k. Thus, we conclude that α induces a homomorphism between cohomology groups

$$\alpha^\star : H^\bullet(C^\bullet) \longrightarrow H^\bullet(A^\bullet), \tag{3.33}$$

defined by $(a^k)^\star([c]) = [\alpha^k(c)]$ for $[c] \in H^k(C^\bullet)$. As an example, recall, from de Rham cohomology, given a smooth map between M and N, we get a map $f^* : \Omega^k(N;\mathbb{R}) \to \Omega^k(M;\mathbb{R})$ (here we think of $\Omega^k(N;\mathbb{R})$ as C^k and $\Omega^k(M;\mathbb{R})$ as A^k). Which, in turn, induces a homomorphism

$$f^\star : H^\bullet_{dR}(N) \longrightarrow H^\bullet_{dR}(M).$$

Definition 3.2.11. Denoting by 0 the trivial abelian group (and the cochain complex formed from it), we say that a sequence of cochain maps,

$$0 \longrightarrow C_1^\bullet \xrightarrow{\alpha} C_2^\bullet \xrightarrow{\beta} C_3^\bullet \longrightarrow 0, \tag{3.34}$$

is a **short exact sequence** if, for each k, the sequence

$$0 \longrightarrow C_1^k \xrightarrow{\alpha^k} C_2^k \xrightarrow{\beta^k} C_3^k \longrightarrow 0$$

is exact. That is, if the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_1^{k-1} \arrow[r, "\alpha^{k-1}"] \arrow[d, "\delta_1^{k-1}"'] & C_2^{k-1} \arrow[r, "\beta^{k-1}"] \arrow[d, "\delta_2^{k-1}"'] & C_3^{k-1} \arrow[r] \arrow[d, "\delta_3^{k-1}"'] & 0 \\
0 \arrow[r] & C_1^{k} \arrow[r, "\alpha^{k}"] \arrow[d, "\delta_1^{k}"'] & C_2^{k} \arrow[r, "\beta^{k}"] \arrow[d, "\delta_2^{k}"'] & C_3^{k} \arrow[r] \arrow[d, "\delta_3^{k}"'] & 0 \\
0 \arrow[r] & C_1^{k+1} \arrow[r, "\alpha^{k+1}"] \arrow[d] & C_2^{k+1} \arrow[r, "\beta^{k+1}"] \arrow[d] & C_3^{k+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots & ,
\end{tikzcd}$$

commutes everywhere.

Now, α and β both induce maps in cohomology, but (as was the case for homology) it need not be the case that the sequences (for all k)

$$0 \longrightarrow H^k(C_1^\bullet) \xrightarrow{\alpha^\star} H^k(C_2^\bullet) \xrightarrow{\beta^\star} H^k(C_3^\bullet) \longrightarrow 0,$$

are exact. Hence, short exact sequences do not induce short exact sequences in cohomology either. However, they do (as in homology) induce long exact sequences.

Theorem 3.2.12. *Let $0 \to C_1^\bullet \xrightarrow{\alpha} C_2^\bullet \xrightarrow{\beta} C_3^\bullet \to 0$ be a short exact sequence of cochain complexes. Then, there exists (connecting) homomorphisms (for all k)*

$$\partial^\star : H^k(C_3^\bullet) \longrightarrow H^{k+1}(C_1^\bullet), \tag{3.35}$$

such that the following sequence is exact:

$$\begin{tikzcd}
\cdots & H^{k+1}(C_1^\bullet) \arrow[l, "\alpha^\star"'] & H^k(C_3^\bullet) \arrow[l, "\partial_\star"'] & H^k(C_2^\bullet) \arrow[l, "\beta^\star"'] & H^k(C_1^\bullet) \arrow[l, "\alpha^\star"'] & \\
 & H^{k-1}(C_3^\bullet) \arrow[urrr, "\partial_\star"] & H^{k-1}(C_2^\bullet) \arrow[l, "\beta^\star"'] & H^{k-1}(C_1^\bullet) \arrow[l, "\alpha^\star"'] & H^{k-2}(C_3^\bullet) \arrow[l, "\partial_\star"'] & \cdots \arrow[l, "\beta^\star"']
\end{tikzcd} \tag{3.36}$$

Proof. The proof is beyond the scope of this thesis, so we refer the reader to any book on algebraic topology. $\square$

So, recapping, we have been able to develop the algebraic tools necessary to study the cohomological groups of certain topological spaces. In the next section, we look at this topological side of cohomology. In particular, we (very briefly) define the singular cohomology of a topological space.

3.2.3 Singular Cohomology

Let us first remind the reader of the definition of singular homology (see section 3.1.2). For any topological space T, we introduced the notion of singular n-chains $C_n(T)$ (Recall, we are abbreviating $C_n(T;\mathbb{Z})$ by $C_n(T)$). An element $s \in C_n(T)$ looks like

$$s = \sum_{\sigma_n : \triangle^n \to T} z_{\sigma_n}\ \sigma_n \qquad (z_{\sigma_n} \in \mathbb{Z}),$$

such that only finitely many z_{σ_n} are nonzero, where σ_n maps the n-simplex into T; i.e., a singular n-chain can be though of as a map of a collection of n-simplicies into the space T. Furthermore, $C_n(T)$ becomes a group with the action taken to be integer multiplication. Next, we defined two subgroups of $C_n(T)$, the group of n-cycles $Z_n(T) := \ker(\partial_n)$ and the group of n-boundaries $B_n(T) := \mathrm{im}(\partial_{n+1})$. From here, we defined the n-th singular homology group of T as the quotient group formed by taking all n-cycles and modding out by all n-boundaries,

$$H_n(T) := \frac{Z_n(T)}{B_n(T)}.$$

Now, we will define the singular cohomology groups of the same topological space T. Roughly speaking, a singular n-cochain on a topological space T is a linear functional on the abelian group $C_n(T)$ of singular n-chains. To be more precise, consider the following definitions.

Definition 3.2.13. Let $C_n(T)$ be the group of singular n-chains with integer coefficients on some topological space T. Then, we define the space of **singular n-cochains** (with coefficients in some abelian group $\mathbb{F}$) on T to be the collection

$$C^n(T;\mathbb{F}) = \mathrm{Hom}(C_n(T), \mathbb{F}). \tag{3.37}$$

Hence, an element of $C^n(T;\mathbb{F})$ is a mapping from $C_n(T)$ to $\mathbb{F}$ (which can be though of as a dual object to $s \in C_n(T)$).

Remark 3.2.14. Note, $C^n(T;\mathbb{F})$ forms an abelian group. We will need this later on when we want to discuss torsion elements. Also, when T is a discrete space(such as a finite group) and $\mathbb{F}$ is free, the singular chains $C_n(T;\mathbb{F})$ (with coefficients in $\mathbb{F}$) are finite rank free $\mathbb{F}$-modules. Furthermore, this implies that the cochains $C^n(T;\mathbb{F})$ are free abelian. We will use this property later on (see proposition 7.5.4).

Next, we define the coboundary operator of our singular cohomology theory so as to be able to define singular n-cocycles and singular n-coboundaries.

Definition 3.2.15. We define the coboundary operator δ of singular cohomology by

$$\langle \delta\alpha, c\rangle = (-1)^n \langle \alpha, \partial c\rangle, \tag{3.38}$$

where α is a n-cochain, c a n-chain, and $\langle \cdot, \cdot \rangle : C^n(T) \otimes C_n(T) \to \mathbb{F}$ is the bilinear natural pairing between singular cohomology and singular homology.

With the aid of the boundary operator, we define the n-cocycles $Z^n(T;\mathbb{F})$, n-coboundaries $B^n(T;\mathbb{F})$ and singular cohomology groups $H^n(T;\mathbb{F})$ as would be expected. In particular, $\alpha \in C^n(T;\mathbb{F})$ is a singular n-cocyle if

$$\langle \delta\alpha, c\rangle = (-1)^n \langle \alpha, \partial c\rangle = 0,$$

for all $c \in C^n(T)$.

Remark 3.2.16. Note, if $\mathbb{F} = \mathbb{R}$, then we have that

$$H^n(T;\mathbb{R}) = H^n(T;\mathbb{Z}) \otimes \mathbb{R}.$$

Hence, by de Rham, these real singular n-cocycles can be represented by differential forms. Furthermore, for divisible groups $\mathbb{F}$ (think of $\mathbb{R}$ itself), we have that

$$H^n(T;\mathbb{F}) = \mathrm{Hom}(H_n(T), \mathbb{F}). \tag{3.39}$$

That is, $\alpha \in H^n(T;\mathbb{F})$ is a homomorphism $Z_n(T) \to \mathbb{F}$ which vanishes on the boundaries (when $\mathbb{F}$ is divisible). In most of our applications we will restrict to $\mathbb{F} = \mathbb{Z}$ or $\mathbb{F} = \mathbb{R}/\mathbb{Z}$. The reason being due to torsion.

3.3 Torsion and the Universal Coefficient Theorem

Physicists are most familiar with the cohomology theories defined, via differential forms, by de Rham. Every smooth manifold M has an exterior derivative operator d which one uses to define the de Rham complex. The deviation of this complex from exactness is measured by the de Rham cohomology of M. As we noted, in each dimension, this cohomology is a real vector space. Topologists, on the other hand, use singular chains and cochains to define homology and cohomology theories, which then appear as abelian groups. Although the fundamental theorem of de Rham states that the tensor product of these cohomology groups with the real numbers gives the de Rham cohomology groups, *torsion* information is lost in this process.

Definition 3.3.1. Let A be an abelian group. An element $a \in A$ is said to be a **torsion element** if it has finite period. That is, if

$$\underbrace{a + \cdots + a}_{z} = z \cdot a = 0,$$

for some $z \in \mathbb{Z}$.

Remark 3.3.2. Let us again reiterate that the de Rham cohomology cannot detect torsion, by considering the following chain (no pun intended) of logic. First, recalling that any abelian group is isomorphic to a $\mathbb{Z}$-module [2], we could replace 'abelian group' in the previous definition with '$\mathbb{Z}$-module'. Then, since the de Rham groups are not $\mathbb{Z}$-modules, unlike singular homology and cohomology groups, we conclude that they necessarily overlook torsion [13].

It is a straightforward result that the subset of all torsion elements of an abelian group, which we denote $\mathrm{Tor}A$, forms a subgroup. If the only torsion element in an abelian group is the identity, we say that the group is **free** (or torsion-free). The free part of an abelian group, denoted $\mathrm{Free}A$, forms a subgroup of A. Indeed, we have the following short exact sequence

$$0 \longrightarrow \mathrm{Tor}A \longrightarrow A \longrightarrow \mathrm{Free}A \longrightarrow 0, \tag{3.40}$$

which shows $\mathrm{Free}A \cong A/\mathrm{Tor}A$ (as would be expected). Additionally, the previous sequence shows that A projects naturally onto its free part, while there is no such natural projection from A to $\mathrm{Tor}A$.

Remark 3.3.3. A good question to ask is, how do we actually detect the torsion elements in the singular cohomology group $H^n(T;\mathbb{Z})$? Note, if we let R denote the kernel of the natural projection $p : \mathrm{Free}A \to A$, then we have the following short exact sequence (known as the free resolution of A)

$$0 \longrightarrow R \longrightarrow \mathrm{Free}A \longrightarrow A. \tag{3.41}$$

This short exact sequence gives us a way to detect torsion elements in $H^n(T;\mathbb{Z})$ - namely, they are elements of $\ker(\rho)$, where $\rho : H^n(T;\mathbb{Z}) \to H^n(T;\mathbb{R})$. This, in turn, shows us that torsion elements in $H^n(T;\mathbb{Z})$ cannot be represented by differential forms. Also note, given an $\alpha \in H^n(T;\mathbb{Z})$, there is no natural way to identify a torsion part of α unless α is itself a torsion element of $H^n(T;\mathbb{Z})$. Converseley, if one wants to study α modulo torsion, then they can do this naturally by studying the image $\rho(\alpha)$ of α in $H^n(T;\mathbb{R})$. We will use these techniques later on when we define what a general topological action looks like.

We would like to end our discussion of torsion by showing that there exists a link between torsion in singular homology groups of degree $n-1$ and the torsion in the singular cohomology groups of degree n, since we will have to use this result later. To begin, one can show (see [13] page 193), that any abelian group A induces the two following exact sequences

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}(A,\mathbb{F}) \longrightarrow \mathrm{Hom}(F,\mathbb{F}) \xrightarrow{i^\star} \mathrm{Hom}(R,\mathbb{F}) \\ \\ R \otimes \mathbb{F} \xrightarrow{i \otimes id_{\mathbb{F}}} F \otimes \mathbb{F} \longrightarrow A \otimes \mathbb{F} \longrightarrow 0, \end{gathered} \tag{3.42}$$

where R is defined above, F is the free part of A and $i : R \to F$.

[2]For the reader unsure of what a $\mathbb{Z}$-module is, just think of the triple $(M, R, \cdot)$, where M is an abelian group, R is a ring, and $\cdot$ is a 'scalar' multiplication on M by R, $\cdot : R \times M \to M$ (plus some axioms for the scalar multiplication); i.e., a module is a vector space whose scalars can lie in some ring R, rather than having to come from a field.

Definition 3.3.4. Define:

(a) $\mathrm{Ext}(A, \mathbb{F}) := \mathrm{coker}(i^*) = \mathrm{Hom}(R, \mathbb{F})/\mathrm{im}(i^*)$,

(b) $\mathrm{Tor}(A, \mathbb{F}) := \ker(i) \otimes id_{\mathbb{F}}$.

Remark 3.3.5. Hence, Ext and Tor measure the obstruction from the two previous exact sequences from being short exact.

We can now state the **universal coefficient theorem** (which relates homology and cohomology with arbitrary coefficients).

Theorem 3.3.6 (**Universal Coefficient Theorem**)**.** *For any space T and abelian group $\mathbb{F}$:*

(*a*) *the homology of T with coefficients in $\mathbb{F}$ has a splitting*

$$H_n(T; \mathbb{F}) \cong H_n(T; \mathbb{Z}) \otimes \mathbb{F} \;\oplus\; Tor(H_{n-1}(X; \mathbb{Z}), \mathbb{F}); \tag{3.43}$$

(*a*) *the cohomology of T with coefficients in T has a splitting*

$$H^n(T; \mathbb{F}) \cong Hom(H_n(T; \mathbb{Z}), \mathbb{F}) \oplus Ext(H_{n-1}(X; \mathbb{Z}), \mathbb{F}); \tag{3.44}$$

We now state our goal in the form of a corollary.

Corollary 3.3.7. *Let T be a reasonable topological space. Then,*

$$TorH^n(T; \mathbb{Z}) \cong TorH_{n-1}(T; \mathbb{Z}), \qquad FreeH^n(T; \mathbb{Z}) \cong FreeH_n(T; \mathbb{Z}). \tag{3.45}$$

Remark 3.3.8. Note, these isomorphisms are not natural (see the discussion in Freed [20] for the natural maps).

Proof. This follows from setting $\mathbb{F}$ to $\mathbb{Z}$ in part (b) of theorem 3.3.6. In this case, we have that

$$H^n(T; \mathbb{Z}) \cong \mathrm{Free}H_n(T; \mathbb{Z}) \oplus \mathrm{Tor}H_{n-1}(T; \mathbb{Z}). \tag{3.46}$$

Hence $\mathrm{Tor}H^n(T; \mathbb{Z}) \cong \mathrm{Tor}H_{n-1}(T; \mathbb{Z})$, while $\mathrm{Free}H^n(T; \mathbb{Z}) \cong \mathrm{Free}H_n(T; \mathbb{Z})$. □

Although our discussion of torsion and its properties has been terse at best, the reader will gain a feel for why/how torsion plays a crucial role in the study of topological field theories in what follows (notably in chapter 8).

3.4 Derived Functors

As we have seen, short exact sequences can give rise to long exact sequences. The notion of a derived functor allows for a clarification into how this works in general.

To begin, let $\mathbf{A}$ and $\mathbf{B}$ be two (abelian) categories - think of the category of exact sequences - and let $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ be a functor.

Definition 3.4.1. An **exact functor** $\mathcal{F} : \mathbf{A} \to \mathbf{B}$ is a functor between two abelian categories $\mathbf{A}$ and $\mathbf{B}$ which preserves exact sequences. In particular, let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $\mathbf{A}$. Then, we say $\mathcal{F}$ is:

- **left exact** if $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ is exact,
- **right exact** if $\mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$ is exact,
- **exact** if $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$ is exact.

As we have just seen, if $\mathcal{F}$ is a left exact functor and if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathbf{A}$, then applying $\mathcal{F}$ yields the short exact sequence

$$0 \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C)$$

in $\mathbf{B}$. Now, one might wonder whether or not this exact sequence can be continued to the right? It turns out that if $\mathbf{A}$ is nice (which we will explain in a moment) then we can extend this exact sequence to the right, with the help of the *right derived functor* of $\mathcal{F}$.

When $\mathbf{A}$ is nice, for all $i \in \mathbb{Z}_+$, we get a (right derived) functor $\mathcal{R}^i\mathcal{F} : \mathbf{A} \to \mathbf{B}$ such that the previous exact sequence continues as

$$0 \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C) \longrightarrow \mathcal{R}^1\mathcal{F}(A) \longrightarrow \mathcal{R}^1\mathcal{F}(B) \longrightarrow \mathcal{R}^1\mathcal{F}(C) \longrightarrow \mathcal{R}^2\mathcal{F}(A) \longrightarrow \cdots.$$

Remark 3.4.2. Note, $\mathcal{F}$ is exact if and only if $\mathcal{R}^1\mathcal{F} = 0$. Therefore, we can think of the right derived functor as measuring how exact $\mathcal{F}$ is. Also, in order to induce the long sequence we must assume that for every object A in $\mathbf{A}$ there exists an monomorphism $A \to I$, where I is an *injective object* in $\mathbf{A}$ [1]. This is the "nice" condition on $\mathbf{A}$.

Let us now formulate this in a more rigorous fashion. To begin, let $\mathbf{A}$ be a nice abelian category and consider some object A in $\mathbf{A}$. Since $\mathbf{A}$ is nice we can construct the following long exact sequence, known as the *injective resolution* of A,

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

where I^j are injective objects. Now, applying $\mathcal{F}$ gives the chain complex

$$0 \longrightarrow \mathcal{F}(I^0) \longrightarrow \mathcal{F}(I^1) \longrightarrow \mathcal{F}(I^2) \longrightarrow \cdots.$$

This is NOT exact, and so we can get nontrivial homology from it. For example, at the i-th position, $H_i = \ker($ of the map from $\mathcal{F}(I^i))/\mathrm{im}($of the map to $\mathcal{F}(I^i))$ may not be trivial. We denote H_i by $\mathcal{R}^i\mathcal{F}(A)$ and call it the **right derived functor** of $\mathcal{F}$ acting on A.

Furthermore, since $\mathcal{F}$ is left exact (i.e., $0 \to \mathcal{F}(A) \to \mathcal{F}(I^0) \to \mathcal{F}(I^1)$ is exact) we have that $\mathcal{R}^0\mathcal{F}(A) = \mathcal{F}(A)$.

Alternatively, let **A** be a category in which every object has an epimorphism $P \to A$, where P is a projective object [1], and let $\mathcal{F}$ be a right exact functor. Then, for any object A in **A**, we can construct the *projective resolution*

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0,$$

apply $\mathcal{F}$, and then finally compute the homology, which we denote by $\mathcal{L}_i\mathcal{F}(A)$. Note, as before, $\mathcal{L}_0\mathcal{F}(A) = \mathcal{F}(A)$. We call $\mathcal{L}$ the **left derived functor** of $\mathcal{F}$. Incidently, applying this to the short exact sequence $0 \to A \to B \to C \to 0$ gives

$$\cdots \mathcal{L}_2\mathcal{F}(C) \longrightarrow \mathcal{L}_1\mathcal{F}(A) \longrightarrow \mathcal{L}_1\mathcal{F}(B) \longrightarrow \mathcal{L}_1\mathcal{F}(C) \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C) \longrightarrow 0.$$

We will use these results later on when we need to tensor $\otimes$ an abelian group to an exact sequence. As we will see, $\otimes$ is right exact and thus will induce a left derived functor, called Tor. In particular, look at proposition 7.5.4.

Chapter 4

Fibre Bundles

4.1 Motivation ("Glued Fibre Bundles")

The construction of a fibre bundle is motivated by the desire to give a manifold structure to the object coming from placing some manifold, usually denoted by F, at each point of another manifold, usually denoted by M. As an example of this type of object in physics, consider a classical $\mathbb{R}$-valued scalar field theory defined on a (spacetime) manifold M. In this case, at all $p \in M$ there 'lives' some function ϕ which takes its values in $\mathbb{R}$, $p \mapsto \phi(p) \in \mathbb{R}$. Hence, at each point of M one is attaching another manifold, namely the real line $\mathbb{R}$.

The easiest way to create a manifold from combining two manifolds, a la the above manner, is by taking their Cartesian product $M \times F$; this structure is called a trivial (or product) bundle. Nevertheless, there are other, less trivial, ways to achieve this. For example, in the case where $F = [0, 1]$ and $M = S^1$, one could construct, in particular, a cylinder (which has the product bundle structure) or a Möbius band (see figure 4.1). Upon further inspection, one can see a key feature that is present in both the cylinder and the Möbius band; a feature which is inherent in all fibre bundles. That of local triviality, i.e. any small region of the combined manifolds looks like the product $U_\alpha \times F$, where $U_\alpha \subset M$ is an open subset of M. With this in mind, we begin with the general construction of a (glued) fibre bundle.

Figure 4.1: Two fibre bundles with $M = S^1$ and $F = [0, 1]$.

To start, let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of some arbitrary manifold M, where I is an indexing set,

$$M = \bigcup_{\alpha \in I} U_\alpha \ .$$

Next, for each of the open covers, U_α, construct a product manifold $U_\alpha \times F$. Now, to construct a fibre bundle, take the locally trivial bundles, $\{U_\alpha \times F\}_{\alpha \in I}$, and "glue" them together via the following procedure.

First, suppose there is given:

- a Lie group G, called the *structure group*, along with a smooth and faithful[1] left action L_G of G on F (for $g \in G$ and $f \in F$, $L_g(f) = g \cdot f$ is defined[2]),
- and, for each $\alpha, \beta \in I$, a smooth function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, called a *transition function*.

Then, from this data, one can construct a "glued" space

$$E := \left(\coprod_{\alpha \in I} U_\alpha \times F \right) \Big/ \sim \ ,$$

where $(x, f) \in \{U_\alpha \cap U_\beta\} \times F$ is $\sim$-equivalent to $(x', f') \in \{U_\alpha \cap U_\beta\} \times F$ if $x = x'$ and $f' = g_{\alpha\beta}(x) \cdot f$, with $\cdot$ denoting the action of G on F. For example, the Möbius band is constructed via this procedure by taking $F = [0, 1]$, $M = S^1$, while for the structure group one takes the discrete group on two elements and the action of the order 2 element is given by flipping the fibre $F = [0, 1]$ around the point 0.5 (see figure 4.1). As a side remark, note that the structure group, G, has somewhat been artificially introduced. In fact, one can, as will be used later when we give the standard definition of a fibre bundle, take for G the group of diffeomorphisms of F, Diff(F). However, in this case the structure group is usually to large to be of use. In particular, when defining characteristic classes of bundles one usually restricts the structure group to be a Lie group [35].

The clever reader will see that the above construction alone does not guarantee E will have the structure of a manifold. For instance, the definition of the glued space E still allows for pinches (see figure 4.2); places in E where $(x, f) \in U_\alpha \times F$ is identified with $(x, f') \in U_\alpha \times F$ when $f \neq f'$. Other problems arise as well, thus forcing further restrictions on the transition functions to insure that E has the structure of a manifold.

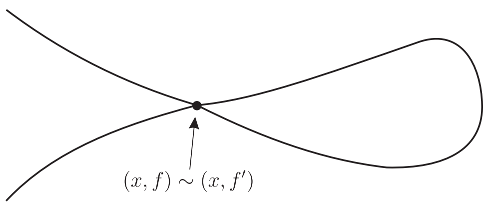


Figure 4.2: Breakdown of the manifold structure.

The problem of pinching is solved by requiring, on each $U_\alpha \times F$, $(x, f) \sim (x, f')$ if and only if $f = f'$. Therefore, from the definition of the glued space, this implies that

[1] A group action is called faithful if the only element in the group which acts as the identity is the identity itself. That is, if $g \cdot f = f$ then $g = id_G$.

[2] By $\cdot$ we mean the group action on f.

$g_{\alpha\alpha}(x) \cdot f = f$. Hence, since the action of G on F is faithful, the first extra condition on the transition functions is that, for all $\alpha \in I$ and $x \in M$,

$$g_{\alpha\alpha}(x) = id_G. \tag{4.1}$$

Along with mapping to the identity of G when not on an overlap, the transition functions should obey the **cocycle relation**. Namely, for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$ one has that

$$g_{\beta\gamma}(x) \cdot g_{\alpha\beta}(x) = g_{\alpha\gamma}(x). \tag{4.2}$$

The interpretation of the cocycle relation is as follows. Let $x \in U_\alpha \cap U_\beta \cap U_\gamma$, and assume that the element $(x, f) \in U_\alpha \times F$ is glued to $(x, f') \in U_\beta \times F$, via $g_{\alpha\beta}(x)$, while at the same time $(x, f') \in U_\beta \times F$ is glued to some element $(x, f'') \in U_\gamma \times F$, via $g_{\beta\gamma}(x)$. Furthermore, it is only logical to assume that one can also glue $(x, f) \in U_\alpha \times F$ to $(x, f'') \in U_\gamma \times F$, via $g_{\alpha\gamma}(x)$, and that these gluings should respect each other, or $g_{\beta\gamma}(x) \cdot g_{\alpha\beta}(x) = g_{\alpha\gamma}(x)$.

Finally, we require $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

As advertised, we will now show that if the transition functions obey the two relations given by equations (4.1) and (4.2) then E is in fact a topological manifold.

Proposition 4.1.1. *Let M and F be two smooth manifolds and let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of M along with G-valued functions defined on their overlaps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Then, the space defined by*

$$E := \left(\coprod_{\alpha \in I} U_\alpha \times F \right) \Big/ \sim \ ,$$

where $(x, f) \in \{U_\alpha \cap U_\beta\} \times F$ is $\sim$-equivalent to $(x', f') \in \{U_\alpha \cap U_\beta\} \times F$ if $x = x'$ and $f' = g_{\alpha\beta}(x) \cdot f$, is a topological manifold when for each $x \in U_\alpha$ one has that $g_{\alpha\alpha}(x) = id_G$ along with the cocycle relation; for each $x \in U_\alpha \cap U_\beta \cap U_\gamma$, the transition functions obey $g_{\beta\gamma}(x) \cdot g_{\alpha\beta}(x) = g_{\alpha\gamma}(x)$.

Proof. Fix a point $p \in E$ and its preimage $p_\alpha \equiv \rho^{-1}(p) \in U_\alpha \times F$ under the projection $\rho : U_\alpha \times F \to E$. Now, let $V \subset U_\alpha \times F$ be an open neighborhood of p_α which is homeomorphic to $\mathbb{R}^n$, for some n. Since no two points in V can be glued together, the restriction of ρ to V gives a homeomorphism. Thus, under the projection, the image of V in E is an open neighborhood of p homeomorphic to $\mathbb{R}^n$. Hence, for each $p \in E$ there exists an open neighborhood of p which is homeomorphic to $\mathbb{R}^n$, i.e. E is a topological manifold.

Note, using the fact that the U_α's are open while considering preimages of compact subsets of M, it can be shown that E is, in fact, a smooth manifold. □

So, to review, a glued fibre bundle is a tuple $(M, I, \{U_\alpha\}_{\alpha \in I}, F, G, L_G, \{g_{\alpha\beta}\}_{\alpha,\beta \in I})$, where:

- M and F are manifolds,
- I is an indexing set,
- $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M,

- G is a group, known as the structure group (recall here that we could generalize this to the diffeomorphism group on F), along with a faithful smooth left G-action, denoted L_G, on F,
- and $\{g_{\alpha\beta}\}_{\alpha,\beta\in I}$ is the set of transition functions, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, obeying the relations given in (4.1) and (4.2).

Before we move on to the standard definition of a fibre bundle, let us first view a couple examples of glued fibre bundles.

Example 4.1.2 (**Trivial Bundle**). Consider the case where the structure group acts simply as the identity on the fibre, i.e. for all $p \in M$, $g_{\alpha\beta}(p) = id_G$. In this case the transition map glues $(x, f) \in U_\alpha \times F$ to $(x, f) \in U_\beta \times F$, and so, in the end, we get a product, or globally trivial, bundle.

Example 4.1.3. As was mentioned earlier, the Möbius band is a glued fibre bundle over $M = S^1$ with fibre $F = [0, 1]$. In this case the structure group G is a discrete group on two elements, and the action of the element of order 2 is given by flipping the fibre around the point 1/2. We can take, for an open cover of M, two open intervals U_1 and U_2 which have two regions in common (see figure 4.3). Then, for the transition functions we take

$$g_{12}(p) = g_{21}(p)^{-1} = \begin{cases} e & \text{if } p \in O_1, \\ r & \text{if } p \in O_2. \end{cases}$$

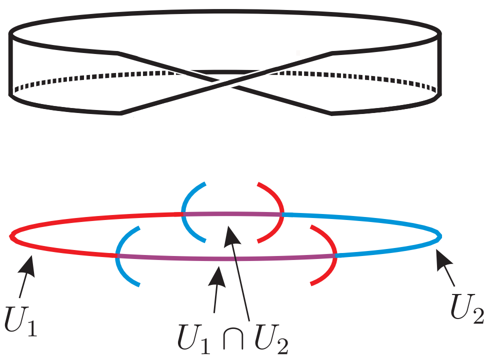


Figure 4.3: Möbius band fibre bundle. Note that the base manifold has been colored to make the distinction between the open covers easier to see.

Example 4.1.4 (**Pullback Bundle**). Suppose we have two manifolds M and N along with a map $f : M \to N$, which is at least continuous and surjective (typically these maps will either be homeomorphisms or diffeomorphisms). Furthermore, let us assume that we are given a glued fibre bundle $(M, I, \{U_\alpha\}_{\alpha\in I}, F, G, L_G, \{g_{\alpha\beta}\}_{\alpha,\beta\in I})$ over M. Then we can use the bundle datum over M to construct a glued bundle over N as follows: First, for all $\alpha \in I$, define $V_\alpha := f^{-1}(U_\alpha)$. By continuity of f, we have that V_α is open. While, by surjectivity, the collection $\{V_\alpha\}_{\alpha\in I}$ forms an open cover of N. Next, for $\alpha, \beta \in I$, let $k_{\alpha\beta} : V_\alpha \cap V_\beta \to G$ be transition functions defined by $g'_{\alpha\beta} := g_{\alpha\beta} \circ f$. Finally, due to the continuity of f, the relations obeyed by the original transition functions extend to the set $\{g'_{\alpha\beta}\}_{\alpha,\beta\in I}$. And so, by taking F, G and L_G as in the case of the original bundle over M, we are left with a glued fibre bundle over N, $(N, I, \{V_\alpha\}_{\alpha\in I}, F, G, L_G, \{g'_{\alpha\beta}\}_{\alpha,\beta\in I})$, known as the *pullback bundle* of the original bundle along f.

There is a natural map from the total space $f^*(E)$ of the pullback to the total space E of the original bundle. On the level of disjoint unions it is given by $(x, a) \mapsto (f(x), a)$. This commutes with the gluing map by construction of the transition functions. Hence the map descends to the total spaces.

4.2 (Standard) Fibre Bundles

Although the previous definition of a glued fibre bundle is natural, it is sometimes too bulky to work with. Moreover, this definition seems to have more information than necessary. For instance, since the goal was to construct a manifold, E, over M which locally looks like $M \times F$, we should only care about the relationship between E and M, and not rely on any open covering of M, or on any gluing rules. We now give a definition of a fibre bundle which is better suited for many purposes.

Definition 4.2.1. A (standard) **fibre bundle**, $E \xrightarrow{\pi} M$, is a tuple (E, M, F, π), where:

(1) E (called the **total space**), M (called the **base space**), and F (called the **fibre**) are all manifolds[3],

(2) $\pi : E \to M$ (called the **projection map**) is a (continuous) surjection,

such that for each $x \in M$ there exists an open neighborhood $U_x \subset M$ and a continuous map $\rho_x : \pi^{-1}(U_x) \to U_x \times F$ which makes the diagram

$$\begin{tikzcd}
\pi^{-1}(U_x) \arrow[rr, "\rho_x"] \arrow[dr, "\pi"'] & & U_x \times F \arrow[dl, "\mathrm{proj}_1"] \\
& U_x &
\end{tikzcd}$$

commute (here by proj_1 we mean the projection onto the first coordinate).

Remark 4.2.2. The last condition is known as a *local triviality* and the maps ρ_x are called *local trivializations*. Thus, if $E \xrightarrow{\pi} M$ is a fibre bundle with fibre F then, for each $x \in M$, $\pi^{-1}(x) \cong F$; i.e., every *fibre over* x, $\pi^{-1}(x)$, "looks like" F. We will often denote the fibre over x, $\pi^{-1}(x)$, by E_x.

Note, if we further restrict π to be C^∞ then we call the fibre bundle a **differentiable fibre bundle**. From now on, we will assume all bundles are differentiable bundles while simply calling them bundles.

[3]It will be customary to take both M and F smooth, unless stated, while smoothness of E follows from M, as stated in the proof of proposition 4.1.1.

Atlases

The goal of this section is to show, explicitly, the correspondence between glued fibre bundles and standard fibre bundles. That is, we want to describe the extra information that a glued fibre bundle contains, as compared to a standard one. To begin, consider the following definition(s).

Definition 4.2.3. Let $E \xrightarrow{\pi} M$ be a fibre bundle, then:

(1) An **atlas** of E is an open cover $\{U_\alpha\}_{\alpha \in I}$ of M, together with a collection of diffeomorphisms $\{\rho_\alpha\}_{\alpha \in I} : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ making the following diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\rho_x"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{proj}_1"] \\ & U_\alpha & \end{tikzcd}$$

commute.

(2) The **transition functions** of an atlas $\{U_\alpha\}_{\alpha \in I}$ are functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$, defined by

$$\rho_\beta \circ \rho_\alpha^{-1}(z, f) = (z, g_{\alpha\beta}(z) \cdot f).$$

(3) An atlas $\{U_\alpha\}_{\alpha \in I}$ has **structure group** G if G is a Lie group with a smooth and faithful left action L_G on F such that, for all $\alpha, \beta \in I$ and $z \in U_\alpha \cap U_\beta$, the transition function $g_{\alpha\beta}(z)$ acts as an element of G; i.e., the transition map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ defines a (smooth) map from $U_\alpha \cap U_\beta$ into the Lie group $G \subset \mathrm{Diff}(F)$ and the corresponding element in G acts on F faithfully from the left. An atlas with a structure group G is called a **G-atlas**.

With these notions behind us, we are now in a position to show that a glued fibre bundle is really just a (standard) fibre bundle with a choice of structure group G and corresponding G-atlas.

We will begin by constructing a fibre bundle $E \xrightarrow{\pi} M$ from a glued fibre bundle $(M, I, \{U_\alpha\}_{\alpha \in I}, F, G, L_G, \{g_{\alpha\beta}\}_{\alpha,\beta \in I})$. To proceed, note that all we must show is how to construct the bundle projection map $E \xrightarrow{\pi} M$. Indeed, we already know how to construct the total space E, and we know that the conditions on the transition functions guarantee that E is a manifold. So, let $p \in E$. Then, for some $\alpha \in I$, take a point $(x, f) \in U_\alpha \times F$ in the preimage of p under the gluing and define $\pi(p) := x$. Note that since gluing only identifies points with the same base coordinate, this is well-defined (x depends only on p and not on the choice of the preimage). We now claim that this construction of $E \xrightarrow{\pi} M$ gives a fibre bundle with atlas $\{U_\alpha\}_{\alpha \in I}$ and structure group G.

Proposition 4.2.4. *The above construction for $E \xrightarrow{\pi} M$ gives a fibre bundle with atlas $\{U_\alpha\}_{\alpha \in I}$ and structure group G from the glued fibre bundle given by*

$$(M, I, \{U_\alpha\}_{\alpha \in I}, F, G, L_G, \{g_{\alpha\beta}\}_{\alpha,\beta \in I}).$$

Proof. It suffices to show that the above construction satisfies local triviality. So, begin by taking some $x \in M$. Then, since $\{U_\alpha\}_{\alpha \in I}$ is an open covering of M, there exists some $\alpha \in I$ such that $x \in U_\alpha$. To make our notation more pleasant for the reader during this proof, denote $U_x \equiv U_\alpha$ along with denoting the restriction of the map to $U_x \times F$ by ρ'_x. Now, since gluing doesn't glue together points of $U_\alpha \times F$, ρ'_α is a diffeomorphism onto its image. However, this image is precisely $\pi^{-1}(U_\alpha)$. Hence, $\rho'_x : U_x \times F \to \pi^{-1}(U_x)$ is a diffeomorphism. From the construction of the projection map π, we see that this diffeomorphism respects it; that is, the diagram in definition 4.2.1 commutes. Implying that, we have local triviality.

As an aside, note that, by construction of the local trivializations ρ', the transition functions of the atlas act in the same way as the transition functions of the glued bundle, i.e. by elements of G. So the atlas, in fact, has a G-structure. □

We will now show that a glued fiber bundle can be constructed from a fiber bundle that has an atlas and a structure group, consequently showing a $1:1$ correspondence between glued fibre bundles and fibre bundles with the additional structure of an atlas and structure group.

Proposition 4.2.5. *Let $E \xrightarrow{\pi} M$ be a fibre bundle with atlas $\{U_\alpha\}_{\alpha \in I}$ and structure group G. Then, using the transition functions of the atlas as gluing functions gives a glued fiber bundle.*

Proof. It remains to show that the transition functions satisfy the necessary relations:

- $g_{\alpha\alpha}(x) = id_G$,
- $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$,
- $g_{\beta\gamma}(x) \cdot g_{\alpha\beta}(x) = g_{\alpha\gamma}(x)$.

However, these two properties follow immediately from the definition of the transition functions given in part (2) of definition 4.2.3. □

Thus, to recap, we have a $1:1$ correspondence between glued fibre bundles and standard fibre bundles with atlas and structure group. For the remainder, unless stated, we will assume all bundles are standard fibre bundles which have the additional information of an atlas and structure group.

4.3 Associated Bundles

As was previously mentioned, there is nothing special about the fibre F in a fibre bundle $E \xrightarrow{\pi} M$. For instance, if G acts on a different manifold F' as a Lie transformation group, then we can construct a different fibre bundle $E' \xrightarrow{\pi} M$ with the same transition functions but different fibre F'. Stated differently, let $\xi = (E, F, \pi, M, G)$ be a fibre bundle with fibre F. Now, take another manifold F' which also has a smooth and faithful left action of G. Then, keeping the same base manifold M, covering, and transition functions, we can consider a new bundle with fiber F', $\xi' = (E', F', \pi, M, G)$. The bundle ξ' is called the **associated bundle** of ξ. For us, the most important associated bundle is the one where we replace the fibre F by the Lie group G itself, giving what is called a principal G-bundle.

Principal G-Bundles

Definition 4.3.1. Let G be a Lie group. Then a fibre bundle, which we denote by $G \hookrightarrow P \xrightarrow{\pi} M$, with fibre and structure group given by G is called a **principal G-bundle** if the action of G (as the structure group) on itself (as the fibre) is given by left translation, i.e. $L_g : x \to g \cdot x$ where $x, g \in G$.

Remark 4.3.2. In the case of a principal bundle, we denote the total space by P and the fibre over a point $x \in M$, $\pi^{-1}(x)$, by P_x.

Principal bundles are the most important of all bundles. They are used in everything from Chern-Weil theory to the main topic of this manuscript, Chern-Simons theory. Furthermore, their use in physics (particularly gauge theories) is invaluable due to the fact that they allow for one to discuss local symmetries (gauge transformations) on fields in a mathematically rigorous setting. Although this topic will be a recurring theme in the sequel, the readers wishing to learn the mathematical machinery behind gauge theories are directed to the wonderful book by J. Baez [8]. Instead, let's carry on with the properties of principal bundles. In particular, we have the following theorem which gives a bit of structure to principal bundles.

Theorem 4.3.3. *Given a principal bundle $G \hookrightarrow P \xrightarrow{\pi} M$ we can define a right action of G on the total space P. That is, we can define a map*

$$R_g : P \times G \longrightarrow P$$

$$(p, g) \longmapsto p \cdot g,$$

such that, for each $g, h \in G$ and $p \in P$, $(p \cdot g) \cdot h = p \cdot (g \cdot h)$. Also, the action takes each fibre, G, onto itself and is free, i.e. if $u \cdot g = u$ then $g = id_G$. Further, the quotient space $P/G \cong M$. Conversely, any fibre bundle with such a right G-action on the total space is a principal G-bundle.

Proof. See pages 236-237 of [35]. □

Remark 4.3.4. It should be noted that a main ingredient to the proof of this theorem is that a principal bundle is trivial if and only if it admits a global section (this idea will be taken up in the next section). This is not however the case for other bundles. For example, a vector bundle[4] always admits a global section, namely the trivial zero section, regardless whether it is trivial or not.

4.4 Sections

Roughly speaking, a *section* of a fibre bundle is a trivialization of the bundle. As noted in the previous section, any principal bundle which admits a section is trivial (i.e.

[4]A vector bundle can be thought of as a fibre bundle whose fibre is a vector space V and whose structure group is $GL(V)$. The theory of vector bundles is very rich due to the possibility of applying linear algebra at the fibre level.

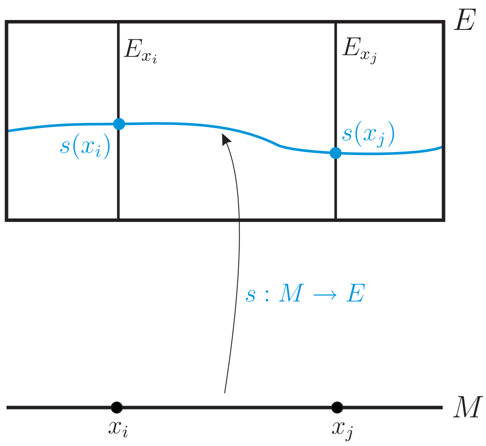


Figure 4.4: Section on a fibre bundle $E \xrightarrow{\pi} M$.

has the product topology $P = M \times G$). Conversely, any principal bundle which is trivial admits a section. This last assertion leads one to believe that, since a principal bundle is always locally trivial, there will always exist local sections (whatever they may be defined as). This is in fact true not only for principal bundles but for general fibre bundles. So, without wasting any more time, let us give a precise definition of sections.

To begin, let $E \xrightarrow{\pi} M$ be some fibre bundle.

Definition 4.4.1. A map $s : M \to E$ is called a (global) **section** of the bundle $E \xrightarrow{\pi} M$ if it commutes the following diagram

$$\begin{tikzcd}
& E \arrow[dr, "\pi"] & \\
M \arrow[ur, "s"] \arrow[rr, "id_M"'] & & M
\end{tikzcd} ,$$

that is, if $s \circ \pi = id_M$. Stated differently, a section is a map which associates to each $p \in E$ an element of the fibre over p, E_p, and obeys the previous commutative diagram.

Remark 4.4.2. We can use the notion of a section to generalize other objects. For instance, we can think of a vector field as a section of the tangent bundle. To be more precise, a vector field $X : M \to TM$ associates to each $x \in M$ a tangent vector living in T_xM. In addition to vector fields, one can think of a 1-form as a section of the cotangent bundle.

As advertised, one has the following theorem.

Theorem 4.4.3. *A principal G-bundle is trivial if and only if it admits a global section.*

Proof. Let $G \hookrightarrow P \xrightarrow{\pi} M$ have a section $s : M \to P$. Then, the map $\varphi : M \times G \to P$, which is defined by $\varphi(x, g) := s(x) \cdot g$, gives an isomoprhism $P \cong M \times G$. $\square$

In view of the preceding theorem, we see that principal bundles should admit local sections, since all fibre bundles are locally trivial. Thus, we define a **local section** as a global section which is restricted to an open subset $U_\alpha \subset M$ of the base manifold, $s|_{U_\alpha} \equiv s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$. Any bundle admits local sections, however one can not always construct global sections out of these local ones. In fact, one of the major problems in modern differential geometry is finding the obstructions against extending such local sections to global sections [13]. We will supply some of the tools needed to answer this question when we begin the study of characteristic classes.

4.5 Bundle Maps and Gauge Transformations

Given two fibre bundles we can construct a map between them, known as a *bundle map*.

Definition 4.5.1. Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be two fibre bundles. Then, a **bundle map** $\varphi : E \to E'$ is a smooth map which preserves the fibres; that is, if $e_1, e_2 \in \pi^{-1}(x)$ then $\varphi(e_1), \varphi(e_2) \in \pi'^{-1}(y)$ for some $y \in M'$.

Remark 4.5.2. A bundle map, $\varphi : E \to E'$, induces a map between the two base spaces, $\hat{\varphi} : M \to M'$. Hence, one could really consider a bundle map as the pair $(\varphi, \hat{\varphi})$.

Category of Bundles

Using bundle maps, we can construct the category of fibre bundles (see chapter 2 for a review of category theory). In particular, we define the category of fibre bundles, denoted Bund, as the category whose objects are fibre bundles and whose morphisms are bundle maps. Thus, an element of Obj($Bund$) is a fibre bundle $E \xrightarrow{\pi} M$ and an element in $\mathrm{Hom}_{Bund}(E, E')$ is a bundle map from $E \xrightarrow{\pi} M$ to $E' \xrightarrow{\pi'} M'$.

When the base spaces are the same, $M = M'$, and when $\hat{\varphi} = id_M$, we call the bundle map a **bundle morphism**. Using this we can construct the category of bundles over M. Let Bund_M denote the category whose objects are bundles over M and whose morphisms are bundle morphisms.

Remark 4.5.3. Bundle isomorphisms (i.e. when $\hat{\varphi} = id_M$ and $\varphi : E \to E'$ is an diffeomorphism) give us a notion of equivalence classes of fibre bundles over M.

In the case of principal bundles, bundle maps gets refined so as to preserve the extra structure which is present.

Definition 4.5.4. Let $G \hookrightarrow P \xrightarrow{\pi} M$ and $G \hookrightarrow P' \xrightarrow{\pi'} M'$ be two principal G-bundles over M and M', respectively. A **G-bundle map** $\varphi : P \to P'$ is a smooth map which commutes with the right action of G. Furthermore, a G-bundle map induces a map from M to M', which we denote by $\hat{\varphi}$.

Remark 4.5.5. As before, when $M = M'$ and $\hat{\varphi} = id_M$ we call the G-bundle map a **G-bundle morphism**. Additionally, we can construct the category of principal G-bundles, GBund in exactly the same way as before.

Gauge Transformations

Definition 4.5.6. Let φ be a G-bundle map and let $M = M'$ and $P = P'$. Then if $\hat{\varphi} = id_M$, we call $\varphi : P \to P$ a **gauge transformation** on P.

Thus, a gauge transformation is a smooth transformation from a fiber P_p back to itself or, since the fiber over a point is equivalent to a copy of G, φ can be thought of as an element of the automorphism group $\mathrm{Aut}(G)$ defined on G.

We get the usual interpretation of a gauge transformation (at least to physicists) with the following fact.

Fact 4.5.7. A gauge transformation $\varphi : P \to P$ can be identified with mappings

$$\hat{g}_\varphi : P \longrightarrow G,$$

which commute with the right G-action. Indeed, to the bundle automorphism φ, we assign the mapping $\hat{g}_\varphi : P \to G$ defined by $\varphi(p) = p \cdot \hat{g}_\varphi(p)$, for $p \in P$. Furthermore, to the mapping $\hat{g} : P \to G$ we assign the gauge transformation $\varphi_{\hat{g}} : P \to P$ defined by $\varphi_{\hat{g}}(p) = p \cdot \hat{g}(p)$. And so, we have a bijection between gauge transformations on $G \hookrightarrow P \xrightarrow{\pi} M$ and the set of maps $\hat{g} : P \to G$. If our principal G-bundle is trivial, then we can identify a gauge transformation with a mapping from M to G, $g : M \to G$. Thus, which is traditionally taught to physicists, a (local) gauge transformation can be thought of as a mapping from spacetime, M, to a Lie group G.

Gauge transformations are an extremely important entity in physics. However, we will end our discussion of gauge transformations here, thus keeping it purely mathematical, while the reader wishing for a more physical background on the importance of gauge transformations (and gauge theories) is directed to [8].

Universal Bundles and Classifying Spaces

We now introduce the concepts of universal bundles and classifying spaces. However, the reader should note that our discussion here is by no means complete (or even completely rigorous), we only want to discuss the topics that will be of use later.

There exists a principal G-bundle, denoted by $G \hookrightarrow EG \xrightarrow{\pi'} BG$ and called the *universal bundle*, with the amazing property that any principal G-bundle P over a manifold M allows a bundle map into this universal bundle, and any two such maps are smoothly homotopic. Said another way, every principal G-bundle over M can be realized as the pullback[5] of the universal bundle along some map

$$\gamma : M \longrightarrow BG,$$

and any two such pullbacks along homotopy equivalent maps γ_1 and γ_2 realize the same principal G-bundle. That is, the different components of $Map(M, BG)$ (also denoted $[M, BG]$) - where each distinct element in $Map(M, BG)$ is a homotopy equivalence class of maps from M to BG - correspond to different principal G-bundles over M. We call BG the

[5]See example 4.1.4 for the construction of pullback bundles.

classifying space and γ the *classifying map.* The topology of E is completely determined by the homotopy class of γ. Finally, it can be shown that any contactable space which has a free action of G is a realization of EG. The classifying space BG of a compact group G is usually infinite dimensional, as the specific examples $B\mathbb{Z}_2 = \mathbb{R}\mathbf{P}^\infty$, $BU(1) = \mathbb{C}\mathbf{P}^\infty$, and $BSU(2) = \mathbb{H}\mathbf{P}^\infty$ show (where, for example, by $\mathbb{R}\mathbf{P}^\infty$ we mean the infinite-dimensional real projective space).

The universal bundle is important in the fact that it allows for one to completely generalize the ideas of bundles. That is, one can define an object on the universal bundle and then, if they wish, use the mappings to pull the information back to a specific principal G-bundle. Thus, they get a "global" definition that holds for all principal G-bundles over M, rather than having to work with each specific bundle. For example, we will see later on that the n-dimensional Chern-Simons action is given by an element of the cohomology group[6] $H^{n+1}(BG;\mathbb{Z})$ and to discuss a specific theory (i.e. define the CS action on a specific manifold) we simply take the pullback of this element along the induce map between M and BG. And so, using the universal bundle, we can discuss a general Chern-Simons action and then when we want to work out the theory for a specific principal G-bundle we simply use the bundle maps to pull the action back to that bundle.

Having cut our teeth on bundles, our next objective is to discuss connections on bundles. This topic takes up the next chapter.

[6] See chapter 3.

Chapter 5

Connections on Fibre Bundles

Given a path γ in the total space E we can always project it down to a path in the base space using the bundle projection map, $\pi(\gamma)$. If the path γ is transversal to the fibers, i.e. the tangent vectors to the path never fall into the tangent spaces along the fibers, then the projection $\pi(\gamma)$ will be a smooth path in the base. Now, suppose we would like to reverse this process so as to be able to lift paths from the base space to the total space of the bundle. We will see that in order to carry out this process we must appoint a *connection* on the total space E.

5.1 Ehresmann Connections

The procedure will be to lift tangent vectors defined on the base manifold to the total space and then integrate the obtained vector field, thus producing curves. To begin, let $x \in M$ and let $u \in E_x$, where E_x is the fibre over x. Then, via the derivative of the bundle projection map $d\pi : T_uE \to T_xM$, we can lift a tangent vector $v \in T_xM$, living in the tangent space of M at x, to a tangent vector $\hat{v} \in T_uE$, living in the tangent space of E at the point u. Said another way, there exists a $\hat{v} \in T_uE$ such that $v = d\pi(\hat{v})$. However, since $d\pi$ is not injective[1], the lift of v to T_uE is not unique; specifically, there could exist tangent vectors $\hat{v} \neq \hat{v}'$ such that $v = d\pi(\hat{v}) = d\pi(\hat{v}')$.

Alternatively, what if instead we lift v to a preferred subspace H_u of T_uE which is transversal to the tangent space along the fibre[2], T_uE_x (see figure 5.1)? If this were the case, that is if we lifted v to $H_u \subset T_uE$, the lift would be unique, which can be seen as follows. First, by definition of the kernel of a map, we have

$$\ker(d\pi) = \big\{ y \in T_eE \mid d\pi(y) = 0 \big\}. \tag{5.1}$$

Inspecting the right hand side of (5.1), it is clear that $\ker(d\pi) = T_uE_x$. Therefore, $\ker(d\pi|_{H_u}) = \emptyset$ which, in turn, implies that the restriction of $d\pi$ to H_u gives an injection.

[1] Recall that the only restriction we place on the bundle projection $\pi : E \to M$ is that it be surjective. Thus, to keep generality, we cannot restrict $d\pi$ to be an injection.

[2] This is equivalent to saying $T_uE = H_u \oplus T_uE_x$.

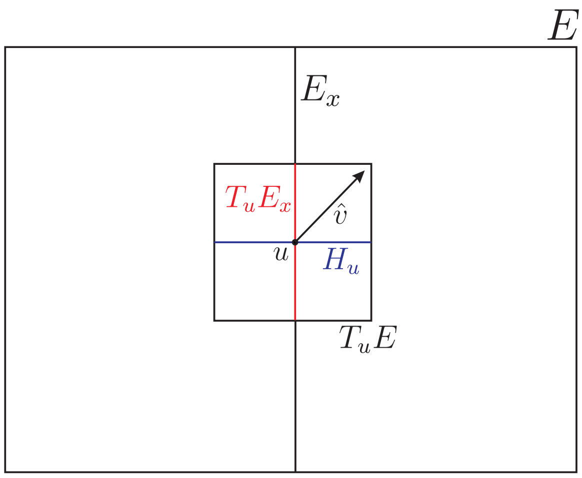


Figure 5.1: Decomposition of the tangent space, T_uE, at a point $u \in E$.

Additionally, since $d\pi|_{H_u}$ is linear (derivative mapping) and since $\dim(H_u) = \dim(T_xM)$, we have, by the rank-nullity theorem[3], that $d\pi|_{H_u}$ is a surjection. Hence, $d\pi|_{H_u} : H_u \to T_xM$ is an isomorphism. Consequently, the lift of v to $H_u \subset T_uE$ is unique.

So, to lift tangent vectors from the base space to the total space we need, at each point $x \in M$, a choice of a subspace of the tangent space transversal to the tangent space along the fiber. Of course since we would like the lifts of our paths to be smooth, we need these choices of subspaces to depend smoothly on the point $x \in M$. More compactly, to lift tangent vectors uniquely we need a choice of a subbundle of the tangent bundle transversal to the tangent bundle along the fibers, i.e. we need a *transversal distribution.*

Definition 5.1.1. A subbundle of the tangent bundle which is transversal to the tangent bundle along the fibers is called a **transversal distribution**, ∇. Thus, if our bundle has a transversal distribution, then, at each $x \in M$, we can pick a subspace $H_u \subset T_uE$ such that

$$H_u \oplus T_uE_x = T_uE,$$

as long as the choices vary smoothly over M.

Recapping, given a path in the base space and a transversal distribution, we have learned how to construct a vector field over this path in the total space. However, the goal was to find a curve in E which is tangent to this vector field at every point; that is, we want to be able to integrate this vector field (see figure 5.2).

To proceed, let $E \xrightarrow{\pi} M$ be a fibre bundle and fix a transversal distribution ∇ on E. Also, recall that if $f : I \to M$ is a path in M, here $I \subset \mathbb{R}$, then the pullback $f^*(E)$ defines a fibre bundle over I, $f^*(E) \xrightarrow{\pi'} I$ (see example 4.1.4), along with a map $\tilde{f} : f^*(E) \to E$. Furthermore, we can define a transversal distribution on $f^*(E)$ by

$$\nabla_{f^*(E)} := d\tilde{f}^{-1}(\nabla),$$

[3]That is, $\dim\big(H_u\big) = \dim\big(\ker(d\pi|_{H_u})\big) + \dim\big(\mathrm{im}(d\pi|_{H_u})\big)$.

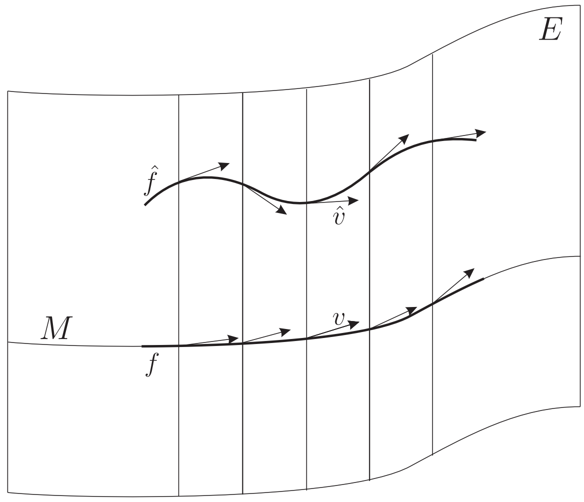


Figure 5.2: Lifting of a path f defined on M to a path $\hat{f}$ defined on E.

where $d\tilde{f}^{-1}(\nabla)$ is the preimage of ∇ under $d\tilde{f}$. Now, let X be the lift (with respect to $\nabla_{f^*(E)}$) of the standard unit vector field on I, $\frac{d}{dt}$, to $f^*(E)$. Then, $\nabla_{f^*(E)}$ is called *integrable* if for each $u \in f^*(E)$ there exists a smooth section $s : I \to f^*(E)$ such that

$$(s \circ \pi')(u) = u,$$

along with, for each $t \in I$,

$$\frac{d}{dt}s\Big|_t = X_{s(t)}.$$

And so, given an integrable transversal distribution, we get our desired path.

Definition 5.1.2. A transversal distribution ∇ is called **integrable along arcs** if its pullback along any path is integrable.

We now arrive at the definition of a connection on a fibre bundle.

Definition 5.1.3. By a **(Ehresmann) connection**, which is also denoted by ∇, we mean a transversal distribution which is integrable along arcs.

Note that a connection on a fiber bundle can be thought of as a local trivialization of the total space, up to first order. This need not extend to a trivialization up to second order, the obstruction against doing so being the curvature of the connection. There are locally no higher order issues; if that obstruction vanishes (in which case we say that the connection is flat), the connection yields a canonical local trivialization. Hence, among other things, a connection measures the 'twistedness' of a fibre bundle. Alternatively, this can be seen as follows. Suppose that we have a fibre bundle $E \xrightarrow{\pi} M$ with a connection ∇. Then, at each point $u \in E_x$, we have a decomposition of the tangent space, T_uE, into a 'horizontal' part, H_u, and a 'vertical' part, T_uE_x. Now, consider a product bundle. In this case the total space is given by $E = M \times F$ and so at any $u = (m,p) \in E$ the tangent space is written as

$$T_uE = T_bM \oplus T_pF,$$

namely, the direct sum of two subspaces, one in the base direction and the other in the fibre direction. However, it is clear that, locally, $H_u = T_b M$ and $T_u E_x = T_p F$. In other words, a fibre bundle with connection is, up to first order, locally trivial. Conversely, any fibre bundle is locally trivial. Indeed, we have the following.

Theorem 5.1.4. *Any arbitrary fibre bundle admits a connection.*

Proof. Since we have seen (see remark at the end of the proof of proposition 4.1.1) that the total space is smooth, we can associate to it a Riemannian metric. Now, for each $u \in E$, define the subspace H_u to be the orthogonal complement of $T_u E_x$ with respect to the metric. This choice is smooth since a Riemannian metric varies smoothly over all $u \in E$. □

Aside 5.1.5. We can gain an algebraic interpretation of a connection on a fibre bundle as follows. For a fibre bundle $E \to M$ we have the following short exact sequence (see the discussion in 3.1.1)

$$0 \longrightarrow \ker(d\pi) \xrightarrow{\iota} TP \xrightarrow{d\pi} TM \longrightarrow 0 \tag{5.2}$$

where $\iota : \ker(d\pi) \hookrightarrow TE$. Now, we think of a connection, which we will denote by A, as some mapping $\nabla : TP \to \ker(d\pi)$ which splits the above short exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & \ker(d\pi) \arrow[r, "\iota"] & TP \arrow[l, dashed, bend right, "\nabla"'] \arrow[r, "d\pi"] & TM \arrow[r] & 0. \end{tikzcd}$$

That is, we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(d\pi) \arrow[r, "\iota"] \arrow[d, "id"] & TP \arrow[r, "d\pi"] \arrow[d, "\cong"] & TM \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & \ker(d\pi) \arrow[r, "i"] & \ker(d\pi) \oplus TM \arrow[r, "p"] & TM \arrow[r] & 0 \end{tikzcd}$$

where $i : \ker(d\pi) \hookrightarrow \ker(d\pi) \oplus TM$ is an injection and $p : \ker(d\pi) \oplus TM \to TM$ is a projection. Hence, a connection defines an isomorphism

$$TP \cong \ker(d\pi) \oplus TM,$$

as we have already seen.

G-Connections

As was mentioned before on several occasions, the most important types of bundles (at least for us) are the principal bundles. Furthermore, since connections play an equally important role in what follows, we would like to marry these two ideas. That is, we now wish to construct a connection on a principal bundle which is comparable with the extra structure present on a principal bundle.

We have seen that for any principal G-bundle, $G \hookrightarrow P \xrightarrow{\pi} M$, one can define a right action of G on P, $R_G : P \times G \to P$. Additionally, one can promote this action to a right action on TP by taking the derivative of the action map, dR_G. Now, if our connection is to respect this right G-action, we must impose further restrictions on the transversal distribution.

Definition 5.1.6. Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle. A **G-connection** on P is a rule to assign a subspace H_u of T_uP, at each $u \in P$, such that:

(1) H_u is transversal to the fibre. That is, for $x \in M$ and $u \in P$, $T_uP = H_u \oplus V_u$, where V_u is the tangent space along the fibre containing p.

(2) The assignment of H_u depends smoothly on $u \in P$.

(3) H_u is invariant under the right G-action, i.e. $dR_g(H_u) = H_{u \cdot g}$.

Equivalently, a G-connection on P is a transversal distribution that satisfies $dR_g(H_u) = H_{u \cdot g}$.

Remark 5.1.7. Condition (3) above gives us integrability along arcs and so, any transversal distribution which satisfies this conditions is in fact a connection. Also, note that, given a G-connection, one can define parallel displacement without forcing the fibre to be compact. Hence, only for compact fibred fibre bundles is it guaranteed to have parallel displacement, while principal G-bundles with both compact and non-compact fibres are guaranteed to have parallel displacement.

Henceforth, all G-connections on principal G-bundles will simply be called connections, rather than the somewhat laborious G-connection terminology.

5.2 Connection Forms

The previous definition of a connection was of a purely geometric nature. We will now give an alternative definition of a connection, which will be shown to be equivalent to the first definition, in terms of a $\mathfrak{g}$-valued one-form known as the *connection form*.

5.2.1 Intermezzo: Vector-Valued Forms

For us, most of the forms we encounter will take values, not in $\mathbb{R}$, but rather in some generic vector space $\mathcal{V}$ (for example, $\mathbb{C}$ or a Lie algebra $\mathfrak{g}$). In this section we review the basic properties of such $\mathcal{V}$-valued differential forms. In particular, the wedge product of two forms.

To begin, let E denote some n-dimensional vector space, with basis $\{e_1, ..., e_n\}$, and let $\mathcal{V}$ denote some m-dimensional vector space, with basis $\{T_1, ..., T_m\}$. Without loss of generality, we will assume that E has an orientation along with inner product, which we denote by g. Now, a map

$$\mathcal{A} : \underbrace{E \times \cdots \times E}_{k} \longrightarrow \mathcal{V}, \tag{5.3}$$

is said to be *k-multilinear* if it is linear in each argument. Furthermore, the set of all such k-multilinear maps, denoted $\mathcal{T}^k(E; \mathcal{V})$, forms a vector space with the usual pointwise operations for addition and multiplication. For our convenience, we set $\mathcal{T}^0(E; \mathcal{V}) := \mathcal{V}$.

N.B. 5.2.1. The elements of $\mathcal{T}^k(E; \mathcal{V})$ are called **covariant $\mathcal{V}$-valued tensors of rank k**.

It is well-known that if $\{T_1, ..., T_m\}$ is a basis for $\mathcal{V}$, then any $\mathcal{A} \in \mathcal{T}^k(E;\mathcal{V})$ can be uniquely written as the sum

$$\mathcal{A} = \sum_{i=1}^{m} A^i T_i, \tag{5.4}$$

where $A^i \in \mathcal{T}^k(E;\mathbb{R})$ are the usual elements in the dual space $E^* \otimes \cdots \otimes E^*$ of $E \otimes \cdots \otimes E$. That is, we can think of $\mathcal{T}^k(E;\mathcal{V})$ as the space

$$\mathcal{T}^k(E;\mathcal{V}) = \mathcal{T}^k(E;\mathbb{R}) \otimes \mathcal{V}. \tag{5.5}$$

We can further split the space $\mathcal{T}^k(E;\mathcal{V})$ into its symmetric subspace, denoted $\mathcal{S}^k(E;\mathcal{V})$, its skew-symmetric subspace, $\Omega^k(E;\mathcal{V})$.

N.B. 5.2.2. The elements of $\Omega^k(E;\mathcal{V})$ are called **$\mathcal{V}$-valued k-forms** (or $\mathcal{V}$-valued differential forms of degree k). Hence, the differential forms we are most accustomed to lie in the space $\Omega^k(E;\mathbb{R})$. Additionally, it also holds that any $\alpha \in \Omega^k(E;\mathcal{V})$ can be uniquely written as

$$\alpha = \sum_{i=1}^{m} \alpha^i T_i, \tag{5.6}$$

where $\alpha^i \in \Omega^k(E;\mathbb{R})$ are the usual differential k-forms.

Tensor and wedge products for $\mathbb{R}$-valued forms depend, for their definition, on the multiplicative structure given on $\mathbb{R}$. And so, unless some kind of multiplication is defined on $\mathcal{V}$ then there is no chance of constructing such products (tensor and wedge products) between $\mathcal{V}$-valued forms. Let us now assume that $\mathcal{V}$ has some such bilinear pairing and show how one generalizes the tensor product and wedge product to forms which take their values in some arbitrary vector field. So to begin, suppose that $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are real vector spaces and that there exists a bilinear map $\rho : \mathcal{U} \times \mathcal{V} \to \mathcal{W}$. Next, let $\mathcal{A} \in \mathcal{T}^j(E;\mathcal{U})$ and $\mathcal{B} \in \mathcal{T}^k(E;\mathcal{V})$. Then, we define the **$\rho$-tensor product** $\mathcal{A} \otimes_\rho \mathcal{B} \in \mathcal{T}^{j+k}(E;\mathcal{W})$ by

$$\big(\mathcal{A} \otimes_\rho \mathcal{B}\big)\big(v_1, ..., v_j, v_{j+1}, ..., v_{j+k}\big) := \rho\Big(\mathcal{A}(v_1, ..., v_j), \mathcal{B}(v_{j+1}, ..., v_{j+k})\Big), \tag{5.7}$$

for any $v_1, ..., v_{j+l} \in E$. Additionally, if $\alpha \in \Omega^j(E;\mathcal{U})$ and $\beta \in \Omega^k(E;\mathcal{V})$, then their **$\rho$-wedge product** $\alpha \wedge_\rho \beta \in \Omega^{j+k}(E;\mathcal{W})$ is defined as

$$\big(\alpha \wedge_\rho \beta\big)\big(v_1, ..., v_{j+k}\big) := \frac{1}{j!k!} \sum_{\sigma \in S_{j+k}} (-1)^\sigma (\alpha \otimes_\rho \beta)\Big(v_{\sigma(1)}, ..., v_{\sigma(j+k)}\Big), \tag{5.8}$$

where the sum is over all permutations $\sigma \in S_{j+k}$ of $\{1, ..., j+k\}$. Hence, if $\{U_1, ..., U_q\}$ is a basis for $\mathcal{U}$, with $\alpha = \sum_i \alpha^i U_i$, and $\{T_1, ..., T_m\}$ is a basis for $\mathcal{V}$, with $\beta = \sum_l \beta^l T_i$, then

$$\alpha \wedge_\rho \beta = \sum_{i=1}^{q} \sum_{l=1}^{m} (\alpha^i \wedge_{\mathbb{R}} \beta^l) \rho(U_i, T_l), \tag{5.9}$$

where by $\wedge_{\mathbb{R}}$ we mean the usual wedge product for $\mathbb{R}$-valued differential forms. One should note that there is no reason for $\rho(U_i, T_l)$ $(i = 1, ..., q,\ l = 1, ..., m)$ to constitute as a basis for $\mathcal{W}$[4]. Thus, the $\alpha^i \wedge_{\mathbb{R}} \beta^l$ cannot be regarded as the components of $\alpha \wedge_\rho \beta$.

Finally, we want to consider an example that is relevant to what follows. Let us suppose that $\mathcal{U} = \mathcal{V} = \mathcal{W} = \mathfrak{g}$, where $\mathfrak{g}$ is some Lie group. In this case, for the bilinear map ρ we take the usual Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We also denote the $[\cdot,\cdot]$-wedge product as $[\wedge]$; i.e.,

$$A \wedge_{[\cdot,\cdot]} B \equiv [A \wedge B]. \tag{5.10}$$

Thus, for any $A \in \Omega^j(E;\mathfrak{g})$ and any $B \in \Omega^k(E;\mathfrak{g}$, we have

$$[A \wedge B](v_1, ..., v_{j+k}) = \frac{1}{j!k!} \sum_{\sigma \in S_{j+k}} (-1)^\sigma \Big[A\big(v_{\sigma(1)}, ..., v_{\sigma j}\big), B\big(v_{\sigma(j+1)}, ..., v_{\sigma j+k}\big)\Big]. \tag{5.11}$$

While if we take $\{T_1, ..., T_m\}$ to be a basis of our Lie algebra $\mathfrak{g}$, then we can write $A = \sum_i \alpha^i T_i$ and $B = \sum_l \beta^l T_l$ (here $\alpha^i, \beta^l \in \Omega^j(E;\mathbb{R})$) and

$$[A \wedge B] = \sum_{i=1}^m \sum_{l=1}^m (\alpha^i \wedge_{\mathbb{R}} \beta^l)[T_i, T_l]. \tag{5.12}$$

Note, since $[\cdot,\cdot]$ is closed in $\mathfrak{g}$, $[A\wedge B](v_1, ...v_{j+k}) \in \mathbb{R}\otimes\mathfrak{g} \cong \mathfrak{g}$. We will use these vector-valued forms frequently in what follows.

Maurer-Cartan Form

Before defining a connection form, we first make a brief digression to remind the reader of the Maurer-Cartan form of a Lie group. To begin, let G be a Lie group with Lie algebra $\mathfrak{g} \equiv T_eG$, where $e \in G$ is the identity element in G. Next, for each $g \in G$ denote by $L_g : G \to G$ the left action of G, $L_g(h) = g \cdot h$. Then, we have the following definition.

Definition 5.2.3. The **Maurer-Cartan form**, $\theta \in \Omega^1(G;\mathfrak{g})$, on G is a $\mathfrak{g}$-valued one-form defined by

$$\theta\Big|_{T_gG}(v) = dL_{g^{-1}}(v) \in T_eG \equiv \mathfrak{g}, \tag{5.13}$$

for all $g \in G$ and $v \in T_gG$.

Indeed, $\theta : T_gG \to T_eG$ is linear in each tangent space T_gG and its dependence on g is smooth. Also, since it only has one argument, antisymmetry is vacuous - hence it is a one-form.

Proposition 5.2.4. *The Maurer-Cartan form is invariant under the left action of G; that is to say, for $\forall\ g \in G$*

$$L_g^*(\theta) = \theta,$$

which is interpreted as follows. Let $v \in T_hG$ for any $h \in G$ then, we have that

$$L_g^*(\theta)(v) = L_g^*\big(\theta(v)\big) = \theta(v) \in \mathfrak{g}.$$

[4]If one considers the case where $\mathcal{U} = \mathcal{V} = \mathcal{W}$ then they would conclude that there are too many $\rho(U_i, T_l)$'s to be a basis for $\mathcal{W}$.

Proof. We are required to show that $L_g^*(\theta) = \theta$ for all $g \in G$. To proceed, let $v \in T_hG$ then consider

$$\begin{aligned} L_g^*(\theta(v)) &= \theta(dL_g(v)), \\ &= dL_{(gh)^{-1}}(dL_g(v)), \\ &= dL_{h^{-1}}(dL_{g^{-1}}(dL_g(v))), \\ &= dL_{h^{-1}}(v), \\ &= \theta(v), \end{aligned}$$

where in the second line the Maurer-Cartan acts as $dL_{(gh)^{-1}}$ on $dL_g(v)$, since if $v \in T_hG$ then $dL_g(v) \in T_{gh}G$. □

Remark 5.2.5. Since the Maurer-Cartan form is a left invariant one-form it implies that it is uniquely determined by its value at T_eG (see page 19 of [36]).

Proposition 5.2.6. *Under the right action of G, the Maurer-Cartan form transforms in the adjoint representation,*

$$R_g^*(\theta) = ad_{g^{-1}}(\theta),$$

for all $g \in G$. We interpret the above expression as follows. For any $v \in T_hG$ and $h \in G$,

$$R_g^*(\theta)(v) = R_g^*\big(\theta(v)\big) = ad_{g^{-1}}\big(\theta(v)\big) \in \mathfrak{g},$$

where $ad : \mathfrak{g} \to \mathfrak{g}$ is the derivative of the usual $Ad : G \to G$ map at T_eG.

Proof. To begin, let $v \in T_hG$. Then we have

$$\begin{aligned} R_g^*(\theta(v)) &= \theta(dR_g(v)), \\ &= dL_{(hg)^{-1}}(dR_g(v)), \\ &= dL_{g^{-1}}(dL_{h^{-1}}(dR_g(v))), \\ &= dL_{g^{-1}}(dR_g(dL_{h^{-1}}(v))), \\ &= (dL_{g^{-1}} \circ dR_g)(\theta(v)), \\ &= ad_{g^{-1}}(\theta(v)), \end{aligned}$$

where in the second line the Maurer-Cartan acts as $dL_{(hg)^{-1}}$, since if $v \in T_hG$ then $dR_g(v) \in T_{hg}G$, wile in the last line we identified $dL_{g^{-1}} \circ dR_g \equiv ad_{g^{-1}}$. □

Thus, we say that θ is type ad_G and write $\theta \in \Omega^1_{ad}(G; \mathfrak{g})$. In fact, we call any object which transforms in this manner a type ad_G object.

The Maurer-Cartan form obeys the **structure equation** (also known as the **Maurer-Cartan equation**),

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0, \tag{5.14}$$

where $[\cdot \wedge \cdot]$ is defined as follows. Let A and B be $\mathfrak{g}$-valued one-forms. We can write $A = \sum \alpha^I T_I$ and $B = \sum \beta^I T_I$, where I is a multi-index, $\alpha, \beta \in \Omega^1(G)$ and $\{T\}$ is a basis for $\mathfrak{g}$. Then, define $[A \wedge B]$ by

$$[A \wedge B] = \sum_{I,J} (\alpha^I \wedge_{\mathbb{R}} \beta^J)[T_I, T_J],$$

with $\wedge_{\mathbb{R}}$ the usual wedge product and $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ the usual (Lie bracket) multiplication defined on $\mathfrak{g}$. For a more in depth explanation see section 5.2.1.

With this review behind us, let's begin the study of connections forms on a principal G-bundle.

Construction of Connection Forms

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle and let $\mathfrak{g}$ denote the Lie algebra of G. Recall that, for all $x \in M$, a connection gives a way to separate the tangent space T_pP, at a point $p \in P_x$ ($\equiv \pi^{-1}(x)$), into the direct sum of two subspaces; a horizontal space, which we denote by H_p, and a vertical space, which we denote by T_pP_x. We now want to construct an isomorphism between the vertical space T_pP_x and the Lie algebra $\mathfrak{g}$. This will then allow for the construction of connection forms.

In definition 4.2.1 we saw that a fibre bundle, in particular a principal G-bundle, assigns to each $x \in M$ the pair (U_x, ρ_x), where U_x is an open neighborhood $U_x \subset M$ and ρ_x is a diffeomorphism $\rho_x : P_x \to U_x \times G$ making the following diagram

$$\begin{tikzcd} P_x \arrow[rr, "\rho_x"] \arrow[dr, "\pi"'] & & U_x \times G \arrow[dl, "\mathrm{proj}_1"] \\ & U_x & \end{tikzcd}$$

commute. It follows that, for each $x \in M$ and $p \in P_x$, there exists an isomorphism $\tau_p : P_x \to G$. This isomorphism is unique, up to an element of G acting on the left [35]; that is, if $\tau_p' : P_x \to G$ is a diffeomorphism from another trivialization, then

$$\tau_p' = L_g \circ \tau_p,$$

for some $g \in G$. Consequently, if we restrict the image of the differential of τ_p to only left-invariant vector fields, i.e. elements of $\mathfrak{g}$, then the mapping $d\tau_p$ will be unique. Thus, for any $p \in P_x$ there exists a unique isomorphism $T_pP_x \cong \mathfrak{g}$, giving the desired result. For later use, note that the pullback of the Maurer-Cartan form on G with respect to the isomorphism τ_p gives a 1-form $\theta_x \in \Omega^1(P_x;\mathfrak{g})$ on the fibre P_x, $\theta_x = \tau_p^*(\theta)$, such that

$$\begin{aligned} R_g^*(\theta_x) &= ad_{g^{-1}}(\theta_x), \\ d\theta_x + \tfrac{1}{2}\left[\theta_x \wedge \theta_x\right] &= 0. \end{aligned} \tag{5.15}$$

Hence $\theta_x \in \Omega^1_{ad}(P_x;\mathfrak{g})$. Also, it can be shown that the form θ_x is independent of the choice of $p \in P_x$ and so, the absence of p in the notation for θ_x is justified.

Now, if $G \hookrightarrow P \xrightarrow{\pi} M$ has a connection then for each $p \in P_x$ we assign the subspace H_p in such a way that $T_pP = H_p \oplus T_pP_x$. Also, we can define a projection map

$$(T_pP \equiv) H_p \oplus T_pP_x \longrightarrow T_pP_x (\equiv \mathfrak{g}).$$

Hence, we have a mapping $\omega : T_pP \to \mathfrak{g}$ for all $p \in P$. Conversely, if we have such a ω defined on P, then

$$H_p = \left\{X \in T_pP \mid \omega(X) = 0\right\}, \tag{5.16}$$

i.e. $H_p = \ker(\omega)$. Furthermore, since H_p depends smoothly on p, it implies that ω also depends smoothly on p. So, ω is in fact a one-form,

$$\omega \in \Omega^1(P; \mathfrak{g}).$$

This Lie algebra-valued 1-form is known as a **connection form**.

Let $\iota : P_x \hookrightarrow P$ be the natural inclusion mapping and let $p \in P$. Then, given a connection form ω on P, the pullback of ω (with respect to $\iota_p^* : P \to T_p P_x$) gives

$$\iota_p^*(\omega) = \theta_x,$$

where θ_x is the pullback of the Maurer-Cartan form from TG to TP_x. Note that θ_x does not depend on the choice of p and so defines a one-form on TP_x. Additionally, it can be shown (see [35] page 264) that the connection form ω obeys

$$R_g^*(\omega) = ad_{g^{-1}}(\omega).$$

Therefore, $\omega \in \Omega^1_{ad}(P; \mathfrak{g})$.

Theorem 5.2.7. *If $G \hookrightarrow P \xrightarrow{\pi} M$ is a principal G-bundle with connection ∇, then a one-form $\omega \in \Omega^1_{ad}(P; \mathfrak{g})$, called the connection form, is defined on P satisfying*

$$\iota_p^*(\omega) = \theta_x. \tag{5.17}$$

Proof.

($\Rightarrow$) This has already been shown.

($\Leftarrow$) For $u \in P$, set

$$H_u = \{X \in T_u P \mid \omega(X) = 0\}.$$

To see that $u \mapsto H_u$ gives a connection consider the following. If ω is a connection form, then, by (5.17), $\omega(X) = X$ for any $X \in T_u P_x (\equiv \mathfrak{g})$. Thus, we see that H_u is transversal to the fibre and that $H_u \oplus T_u P_x = T_u P$. Also, since $\omega \in \Omega^1_{ad}(P; \mathfrak{g})$ (hence of type ad_G), we see that $dR_g H_u = H_{u \cdot g}$. Finally, since ω is a one-form, we see that H_u depends smoothly on u.

□

In short, we have just shown that a connection on P can be identified with a type ad_G one-form $\omega \in \Omega^1_{ad}(P; \mathfrak{g})$ such that

$$\iota_p^*(\omega) = \theta_x.$$

Let us now state a proposition that will allow for one to glue connections over manifolds with boundary along boundaries where the connections agree.

Proposition 5.2.8. *Suppose $G \hookrightarrow P \xrightarrow{\pi} M$ is a principal bundle over an oriented manifold M, and $\Sigma \hookrightarrow M$ is an oriented codimension one submanifold of M. Let M^{cut} be the manifold obtained by cutting M along Σ (see figure 5.3). There exists a gluing map $\hat{g} : M^{cut} \to M$ which is a diffeomorphism off of Σ and maps two distinct submanifolds Σ_1, Σ_2*

of ∂M^{cut} diffeomorphically onto Σ. Furthermore, let $P^{cut} \equiv \hat{g}^*(P)$ be the "cut" bundle (i.e. the pullback bundle along $\hat{g}$, see example 4.1.4) and denote the induced map, coming from the pullback bundle construction (see example 4.1.4) between P^{cut} and P by g. Now, suppose ω^{cut} is a connection on P^{cut} such that there exists a connection η on $P|_\Sigma$ with $g^*(\eta) = \omega|_{\Sigma_1 \sqcup \Sigma_2}$. Then, η extends to a connection ω on P over M such that $g^*(\omega)$ is gauge equivalent to ω^{cut}.

Remark 5.2.9. This proposition tells us that we can glue smoothly if we make a gauge transformation. It also asserts that $\omega^{\text{cut}}|_{\Sigma_1}$ and $\omega^{\text{cut}}|_{\Sigma_2}$ agree under the identification of the bundles.

Proof. See [23]. □

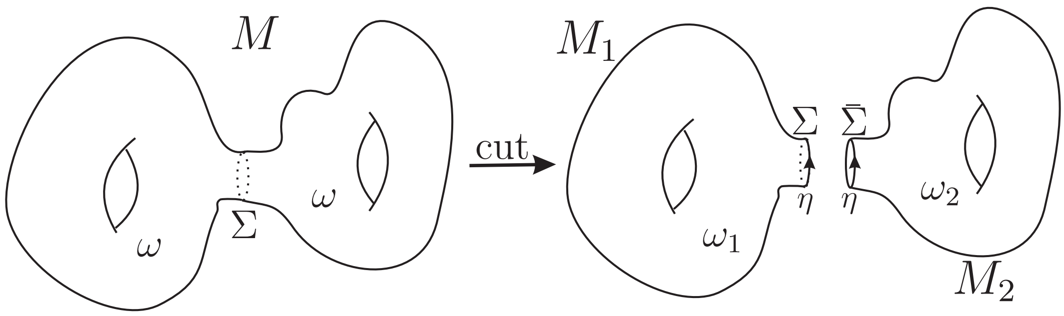


Figure 5.3: Gluing Connections.

To end this section we will give an example of how to construct a connection form ω on M. First, consider a product bundle, $P = M \times G$, and define, for each $p \in M \times G$, the projection map onto the second component, $\bar{\iota}_p : M \times G \to G$. We can use $\bar{\iota}$ to pull back the Maurer-Cartan form from G to the total space $M \times G$. We define our connection form on $M \times G$ to be

$$\omega := \bar{\iota}_p^*(\theta).$$

Now, if our principal bundle is not trivial then we can take an open covering of M, $\{U_\alpha\}_{\alpha \in I}$, such that $U_\alpha \times G$ is a trivial bundle. Next, we define the connection form on $U_\alpha \times G$ as the pullback of θ by the restriction of $\bar{\iota}_p$ to $U_\alpha \times G$ (i.e. $p \in U_\alpha \times G$). So, on each trivial bundle we have a connection defined, namely the pullback of θ. To turn these connections into a connection form on the whole of P we take a partition of unity $\{f_\alpha\}_{\alpha \in I}$ subordinate to $\{U_\alpha\}_{\alpha \in I}$ (i.e. every f_α has compact support on U_α in addition to $\sum_\alpha f_\alpha = 1$) and define the connection form ω on P to be

$$\omega := \sum_\alpha f_\alpha \omega_\alpha,$$

where $\omega_\alpha = \bar{\iota}_{p \in U_\alpha}^*(\theta)$.

Space of Connection Forms

The main result of this section is that the space of connections on a principal G-bundle P is not a vector space. Instead this space is an *affine space*. To begin, let us recall the definition of an affine space.

Definition 5.2.10. A set S is an **affine space** over a vector space V if it has a mapping $\gamma : S \times S \to V$, denoted by $\gamma(a, b) = a - b$, such that

(1) $(a - b) + (b - c) = a - c$, and

(2) $\gamma_b : S \to V$, where γ_b is defined by $\gamma_b(a) := b - a$, is a bijection.

Said differently, an affine space is a set S which has a simply transitive action of the abelian group V defined on it, or a *V-torsor*.

To see that the space of connections, which we denote by $\mathcal{A}(P) \subset \Omega^1_{ad}(P; \mathfrak{g})$ (i.e. an element of $\omega \in \mathcal{A}(P)$ is a type ad_G $\mathfrak{g}$-valued 1-form such that $\iota_p^*(\omega) = \theta_x$), is not a vector space consider the addition of two connections. Let ω_1 and ω_2 be two elements of $\mathcal{A}(P)$. Then, we have that

$$\begin{aligned} \iota_p^*(\omega_1 + \omega_2) &= \iota_p^*(\omega_1) + \iota_p^*(\omega_2), \\ &= \theta_x + \theta_x, \\ &= 2\theta_x. \end{aligned} \tag{5.18}$$

While for $\omega_1 + \omega_2$ to be a connection, the right-hand side should equal θ_x. And so, we see that the sum of two connections is not a connection. Hence the space of connections on P, $\mathcal{A}(P)$, is not a vector space.

Now, what about the difference of two connections? As before, let ω_1 and ω_2 be connections on P, and consider

$$\begin{aligned} \iota_p^*(\omega_1 - \omega_2) &= \iota_p^*(\omega_1) - \iota_p^*(\omega_2), \\ &= \theta_x - \theta_x, \\ &= 0. \end{aligned}$$

This means that the difference between the two connections is the pullback of a $\mathfrak{g}$-valued one-form defined on M, i.e.

$$\omega_1 - \omega_2 = \pi^*(\eta),$$

with $\eta \in \Omega^1(M; \mathfrak{g})$. Which, by reviewing definition 5.2.10, implies that the space of connections on P is an affine space over $\Omega^1(M; \mathfrak{g})$. In summary, we have just proven the following theorem.

Theorem 5.2.11. *The space of connections on P, $\mathcal{A}(P)$, is an affine space over $\Omega^1(M; \mathfrak{g})$.*

Proof. □

5.3 Curvature Forms

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a trivial bundle, that is $P = M \times G$, and let the connection form on P, ω, coincide with the pullback of the Maurer-Cartan form, $\omega = \bar{\iota}_p^*(\theta)$. Then, by the Maurer-Cartan equation, we have that

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = 0. \tag{5.19}$$

However, if our principal bundle is not trivial we cannot define the connection form as the pullback of the Maurer-Cartan, (except locally). Consequently, for a non-trivial principal bundle, ω does not obey the previous equation (except locally on a trivialization). Indeed, instead of (5.19) we have that

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = \text{“stuff”}.$$

We label this “stuff” as Ω and call it the **curvature** (**curvature form**). Hence, for any principal G-bundle with connection form ω, one has that

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = \Omega, \tag{5.20}$$

which is known as the **structure equation**. Note that $\Omega \in \Omega^2(P; \mathfrak{g})$. From the structure equation, we see that the deviation from a trivial structure is measured by Ω, hence the name curvature.
Furthermore, the curvature form obeys the following properties.

Proposition 5.3.1. *For a connection form ω with associated curvature form Ω, the following identities hold:*

(1) $\Omega\big|_{TP_x} = 0$, *i.e. the curvature form annihilates any vector lying in the tangent space along the fibre.*

(2) *The curvature form is of type ad_G,*

$$R_g^*(\Omega) = ad_{g^{-1}}(\Omega).$$

(3) *The curvature form satisfies the Bianchi identity, namely*

$$d\Omega + [\omega \wedge \Omega] = 0.$$

Remark 5.3.2. Property (2) says that $\Omega \in \Omega^2_{ad}(P; \mathfrak{g})$.

Proof. (1) Let $\iota : P_x \hookrightarrow P$ be the natural inclusion from before, then, for any $p \in P$, $\Omega|_{TP_x} = \iota_p^*(\Omega)$. Thus,

$$\begin{aligned}
\Omega\big|_{TP_x} &= \iota_p^*\left(d\omega + \frac{1}{2}[\omega \wedge \omega]\right), \\
&= \iota_p^*(d\omega) + \frac{1}{2}\iota_p^*([\omega \wedge \omega]), \\
&= d(\iota_p^*(\omega)) + \frac{1}{2}[\iota_p^*(\omega) \wedge \iota_p^*(\omega)], \\
&= d\theta_x + \frac{1}{2}[\theta_x \wedge \theta_x],
\end{aligned}$$

which vanishes by the Maurer-Cartan equation (see (5.15)). Therefore, $\Omega\big|_{TP_x} = 0$ as desired.

(2) We have

$$\begin{aligned}
R_g^*(\Omega) &= R_g^*\left(d\omega + \frac{1}{2}[\omega \wedge \omega]\right), \\
&= d(R_g^*(\omega)) + \frac{1}{2}[R_g^*(\omega) \wedge R_g^*(\omega)], \\
&= d(ad_{g^{-1}}(\omega)) + \frac{1}{2}[ad_{g^{-1}}(\omega) \wedge ad_{g^{-1}}(\omega)], \\
&= ad_{g^{-1}}\left(d\omega + \frac{1}{2}[\omega \wedge \omega]\right), \\
&= ad_{g^{-1}}(\Omega).
\end{aligned}$$

(3) This follows from a straightforward but rather tedious calculation.

□

Gauge Transformations Since we need it later on, let us describe the transformation properties of the connection and curvature forms under a gauge transformation. To begin, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle with connection form ω and curvature form Ω. Now, let $\varphi \in \mathrm{Aut}(P)$ be a gauge transformation with associated map $\hat{g}_\varphi : P \to G$, defined by $p \cdot \hat{g}_\varphi(p) = \varphi(p)$, and let $\phi = \hat{g}_\varphi^*(\theta)$, then:

Theorem 5.3.3. *Under a gauge transformation $\varphi : P \to P$, the connection form and curvature form obey*

$$\begin{aligned}
\varphi^*(\omega) &= ad_{\hat{g}_\varphi^{*-1}}(\omega) + \phi, \\
\varphi^*(\Omega) &= ad_{\hat{g}_\varphi^{*-1}}(\Omega).
\end{aligned} \tag{5.21}$$

Proof. See [23]. □

5.3.1 Universal Connections and Curvatures

We briefly mention the concepts of universal connection forms ω_u and universal curvature forms Ω_u. As we have seen in section 4.5, there exists certain universal bundles $EG \to BG$ with the property that any principal G-bundle over some manifold M can be realized as a pullback bundle of EG, via the classifying map $\gamma : M \to BG$. Furthermore, any two principal G-bundles over M which are isomorphic, must have classifying maps which are homotopy equivalent. Now, since $EG \to BG$ is a fibre bundle (in fact a principal bundle) one might ask if there exists connection forms on EG? This question was answered in the affirmative for compact Lie groups G by Narasinhan and Ramanan [38]. What is more, they were able to show that any connection form ω on P comes from the pullback of some universal connection form ω_u on EG via the pullback of the induced[5] bundle map

[5] Recall that given a bundle map between base spaces (or total spaces) it induces a map between the total spaces (or base spaces).

between total spaces $\gamma' : P \to EG$. Furthermore, as was the case for connection forms on P, one can associate a universal curvature form Ω_u to each ω_u. We will see later on that these universal connections and curvatures come up quite naturally when discussing the Chern-Simons action on general (i.e., non-trivial) principal G-bundles.

5.4 Gauge Potentials and Field Strengths

Although the previous sections might have seemed mysterious to the physicist reading along, this next section will tie the previous notions of connections and curvature together with concepts that are familiar to many physicists - that being the gauge potential, $\mathcal{A}$, and field strength tensors $\mathcal{F}$.

Gauge Potential

Roughly speaking, a *gauge potential* $\mathcal{A}$ (such as the gauge potential in E&M) is the pullback of a connection form on P to M, while the *field strength* tensor $\mathcal{F}$ (for example the stress-energy tensor in E&M) is nothing but the pullback of the curvature form.

To be more precise, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal bundle and let $s : M \to P$ be a section on P. Then, using this section, we can pullback the connection form ω to give a "connection form", or **gauge potential**, on M,

$$\mathcal{A} := s^*(\omega).$$

Now, if our principal bundle is not trivial then this (global) section does not exist. However, there do exists local sections, $s_\alpha = s|_{U_\alpha}$, and we can use these local sections to define (local) gauge potentials. Hence, gauge potentials are local entities defined on our spacetime M and only if P is trivial do these gauge potentials turn into global objects.

From the definition of a principal bundle, we know that there will exist certain U_α's which have nonzero overlaps. So, in this case, how does the gauge potential on one open subset relate to the gauge potential on the other? State differently, if U_α and U_β are two open subsets of M such that $U_\alpha \cap U_\beta \neq \emptyset$, then how does $\mathcal{A}_\alpha$ relate to $\mathcal{A}_\beta$? To answer this question, begin by letting $\{U_\alpha\}_{\alpha \in I}$ be an open covering of M and denote the local section (or **gauge**) corresponding to the open set U_α by s_α; that is, $s_\alpha : U_\alpha \to P$. Additionally, let U_β be another open cover such that $U_\alpha \cap U_\beta \neq \emptyset$. Then, on each open cover U_α and U_β there exists a local gauge potential, $\mathcal{A}_\alpha := s_\alpha^*(\omega)$ and $\mathcal{A}_\beta := s_\beta^*(\omega)$ respectively, and on the overlap $U_\alpha \cap U_\beta$ we can find a relation between these two gauge potentials. Indeed, since the two gauge potentials are defined as the pullback of the connection form on P and since the same connection form is used in the definition, the only difference between $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ is the sections used - s_α for $\mathcal{A}_\alpha$ and s_β for $\mathcal{A}_\beta$. Furthermore, since s_α is related to s_β by a gauge transformation (on the overlap), we see that the correct relation for $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ is given by pulling back the gauge transformation relation for ω to M. That is, given two gauge potentials $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ defined on some open set $U_\alpha \cap U_\beta$ they are related via

$$\mathcal{A}_\beta = ad_{g_{\alpha\beta}^{-1}}(\mathcal{A}_\alpha) + g_{\alpha\beta}^*(\theta), \tag{5.22}$$

where by $g_{\alpha\beta}^{-1}$ we mean the inverse of the element in G mapped to by the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, while $g_{\alpha\beta}^*(\theta)$ is the pullback of the Maurer-Cartan form via the transition function. When G is a matrix Lie group, this relation reduces to

$$\mathcal{A}_\beta = g_{\alpha\beta}^{-1} \cdot \mathcal{A}_\alpha \cdot g_{\alpha\beta} + g_{\alpha\beta}^{-1} \cdot dg_{\alpha\beta}. \tag{5.23}$$

Equation (5.22) is known as the *compatibility condition* (it is just the gauge transformation rule for the gauge potentials).

Example 5.4.1 ($G = U(1)$). Let $G = U(1)$ and let U_α and U_β be two open covers on M such that $U_\alpha \cap U_\beta \neq \emptyset$. Then, the transition functions are given by mappings $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ such that $p \in U_\alpha \cap U_\beta \mapsto e^{if(p)}$, where f is an $\mathbb{R}$-valued function. And so, using the compatability condition for matrix groups (namely (5.23)), we get

$$\begin{aligned}
\mathcal{A}_\beta(p) &= g_{\alpha\beta}^{-1}(p) \cdot \mathcal{A}_\alpha(p) \cdot g_{\alpha\beta}(p) + g_{\alpha\beta}^{-1}(p) \cdot dg_{\alpha\beta}(p), \\
&= e^{-if(p)} \mathcal{A}_\alpha(p) e^{if(p)} + e^{-if(p)} d(e^{if(p)}), \\
&= \mathcal{A}_\alpha(p) + i df(p).
\end{aligned} \tag{5.24}$$

Thus, by taking $\mathcal{A} = iA$ we get the usual rule from E&M. Although not entirely the defining reason, we see from this calculation that E&M is represented by a $U(1)$ gauge theory.

Remark 5.4.2. Note that the connection form ω is defined globally over P while the gauge potential is only defined locally over M (that is, unless P is trivial). Also, although there may be many connection forms on P they share the same global information, while since $\mathcal{A}$ is only defined locally it cannot give any global information. However, the collection of gauge potentials $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ along with the compatability condition does give the global information contained in ω.

N.B. 5.4.3. If the gauge potential is given by $\mathcal{A} = s^*(\omega)$ then we say that the gauge potential is in gauge s. And so, we say that $\mathcal{A}_\alpha$ is the gauge potential in gauge s_α while $\mathcal{A}_\beta$ is the gauge potential in gauge s_β and that the two are related via a gauge transformation, i.e. a transformation from gauge s_α to s_β.

Field Strengths

We now turn our attention to defining the (local) *field strength* $\mathcal{F}$ on M. So, to begin, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle with connection form ω and corresponding curvature form Ω. Then, given a section $s : M \to P$, we define the **field strength** as $\mathcal{F} := s^*(\Omega)$. Hence,

$$\begin{aligned}
\mathcal{F} &= s^* \left(d\omega + \frac{1}{2}[\omega \wedge \omega] \right), \\
&= d(s^*(\omega)) + \frac{1}{2}[s^*(\omega) \wedge s^*(\omega)], \\
&= d\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}].
\end{aligned}$$

Pulling back the transformation for the curvature under a gauge transformation gives us the compatability condition for the field strength, namely

$$\mathcal{F}_\beta = ad_{g_{\alpha\beta}^{-1}}(\mathcal{F}_\alpha). \tag{5.25}$$

If G is a matrix group the compatability condition becomes

$$\mathcal{F}_\beta = g_{\alpha\beta}^{-1} \cdot \mathcal{F}_\alpha \cdot g_{\alpha\beta}.$$

Thus, in the case where $G = U(1)$ we have that $\mathcal{F}_\alpha = e^{-if(p)}\mathcal{F}_\beta e^{if(p)} = \mathcal{F}_\beta$. And so, our local field strengths glue together to give a globally defined field strength - this is the case whenever our structure group G is abelian as can be seen from (5.25).

As we will see in subsequent chapters, these local gauge potentials and local field strengths will play an important role. Finally, note that any gauge potential $\mathcal{A}$ on M (which is the base space of a principal G-bundle $P \to M$) can be obtained as

$$\mathcal{A} = \gamma^*(\mathcal{A}_u),$$

where γ is the classifying map $M \to BG$, BG is the base space of the universal bundle and $\mathcal{A}_u$ is a the universal gauge potential defined on BG. Furthermore, as was the case before for field strengths, one can pullback the universal curvature Ω_u from EG to BG, thus giving one the notion of a universal field strength $\mathcal{F}_u$ associated to the universal gauge potential $\mathcal{A}_u$ (see [38]).

Chapter 6

Characteristic Classes

A *characteristic class* associates, to each principal bundle over M, a (globally defined) cohomology class of M; where the cohomology class measures the extent to which the bundle is “twisted”. Said another way, characteristic classes are global invariants of principal bundles which measure the deviation of the bundle's total space P from a global product structure. The goal of this chapter is to see how one constructs these global invariants, while in later sections we will see how they arise in mathematical physics. In particular, how they arise in the study of topological quantum field theories.

6.1 Motivation (Classification of U(1)-Bundles)

Because a characteristic class is a cohomology class measuring the twistedness of a bundle, we want to look at globally defined objects on M constructed from the curvature forms Ω (since they give us a measure of non-triviality) and the cohomology classes that they define.

Let us begin by considering a principal G-bundle, $G \hookrightarrow P \xrightarrow{\pi} M$, over a manifold M with connection form ω (see section 5.2). Further, let's restrict to the case where the structure group G is given by $U(1)$. Since $\mathfrak{u}(1)$ is an one-dimensional, and hence abelian, Lie algebra, we have that for any $\mathfrak{u}(1)$-valued one-form, $\omega = \alpha T$,

$$[\omega \wedge \omega] \equiv \omega \wedge_{[\cdot,\cdot]} \omega = (\alpha \wedge_{\mathbb{R}} \beta)[T, T] = 0,$$

where the singleton $\{T\}$ gives the basis for $\mathfrak{u}(1)$ and $\alpha, \beta \in \Omega^1(P)$. Using this simple derivation, we can rewrite the Maurer-Cartan (or structure) equation, see (5.14), as

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] = d\omega.$$

Thus, the curvature form Ω is nothing more than the exterior derivative of the connection form; this, in fact, holds anytime the structure group is an abelian connected Lie group, since then the corresponding Lie algebra will also be abelian. Additionally, for any local section $s : U_x \to \pi^{-1}(U_x)$ we can pull the connection and curvature forms back down to the open subset $U_x \subset M$ of the base manifold M. We write this as (see page 44 of [36])

$s^*(\omega) \equiv \mathcal{A} = -iA$ and $s^*(\Omega) \equiv \mathcal{F} = -iF$, where A and F are $\mathbb{R}$-valued forms[1] on U_x, and we call $\mathcal{A}$ the local gauge potential and $\mathcal{F}$ the local field strength.

Now, we would like to extend the local field strength, defined on U_x, to a field strength which is defined globally on M. To do this, recall that if U_α and U_β are open subsets of M which have a non-empty intersection and if $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ as well as $s_\beta : U_\beta \to \pi^{-1}(U_\beta)$ exist, then we can pull back the curvature form to the open subsets U_α and U_β via s_α and s_β, respectively. On the intersection $U_\alpha \cap U_\beta$, these two field strengths are related by (see (5.25))

$$\mathcal{F}_\alpha = ad_{\left(g_{\alpha\beta}(p)\right)^{-1}}\left(\mathcal{F}_\beta\right),$$

where $p \in U_\alpha \cap U_\beta$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is a transition function. Restricting G to be a matrix Lie group gives

$$\mathcal{F}_\alpha = g_{\alpha\beta}^{-1}(p) \cdot \mathcal{F}_\beta \cdot g_{\alpha\beta}(p),$$

where by $\cdot$ we mean matrix multiplication. Further restricting to the case in which G is abelian, we are left with

$$\mathcal{F}_\alpha = \mathcal{F}_\beta.$$

And so, we see that for a principal G-bundle with abelian structure group the local field strengths agree on the overlap regions, giving us a globally defined field strength on M which we also denote by $\mathcal{F}$. Moreover, since $\mathcal{F} = d\omega$, one has that $d\mathcal{F} = 0$, which, in turn, implies that F is closed. Therefore $\frac{1}{2\pi}F$ determines an element in $H^2_{dR}(M;\mathbb{R})$; the factor of $\frac{1}{2\pi}$ ensures integrality of the cohomology class, in the sense that its periods are always integers.

We now want to show that any other closed two-form $\frac{1}{2\pi}F'$, coming from a different connection form defined on $U(1) \hookrightarrow P \xrightarrow{\pi} M$, differs from $\frac{1}{2\pi}F$ by at most a closed two-form. This would then tell us: firstly, any two $\frac{1}{2\pi}F$ and $\frac{1}{2\pi}F'$ are cohomologous and secondly, the cohomology class determined in this way is independent of the choice of connection, thus giving a characteristic class of the $U(1)$-bundle. To proceed, suppose that along with the prior connection form, ω, there exists another connection form on the $U(1)$-bundle, which we denote by ω', and its corresponding curvature form Ω'. By similar reasoning, this new curvature form gives rise to two global two-forms on M, namely $\mathcal{F}'$ and F'. Also, since the space of connections $\mathcal{A}(P)$ on P is affine (see theorem 5.2.11), the element $\omega - \omega'$ is again a connection, i.e. $\omega - \omega' = \tau \in \mathcal{A}(P)$; implying that $d\omega - d\omega' = d\tau$, or, equivalently in our case,

$$\Omega - \Omega' = d\tau.$$

Furthermore, since $\tau \in \mathcal{A}(P) \subset \Omega^1_{ad}(P;\mathfrak{u}(1))$ and since $U(1)$ is an abelian matrix group, we have that

$$R_g^*(\tau) = ad_{g^{-1}}(\tau) = g^{-1} \cdot \tau \cdot g = \tau,$$

for all $g \in G$. Now consider the following lemma (Lemma 1 page 294 [26]).

Lemma 6.1.1. *A q-form φ on P projects down to a unique q-form, say $\bar{\varphi}$, on M (i.e. we have that $\pi^*(\bar{\varphi}) = \varphi$), if:*

---

[1] Note that the $\mathbb{R}$-valued one-form A is the usual vector potential from electromagnetism, while the $\mathbb{R}$-valued two-form F is the electromagnetic tensor.

(*a*) $\varphi(X_1, ..., X_q) = 0$, *whenever at least one of the X_i's is vertical*[2], *and*

(*b*) $\varphi(R_g(X_1), ..., R_g(X_1)) = \varphi(X_1, ..., X_1)$, *for the right action R_g of $g \in G$.*

Proof. See page 294 of [26]. □

Thus, since $\tau \in \mathcal{A}(P)$ along with $ad_{g^{-1}}(\tau) = \tau$, it follows, by lemma 6.1.1, that τ projects down to a unique form on M , i.e. there exists a unique $\mathfrak{u}(1)$-valued one-form $\bar{\tau}$ on M such that $\tau = \pi^*(\bar{\tau})$, where $\pi : P \to M$ is the bundle projection map. Accordingly, since our structure group is abelian, the curvature forms project down, $\Omega = \pi^*(\mathcal{F})$ and $\Omega' = \pi^*(\mathcal{F}')$, as well. Therefore, from $\Omega - \Omega' = d\omega - d\omega' = d\tau$, we get that

$$\pi^*(\mathcal{F}) - \pi^*(\mathcal{F}') = d\left(\pi^*(\bar{\tau})\right),$$

or

$$\pi^*(\mathcal{F} - \mathcal{F}') = \pi^*(d\bar{\tau}).$$

However, if a projection to M exists it must be unique, and so

$$\mathcal{F} - \mathcal{F}' = d\bar{\tau},$$

leading to the difference between $\frac{1}{2\pi}F$ and $\frac{1}{2\pi}F'$ being a closed two-form.

Thus, we have just shown that, in particular, the elements $\frac{1}{2\pi}F$ and $\frac{1}{2\pi}F'$ are cohomologous and thereby determine the same element of $H^2_{dR}(M;\mathbb{R})$. This unique cohomology class, constructed in the prior manner from an arbitrary connection form, is called the **1$^{\mathbf{st}}$ Chern class** of the principal $U(1)$-bundle and is usually denoted by

$$c_1(P) \equiv \left[\frac{1}{2\pi}F\right] \in H^2_{dR}(M;\mathbb{R}).$$

The most remarkable property of the 1$^{\text{st}}$ Chern class is that it is determined entirely by the bundle itself; it does not depend on the particular choice of connection form from which it is constructed. In fact, the 1$^{\text{st}}$ Chern class $c_1(P)$ is actually characteristic of the bundle in the sense that any two principal $U(1)$-bundles P and P' over the same manifold M which are equivalent have the same 1$^{\text{st}}$ Chern class (i.e., by negation, if $c_1(P) \neq c_1(P')$ then P is not isomorphic to $P'P$). Characteristic classes are very powerful tools in the classification of principal bundles and, as will become apparent later, they play a significant role in the construction of topological field theories.

The next step is to generalize the preceding construction to the case where G is no longer abelian. This does not seem promising since, basically, every idea and calculation in the prior paragraphs depended on the commutativity of $U(1)$. For example, in order to construct a field strength which was globally defined on M, we had to insure that each local field strength agreed on the overlap regions with its corresponding neighbors. However, in general, such local field strengths are related by $\mathcal{F}_\alpha = ad_{g^{-1}}(\mathcal{F}_\beta)$ and so, when the structure group is non-abelian, it is not true that $\mathcal{F}_\alpha = \mathcal{F}_\beta$. The way that we will get

[2]Recall, from section 5.1, that if a vector $X \in T_uP$ has any vertical component then its projection down to a tangent spact in M is not unique.

around this roadblock is by only considering specific polynomials in the field strengths which are constant on the G-orbits (i.e. polynomials that are immune to the replacement of $\mathcal{F}$ by $ad_{g^{-1}}(\mathcal{F})$), known as *invariant polynomials*. Then, in terms of these polynomials, we will be able to define global field strengths on the base manifold of a principal G-bundle for arbitrary G.

6.2 Invariant Polynomials

We will follow chapter 11 of [26] in what follows.

To begin, let $\mathcal{V}$ be a finite dimensional vector space. For $k \geq 1$, a map

$$f : \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{k} \longrightarrow \mathbb{R}$$

is called *k-multilinear and symmetric* if it is linear in each variable separately as well as obeying $f(v_{\sigma(1)}, ..., v_{\sigma(k)}) = f(v_1, ..., v_k)$ for all permutations $\sigma \in S_k$ and $v_i \in \mathcal{V}$. We denote the set of all such mappings by $\mathcal{S}^k(\mathcal{V})$. Note that $\mathcal{S}^k(\mathcal{V})$ has the pointwise structure of a vector space and we define

$$\mathcal{S}^\bullet(\mathcal{V}) := \bigoplus_{k=0}^{\infty} \mathcal{S}^k(\mathcal{V}),$$

with $\mathcal{S}^0(\mathcal{V}) = \mathbb{R}$. For $f \in \mathcal{S}^m$ and $g \in \mathcal{S}^n$ we can construct an element in $\mathcal{S}^{m+n}$, $f \star g$, by setting

$$\big(f \star g\big)(v_1, ..., v_m, v_{m+1}, ..., v_n) = \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} f(v_{\sigma(1)}, ..., v_{\sigma(m)}) g(v_{\sigma(m+1)}, ..., v_{\sigma(n)}). \tag{6.1}$$

The $\star$ multiplication can be extended to all of $\mathcal{S}^\bullet(\mathcal{V})$; resulting in giving $\mathcal{S}^\bullet(\mathcal{V})$ the structure of a graded commutative algebra.

Let us now explore how we can explicitly write any $f \in \mathcal{S}^k(\mathcal{V})$. Suppose $\{e_1, ..., e_n\}$ is a basis for $\mathcal{V}$ and $\{\chi_1, ..., \chi_n\}$ a basis for the dual space $\mathcal{V}^*$, then we can uniquely write any $f \in \mathcal{S}^k(\mathcal{V})$ as[3]

$$f = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1 \cdots i_k} \chi_{i_1} \otimes \cdots \otimes \chi_{i_k}, \tag{6.2}$$

or, using the Einstein summation convention,

$$f = a_{i_1 \cdots i_k} \chi^{i_1} \otimes \cdots \otimes \chi^{i_k}.$$

In particular, using the expression for f, found in (6.2), and remembering that the dual basis acts on the normal basis for $\mathcal{V}$ as $\chi_i(e_j) = \delta_{ij}$, it is easy to see that if we write $v_1 = x_1^{i_1} e_{i_1}, ..., v_k = x_k^{i_k} e_{i_k}$, then

$$f(v_1, ..., v_k) = a_{i_1 \cdots i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}.$$

[3]Note the we must also impose that the forthcoming expansion is symmetric in $i_1, ..., i_k$.

We can use the elements of $\mathcal{S}^k(\mathcal{V})$ to define homogeneous polynomials of degree k on $\mathcal{V}$ as follows. First, denote by $\triangle^k$ the diagonal map from the vector space $\mathcal{V}$ to the vector space constructed from the Cartesian product of $\mathcal{V}$ with itself k times; i.e. the map $\triangle^k : \mathcal{V} \to \mathcal{V}^{\times k}$ defined by

$$v \longmapsto (\underbrace{v, ..., v}_{k}).$$

Then, for $f \in \mathcal{S}^k(\mathcal{V})$, define $\tilde{f} : \mathcal{V} \to \mathbb{R}$ by

$$\tilde{f}(v) := \big(f \circ \triangle^k\big)(v) = f(v, ..., v). \tag{6.3}$$

And so, if $v = x^i e_i$, we have that $\tilde{f}(v) = a_{i_1 \cdots i_k} x^{i_1} x^{i_2} \cdots x^{i_k}$. Hence, $\tilde{f}$ is a homogeneous polynomial of degree k in the components of v. Furthermore, the collection of all such homogeneous polynomials of degree k on $\mathcal{V}$, with its pointwise structure, is denoted $\mathcal{P}^k(\mathcal{V})$ and we define

$$\mathcal{P}^\bullet(\mathcal{V}) := \bigoplus_{k=0}^{\infty} \mathcal{P}^k(\mathcal{V}),$$

with $\mathcal{P}^0(\mathcal{V}) = \mathbb{R}$. It is possible to turn $\mathcal{P}^\bullet(\mathcal{V})$ into a graded commutative algebra by defining, for $\tilde{f} \in \mathcal{P}^m(\mathcal{V})$ and $\tilde{g} \in \mathcal{P}^n(\mathcal{V})$, the product $\tilde{f} \diamond \tilde{g} \in \mathcal{P}^{m+n}(\mathcal{V})$ as

$$(\tilde{f} \diamond \tilde{g})(v) = \tilde{f}(v)\tilde{g}(v), \tag{6.4}$$

and then extending $\diamond$ to all of $\mathcal{P}^\bullet(\mathcal{V})$.

We have just seen that given any element $f \in \mathcal{S}^\bullet(\mathcal{V})$ it defines an element $\tilde{f} \in \mathcal{P}^\bullet(\mathcal{V})$. However, not only do the elements of $\mathcal{S}^\bullet(\mathcal{V})$ define elements in $\mathcal{P}(^\bullet\mathcal{V})$, but we, in fact, have the following observation.

Theorem 6.2.1. *The identification $f \mapsto \tilde{f}$ gives an algebra isomorphism between $\mathcal{S}^\bullet(\mathcal{V})$ and $\mathcal{P}^\bullet(\mathcal{V})$.*

Proof. See [36]. □

To recap, any symmetric k-multilinear function on $\mathcal{V}^{\times k}$ defines a unique homogeneous polynomial of degree k on $\mathcal{V}$.

Let us now restrict to a case more appropriate for our purposes. Suppose that G is a Lie group with a representation $\rho : G \to GL(\mathcal{V})$ on $\mathcal{V}$. Now, for each $k \geq 1$, let $\mathcal{S}^k_\rho(\mathcal{V})$ and $\mathcal{P}^k_\rho(\mathcal{V})$ denote the subspaces of $\mathcal{S}^k(\mathcal{V})$ and $\mathcal{P}^k(\mathcal{V})$, respectively, which are invariant under ρ. Precisely, if $f \in \mathcal{S}^k_\rho(\mathcal{V})$ and $\tilde{f} \in \mathcal{P}^k_\rho(\mathcal{V})$, we have

$$f\big(\rho_g(v_1), ..., \rho_g(v_k)\big) = f(v_1, ..., v_k), \tag{6.5}$$

along with

$$\tilde{f}\big(\rho_g(v)\big) = \tilde{f}(v), \tag{6.6}$$

for all $g \in G$ and $v_1, ..., v_k, v \in \mathcal{V}$.

As was already alluded to earlier, we want polynomials which are constant on the G-orbits so that we may define a global two-form on M, which is the pullback of the curvature form defined on E. So, let us further restrict the previous paragraph to the case where $\mathcal{V}$ is given by the Lie algebra of G, $\mathcal{V} = \mathfrak{g}$, and where $\rho : G \to GL(\mathfrak{g})$, $g \mapsto ad_g$, is the adjoint representation of G on $\mathfrak{g}$. Then, any $f \in \mathcal{S}^k_{ad}(\mathfrak{g})$ is a symmetric multilinear map $f : \mathfrak{g}^{\times k} \to \mathbb{R}$ such that

$$f\big(ad_{g^{-1}}(X_1), ..., ad_{g^{-1}}(X_k)\big) = f\big(X_1, ..., X_k\big),$$

for all $g \in G$ and $X_i \in \mathfrak{g}$. Consequently, for any $\tilde{f} \in \mathcal{P}^k_{ad}(\mathfrak{g})$, we have that $\tilde{f}(ad_{g^{-1}}(X)) = \tilde{f}(X)$, where g ranges over all of G and $X \in \mathfrak{g}$. It is customary to denote $\mathcal{S}^k_{ad}(\mathfrak{g})$ by $I^k(G)$, which we will follow, along with defining

$$I^\bullet(G) := \bigoplus_{k=0}^{\infty} I^k(G).$$

Recalling that the algebras $\mathcal{S}^\bullet(\mathcal{V})$ and $\mathcal{P}^\bullet(\mathcal{V})$ are isomorphic (see theorem 6.2.1), one may wonder if there exists an isomorphism between the subalgebras $I^\bullet(G)$ ($\equiv \mathcal{S}^\bullet_{ad}(\mathfrak{g})$) and $\mathcal{P}^\bullet_{ad}(\mathfrak{g})$? The answer to this question is yes. However, we can, in fact, generalize this even further.

Theorem 6.2.2. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the algebra $\mathcal{S}^\bullet_\rho(\mathfrak{g})$ of ρ-invariant symmetric multilinear mappings of $\mathfrak{g}$ into $\mathbb{R}$ is isomorphic with the algebra $\mathcal{P}^\bullet_\rho(\mathfrak{g})$ of ρ-invariant polynomial functions on $\mathfrak{g}$.*

Proof. The isomorphism $f \mapsto \tilde{f}$, see theorem 6.2.1, clearly carries $\mathcal{S}^k_\rho(\mathcal{V})$ into $\mathcal{P}^k_\rho(\mathcal{V})$. Furthermore, each of the products we have defined, namely the $\star$ product for $\mathcal{S}^\bullet(\mathcal{V})$ and the $\diamond$ product for $\mathcal{P}^\bullet(\mathcal{V})$, preserves ρ-invariance. Therefore, the subspaces

$$\mathcal{S}^\bullet_\rho(\mathcal{V}) := \bigoplus_{k=0}^{\infty} \mathcal{S}^k_\rho(\mathcal{V})$$

and

$$\mathcal{P}^\bullet_\rho(\mathcal{V}) := \bigoplus_{k=0}^{\infty} \mathcal{P}^k_\rho(\mathcal{V})$$

of $\mathcal{S}^\bullet(\mathcal{V})$ and $\mathcal{P}^\bullet(\mathcal{V})$, respectively, are isomorphic as (sub)algebras. $\square$

In particular, we have the following corollary.

Corollary 6.2.3. *Let G be a Lie group. Then the algebra $I^\bullet(G)$ of ad_G-invariant symmetric multilinear mappings of the Lie algebra $\mathfrak{g}$ into $\mathbb{R}$ may be identified with the algebra of ad_G-invariant polynomial functions on $\mathfrak{g}$.*

Proof. Follows from the proof of theorem 6.2.2 by replacing the general representation ρ with the adjoint representation. $\square$

6.3 The Chern-Weil Homomorphism

The main objective of this section is to define a certain homomorphism from the (sub)algebra $I^\bullet(G)$ into the cohomology algebra given by $H^\bullet_{dR}(M;\mathbb{R})$.

To begin, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle. Also, since any principal bundle comes equipped with a connection, fix one and let ω and Ω be its connection form and curvature form, respectively. We would like to define, for any $f \in I^k(G)$ (here $k \geq 1$), a $2k$-form $f(\Omega)$ on P. To do this, set, at every $p \in P$,

$$f(\Omega)\Big|_{T_pP}(v_1,..,v_{2k}) = \frac{1}{(2k)!}\sum_{\sigma\in S_{2k}}(-1)^\sigma f\Big(\Omega\big(v_{\sigma(1)},v_{\sigma(2)}\big),...,\Omega\big(v_{\sigma(2k-1)},v_{\sigma(2k)}\big)\Big), \quad (6.7)$$

for all $v_1,...,v_{2k} \in T_pP$. Notice that for $k=0$, $f:\mathbb{R}\to\mathbb{R}$ and we will take the corresponding $f(\Omega)$ to be the constant 0-form whose value is f.

Now, since the curvature Ω is tensorial of type ad_G, in addition to being horizontal[4], and since f is invariant under the action of ad_G, it follows that the $2k$-form $f(\Omega)$ is both horizontal and invariant under the right G-action. Thus, by lemma 6.1.1, the $2k$-form $f(\Omega)$ projects down to a unique $2k$-form on M, say $\bar{f}(\mathcal{F})$, where $\mathcal{F}=s^*(\Omega)$ (Note that since $f\in I(G)$, $\bar{f}(\mathcal{F})$ is globally defined on M). In other words, there exists a unique $2k$-form, $\bar{f}(\mathcal{F})$, defined on M which pulls back to $f(\Omega)$ by $\pi^*:\Omega^{2k}_{ad}(M)\to\Omega^{2k}_{ad}(P)$. We can further show that this newly defined $2k$-form is closed; thus defining a cohomology class on M. We collect the previous two results in the following theorem.

Theorem 6.3.1. *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle with connection form ω, curvature form Ω and let $f\in I^k(G)$ (for some $k\geq 1$). Then, the $2k$-form $f(\Omega)$, defined on P by (6.7), projects down to a unique closed $2k$- form $\bar{f}(\mathcal{F})$ on M (i.e., we have that $f(\Omega)=\pi^*\left(\bar{f}(\mathcal{F})\right)$ and $d\left(\bar{f}(\mathcal{F})\right)=0$).*

Proof. Since we have already shown the existence and uniqueness of $\bar{f}(\mathcal{F})$, we are left only to show that it is closed. To begin, notice that $d\left(f(\Omega)\right)=d\big(\pi^*\left(\bar{f}(\mathcal{F})\right)\big)=\pi^*\big(d\left(\bar{f}(\mathcal{F})\right)\big)$, so $d(f(\Omega))$ projects down to $d\left(\bar{f}(\mathcal{F})\right)$. Now, since projections are unique when they exist, it suffices that show that $f(\Omega)$ is closed. To proceed, we recall a fact (see [26] pages 294-295): If a q-form φ on P projects to a q-form $\bar{\varphi}$ on M, then $d\varphi = D\varphi$; where D is called the *exterior covariant differentiation* and is defined by

$$\big(D\varphi\big)(X_1,...,X_{q+1}) = \big(d\varphi\big)\left(hX_1,...,hX_{q+1}\right), \quad (6.8)$$

where hX_i is the horizontal component of X_i with respect to the connection ω. Thus, we have that $d(f(\Omega))=D(f(\Omega))$ and so, $d(f(\Omega))=0$ when any of its arguments are vertical. Consequently, we need only investigate the case where all of the arguments are horizontal. To proceed, let $\{T_a\}$ be a basis for $\mathfrak{g}$, then we can write the curvature form Ω as $\Omega=\Omega^aT_a$, where each Ω^a is a two-form (we also are using the Einstein summation convention). Using this we have that

$$f(\Omega)=f(T_{a_1},...,T_{a_k})\Omega^{a_1}\wedge\cdots\wedge\Omega^{a_k},$$

[4]By horizontal we mean that Ω annihilates any vectors living in the tangent space directed along the fibres, i.e. T_uP_x. Equivalently, for a q-form φ, we have that $\varphi(X_1,...,X_q)=0$ if any $X_i\in T_uP_x$.

hence

$$\begin{aligned} d(f(\Omega)) = f(T_{a_1}, ..., T_{a_k})\Big(& d\Omega^{a_1} \wedge \cdots \wedge \Omega^{a_k} + \\ &+ \Omega^{a_1} \wedge d\Omega^{a_2} \wedge \cdots \wedge \Omega^{a_k} + \\ &+ \cdots + \Omega^{a_1} \wedge \cdots \wedge d\Omega^{a_k}\Big). \end{aligned} \tag{6.9}$$

Now, from the Bianchi identity $D\Omega = 0$, we have

$$\begin{aligned} D\Omega &= 0, \\ \Rightarrow D(\omega^a T_a) &= 0, \\ \Rightarrow (D\Omega^a)T_a &= 0, \end{aligned}$$

or that $D\Omega^a = 0$ for all $a = 1, ..., n$. Consequently, each $d\Omega^a$ vanishes when all of its arguments are horizontal and, by (6.9), so does $d(f(\Omega))$. Hence, we have that the $2k$-form $f(\Omega)$ is closed, implying that the $2k$-form $\bar{f}(\mathcal{F})$ is also closed.
Note that since $df(\Omega) = 0$ it defines an element in $H^{2k}_{dR}(P;\mathbb{R})$, while since $d\bar{f}(\mathcal{F}) = 0$ it defines an element in $H^{2k}_{dR}(M;\mathbb{R})$. □

Recapping, we have just shown that for each $f \in I^k(G)$, the $2k$-form $f(\Omega)$ on P projects down to a unique closed $2k$-form $\bar{f}(\mathcal{F})$ on M, and thus defines an element of the $2k^{th}$ de Rham cohomology group $H^{2k}_{dR}(M;\mathbb{R})$. Furthermore, we also have the following property, which will prove useful in the remainder.

Proposition 6.3.2. *Consider a bundle map $(\psi, \hat{\psi})$,*

$$\begin{array}{ccc} E' & \xrightarrow{\psi} & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{\hat{\psi}} & M, \end{array}$$

then for ω a connection in E and $\omega' = \psi^*(\omega)$ the induced connection in E' we have that

$$f(\Omega') = \psi^*\big(f(\Omega)\big),$$

where $f \in I^\bullet(G)$ and Ω' is the curvature form coming from ω' and Ω from ω. Furthermore, by the results in the theorem 6.3.1 both $f(\Omega')$ and $f(\Omega)$ project down to unique $\bar{f}(\mathcal{F}') \in M'$ and $\bar{f}(\mathcal{F}) \in M$ such that

$$\bar{f}(\mathcal{F}') = \hat{\psi}^*\big(\bar{f}(\mathcal{F})\big).$$

Proof. See [26]. □

It also turns out that $f(\Omega)$ does not depend on the choice of connection, cohomologically speaking. To be precise, suppose we have a fixed principal G-bundle, $G \hookrightarrow P \xrightarrow{\pi} M$ and that ω_0 and ω_1 are two connections on the bundle with corresponding curvature forms Ω_0 and Ω_1, respectively. In addition, fix a $f \in I^k(G)$. Then the projections $\bar{f}(\mathcal{F}_0)$ and $\bar{f}(\mathcal{F}_1)$ down to M, of $f(\Omega_0)$ and $f(\Omega_1)$, differ at most by an exact form, and thus, in terms of cohomology, they are equivalent.

Theorem 6.3.3. *Let ω_0 and ω_1 be connection forms on a principal G-bundle $G \hookrightarrow P \xrightarrow{\pi} M$ with curvature forms Ω_0 and Ω_1, respectively, and let $f \in I^k(G)$ for some $k \geq 1$. Then, there exists a unique $(2k-1)$-form φ on M such that*

$$\bar{f}(\mathcal{F}_1) - \bar{f}(\mathcal{F}_0) = d\varphi.$$

Proof. See page 297 of [26]. □

Remark 6.3.4. From theorem 6.3.3, we see that $\bar{f}(\mathcal{F}_1)$ and $\bar{f}(\mathcal{F}_0)$ are cohomologous and, hence, represent the same cohomology class in $H^{2k}_{dR}(M;\mathbb{R})$. In other words, the cohomology class defined by $\bar{f}(\mathcal{F})$ does not depend on the particular choice of connection. This, in turn, implies that this cohomology class is, in fact, a characteristic class of the principal bundle.

Definition 6.3.5. Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle with connection form ω and corresponding curvature form Ω. Denote by $w(P;f)$ the class $H^{2k}_{dR}(M;\mathbb{R})$ defined by the closed $2k$-form $\bar{f}(\mathcal{F})$. That is

$$w(P;f) \equiv \left[\bar{f}(\mathcal{F})\right] \in H^{2k}_{dR}(M;\mathbb{R}). \tag{6.10}$$

We will denote $w(P;f)$ simply by $w(f)$ if the bundle P is clear from the context. For $f \in I^k(G)$, $w(f) \in H^{2k}_{dR}(M;\mathbb{R})$ is called the **characteristic class** for P corresponding to f.

Remark 6.3.6.

(1) Some authors use $c(P)$ to denote $w(P;f)$.

(2) From proposition 6.3.2 we see that given a bundle map $(\psi,\hat{\psi}) : (P,\pi,M) \to (P',\pi',M')$ between the two principal G-bundles, the characteristic classes obey

$$c(P) = \hat{\psi}^*\big(c(P')\big).$$

This property proves extremely useful, allowing for one to define a characteristic class on some bundle, say P', from a characteristic class defined on some bundle P along with a bundle map from P' to P. In particular, as we will see, we could define a characteristic class on the universal bundle EG and then pull it back to give a characteristic class on P.

(3) If $c(P) \in H^m_{dR}(M;\mathbb{R})$, then $c(P)$ is said to have degree m.

(4) The set of characteristic classes (with $\mathbb{R}$ coefficients) is a ring denoted by $H^\bullet_G$ or $H^\bullet_G(\mathbb{R})$.

Let us take a moment to remind the reader of our present situation: We are given a principal G-bundle $G \hookrightarrow P \xrightarrow{\pi} M$. Now, since we can always fix a connection on P we do so and denote the associated connection form by ω and the associated curvature form by Ω. Then, for any $f \in I^k(G)$ we define a closed $2k$-form $\bar{f}(\mathcal{F})$ on the base space M; and hence, an element of $H^{2k}_{dR}(M;\mathbb{R})$. Furthermore, if we fix another connection, with its own connection form ω' and curvature form Ω', then we will define a different closed $2k$-form on M, denoted by $\bar{(}\mathcal{F}')$. However, we have just seen that $\bar{f}(\mathcal{F})$ and $\bar{f}(\mathcal{F}')$ are cohomologous, implying that $\bar{f}(\mathcal{F})$ and $\bar{f}(\mathcal{F}')$ determine the same de Rham cohomology class, which we denote by $w(f)$. Thus, for any $f \in I^k(G)$ we define an element $w(f) \in H^{2k}_{dR}(M;\mathbb{R})$ for all $k \geq 1$. And so, we have, in fact, defined a mapping

$$w : I^k(G) \longrightarrow H^{2k}_{dR}(M;\mathbb{R}), \tag{6.11}$$

which assigns to each $f \in I^k(G)$ the cohomology class $\big[\bar{f}(\mathcal{F})\big] \in H^{2k}_{dR}(M;\mathbb{R})$, where $\mathcal{F}$ is the field strength of the curvature form Ω associated to ANY connection on P. This map can be further extended to an algebra homomorphism, known as the **Chern-Weil homomorphism**, by noting that

$$w\big(f \star g\big) = w(f) \wedge w(g),$$

for $f, g \in I^\bullet(G)$. Thus, the Chern-Weil homomorphism is a way of relating the curvature of M to its de Rham cohomology classes, i.e. relating geometry with algebraic topology.

We end this section with a theorem (whose proof we omit), due to Cartan, which states that when G is compact the Chern-Weil homomorphism extends to an isomorphism.

Theorem 6.3.7. *Let G be a compact Lie group. Then*

$$w : I^k(G) \longrightarrow H^{2k}_G(\mathbb{R})$$

is an isomorphism.

Proof. See [14]. ☐

Example: Chern Classes

As an example of the previous constructions, we will discuss Chern classes. First, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle with connection form ω with curvature form Ω and let $f_k \in I^k(G)$ be the ad-invariant polynomial defined by

$$\begin{aligned} \det\left(\lambda I + \frac{i}{2\pi}A\right) &= \sum_k f_k(A)\lambda^k, \\ &= f_0(A)\lambda^k + f_1(A)\lambda^{k-1} + \cdots + f_{k-1}(A)\lambda + f_k(A), \end{aligned} \tag{6.12}$$

where $A \in \mathfrak{g}$. Then, the corresponding characteristic class

$$c_k(P) \equiv w(P; f_k) = [\bar{f}_k(\mathcal{F})],$$

is called the k^{th} **Chern Class** of P, while

$$c(P) = \sum_k c_k(P),$$

is known as the **total Chern class** of P. For example, when G is a matrix group, we can write the first few Chern classes as [36] (in what follows, by Tr we mean the trace of a matrix):

- $c_1(P) = \frac{i}{2\pi} Tr(\mathcal{F})$,
- $c_2(P) = \frac{-1}{8\pi^2}\big[Tr(\mathcal{F}) \wedge Tr(\mathcal{F}) - Tr(\mathcal{F} \wedge \mathcal{F})\big]$,
- $c_3(P) = \frac{-i}{48\pi^3}\big[Tr(\mathcal{F}) \wedge Tr(\mathcal{F}) \wedge Tr(\mathcal{F}) - 3Tr(\mathcal{F} \wedge \mathcal{F}) + 2Tr(\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F})\big]$.

Furthermore, when $G = SU(2)$ we get (since $SU(2)$ is traceless):

- $c_1(P) = 0$,
- $c_2(P) = \frac{1}{8\pi^2} Tr(\mathcal{F} \wedge \mathcal{F})$,
- $c_3(P) = \frac{i}{48\pi^3}\big[3Tr(\mathcal{F} \wedge \mathcal{F}) - 2Tr(\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F})\big]$.

In addition to the previous results, note that when $G = U(1)$ the field strength $\mathcal{F}$ is a 1×1 matrix, and so, projecting the first Chern class down to M gives

$$c_1(P) = \frac{i}{2\pi}\mathcal{F},$$

which is in agreement with the formula we defined in section 6.1 for the 1^{st} Chern class when $\mathcal{F} = -iF$. As a final remark, the reader should remember the expression for the 2^{nd} Chern class, since it will show up when we define the Chern-Simons form.

6.4 Characteristic Numbers

Perhaps the reader has notice that since characteristic classes are cohomology classes defined on the base manifold M, they can then be integrated over submanifolds of M. These integrals are usually termed *characteristic numbers*. The practicality of characteristic numbers comes from the following observation.

Theorem 6.4.1. *Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal bundle, $f \in I^k(G)$ (with $1 \leq k \leq 1/2\dim(M)$), N a compact, oriented submanifold of M, where* $\dim(N) = 2k$ *and $\iota : N \hookrightarrow M$ the inclusion map. Then, the integral of the $2k$-cocycle $\bar{f}(\mathcal{F})$ over N,*

$$\int_N \iota^*\big(\bar{f}(\mathcal{F})\big),$$

does not depend on the choice of connection ω. Additionally, if N_1 and N_2 are two compact, oriented submanifolds of M such that N_1 can be smoothly deformed into N_2, then

$$\int_{N_1} \iota^*\big(\bar{f}(\mathcal{F})\big) - \int_{N_2} \iota^*\big(\bar{f}(\mathcal{F})\big) = 0.$$

Proof. The proof is irrelevant for what follows, however one can consult [33] to get some hints. □

Remark 6.4.2. When f correspond to the Chern classes, the integral of $\bar{f}(\mathcal{F})$ over N is an integer. Hence we say that the Chern classes are *integral cohomology classes.*

When one takes $N = M$ (here we assume $\dim(M) = 2n$) and $f \in I^n(G)$, then the integral of $\bar{f}(\mathcal{F}) \in C^{2n}(M;\mathbb{R})$ over M gives a number, called the **characteristic number**. For example, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle over M and let M be a compact, oriented manifold of $\dim(M) = 2n$. Then the n^{th} Chern class $c_n(P)$ is an element of $H^{2n}_{dR}(M;\mathbb{R})$ and so can be integrated over M,

$$\int_M c_n(P).$$

Since the Chern class is integral, integrating $c_n(P)$ gives an element in $\mathbb{Z}$ which we call the *Chern number*. It should be noted that characteristic numbers (in particular Chern numbers) for equivalent principal bundles are equal.

The idea of characteristic numbers will return when we discuss the Chern-Simons action for a non-trivial G-bundle. The reader wishing for a more in depth review of characteristic numbers can consult section 6.4 of [36].

6.5 Universal Characteristic Classes

Before being caught up in the details, the main result of this chapter can be simply stated as follows. Using the bundle map from a principal bundle P to its universal bundle EG, one can pullback certain elements of $H^m_{dR}(BG;\mathbb{R})$ to $H^m_{dR}(M;\mathbb{R})$ to give characteristic classes on M as before.

To begin, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle and let $G \hookrightarrow EG \xrightarrow{\pi'} BG$ be the universal bundle, with classifying space BG (the reader wishing to review the concept of a universal bundle should consult chapter 4). Hence, there exists a bundle map $\varphi : P \to EG$ (with the property that any homotopic deviation corresponds to the same bundle P) and an induced (classifying) map $\hat{\varphi} : M \to BG$. Now, let $f \in I^k(G)$ be an invariant polynomial and let Ω_u be the *universal curvature* associated to the *universal connection* ω_u. That is, ω_u is a connection on EG with the property that any connection ω on P can be obtained as $\omega = \varphi^*(\omega_u)$, while Ω_u is the curvature of this connection,

$$\Omega_u = d\omega_u + \frac{1}{2}[\omega_u \wedge \omega_u].$$

As was previously explained, we can define the $2k$-cocycle $f(\Omega_u) \in C^{2k}(EG;\mathbb{R})$ and the de Rham cohomology class, $[f(\Omega_u)] \in H^{2k}_{dR}(EG;\mathbb{R})$, it represents. Then, by lemma 6.1.1, we get a $2k$-cocyle on BG, $\bar{f}(\mathcal{F}_u) \in C^{2k}(BG;\mathbb{R})$, and its corresponding class $[\bar{f}(\mathcal{F}_u)] \in H^{2k}_{dR}(BG;\mathbb{R})$. Furthermore, from theorem 6.3.3, given any other curvature Ω'_u on EG (corresponding to ω'_u), the $2k$-cocycles $\bar{f}(\mathcal{F}_u)$ and $\bar{f}(\mathcal{F}'_u)$ represent the same class in $H^{2k}_{dR}(BG;\mathbb{R})$. Hence, $[\bar{f}(\mathcal{F}_u)] \in H^{2k}_{dR}(BG;\mathbb{R})$ gives a characteristic class on BG, known as a **universal characteristic class**, which we denote by $w(EG;f)$. Now, from property (2) of remark 6.3.6, we can pull this class back to M, via the classifying map $\hat{\varphi}$, giving the characteristic class on M defined by

$$w(P;f) := \hat{\varphi}^*\big(w(EG;f)\big) \in H^{2k}_{dR}(M;\mathbb{R}). \tag{6.13}$$

To be exact, first construct a $2k$-cocylce on M by pulling $\bar{f}(\mathcal{F}_u) \in C^{2k}(BG;\mathbb{R})$ back via $\hat{\varphi}$,

$$\bar{f}(\mathcal{F}) := \hat{\varphi}^*(\bar{f}(\mathcal{F}_u).$$

Then, denote the (characteristic) cohomology class which this cocycle represents in $H^{2k}_{dR}(M;\mathbb{R})$ by $w(P;f)$, or $[\bar{f}(\mathcal{F})]$.

This allows for one to simply work with universal characteristic classes on EG, rather than working with a class on each type of principal G-bundle; a trait which lets one quickly generalize properties of $w(P;f)$ to all G-bundles, that is, if the property they are proving survives a pullback map.

Part II

TOPOLOGICAL FIELD THEORIES

Chapter 7

Classical Chern-Simons Theory

One of the more remarkable results of the last 25 years is due to Edward Witten, who showed that a variation on the quantization of Chern-Simons theory gives the Jones polynomial of links [42]. This is very surprising, since it gives a relationship between seemingly different areas - field theory, geometry, and low-dimensional topology. This area of research has grown considerably over the years. In particular, Reshetikhin and Turaev [39] constructed, combinatorically, a topological quantum field theory (TQFT) which, in particular, gives the Jones polynomial. It is conjectured that this is the TQFT coming from the quantum Chern-Simons theory, evading proof due to the fact that certain quantities of the Chern- Simons theory are not rigorously defined. This would be a very interesting result since the two approaches are very different and belong to different areas. In this way, Chern-Simons theory is a meeting point for many branches of mathematics and physics.

7.1 Topological Field Theories

Typically in the eyes of physicists, a topological field theory is thought of as a theory whose Lagrangian is independent of the spacetime (metric) on which the theory is defined; although there do exist topological theories which have Lagrangians depending on the metric, the expectation values and correlation functions of these theories do not, thus making them topological in nature. As an example of a field theory constructed from a metric-free Lagrangian, we will consider the topological theory known as Chern-Simons theory.

The Chern-Simons theory is a geometrical construction on principal G-bundles. It arose as a gauge theory in physics - a field theory which is invariant under a certain group of symmetries (the gauge group). Such a theory starts out with a *spacetime*, a manifold on which the physical system lives. In our discussion the spacetime is the base manifold M. A *field*, in Chern-Simons theory, consists of a connection on some principal G-bundle over the spacetime manifold.

The next ingredient is the *Lagrangian*, a functional on the space of fields. This will be the Chern-Simons form defined in the next section; it will depend on the extra datum of a connection. In cases where the bundle is trivial, we may take a pullback of the Lagrangian to the spacetime along a field (i.e., a section of the bundle). The integral over

M of this pulled-back Lagrangian will give a number called the *action*. Thus, the action gives a number for every field. The critical points of the action are the classically allowed fields. In the case of an electric field, for example, one obtains Maxwell's equations in this way. For the case in which the bundle is non-trivial we can still define the Chern-Simons action but, as we will see, it is no longer the integral over M of the Chern-Simons form.

In the next section we describe the Lagrangian in the Chern-Simons theory, followed by defining the action function in the subsequent sections. We will rely heavily on the material already developed. In particular, knowledge of principal bundles, characteristic classes and (co)homological algebra is a prerequisite for understanding this chapter. The theory is investigated further in [23].

7.2 Chern-Simons Form

The goal of this section is to define the *Chern-Simons form* or, as usually called in the physics literature, the *Chern-Simons Lagrangian.* From this Lagrangian one usually defines the action functional, which can then by used to give a quantum version of the theory. So, without further ado, let us begin[1].

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G bundle with connection form ω and associated curvature form Ω. Furthermore, let $\langle \Omega \wedge \Omega \rangle$ be the Chern-Weil 4-form, defined on P, associated to the ad_G-invariant, symmetric, bilinear form $\langle \cdot \wedge \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. Note that, since Ω transforms in the adjoint representation of G under a gauge transformation and since $\langle \cdot \wedge \cdot \rangle \in I^2(G)$, we deduce that $\langle \Omega \wedge \Omega \rangle$ is gauge invariant. Roughly speaking, the Chern-Simons form is given by the antiderivative of the Chern-Weil 4-form $\langle \Omega \wedge \Omega \rangle$. To be more precise:

Definition 7.2.1. For a principal G-bundle $G \hookrightarrow P \xrightarrow{\pi} M$ with connection form ω and curvature form Ω, the **Chern-Simons form** $\alpha(\omega) \in \Omega^3(P; \mathbb{R})$ on P is defined by

$$\alpha(\omega) := \langle \omega \wedge \Omega \rangle - \frac{1}{6} \langle \omega \wedge [\omega \wedge \omega] \rangle. \tag{7.1}$$

Remark 7.2.2. The Chern-Simons form satisfies the following properties:

Proposition 7.2.3.

(1) *Let $\iota_p : P_x \hookrightarrow P$ be the natural inclusion map, then, for all $x \in M$ and $p \in P_x$,*

$$\iota_p^*(\alpha(\omega)) = -\frac{1}{6} \langle \theta_x \wedge [\theta_x \wedge \theta_x] \rangle,$$

where by θ_x we mean the pullback of the Maurer-Cartan form θ from G to P_x (see section 5.2).

(2) *The Chern-Simons form is right G-invariant,*

$$R_g^*(\alpha(\omega)) = \alpha(\omega).$$

[1] The reader wishing to review the basics of the Chern-Weil theory (or at least what is assumed here) should consult chapter 6.

(3) *The Chern-Simons form is the anti-derivative of the Chern-Weil* 4*-from* $\langle \Omega \wedge \Omega \rangle$,

$$d\alpha = \langle \Omega \wedge \Omega \rangle.$$

(4) *For a bundle map* $\varphi : P \to P'$, $\varphi^*(\alpha(\omega)) = \alpha(\varphi^*(\omega))$.

(5) *As a corollary to the previous result, let* $\varphi : P \to P$ *be a gauge transformation with associated map* $\hat{g}_\varphi : P \to G$, *defined by* $\varphi(p) = p \cdot \hat{g}_\varphi(p)$, *and let* $\phi = \hat{g}_\varphi^*(\theta)$. *Then*

$$\varphi^*(\alpha) = \alpha + d\left(\langle ad_{g^{-1}}(\omega) \wedge \phi \rangle\right) - \frac{1}{6}\langle \phi \wedge [\phi \wedge \phi] \rangle.$$

Proof.

(1) From theorem 5.2.7 and part (1) of proposition 5.3.1, we see that

$$\begin{aligned}
\iota_p^*(\alpha) &= \iota_p^*\left(\langle \omega \wedge \Omega \rangle - \frac{1}{6}\langle \omega \wedge [\omega \wedge \omega] \rangle\right), \\
&= \langle \iota_p^*(\omega) \wedge \iota_p^*(\Omega) \rangle - \frac{1}{6}\langle \iota_p^*(\omega) \wedge [\iota_p^*(\omega) \wedge \iota_p^*(\omega)] \rangle, \\
&= \langle \theta_x \wedge 0 \rangle - \frac{1}{6}\langle \theta_x \wedge [\theta_x \wedge \theta_x] \rangle, \\
&= -\frac{1}{6}\langle \theta_x \wedge [\theta_x \wedge \theta_x] \rangle.
\end{aligned}$$

(2) Follows directly from ω and Ω being of type ad_G under R_g along with $\langle \cdot \wedge \cdot \rangle \in I^{2k}(G)$ as well as $R_g^*([\cdot \wedge \cdot]) = [R_g^*(\cdot) \wedge R_g^*(\cdot)]$.

(3) See [23] page 11.

(4-5) Straight forward calculations, albeit extremely tedious for (5).

□

After having constructed a Lagrangian, the next step in any field theory is to define the action.

7.3 Chern-Simons Action (Trivial Bundle)

We will begin the construction of the action functional with the case in which $G \hookrightarrow P \xrightarrow{\pi} M$ is a trivial principal G-bundle over a closed (i.e. M is compact without boundary) three-dimensional oriented manifold M, along with assuming that G be compact, connected and simply connected (we discuss the case where G is finite - the so-called Dijkgraaf-Witten theories - in the next chapter). The reason for imposing the restriction on G to be connected and simply connected will become clear later (although the reader who is too impatient to wait can see theorem 7.3.4). Thus, let M be a closed, oriented 3-manifold, G a compact, connected, simply connected Lie group and ω a connection form on P with associated curvature form Ω. From the previous section we know that we can associate to the connection form ω the Chern-Simons form $\alpha(\omega) \in \Omega^3(P)$. Now, since P is trivial, we can use a global section $s : M \to P$ to pull the Chern-Simons form (or Lagrangian) down to M. Furthermore, since M is orientable, we can integrate $s^*\big(\alpha(\omega)\big)$ over M; this is how the *Chern-Simons action* is constructed.

Definition 7.3.1. For the case in which M is a closed oriented manifold, with $\dim(M) = 3$, and G is a compact, connected and simply connected Lie group, the 3-dimensional **Chern-Simons action** is defined to be

$$\begin{aligned} S_{M,P}(s,\omega) &:= \frac{k}{8\pi^2}\int_M s^*\left(\alpha(\omega)\right), \qquad (7.2)\\ &= \frac{k}{8\pi^2}\int_M \left\{\langle s^*(\omega)\wedge s^*(\Omega)\rangle - \frac{1}{6}\langle s^*(\omega)\wedge [s^*(\omega)\wedge s^*(\omega)]\rangle\right\},\\ &\equiv \frac{k}{8\pi^2}\int_M \left\{\langle \mathcal{A}\wedge\mathcal{F}\rangle - \frac{1}{6}\langle\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle\right\}, \end{aligned}$$

where $\mathcal{A} \equiv s^*(\omega)$ is the local gauge potential associated to ω, $\mathcal{F} \equiv s^*(\Omega)$ is the local field strength associated to Ω and $k \in \mathbb{Z}$ is called the *level* of the theory.

N.B. 7.3.2. We say that $\mathcal{A}$ and $\mathcal{F}$ are in gauge s, while $\bar{\mathcal{A}} \equiv \bar{s}^*(\omega)$ and $\bar{\mathcal{F}} \equiv \bar{s}^*(\Omega)$ would be in gauge $\bar{s}$, etc. Hence, as in any gauge theory, the local field strength and gauge potential depend on which gauge is chosen (or fixed).

Remark 7.3.3. One should keep in mind that even though we write the Chern-Simons action as depending on the connection form (which it does), it is defined on the base manifold M and NOT on P. In fact, some authors like to write the Chern-Simons action as $S_{M,P}(s,\mathcal{A})$ to help the readers not mix up this subtle detail. Also, this action functional can be used to define a topological quantum field theory by setting the partition function $Z(M)$ to be

$$Z(M) = \int e^{2\pi i S_{M,P}(s,\omega)}\mathcal{D}\mathcal{A}. \qquad (7.3)$$

Note that the parameter k must be integer so that the integrand in the path-integral is single-valued. Additionally, in the above expression for the partition function, the integral is taken over the infinite-dimensional space of connections on the total space P modulo gauge transformations, $\mathcal{A}(P)/\mathcal{G}(\mathcal{P})$, and one of the harder problems in mathematical physics is giving a (well-)defined meaning to $\mathcal{D}\mathcal{A}$. We will come back to this when we treat the quantization of the Chern-Simons theory. Note, we integrate over the quotient (moduli) space of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$ so as to not integrate over all field configurations, but only those physically distinct configurations not related by gauge symmetries.

The astute reader might object to the previous definition since it requires for the principal bundle P to be trivial. Although generally their objection is valid, in the case when G is a connected, simply connected, compact Lie group we have the following theorem.

Theorem 7.3.4. *Let G be a compact, connected, simply connected Lie group and let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle over M. Furthermore, let M be of dimension ≤ 3. Then, there exists a global section $s : M \to P$, hence P is a product bundle $P = M \times G$.*

Proof. The proof to the above statement is found in almost any book on obstruction theory, or see [30]. □

So, as long as we work in the realm where G is simply connected and where $\dim(M) \leq 3$, we can use (7.2) as the general definition of the Chern-Simons action. However, once we move away from either G being simply connected or $\dim(M) \leq 3$ more work is required to define the action functional. We will take up that task in the next section.

Example 7.3.5 (**Chern-Simons action for** $SU(2) \hookrightarrow P \xrightarrow{\pi} M$ **bundle**). Let us work out the details of the Chern-Simons action on a principal $SU(2)$-bundle over a closed 3-manifold M. To proceed, from theorem 7.3.4 we see that since M is of $\dim(M) \leq 3$ and since $SU(2)$ is a compact, connected, simply connected Lie group, the bundle is trivial. Hence, it admits a global section which we can use to pull the Chern-Simons 3-form down to M and integrate. Furthermore, in the case where $G = SU(2)$, we take the bilinear form $\langle \cdot \wedge \cdot \rangle : \mathfrak{su}(2) \otimes \mathfrak{su}(2) \to \mathbb{R}$ to be

$$\langle a \wedge b \rangle := Tr(ab), \tag{7.4}$$

where $a, b \in \mathfrak{su}(2)$ and Tr is the trace. With this, the Chern-Simons action functional becomes the usual

$$S_{M,P}(s,\omega) = \frac{k}{8\pi^2} \int_M Tr\left(\mathcal{A} \wedge_{\mathbb{R}} d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge_{\mathbb{R}} \mathcal{A} \wedge_{\mathbb{R}} \mathcal{A}\right), \tag{7.5}$$

where $\mathcal{A} \in \Omega^1_{ad}(M; \mathfrak{su}(2))$ is the local gauge potential on M in gauge s (that is, $\mathcal{A} = s^*(\omega)$). This result follows immediately from the succeeding theorem in addition to choosing a bounding manifold of M along with a trivial bundle extension over the bounding manifold.

Theorem 7.3.6. *Let $SU(2) \hookrightarrow P \xrightarrow{\pi} M$ be a principal $SU(2)$-bundle over a closed, orientable manifold M with connection form ω, curvature form Ω, and let $s : M \to P$ be a (global) section such that, in this fixed gauge s, $\mathcal{A} \equiv s^*(\omega)$ is the local gauge potential along with $\mathcal{F} \equiv s^*(\Omega)$ the local field strength. Then,*

$$Tr(\mathcal{F} \wedge_{\mathbb{R}} \mathcal{F}) = d\left(Tr\Big(\mathcal{A} \wedge_{\mathbb{R}} d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge_{\mathbb{R}} \mathcal{A} \wedge_{\mathbb{R}} \mathcal{A}\Big)\right). \tag{7.6}$$

Proof. See [36] section 6.4. □

This concludes our example, for a deeper look at explicit examples of the Chern-Simons action one can consult [36] and [37], to name a few.

The next issue to discuss is the dependence of the Chern-Simons action on the particular section which is used to pull the Lagrangian down to M. Since any two sections $s_1, s_2 : M \to P$ of a principal G-bundle are related by a gauge transformation (see section 4.5), we would like to see how such a transformation affects the Chern-Simons action (7.2). So, to begin, let $\varphi : P \to P$ be a gauge transformation with associated map $\hat{g}_\varphi : P \to G$, defined by $p \cdot \hat{g}_\varphi(p) = \varphi(p)$, and let $\phi = \hat{g}^*_\varphi(\theta)$ be the pullback of the Maurer-Cartan form θ on G to P. Furthermore, let $\phi_g = s^*(\phi)$ be the pullback of the Maurer-Cartan form to M, then we have (omitting the factor of $k/8\pi^2$),

$$\begin{aligned}
S_{M,P}(\varphi \circ s, \omega) &= \int_M (\varphi \circ s)^*(\alpha(\omega)), \\
&= \int_M \alpha(\omega) \circ (\varphi \circ s)_*, \\
&= \int_M \alpha(\omega) \circ \varphi_* \circ s_*, \\
&= \int_M s^*\big(\varphi^*(\alpha(\omega))\big), \\
&= \int_M s^*\big(\alpha + d\langle ad_{g^{-1}}(\omega) \wedge \phi\rangle - \frac{1}{6}\langle \phi \wedge [\phi \wedge \phi]\rangle\big), \\
&= \int_M s^*(\alpha) + \int_M d\big(s^*(\langle ad_{g^{-1}}(\omega) \wedge \phi\rangle)\big) - \frac{1}{6}\int_M s^*(\langle \phi \wedge [\phi \wedge \phi]\rangle).
\end{aligned}$$

Notice how we can rewrite the middle term, using Stokes' theorem, as a surface integral over ∂M. However, it has been assumed that M is closed, thus $\partial M = \emptyset$. Consequently, this middle term vanishes and we are left with

$$\begin{aligned}
S_{M,P}(\varphi \circ s, \omega) &= S_M(s, \omega) - \frac{1}{6}\int_M s^*(\langle \phi \wedge [\phi \wedge \phi]\rangle), \\
&= S_{M,P}(s, \omega) - \frac{1}{6}\int_M \langle \phi_g \wedge [\phi_g \wedge \phi_g]\rangle.
\end{aligned}$$

We can further simplify the expression due to the following fact, which we call the *integrality condition.*

Fact 7.3.7. It is known that one can choose the form $\langle \cdot \wedge \cdot \rangle$ in such a way that forces the integral $-\frac{1}{6}\int_M \langle \phi_g \wedge [\phi_g \wedge \phi_g]\rangle$ to always take its values in $\mathbb{Z}$. To be a bit more precise, if we take for $\langle \cdot \wedge \cdot \rangle$ a multiple of the Killing form on $\mathfrak{g}$, then $-\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta]\rangle$ represents an integral class in $H^3(G; \mathbb{R})$ and hence

$$-\frac{1}{6}\int_M \langle \phi_g \wedge [\phi_g \wedge \phi_g]\rangle \in \mathbb{Z}.$$

This remarkable fact implies that the 3-dimensional Chern-Simons action on a closed 3-dimensional manifold M is independent of the chosen section $s : M \to P$ (up to an integer); i.e.,

$$S_{M,P}(s, \omega) = S_{M,P}(\omega) \qquad (\text{mod } 1) \tag{7.7}$$

is independent of s. And so, we can view the Chern-Simons action as a (well-defined) mapping from the space of connections on P to $\mathbb{R}/\mathbb{Z}$,

$$S_{M,P} : \mathcal{A}(P) \longrightarrow \mathbb{R}/\mathbb{Z}. \tag{7.8}$$

Remark 7.3.8. We will often write the action functional as

$$e^{2\pi i S_{M,P}(\omega)}. \tag{7.9}$$

Thus, we have that (since addition now becomes multiplication)

$$e^{2\pi i S_{M,P}} : \mathcal{A}(P) \longrightarrow \mathbb{T},$$

where $\mathbb{T}$ is the circle group.

Although our construction of the Chern-Simons theory has, so far, only been treated classical (hence not yet a TQFT), it does satisfy several similar properties (see 8.1).

Proposition 7.3.9. *Let M be a closed oriented 3-manifold, P a principal G-bundle over M, G a compact, connected and simply connected Lie group and let ω be a connection form on P. Then, the action $e^{2\pi i S_{M,P}(\omega)}$ satisfies the following properties:*

(a) ***Functoriality:*** *Let P and P' be two trivial principal bundles over M and M', respectively. Furthermore, let $\varphi : P' \to P$ be a bundle map with $\hat{\varphi} : M' \to M$ the corresponding map on the base spaces which is also orientation preserving. Now, if ω is a connection form on P we can then use φ to pull it back to a connection on P', thus defining a Chern-Simons form on P'. Additionally, since P' is trivial we can use a global section $s' : M' \to P'$ to pull $\alpha\big(\varphi^*(\omega)\big)$ down to M', which we can then use to define a Chern-Simons action on M', $e^{2\pi i S_{M',P'}(\varphi^*(\omega))}$. The claim is that this induced Chern-Simons action on M' is equivalent to the original Chern-Simons action on M, or*

$$e^{2\pi i S_{M,P}(\omega)} = e^{2\pi i S_{M',P'}(\varphi^*(\omega))}. \tag{7.10}$$

Remark 7.3.10. It follows from functoriality of the Chern-Simons action that it induces the action

$$S_{M,P} : \mathcal{A}(P)/\mathcal{G}(P) \longrightarrow \mathbb{R}/\mathbb{Z}, \tag{7.11}$$

hence $e^{2\pi i S_{M,P}} : \mathcal{A}(P)/\mathcal{G}(P) \to \mathbb{T}$.

(b) ***Orientation:*** *Denote by $\bar{M}$ the manifold M with opposite orientation, then the action over $\bar{M}$ is the inverse of the action over M, or*

$$e^{2\pi i S_{M,P}(\omega)} = e^{-2\pi i S_{\bar{M},P}(\omega)}. \tag{7.12}$$

(c) ***Multiplicativity:*** *Over a disjoint union, the action is multiplicative; i.e., if $M = \sqcup_{i=1}^{n} M_i$ and ω_i are connections over M_i, then*

$$e^{2\pi i S_{M,P}(\sqcup_{i=1}^{n}\omega_i)} = e^{2\pi i S_{M_1,P}(\omega_1)} e^{2\pi i S_{M_2,P}(\omega_2)} \cdots e^{2\pi i S_{M_n,P}(\omega_n)}. \tag{7.13}$$

Proof. Since the orientation and multiplicativity axioms follow from the usual properties of integration of differential forms over oriented manifolds, we only need to address the functoriality property. However, the functoriality property follows from the functoriality of characteristic classes (see proposition 6.3.2). $\square$

Let's now generalize things a bit and allow M to have a boundary. In this case, we define the Chern-Simons action functional in exactly the same way as before. That is, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a trivial principal G-bundle over a compact, but not closed, manifold

M with compact, connected and simply connected Lie group G and let ω be a connection form on P with curvature form Ω. Then, the Chern-Simons action is defined by

$$\begin{aligned} S_{M,P}(s,\omega) &:= \frac{k}{8\pi^2}\int_M s^*(\alpha(\omega)), \\ &\equiv \frac{k}{8\pi^2}\int_M \left\{ \langle \mathcal{A}\wedge\mathcal{F}\rangle - \frac{1}{6}\langle \mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle \right\}, \end{aligned}$$

where $s : M \to P$ is a globally defined section. Even though we are able to retain the previous definition for the Chern-Simons action, we are not so lucky when it comes to showing that the action is independent of the particular section s. Indeed, we must replace equation (7.7) with the following.

Remark 7.3.11. Once again, in what that follows, we will omit the factor $k/8\pi^2$, unless otherwise stated.

Proposition 7.3.12. *Let $\varphi : P \to P$ be a gauge transformation with associated map $\hat{g}_\varphi : P \to G$, defined by $p \cdot \hat{g}_\varphi(p) = \varphi(p)$, and let $\phi = \hat{g}_\varphi^*(\theta)$. Furthermore, let $\phi_g = s^*(\phi)$, then*

$$S_{M,P}(\varphi\circ s,\omega) = S_{M,P}(s,\omega) + \int_{\partial M}\langle ad_{g^{-1}}(s^*(\omega))\wedge\phi_g\rangle - \frac{1}{6}\int_M \langle \phi_g\wedge[\phi_g\wedge\phi_g]\rangle, \tag{7.14}$$

or, viewing the action as an element in $\mathbb{T}$, [2]

$$e^{2\pi i S_{M,P}(\varphi\circ s,\omega)} = e^{2\pi i S_{M,P}(s,\omega)} e^{2\pi i\left(\int_{\partial M}\langle ad_{g^{-1}}(s^*(\omega))\wedge\phi_g\rangle - \frac{1}{6}\int_M\langle\phi_g\wedge[\phi_g\wedge\phi_g]\rangle\right)}. \tag{7.15}$$

Remark 7.3.13. Hence, under a gauge transformation, the action $e^{2\pi i S_{M,P}(s,\omega)}$ is multiplied by a phase. Thus, as was alluded to earlier, the (classical) 3-dimensional Chern-Simons action is <u>not</u> gauge invariant - only if we restrict the defining manifold to be closed do we insure a gauge invariant action. Indeed, the phase

$$e^{2\pi i\left(\int_{\partial M}\langle ad_{g^{-1}}(s^*(\omega))\wedge\phi_g\rangle - \frac{1}{6}\int_M\langle\phi_g\wedge[\phi_g\wedge\phi_g]\rangle\right)}$$

"measures" the lack of gauge invariance of our theory and so, only when the phase equals unity does our theory become gauge invariant.

Proof. See the calculations leading up to equation (7.7). □

Whereas before, when M was closed, the middle term in (7.14) vanished due to $\partial M = \emptyset$ and the last term in (7.14) took its values in $\mathbb{Z}$, we cannot say the same is true now; the integrality condition on the form (see fact 7.3.7) guarantees only that the integral over <u>closed</u> manifolds is an integer. However, we can see that the middle term only depends on the restriction of s to the boundary of M. For that matter, we also have:

[2] Note that we really should view the action in "exponential form" (i.e., as $e^{2\pi i S_{M,P}(s,\omega)}$) when our manifold M has a boundary since when $\partial M \neq \{0\}$ the action itself $S_{M,P}(s,\omega)$ is not (well-)defined.

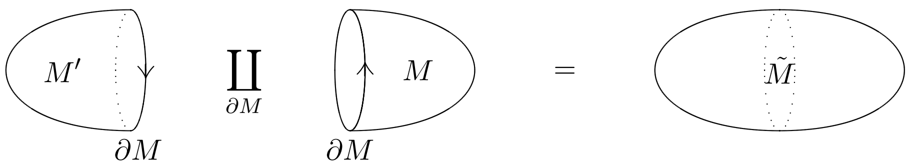
Lemma 7.3.14. *If the form $\langle \cdot \wedge \cdot \rangle$ is integral (i.e. a multiple of the Killing form), then*

$$-\frac{1}{6}\int_M \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle \qquad (mod\ 1)$$

depends only on the restriction of $g = \hat{g}_\varphi \circ s : M \to G$ to ∂M (up to an integer)[3].

Remark 7.3.15. The expression $-\frac{1}{6}\int_M \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle$ is known as the Wess-Zumino-Witten functional and is usually denoted by $W_{\partial M}(g)$ to remind the reader that it only depends on the boundary information of $g : M \to G$.

Proof. Let M' be another compact 3-manifold with boundary equal the boundary of M, $\partial M' = \partial M$, and let $g' : M' \to G$ such that $g'|_{\partial M} = g|_{\partial M}$. Then, we can glue these two 3-manifolds together $M' \coprod_{\partial M} M$, along ∂M, to give a closed manifold $\tilde{M}$:



However, since $\partial M'$ needs its orientation to be reversed before the gluing, the WZW functional for M' picks up an overall minus sign. And so, the difference between the two WZW functionals is a WZW functional over a closed manifold, which by fact 7.3.7 it is an integer,

$$W_{\partial M}(g) - W_{\partial M'}(g') = -\frac{1}{6}\int_{M' \coprod_\Sigma M} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle \in \mathbb{Z}.$$

This implies that, up to an integer, $W_{\partial M}(g)$ only depends on the restriction of $g : M \to G$ to ∂M. $\square$

So, although the Chern-Simons action is not gauge invariant in this case (and hence depends on which section one chooses (or, equivalently stated, gauge is fixed) to pull $\alpha(\omega)$ down to M), we do see that the Chern-Simons action functional depends in a "controlled" manner on the restriction of the section $s : M \to P$ to the boundary, $s|_{\partial M}$, along with the restriction of $s^*(\omega)$ to the boundary, $s^*(\omega)|_{\partial M}$.

7.4 Classical Field Theory Construction (a.k.a. Pre-Quantization)

Before we construct our theory, we first need to introduce the concept of an *invariant section* of a functor.

[3]Recall that if $G \hookrightarrow P \xrightarrow{\pi} M$ is trivial, then any gauge transformation $\varphi \in \mathrm{Aut}(P)$ can be identified with a mapping $g : M \to G$.

7.4.1 Invariant Section Construction

the **invariant section construction**, or **inverse limit construction**, is a construction which allows one to "glue together" several related objects, the precise manner of the gluing process being specified by morphisms between the objects. This will come in handy in a moment when we have a collection of inner product spaces which we want to glue together to form a fibre bundle. Also, as we will see shortly, invariant sections are invariant under gauge transformations (i.e., morphisms of the fields) - a property that must be satisfied by gauge theories.

In the construction of TQFTs on manifolds it is sometimes helpful to assume that each particular manifold has some 'additional' data; e.g., triangulations. On the other hand, the theory will then only produce an invariant of a manifold endowed with this extra structure, and hence, the TQFT is not really a topological invariant. However, we are saved by the invariant section construction, because we can construct a TQFT with additional data and then use the invariant section construction to eliminate the added data, thus giving a true topological invariant.

To begin, let $\mathcal{L}$ be a category whose objects are one-dimensional complex lines and whose morphisms are unitary isomorphisms. Next, suppose that on each line we have some additional data, which we call a 'choice'. For our case, the choice will be which section to pick on the boundary. Now, let $\mathscr{C}$ be the groupoid (see section 2.1) constructed from these choices and isomorphisms of these choices. Further, suppose that $\mathcal{F} : \mathscr{C} \to \mathcal{L}$ is a functor. Then, define $V_{\mathcal{F}}$ to be the inner product space of invariant sections of the functor $\mathcal{F}$: An element $v \in V_{\mathcal{F}}$ is a collection $\{v(C) \in \mathcal{F}(C)\}$ such that if $\psi : C_1 \to C_2$ is a morphism, then v obeys $\mathcal{F}(\psi)\big(v(C_1)\big) = v(C_2)$, while the inner product is induced by the normal inner product on the objects of $\mathcal{L}$. Alternatively stated, a point in the inverse limit is a 'section' of $\mathcal{F}$; that is, a function of the objects in $\mathscr{C}$ such that $v(C) \in \mathcal{F}(C)$. Furthermore, these elements (or functions) are required to be invariant under $\mathscr{C}$ morphisms in the sense that if $\psi : C_1 \to C_2$ is a morphism, then $\mathcal{F}(\psi)\big(v(C_1)\big) = v(C_2)$. Thus, as we previously stated, when we replace the category $\mathscr{C}$ with the category of sections $\mathcal{C}$ we will have that the invariant sections are invariant under the field morphisms, or gauge transformations. Finally, suppose that $\mathscr{C}$ is **connected**; that is, there exists a morphism between any two objects (which is clearly the case for us). Then, it can be shown that either $\dim(V_{\mathcal{F}}) = 0$ or $\dim(V_{\mathcal{F}}) = 1$, the latter occurring if $\mathcal{F}$ has no **holonomy**; i.e., $\mathcal{F}(\psi) = id$ for every automorphism $\psi : C \to C$. As an analogy to differential geometry, think of the objects in $\mathscr{C}$ as points in a space and the invariant sections as fibres of a bundle with connection over this space, and the morphisms as parallel transport. Then, the inner product space $V_{\mathcal{F}}$ is the space of all flat sections. That is, the functor has no holonomy if and only if the connection has no holonomy.

Let us now construct our topological theory.

Roughly speaking, a topological field theory is a functor[4] from the category of fields to the collection of inner product spaces (the precise definition of a field theory will

[4]See chapter 2.

be given later on). We can obtain a field theory from the previous data in the following way. Let $G \hookrightarrow Q \xrightarrow{\bar{\pi}} \Sigma$ be a principal G-bundle over a closed 2-dimensional manifold Σ and fix a connection form η on Q. Then, in order to construct a field theory, we will construct a functor $\mathcal{F}_\eta : \mathcal{C}_\Sigma \to \mathcal{L}$ from the category of sections C_Σ on $\Sigma \to Q$ (objects are sections $\bar{s} : \Sigma \to Q$ and morphisms are gauge transformations) to the category of metrized lines $\mathcal{L}$ (objects are 1-dimensional inner product spaces and morphisms are isometries). Hence, to each connection form η on Q, we construct a line together with its isometries. Thus, one can wonder: if we take the space of connections as a base space, could these lines be pieced together in a smooth manner, so as to yield a line bundle? We will see that the answer to this question is yes and to construct the line bundle over the space of connections we need to use the invariant section construction. Also, recall that the invariant section construction picks out spaces which are invariant under the automorphisms of Q. Hence, elements in the space of invariant sections are invariant under the gauge transformations of Q, $\mathrm{Aut}(Q)$. Finally, we will show that when acting on a connection form ω which is defined on a trivial bundle over a 3-dimensional manifold M with boundary, the theory gives an element $e^{2\pi i S_{M,P}(\omega)} \in L_{\partial M, \partial\omega}$, where $\partial\omega$ is the restriction of ω to the boundary - that is, $s^*(\omega)|_{\partial M}$. Hence, the Chern-Simons action for a 3-manifold with boundary is not a function, but rather a section of the previously mentioned line bundle over the space of connections on the boundary.

Let us first construct the functor from the category of sections to 1-dimensional inner product spaces. To begin, let $G \hookrightarrow Q \xrightarrow{\bar{\pi}} \Sigma$ be a principal G-bundle over the 2-dimensional closed oriented manifold Σ and fix a connection η on Q. Next, from theorem 7.3.4, we see that $G \hookrightarrow Q \xrightarrow{\bar{\pi}} \Sigma$ is trivial and thus admits global sections. So, to each of these sections, $\bar{s} : \Sigma \to Q$, associate a copy of $\mathbb{C}$, denoted by $\mathbb{C}_{\bar{s}}$, and let $\mathbb{C}$ have its standard metric. Additionally, to each gauge transformation $\psi_{\bar{s}_1,\bar{s}_2} : Q \to Q$, which takes a section $\bar{s}_1$ to a section $\bar{s}_2$ (i.e. $\bar{s}_2 = \psi_{\bar{s}_1,\bar{s}_2}(\bar{s}_1)$), assign an isometry from $\mathbb{C}_{\bar{s}_1}$ to $\mathbb{C}_{\bar{s}_2}$ given by multiplication by $c_\Sigma(\bar{s}_1^*(\eta), \hat{g}_\psi \circ \bar{s}_1)$, where $\hat{g}_\psi : Q \to G$ is the map associated with the gauge transformation $\psi_{\bar{s}_1,\bar{s}_2}$ and

$$c_\Sigma(a, g) := e^{2\pi i\left(\int_\Sigma \langle ad_{g^{-1}}(a) \wedge \phi_g\rangle + W_\Sigma(g)\right)}, \tag{7.16}$$

with $a \in \Omega^1_{ad}(\Sigma; \mathfrak{g})$ and $g : \Sigma \to G$. Note that, since $|c_\Sigma(a, g)| = 1$, we indeed have that $c_\Sigma(a, g)$ is an isometry. It is obvious that this construction gives a functor $\mathcal{F}_\eta : \mathcal{C}_\Sigma \to \mathcal{L}$. Indeed, let $\mathcal{C}_\Sigma$ be the category whose objects, $\mathrm{Obj}(\mathcal{C}_\Sigma)$, are sections $\Sigma \to Q$ and whose morphisms, $\mathrm{Hom}_{\mathcal{C}_\Sigma}(\bar{s}_1, \bar{s}_2)$, are gauge transformations $\psi_{\bar{s}_1,\bar{s}_2}$. Next, let $\mathcal{L}$ denote the category whose objects are metrized complex lines (i.e. 1-dimensional inner product spaces) and whose morphisms are isometries. Then, given a connection form η on $G \hookrightarrow Q \xrightarrow{\bar{\pi}} \Sigma$, the functor $\mathcal{F}_\eta : \mathcal{C}_\Sigma \to \mathcal{L}$ is defined by

$$\begin{aligned} \mathcal{F}_\eta(\bar{s}) &= \mathbb{C}_{\bar{s}}, \\ \mathcal{F}_\eta(\psi_{\bar{s}_1,\bar{s}_2}) &= c_\Sigma(\bar{s}_1^*(\eta), \hat{g}_\psi \circ \bar{s}_1). \end{aligned} \tag{7.17}$$

Thus, we have our desired functor, which places lines (i.e., 1-dimensional inner product spaces) over each connection form η. Now, let us construct the line bundle over the space of connections $\mathcal{A}$.

The creation of a line bundle from the previous data is not difficult and is accomplished as follows. First, denote by $L_{\Sigma,\eta}$ the set of functions $f : \Gamma(\Sigma; Q) \to \mathbb{C}$ (here $\Gamma(\Sigma; Q)$ is the space of sections on $G \hookrightarrow Q \xrightarrow{\bar{\pi}} \Sigma$) which satisfy the relation

$$f(\bar{s_2}) = c_{\Sigma}(\bar{s}_1^*(\eta), \hat{g}_{\psi} \circ \bar{s}_1)\big(f(\bar{s_1})\big). \tag{7.18}$$

Now, the fact that $L_{\Sigma,\eta}$ is 1-dimensional follows from $\mathcal{C}_{\Sigma}$ being connected[5] and $\mathcal{F}_{\eta}$ having no holonomy[6]. Finally, the standard inner product on $\mathbb{C}$ induces a Hermitian inner product on $L_{\Sigma,\eta}$, which is given by $(f_1, f_2) := \overline{f_1(\bar{s})} f_2(\bar{s})$. Note, from the construction of $L_{\Sigma,\eta}$, we see that any section $\bar{s} : \Sigma \to Q$ induces a trivialization $L_{\Sigma,\eta} \cong \mathbb{C}$. We call the 1-dimensional inner product spaces, L_{η}, **Chern-Simons lines**. Additionally, as η varies over the space of connections on Q, $\mathcal{A}(Q)$, the lines $L_{\Sigma,\eta}$ fit together smoothly into a Hermitian line bundle [32]. Indeed, we have the following proposition.

Proposition 7.4.1. *Let $G \hookrightarrow Q \xrightarrow{\bar{\pi}} \Sigma$ be a principal bundle over a closed, oriented 2-manifold Σ and let $\eta \in \mathcal{A}(Q)$ be a connection form on Q. Then, as η varies over $\mathcal{A}(Q)$, the assignments*

$$\eta \longmapsto L_{\Sigma,\eta} \tag{7.19}$$

are smooth and form a Hermitian line bundle over $\mathcal{A}(Q)$.

Proof. See [32]. □

To finish this section we will show that our gauge theory maps ω (a connection form defined on $G \hookrightarrow P \xrightarrow{\pi} M$) to $e^{2\pi i S_{M,P}(\omega)} \in L_{\partial M,\partial\omega}$; hence, the Chern-Simons action is really a section in the line bundle $\mathcal{C}_{\partial M} \to \mathcal{A}(\partial M)$. So, to begin let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle over a compact 3-manifold M and let ω be a connection form on P. First, note that $e^{2\pi i S_{M,P}(s,\omega)}$ defines an element in $\mathbb{T}$, and thus $\mathbb{C}$. Hence, we have shown that the action is an element of $\mathbb{C}$ and to show that it further restricts to $L_{\partial M,\partial\omega} \subset \mathbb{C}$ we must show that it obeys the invariant section property; namely, under a gauge transformation, we have that

$$e^{2\pi i S_{M,P}(\varphi\circ s,\omega)} = c_{\partial M}\left(s^*(\omega)|_{\partial M}, g|_{\partial M}\right) e^{2\pi i S_{M,P}(s,\omega)}, \tag{7.20}$$

where $g|_{\partial M}$ is the restriction of $g := \hat{g}_{\varphi} \circ s$ to ∂M. Now, from proposition 7.3.12 we see that, under a gauge transformation $\varphi \in \mathrm{Aut}(P)$, the action obeys

$$e^{2\pi i S_{M,P}(\varphi\circ s,\omega)} = e^{2\pi i S_{M,P}(s,\omega)} e^{2\pi i\left(\int_{\partial M}\langle ad_{g^{-1}}(s^*(\omega))\wedge\phi_g\rangle - \frac{1}{6}\int_M\langle\phi_g\wedge[\phi_g\wedge\phi_g]\rangle\right)}.$$

Which, upon rewriting, gives

$$\begin{aligned} e^{2\pi i S_{M,P}(\varphi\circ s,\omega)} &= e^{2\pi i\left(\int_{\partial M}\langle ad_{g^{-1}}(s^*(\omega))\wedge\phi_g\rangle - \frac{1}{6}\int_M\langle\phi_g\wedge[\phi_g\wedge\phi_g]\rangle\right)} e^{2\pi i S_{M,P}(s,\omega)}, \\ &= c_{\partial M}\left(s^*(\omega)|_{\partial M}, g|_{\partial M}\right) e^{2\pi i S_{M,P}(s,\omega)}, \end{aligned}$$

[5]Connectedness of $\mathcal{C}_{\Sigma}$ follows from the fact that any two sections (and hence, any two objects of $\mathcal{C}_{\Sigma}$) are connected via a gauge transformation; that is, given any two sections $\bar{s}$ and $\bar{s}'$, there exists a gauge transformation $\psi_{\bar{s},\bar{s}'}$ such that $\bar{s}' = \psi_{\bar{s},\bar{s}'}(\bar{s})$.

[6]Recall that this means that for any automorphism $\psi_{\bar{s},\bar{s}}$, then $\mathcal{F}_{\eta}(\psi_{\bar{s},\bar{s}}) = id$.

where in the last line we used (7.16). Thus, $\partial s \to e^{2\pi i S_{M,P}(\omega)}$ defines an invariant section of unit norm

$$e^{2\pi i S_{M,P}(\omega)} \in L_{\partial M, \partial\omega}. \tag{7.21}$$

Additionally, we allow either M or ∂M to equal $\emptyset$ and we set $L_\emptyset = \mathbb{C}$ and $S_\emptyset = 0$.

And so, we see that the classical 3-dimensional Chern-Simons theory associates a line bundle over the space of connections on each principal bundle over a 2-dimensional manifold, while to each connection on a bundle over a 3-dimensional manifold, the theory gives an element in the total space associated to the boundary; i.e., as the connection varies over all connections on the bundle over a 3-manifold, we get a section of the line bundle.

Although our constructed theory is not a quantum theory it does obey several similar properties. In particular, we have the following theorem.

Theorem 7.4.2. *Suppose G is a connected, simply connected, compact Lie group with $\langle \cdot \wedge \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ an ad_G-invariant, symmetric, bilinear form which obeys the integrality condition (see fact 7.3.7). Then, the assignments*

$$\begin{aligned} \eta &\longmapsto L_{\Sigma,\eta}, \\ \omega &\longmapsto e^{2\pi i S_{M,P}(\omega)} \in L_{\partial M,\partial\omega}, \end{aligned} \tag{7.22}$$

previously defined for closed oriented 2-manifolds Σ and compact oriented 3-manifolds M, are smooth and satisfy:

(*a*) ***Functoriality:*** *Let $G \hookrightarrow Q \xrightarrow{\pi} \Sigma$ and $G \hookrightarrow Q' \xrightarrow{\pi'} \Sigma'$ be two principal G-bundles and let $\psi : Q' \to Q$ be a bundle map such that the induced map $\hat{\psi} : \Sigma' \to \Sigma$ is orientation preserving. Then, any connection form η on Q induces an isometry*

$$\psi^* : L_{\Sigma,\eta} \longrightarrow L_{\Sigma',\psi^*(\eta)}. \tag{7.23}$$

Additionally, let $G \hookrightarrow P \xrightarrow{\pi} M$ and $G \hookrightarrow P' \xrightarrow{\pi'} M'$ be two principal G-bundles over M and let $\varphi : P' \to P$ be a bundle map with induced map $\hat{\varphi} : M' \to M$ orientation preserving. If ω is a connection form on P, then

$$\varphi^*\Big|_{\partial P'} \left(e^{2\pi i S_{M,P}(\omega)}\right) = e^{2\pi i S_{M',P'}\left(\varphi^*(\omega)\right)}, \tag{7.24}$$

where $\varphi^*|_{\partial P'}$ is the restriction of $\varphi : P' \to P$ to the bundles over the boundary.

(*b*) ***Orientation:*** *Let $\bar{\Sigma}$ denote the manifold Σ but with opposite orientation. There is a natural isometry*

$$L_{\bar{\Sigma},\eta} \cong L^*_{\Sigma,\eta}, \tag{7.25}$$

where by $L^*_{\Sigma,\eta}$ we mean the dual vector space to $L_{\Sigma,\eta}$. Furthermore, we have that the action defined on $\bar{M}$ is the complex conjugate of the action defined on M,

$$e^{2\pi i S_{\bar{M},P}(\omega)} = \overline{e^{2\pi i S_{M,P}(\omega)}}, \tag{7.26}$$

which (as long as $S_{M,P}(\omega) \in \mathbb{R}/\mathbb{Z}$) becomes

$$e^{2\pi i S_{\bar{M},P}(\omega)} = e^{-2\pi i S_{M,P}(\omega)}.$$

(c) ***Multiplicativity:*** *If $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_n$ with connections η_i on the subbundle of Q restricted to Σ_i, which we denote by $Q|_{\Sigma_i}$, then*

$$L_{\Sigma,\sqcup_i \eta_i} \cong L_{\Sigma_1,\eta_1} \otimes \cdots \otimes L_{\Sigma_n,\eta_n}. \tag{7.27}$$

While, if M decomposes as the disjoint union $M = \sqcup_{i=1}^n M_i$ with connections ω_i over M_i, then

$$e^{2\pi i S_{\sqcup_i M_i, \sqcup_i P|_{M_i}}(\sqcup_i \omega_i)} = e^{2\pi i S_{M_1,P|_{M_1}}(\omega_1)} \otimes \cdots \otimes e^{2\pi i S_{M_n,P|_{M_n}}(\omega_n)}. \tag{7.28}$$

(d) ***Gluing:*** *Suppose M is a compact, oriented manifold and that $\Sigma \hookrightarrow M$ is a closed oriented codimension one submanifold of M. Let M^{cut} denote the manifold obtained by cutting M along Σ. Then, $\partial M^{cut} = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$. Now, suppose ω is a connection over M with ω^{cut} the induced connection over M^{cut} (see proposition 5.2.8) and η the restriction of ω to $P|_\Sigma$. Then*

$$e^{2\pi i S_{M,P}(\omega)} = \left\langle e^{2\pi i S_{M^{cut},P^{cut}}(\omega^{cut})} \right\rangle_\eta, \tag{7.29}$$

where $\langle \cdot \rangle_\eta$ is the natural contraction from $L_{\partial M,\partial\omega} \otimes L_{\Sigma,\eta} \otimes L^*_{\Sigma,\eta}$ to $L_{\partial M,\partial\omega}$,

$$\langle \cdot \rangle_\eta : L_{\partial M,\partial\omega} \otimes L_{\Sigma,\eta} \otimes L^*_{\Sigma,\eta} \longrightarrow L_{\partial M,\partial\omega}.$$

Remark 7.4.3. From (a), we see that $\eta \mapsto L_{\Sigma,\eta}$ defines a functor $\mathcal{C}_\Sigma \to \mathcal{L}$ and that each M determines an invariant section $e^{2\pi i S_{M,P}(\cdot)}$ of the composite functor $\mathcal{C}_M \to \mathcal{C}_{\partial M} \to \mathcal{L}$. That is, to each principal G-bundle $Q \to \Sigma$ over a closed, oriented 2-dimensional manifold Σ, there is an associated smooth line bundle $L_Q \to \mathcal{A}(Q)$ over the space of connections $\mathcal{A}(Q)$ on Q. Furthermore, the action of gauge transformations $\mathcal{G}(Q)$ on $\mathcal{A}(Q)$ lifts up to the total space L_Q - hence, for any $g \in \mathcal{G}(Q)$ there exists a $\bar{g} \in \mathrm{Aut}(L_Q)$ such that $\pi^*(g) = \bar{g}$, where π is the bundle projection map. Additionally, any bundle $P \to M$ over a compact, oriented 3-dimensional manifold M induces a restriction map $\mathcal{A}(P) \to \mathcal{A}(\partial P)$ which in turn induces a line bundle $L_P \to \mathcal{A}(P)$ over $\mathcal{A}(P)$ (here as well, the gauge transformations on $\mathcal{A}(P)$ lift up to L_P). Consequently, the exponentiated action $e^{2\pi i S_{M,P}(\cdot)}$ determines an invariant section of L_P; that is, the classical action $e^{2\pi i S_{M,P}(\cdot)}$ is a section of the bundle $L_P \to \mathcal{A}(P)$ which is invariant under the gauge transformations (or field morphisms) in the sense that under a gauge transformation the action obeys

$$e^{2\pi i S_{M,P}(\varphi \circ s,\omega)} = c_{\partial M}\left(s^*(\omega)|_{\partial M}, g|_{\partial M}\right) e^{2\pi i S_{M,P}(s,\omega)}.$$

From (b) we see that our theory is unitary, while from (d) we see that our theory is locally defined.

Proof. The proof is technical and offers no new insight into the theory which has been developed. Therefore, we direct the reader to [19]. □

Thus, using the Chern-Simons form we have constructed a field theory in the spirit of, for example, Atiyah's axioms for a topological quantum field theory (see section 8.1). However, everything we have previously done relied on one very important property: all of the bundles we considered were trivial! We have not dealt with the more general case - when we allow for our principal G-bundles to no longer have the global product topology $P = M \times G$. This is the topic of the next section.

7.5 Chern-Simons Action (General Theory)

In this section we define the Chern-Simons action for an arbitrary principal G-bundle. To do this we will rely on certain powerful tools from algebraic topology - in particular, the cohomology theory of topological groups G - which will be developed in section 7.5.2. Following this discussion, we define the general Chern-Simons action. However, let us first begin with a "warm up".

7.5.1 Warm up (somewhat less-general case)

When our principal G-bundle is not trivial, we run into the problem of how to define the Chern-Simons action globally on the base manifold M. Recall, in the case where the bundle P is trivial we defined our Chern-Simons action as the integral, over M, of the pullback of Chern-Simons form, $\alpha(\omega) \in \Omega^3(P;\mathbb{R})$, via a section; this form on M is globally defined since by triviality we have a global section $s : M \to P$. Conversely, when we permit P to be non-trivial we no longer have such a globally defined section. Hence, at best, one can only pull the Chern-Simons form down to local coverings of M. Furthermore, unlike before when we discussed the Chern-Weil theory (in particular invariant polynomials), there is no way to patch the forms together to give a globally defined form on M - since the $\alpha(\omega)$'s do not agree on overlaps, see proposition 7.2.3 part 5. However, this previous sentence holds the key to how one can construct a globally defined Chern-Simons action in the case where P is non-trivial. That is, if we could somehow construct the Chern-Simons form out of an invariant polynomial then we could patch the local pieces together to give a globally defined form.

Perhaps the reader remembers (or they can go look at proposition 7.2.3 part 3) that the Chern-Simons form $\alpha(\omega)$ is the antiderivative of the Chern-Weil form $\langle\Omega\wedge\Omega\rangle$, which just happens to be a form which agrees on the overlaps[7]. Thus, we have a good candidate for what our (more general) Chern-Simons action should be. Unfortunately, upon further inspection we see that the Chern-Weil form is a $(d+1)$-form, where the dimension of M is d, and thereby always vanishes on M. We can overcome this setback by considering "extensions" in dimensions for the principal bundles in question; i.e, by viewing M as the boundary of a $(d+1)$-dimensional manifold, which is itself the base space of some principal bundle, $G \hookrightarrow P' \xrightarrow{\pi'} B$. The goal of this section is to see how to do this and to explicitly work out the details when $\dim(M) = 3$.

To wet our appetite, let us consider the case where P is topologically a product bundle $G \times M$ over some closed, oriented, 3-manifold M and see how we can define the Chern-Simons action of a connection which extends over a 4-manifold B whose boundary is given by M. First, note that (from cobordism theory) any closed, oriented 3-manifold M is the boundary of some 4-manifold B. Furthermore, since P is trivial, we can always extend P to some trivial bundle P' over B. Now, to proceed, let $G \hookrightarrow P \xrightarrow{\pi} M$ be a trivial principal G-bundle with connection form ω, and let B be a compact bounding manifold[8] of

[7] This is true since Ω is type ad_G and since $\langle \cdot \wedge \cdot \rangle \in I^\bullet(G)$.

[8] By this we mean that M extends to B in such a way that $\partial B = M$.

M such that P extends over B to give another principal G-bundle $G \hookrightarrow P' \xrightarrow{\pi'} B$ (recall, since P is trivial there always exists an extension to a trivial P' over B). Additionally, let $\omega' \in \mathcal{A}(P')$ be the extension of $\omega \in \mathcal{A}(P)$ to P' such that the pullback of ω' to $\partial B(\equiv M)$ gives $s^*(\omega)$, i.e.

$$\tilde{s}^*(\omega')\big|_{\partial B} = s^*(\omega),$$

where $\tilde{s} : U' \subset B \to P'$ with $U' \cap \partial B \neq \emptyset$ and $s : M \to P$. For example, if we take $B = M \times [0,1]$, where $[0,1]$ is the unit interval subset of $\mathbb{R}$, then, since B retracts onto M, the bundle P has a unique extension (up to homotopy) to P'. Furthermore, we can extend the connection form by taking (for example) $\omega' = (1-t)\omega_1 - t\omega_2$, where ω_1 is the connection ω on $M \times \{0\}$, ω_2 is the connection ω on $M \times \{1\}$ and $t \in [0,1]$. Now, reverting back to the general case, let Ω' be the curvature form associated with ω'. Then the Chern-Simons action of the restriction of ω' to the boundary M is given by the integral, over B, of the pullback of the Chern-Weil 4-form $\langle \Omega' \wedge \Omega' \rangle$,

$$\begin{aligned} S_{M,P}(\partial\omega') &:= \frac{k}{8\pi^2}\int_B s^*\left(\langle \Omega' \wedge \Omega' \rangle\right) \qquad (\text{mod } 1), \\ &\equiv \frac{k}{8\pi^2}\int_B \langle \mathcal{F}' \wedge \mathcal{F}' \rangle \qquad (\text{mod } 1), \end{aligned} \tag{7.30}$$

where $\partial\omega' \equiv \omega'|_{\partial P'}$ and $\partial P'$ is the restriction of P' to $\partial B = M$, while $\mathcal{F}'$ denotes the local field strength associated to Ω'. That this is the correct expression for the Chern-Simons action follows from proposition 7.2.3 part (3) (in addition to P being trivial so that $s^*(\omega)$ is defined globally on M) and Stokes' theorem.

Remark 7.5.1. Note, as stated earlier, since $\langle \cdot \wedge \cdot \rangle \in I^\bullet(G)$ and since $\mathcal{F}'$ is type ad_G, the 4-form is globally defined on B. Furthermore, since the right-hand side of (7.30) gives an element in $\mathbb{Z}$ when evaluated on a closed 4-manifold, this expression for the Chern-Simons action is independent of the choice of B and the way in which ω is extended to ω'.

Recapping, we have shown how one can generalize the Chern-Simons action by using the techniques of cobordism theory (i.e., by extending the trivial principal bundle P to a trivial bundle P' over B, where $\partial B = M$). However, one must note that the above was only valid for the case in which P was already trivial. Let us now further reduce the restrictions on P and allow for any principal G-bundle.

7.5.2 Group Cohomology and the Chern-Simons Form

Before we proceed with the general case, let's try and view the Chern-Simons form from a cohomological standpoint - viewing the Chern-Simons form as an element of a certain cohomology class will prove itself useful in subsequent material; indeed, the tools developed in this section will be called upon, time and time again, when we define the general Chern-Simons action. In order to achieve our goal, we must introduce the concept of *group cohomology*. The reader wishing to review the basics of cohomology theory should consult chapter 3.

In order to define the cohomology of a topological group G, we first need to review the notion of a universal bundle (see section 4.5). Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a principal G-bundle over a manifold M. Then, there exists another principal G-bundle $EG \to BG$, known as the universal G-bundle (or universal bundle when G is obvious), with the property that $P \to M$ is a pullback bundle of EG with respect to some classifying map $\gamma : M \to BG$; recall from section 4.5 that, in fact, *any* G-bundle over M is the pullback of EG with respect to some classifying map. Furthermore, if there exists another G-bundle P' over M with $\gamma' : M \to BG$ which is homotopic to $\gamma : M \to BG$, then the two principal G-bundles, $P \to M$ and $P' \to M$, are isomorphic (see section 4.5 for the definition of a bundle isomorphism). Hence, different components of the space $[M, BG]$ correspond to different principal bundles over M. Furthermore, it can be shown that (up to homotopy) BG is uniquely determined by requiring EG to be contractible. Equivalently stated, any contractible space with a free action of G is a realization of EG. Examples of classifying spaces BG are: $B\mathbb{Z}_2 = \mathbb{R}\mathbf{P}^\infty$, $BU(1) = \mathbb{C}\mathbf{P}^\infty$, and $BSU(2) = \mathbb{H}\mathbf{P}^\infty$. With the review of universal bundles behind us, we now proceed to define the group cohomology of a topological group G.

Definition 7.5.2. Let G be a compact Lie group. We define the **group cohomology** of G with coefficients in some abelian group $\mathbb{F}$ to be the cohomology of the associated classifying space BG, which we denote by $H^\bullet(BG; \mathbb{F})$, with coefficient group $\mathbb{F}$.

Remark 7.5.3. Note that the group cohomology of G, $H^\bullet(BG; \mathbb{F})$, differs from its cohomology viewed as a manifold, $H^\bullet(G; \mathbb{F})$. However, there is a relation between the two cohomology theories. Indeed, 3-dimensional Chern-Simons theories can be classified by $H^4(BG; \mathbb{Z})$ while 2-dimensional WZW theories are classified by $H^3(G; \mathbb{Z})$. Now, since both theories are intimately related (see section 7.3), there must exist some map between the two cohomologies. It turns out that this is, in fact, true. For instance, in [16] an explicit map $\tau : H^k(BG; \mathbb{F}) \to H^{k-1}(G; \mathbb{F})$ is constructed, where $\mathbb{F}$ is any abelian group of coefficients. The elements of $H^\bullet(BG; \mathbb{Z})$ are called *universal characteristic classes* due to the fact that under the pullback of $\gamma : M \to BG$ any element of $H^\bullet(BG; \mathbb{Z})$ induces a cohomology class in $H^\bullet(M; \mathbb{Z})$ which only depends on the topology of P.

For a compact Lie group G, one can show that its odd group cohomology with real coefficients vanishes,

$$H^{odd}(BG; \mathbb{R}) = 0, \tag{7.31}$$

while the even group cohomology of G is isomorphic to the ring of G-invariant polynomials,

$$H^{even}(BG; \mathbb{R}) \cong I(G). \tag{7.32}$$

Hence, the general Chern-Simons form, the Chern-Weil 4-form $\frac{k}{8\pi^2}\langle \mathcal{F}' \wedge \mathcal{F}' \rangle$, really represents an element of $H^4(BG; \mathbb{R})$ (we will say more on this later). For this differential has integral periods and hence, lies in the image of the natural map $\rho : H^4(BG; \mathbb{Z}) \to H^4(BG; \mathbb{R})$. And so, we can view the general Chern-Simons form algebraically as an element originating from $H^4(BG; \mathbb{Z})$ (at least when P is trivial - we will see that the same holds for general P' in the next section). Defining torsion[9] elements in $H^k(BG; \mathbb{Z})$ as elements of $\ker(\rho)$, where $\rho :$

[9] See chapter 3.

$H^k(BG;\mathbb{Z}) \to H^k(BG;\mathbb{R})$ is the natural map, we see that all of the odd integral cohomology consists completely of torsion. What is more, since $H^k(BG;\mathbb{R}) = \mathrm{Hom}(H_k(BG), \mathbb{R}) = 0$ for odd k and since $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$, we conclude that all odd integral homology $H_{odd}(BG;\mathbb{Z})$ consists completely of torsion as well - remember that $\mathbb{R}$ is characteristic zero. We will use these facts later on to argue for the existence of bundle extensions when P is not trivial. For finite G an even stronger statement holds: all cohomology with real coefficients is finite, $H^\bullet(BG;\mathbb{R}) = 0$, so all integral cohomology is torsion for finite G. We can use this result to prove the following proposition.

Proposition 7.5.4. *Let G be a finite group, then*

$$H^k(BG;\mathbb{Z}) \cong H^{k-1}(BG;\mathbb{R}/\mathbb{Z}), \tag{7.33}$$

for all $k \in \mathbb{N}$.

Proof. To begin, let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups A, B, and C. Then, we can construct another short exact sequence by tensoring a free abelian group to each abelian group in the previous sequence[10]. So, given the previous short exact sequence, we can construct the following short exact sequence

$$0 \longrightarrow C^n(X;\mathbb{Z}) \otimes A \longrightarrow C^n(X;\mathbb{Z}) \otimes B \longrightarrow C^n(X;\mathbb{Z}) \otimes C \longrightarrow 0,$$

where $C^n(X;\mathbb{Z})$ is the free[11] abelian group of n-cocycles on some topological space X with values in $\mathbb{Z}$ (for any n). Thus, since the above short exact sequence holds for all n, we end up with a short exact sequence of cochain complexes (see section 3.2.2, in particular definition 3.2.11)

$$0 \longrightarrow C^\bullet(X;\mathbb{Z}) \otimes A \longrightarrow C^\bullet(X;\mathbb{Z}) \otimes B \longrightarrow C^\bullet(X;\mathbb{Z}) \otimes C \longrightarrow 0,$$

Further, note that $C^n(X;\mathbb{Z}) \otimes A$ is the group of n-cochains with values in A, $C^n(X;\mathbb{Z}) \otimes A = C^n(X;A)$. These arguments imply that the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0,$$

gives another short exact sequence

$$0 \longrightarrow C^\bullet(BG;\mathbb{Z}) \otimes \mathbb{Z} \longrightarrow C^\bullet(BG;\mathbb{Z}) \otimes \mathbb{R} \longrightarrow C^\bullet(BG;\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

[10] Indeed, given a short exact sequence $0 \to A \to B \to C \to 0$, we can form the new long exact sequence $\cdots \to Tor_2^{\mathbb{Z}}(D,C) \to Tor_1^{\mathbb{Z}}(D,A) \to Tor_1^{\mathbb{Z}}(D,B) \to Tor_1^{\mathbb{Z}}(D,C) \to D \otimes A \to D \otimes B \to D \otimes C \to 0$, where D is some abelian group and Tor is called the Tor functor (this is the right derived functor of $\otimes$, see section 3.4). The definition of Tor is inconsequential, except for the fact that for abelian groups, G_1 and G_2, $Tor_n^{\mathbb{Z}}(G_1, G_2) = 0$ when $n \geq 2$ and that $Tor^{\mathbb{Z}}(G_1, G_2) = 0$ if either G_1 or G_2 is free. Hence, the long exact sequence becomes the short exact sequence $0 \to D \otimes A \to D \otimes B \to D \otimes C \to 0$, when A, B, and C are abelian groups and D is a free abelian group.

[11] See remark 3.2.14.

which, using the fact that $C^n(Z;\mathbb{Z}) \otimes A = C^n(X;A)$, can be rewritten as

$$0 \longrightarrow C^\bullet(BG;\mathbb{Z}) \longrightarrow C^\bullet(BG;\mathbb{R}) \longrightarrow C^\bullet(BG;\mathbb{R}/\mathbb{Z}) \longrightarrow 0.$$

This short exact sequence between cochain complexes, in turn, generates the following long exact sequence between cohomologies (see theorem 3.2.12)

$$\cdots \longrightarrow H^{k-1}(BG;\mathbb{R}) \longrightarrow H^{k-1}(BG;\mathbb{R}/\mathbb{Z}) \longrightarrow H^k(BG;\mathbb{Z}) \longrightarrow H^k(BG;\mathbb{R}) \longrightarrow \cdots.$$

Now, using the fact that for a finite group G all (group) cohomology with real coefficients vanishes, we have the collection of exact sequences

$$0 \longrightarrow H^{k-1}(BG;\mathbb{R}/\mathbb{Z}) \longrightarrow H^k(BG;\mathbb{Z}) \longrightarrow 0,$$

for all $k \in \mathbb{N}$. Hence, since each sequence is exact, we have (see part (c) of proposition 3.1.11)

$$H^k(BG;\mathbb{Z}) \cong H^{k-1}(BG;\mathbb{R}/\mathbb{Z}).$$

This proposition will prove itself invaluable when we define the general Chern-Simons action for regular covering spaces (i.e., for principal G-bundles with G finite). □

To close this section, we briefly list the group cohomologies of two Lie groups which are important in physics:

Example 7.5.5 (Group Cohomology).

(a) Unitary Groups $U(N)$: For $U(N)$ there is no torsion and its group cohomology is given by the polynomial ring in the Chern classes c_k of degree $2k$ (see section 6.3)

$$H^\bullet(BU(N);\mathbb{Z}) = \mathrm{Pol}[c_1, ..., c_n].$$

(b) Cyclic Group $\mathbb{Z}_n$: In this case we have

$$H^{odd}(B\mathbb{Z}_n;\mathbb{Z}) = 0,$$

$$H^{even}(B\mathbb{Z}_n;\mathbb{Z}) = \mathbb{Z}_n.$$

7.5.3 General Case

Closed Manifolds ($\partial \mathbf{M} = \mathbf{0}$)

As we have seen, given a trivial principal G-bundle P over a three dimensional manifold M, the Chern-Simons action is (writing $\mathcal{A}$ instead of $s^*(\omega)$)

$$S_{M,P}(\omega) = \frac{k}{8\pi^2}\int_M \left\{\langle \mathcal{A} \wedge \mathcal{F}\rangle - \frac{1}{6}\langle \mathcal{A} \wedge [\mathcal{A} \wedge \mathcal{A}]\rangle\right\}, \tag{7.34}$$

while if M can be extended to some 4-manifold B and P to a trivial bundle P' over B, the action rewrites as

$$S_{M,P}(\partial\omega') = \frac{k}{8\pi^2}\int_B \langle \mathcal{F}' \wedge \mathcal{F}'\rangle \qquad (\text{mod } 1), \tag{7.35}$$

where $\mathcal{F}'$ is the curvature of any gauge field (or gauge potential) $\mathcal{A}'(\equiv \tilde{s}*(\omega'))$ on B which reduces to $\mathcal{A}$ on the boundary $\partial B = M$. It is known, from cobordism theory, that any closed, oriented 3-manifold is the boundary of some four dimensional oriented manifold. Furthermore, one can always extend a trivial bundle over M to a trivial one over B. Hence, when M is as above and P is trivial over M, then (7.35) serves as the general definition of the Chern-Simons action. However, if the bundle P over M is not trivial (i.e., is not topologically a product $M \times G$) then it will in general not be possible to find such an extension of the bundle P'. Consequently, the previous definition needs to be modified.

So, the problem is, how do we fix this inability to extend P to a bundle P' over a bounding manifold B? The solution to this problem is to reduce the restrictions on B (namely that of being a manifold) and allow for B to simply be a singular 4-chain (see chapter 3). To be more concrete, looking at (7.35) we see that the Chern-Simons action really is just an integration of a 4-form over some manifold B. Generalizing this idea, we instead allow for B to be a smooth singular 4-chain - since any 4-form can be integrated over a 4-chain. That is, since we are looking for a 4-chain B with bundle P' that restricts to P at the boundary $\partial B = M$, we are actually trying to find a 4-chain in the classifying space BG which bounds the image of M under the classifying map γ (restricting the universal bundle EG to this chain would then give P') $\gamma(M)$, [18]. So, plain and simple, finding such a 4-chain is equivalent to finding an extension of P to P', for some general P. Note, by allowing B to be a singular chain, we are simplifying our problem; indeed, we have turned our problem of finding a bundle extension P' over some manifold B (which relies on cobordism theory) to that of finding a bundle extension P' over a singular 4-chain (which uses the simpler ideas of homology theory). Thus, the obvious question to ask is when do such 4-chains exist (hence, when can we extend P to P')?

It turns out that the obstruction to the existence of the aforementioned 4-chain is completely 'measured' by the exactness of homology group $H_3(BG;\mathbb{Z})$ (i.e., $H_3(BG;\mathbb{Z}) = 0$). Since then, if $H_3(BG;\mathbb{Z})$ is exact, the image of the fundamental class[12] of M, $[M]$, under γ (denoted $\gamma_*[M] \in H_3(BG;\mathbb{Z})$) vanishes. Which implies that any cycle $\gamma_*(m) \in Z_3(BG;\mathbb{Z})$ representing $\gamma_*[M]$ is given by the boundary of some chain $B \in C_4(BG;\mathbb{Z})$[13]. And so, our next task is to investigate when $H_3(BG;\mathbb{Z})$ vanishes, since then there would be no obstruction to finding such a 4-chain - hence, we could then define our Chern-Simons theory on non-trivial bundles. Note, for any given connected, simply connected Lie group G, the homology group $H_3(BG;\mathbb{Z})$ vanishes, giving us that (7.35) can serve as the general definition of the Chern-Simons action for any principal G-bundle. However, for compact G we need to take this obstruction into account, and that is what we now do.

To begin, recall that, since $H^k(BG;\mathbb{R}) = \mathrm{Hom}(H_k(BG;\mathbb{Z}),\mathbb{R}) = 0$ for odd k and since $\mathrm{Hom}(\mathbb{Z},\mathbb{R}) = \mathbb{R}$ together with $\mathbb{R}$ having characteristic zero, $H_{\mathrm{odd}}(BG;\mathbb{R})$ consists only

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[12]See chapter 3.

[13]Recall, the n-th homology group H_n is defined as the quotient of the group of n-cycles by the group n-boundaries. So, if the n-th homology group vanishes then all cycles can be written as the boundary of some $(n+1)$-chain.

of torsion; that is, there exists some $n \in \mathbb{Z}_+$ such that

$$n \cdot \gamma_*[M] = \underbrace{\gamma_*[M] + \cdots + \gamma_*[M]}_{\text{n times}} = 0.$$

Which implies, for each principal bundle over M,

$$n \cdot \gamma_*[M] = 0,$$

or, equivalently, P can be extended to a bundle P' over the 4-chain B, whose boundary is given by

$$\partial B = \underbrace{M \sqcup \cdots \sqcup M}_{\text{n times}},$$

such that the restriction of P' on all boundary components is isomorphic with P. It is customary to call P' a *principal bundle of order n*. Thus, since $n \cdot \gamma_*[M] = 0$, there is no obstruction to extending P to a bundle P' over the singular 4-chain B, where $\partial B = M \sqcup \cdots \sqcup M$ - hence, we can can define the Chern-Simons action. In fact, since we can always choose some gauge field $\mathcal{A}'$ on B which reduces to $\mathcal{A}$ on ∂B, the right-hand side of (7.35) becomes

$$\frac{k}{8\pi^2} \int_B \langle \mathcal{F}' \wedge \mathcal{F}' \rangle \qquad (\text{mod } 1),$$

where $\mathcal{F}'$ is field strength associated to $\mathcal{A}'$, while the left-hand side (due to the additivity of $S_{M,P}$ on disjoint unions) becomes

$$S_{M,P}(\partial\omega') + \cdots + S_{M,P}(\partial\omega') = n \cdot S_{M,P}(\partial\omega').$$

Consequently, we have that

$$n \cdot S_{M,P}(\partial\omega') = \frac{k}{8\pi^2} \int_B \langle \mathcal{F}' \wedge \mathcal{F}' \rangle \qquad (\text{mod } 1). \tag{7.36}$$

Immediately we face a dilemma, our action (7.36) does not have a precise definition - one can add a multiple of $1/n$ without affecting the action. We must resolve this enigma in a way which is consistent with the properties of the Chern-Simons action, which are given in 7.4.2; namely orientation (i.e. unitarity of our action) and multiplicativity (i.e. factorization). We absolve this ambiguity as follows.

So far the basic defining object of our Chern-Simons action has been the differential form

$$\Delta(\mathcal{F}') := \frac{k}{8\pi^2} \langle \mathcal{F}' \wedge \mathcal{F}' \rangle. \tag{7.37}$$

From the previous section (namely section 7.5.2), we see that $\Delta(\mathcal{F}')$ really represents some class, which we denote by $[\Delta]$, in the de Rham cohomology group $H^4(BG; \mathbb{R})$. Now, as stated before in section 6.4, $[\Delta]$ is integral (that is, it has integral periods) and thus lies in the image of the natural map $\rho : H^4(BG; \mathbb{Z}) \to H^4(BG; \mathbb{R})$. This, in turn, tells us that there exists a class $[\varpi]$ in $H^4(BG; \mathbb{Z})$ with the property that

$$\rho([\varpi]) = [\Delta]. \tag{7.38}$$

Sadly, the choice of class $[\varpi]$ in $H^4(BG;\mathbb{Z})$ is only unique up to torsion - the choice of $[\varpi]$ is unique modulo some torsion element in $H^4(BG;\mathbb{Z})$. Which follows from the fact that torsion elements correspond exactly to elements of $\ker(\rho)$ and hence map to the "identity" element of $H^4(BG;\mathbb{R})$. However, all is not lost. The choice of which particular $[\varpi]$ (such that $\rho([\varpi]) = [\Delta]$) we can take gives us a way to get rid of the ambiguity in (7.36)

Definition 7.5.6. Let B be an oriented singular 4-chain and let P' be a principal G-bundle of order n over B[14]. Furthermore, let $\varpi \in Z^4(BG;\mathbb{Z})$ be some integer-valued cocycle representing $[\varpi] \in H^4(BG;\mathbb{Z})$, and let $[B]$ denote the fundamental class of B. The **general Chern-Simons action** on a bundle of order n is defined as

$$S_{M,P}(\partial\omega') = \frac{1}{n}\left(\int_B \frac{k}{8\pi^2}\langle \mathcal{F}' \wedge \mathcal{F}'\rangle - \langle \gamma^*(\varpi), b\rangle\right) \qquad (\text{mod } 1), \tag{7.39}$$

where $b \in C_4(B;\mathbb{Z})$ represents the fundamental class $[B]$ of B and $\langle\cdot,\cdot\rangle$ is the natural pairing $\langle\cdot,\cdot\rangle : C^k(T) \otimes C_k(T) \to \mathbb{Z}$ (see chapter 3).

Remark 7.5.7. Note, this is the general Chern-Simons action for manifolds without boundaries, $\partial M = 0$. The case where $\partial M \neq 0$ is treated in a moment.

Let us now check the "well-definedness" of our definition:

First, we should check that the action does not depend on which cocycle we choose to represent $[\varpi]$, rather only on the class $[\varpi]$ itself. So, to check this, let ϵ be some integer-valued cochain and consider the shift $\varpi \mapsto \varpi + \delta\epsilon$. Under this shift, the action changes by

$$\begin{aligned}
\delta S_{M,P}(\partial\omega') &= \frac{1}{n}\left(\int_B \frac{k}{8\pi^2}\langle \mathcal{F}' \wedge \mathcal{F}'\rangle - \langle \gamma^*(\varpi + \delta\epsilon), b\rangle\right) - S_{M,P}(\partial\omega'),\\
&= -\frac{1}{n}\langle \gamma^*(\delta\epsilon), b\rangle,\\
&= -\frac{1}{n}\langle \gamma^*(\epsilon), \partial b\rangle,\\
&= -\frac{1}{n}\langle \gamma^*(\epsilon), n \cdot m\rangle,\\
&= -\langle \gamma^*(\epsilon), m\rangle,
\end{aligned}$$

where $m \in C_3(M) \equiv C_3(M;\mathbb{Z})$ represents the fundamental class $[M]$ of M. Now, since ϵ is integer-valued, we have that $\langle \gamma^*(\epsilon), m\rangle \in \mathbb{Z}$, or

$$\delta S_{M,P}(\partial\omega') = -\langle \gamma^*(\epsilon), [M]\rangle = 0 \qquad (\text{mod } 1).$$

Hence, our action does not depend on which cocycle we pick to represent $[\varpi]$, only on the class (up to an integer).

[14] Recall that this implies that P' is a bundle over B with ∂B consisting of n copies of M and that P' is isomorphic to P on each boundary component.

Second, note that for a closed 4-manifolds B

$$\int_B \frac{k}{8\pi^2}\langle \mathcal{F}' \wedge \mathcal{F}' \rangle = \langle \gamma^*(\varpi), b\rangle,$$

and so, in this case,

$$S_{M,P}(\partial\omega') = 0 \qquad (\mathrm{mod}\ 1).$$

This implies that our definition for the action does not depend on the particular choice of B or the way in which we extend the connection over B. In particular, let B_1 and B_2 be two bounding manifolds with $\partial B_1 = \sqcup_{i=1}^n M$ and $\partial B_2 = \sqcup_{i=1}^n M$. Then, gluing these two manifolds together along the boundaries gives an action over a closed manifold, which we have just seen vanishes. Hence, the difference between the two actions, the one over B_1 and the one over B_2, must vanish. Consequently, our definition does not depend on the choice of B. The same arguments show that it doesn't depend on how we extend ω to ω' either.

We next show that our definition for the action is homotopy invariant. To do this let γ_1 and γ_2 be two classifying maps, $\gamma_i : B \to BG$, which are homotopy equivalent; i.e., they are related by a homotopy transformation. Then, since cohomology classes are invariant under homotopy transformations, we have that

$$\gamma_1^*(\varpi) - \gamma_2^*(\varpi) = \delta\epsilon,$$

for some integer cochain ϵ. Now, denote by S_i the action defined with respect to the classifying map γ_i. Then, since the first part of the action (namely $\Delta(\mathcal{F}')$) does not depend on the particular classifying map, the difference between S_1 and S_2, which we denote by $\delta_{hom}S$, is given by

$$\delta_{hom}S = -\frac{1}{n}\langle \gamma_1^*(\varpi) - \gamma_2^*(\varpi), b\rangle = -\frac{1}{n}\langle \delta\epsilon, b\rangle = -\langle \epsilon, m\rangle.$$

Which, as stated before, implies that the action is invariant under a homotopy transformation,

$$\delta_{hom}S = 0 \qquad (\mathrm{mod}\ 1).$$

Finally, one can show the action given in (7.39) is indeed gauge invariant when considering manifolds without boundary. While, as we will see shortly, when the manifold has a boundary the action is gauge invariant up to a 'surface' term $\langle \gamma^*(\nu), \sigma\rangle$, where $\nu \in C^2(BG; \mathbb{R}/\mathbb{Z})$ and $\sigma \in C_2(\Sigma)$ represents the fundamental class $[\Sigma]$ of Σ - this is the WZW term we saw before (see [18]).

Remark 7.5.8. As a side remark, when the level of our theory is zero (i.e., $k = 0$) the Chern-Weil form vanishes, $\Delta(\mathcal{F}') = 0$. Consequently, the class $[\varpi] \in H^4(BG; \mathbb{Z})$ becomes torsion and hence, via the isomorphism $H^4(BG; \mathbb{Z}) \cong H^3(BG; \mathbb{R}/\mathbb{Z})$ (see proposition 7.5.4), $[\varpi]$ defines a torsion class $[\alpha] \in H^3(BG; \mathbb{R}/\mathbb{Z})$. Furthermore, letting $\alpha \in Z^3(BG; \mathbb{R}/\mathbb{Z})$ be a 3-cocycle which represents $[\alpha]$, the level 0 action becomes

$$S_{M,P} = \langle \gamma^*(\alpha), m\rangle. \tag{7.40}$$

This is important for the theories where G is finite; since in this case all connections are flat and so $\Delta(\mathcal{F}') = 0$ always. Thus, the general Chern-Simons action for a finite G (so-called Dijkgraaf Witten theories [9]) is given by (7.40).

To end this section we will show that our general Chern-Simons theory defined by (7.39) is both *unitary* and *additive over disjoint unions*. To show unitarity we note that reversing the orientation of B results in changing b to $-b$, while the integral of $\Delta(\mathcal{F}')$ over B also picks up a minus sign. Therefore, we have just proved the following proposition.

Proposition 7.5.9. *Under a change of orientation the action defined in* (7.39) *becomes negative itself,*

$$S_{\bar{M},P}(\partial\omega') = -S_{M,P}(\partial\omega'). \tag{7.41}$$

Proof. □

Remark 7.5.10. From remark 7.4.3 (b) we see that this implies that our theory is unitary.

Additionally we have the following.

Proposition 7.5.11. *Over a disjoint union, the action is additive.*

Proof. This follows from the linearity of $\langle\cdot,\cdot\rangle : C^k \otimes C_k \to \mathbb{Z}$ and from the usual properties of integrals. □

So, recapping, our general Chern-Simons action is (well-)defined and leads to a unitary theory which obeys factorization (i.e., behaves additively under disjoint union). Furthermore, from our construction we see that any class in $H^{n+1}(BG;\mathbb{Z})$ (equivalently, via the isomorphism $H^{n+1}(BG;\mathbb{Z}) \cong H^n(BG;\mathbb{R}/\mathbb{Z})$, any class in $H^n(BG;\mathbb{R}/\mathbb{Z})$) leads to a n-dimensional topological theory. Indeed, 3-dimensional Chern-Simons theory can be classified by $H^4(BG;\mathbb{Z})$, while 2-dimensional WZW theories are classified by $H^3(G;\mathbb{Z})$ (see the next section).

Manifolds with Boundary ($\partial\mathbf{M} \neq \mathbf{0}$)

In order to properly treat the case where M has a nonempty boundary, we must introduce the concept of *differential characters* [15]. Roughly speaking, a degree k differential character α is just like a k-cocycle (i.e., a homomorphism from the group of singular k-cycles $H_k(T)$ to some coefficient group, which we will take to be $\mathbb{R}/\mathbb{Z}$), but, instead of requiring α to vanish on boundaries, one simply requires that its value on a boundary equal the integral of some $(k+1)$-degree differential form Δ over the bounding chain - that is

$$\langle\alpha, \partial B\rangle = \int_B \Delta \qquad (\mathrm{mod}\ 1). \tag{7.42}$$

To be precise (here restricting to $k = 3$ for relevance to our problem), we define a **differential character** $\alpha \in C^3(BG;\mathbb{R}/\mathbb{Z})$ as the cochain obtained from a mod $\mathbb{Z}$ reduction of a real cochain $\beta \in C^3(BG;\mathbb{R})$ which satisfies

$$\delta\beta = \Delta(\mathcal{F}_u) - \varpi, \tag{7.43}$$

where $\mathcal{F}_u$ is the universal curvature (see section 5.3.1), $\Delta(\cdot)$ is defined in (7.37) and ϖ is a cocycle representing the class $[\varpi] \in H^4(BG;\mathbb{Z})$ defined in (7.38). One should note that, although (7.43) does not guarantee that α be closed or even uniquely defined, it does guarantee that the pullback of α to M via γ, $\gamma^*(\alpha) = \alpha_{\mathcal{A}} \in Z^3(M; \mathbb{R}/\mathbb{Z})$, is a (well-)defined cocycle which is completely determined by $[\varpi] \in H^4(BG; \mathbb{Z})$ and $\mathcal{A}$, see [18]. Using this differential character α, one can rewrite the general Chern-Simons action as

$$S_{M,P} = \langle \alpha_{\mathcal{A}}, m \rangle. \tag{7.44}$$

As is the case in cohomology, one can define a ring of differential characters, $\hat{H}^*(T, \mathbb{R}/\mathbb{Z})$ on some topological space T. Furthermore, when $\dim(T) = k$ and T is closed, the previous definition of $\hat{H}^k$ (namely homomorphisms from $H_k(T)$ to $\mathbb{R}/\mathbb{Z}$ obeying (7.42)) reduces to the usual cohomology group H^k since any $(k+1)$-degree form vanishes on a k-dimensional space - hence, all differential characters vanish on the boundary, thus becoming cocycles. Additionally, from (7.42) we see that: (1) Δ should be closed (vanish under the 'boundary' (exterior derivative) operator), so that acting on the left-hand side with the boundary operator (either on α or ∂B) gives zero, and (2) since $\langle \cdot, \cdot \rangle : C^k \otimes C_k \to \mathbb{Z}$ is integral the class $[\Delta] \in H^{k+1}(T; \mathbb{R})$ should also be integral. Stated differently, $[\Delta]$ will always be the image $\rho([\varpi])$ of some integral class $[\varpi] \in H^{k+1}(T; \mathbb{Z})$ under the natural map $\rho : H^{k+1}(T; \mathbb{Z}) \to H^{k+1}(T; \mathbb{R})$. However, as was the case before for closed manifolds, $H^{k+1}(T; \mathbb{Z})$ can contain torsion, and thus $[\varpi]$ is not completely determined by its image $\rho([\varpi])$. Instead, one can show (see [15]) that, when $T = BG$, the pair (Δ, ϖ) of 4-forms satisfying $[\Delta] = \rho([\varpi])$ determines a unique differential character $\alpha \in C^3(BG; \mathbb{R}/\mathbb{Z})$, via (7.43), which when pulled back to the three manifold $\gamma^*(\alpha)$ becomes a cocycle.

With this business of differential characters taken care of, we now proceed to define the general Chern-Simons action for compact manifolds, which can have non-empty boundaries.

Definition 7.5.12. Let M be an oriented compact manifold and let P be a principal G-bundle over M. Then, we define our general Chern-Simons action as

$$S_{M,P} = \langle \alpha_{\mathcal{A}}, m \rangle, \tag{7.45}$$

where $\alpha_{\mathcal{A}} \in Z^3(M; \mathbb{R}/\mathbb{Z})$ is the pullback of the differential character $\alpha \in C^3(BG; \mathbb{R}/\mathbb{Z})$ from BG to M via γ, $\alpha_{\mathcal{A}} = \gamma^*(\alpha)$.

Remark 7.5.13. Recall, α was defined as the mod $\mathbb{Z}$ reduction of some real cochain β which satisfied $\delta\beta = \Delta(\mathcal{F}_u) - \varpi$.

Let us now check the validity of our definition. In particular, we need to check that: (1) for fixed classifying maps γ our action is (well-)defined, and (2) that the action transforms in the appropriate sense under homotopy (gauge) transformations:

(1) From the defining properties of α, namely being a mod $\mathbb{Z}$ reduction of some real cochain β which satisfies $\delta\beta = \Delta(\mathcal{F}_u) - \varpi$, we see that one still has the possibility of a 'gauge' transformation

$$\alpha \mapsto \alpha + \delta\nu, \tag{7.46}$$

where $\nu \in C^2(BG; \mathbb{R}/\mathbb{Z})$. Now, let us see how this affects our action. First, pulling $\alpha + \delta\nu$ back to M gives $\alpha_{\mathcal{A}} + \delta\nu_{\mathcal{A}}$. Then, plugging this into our expression for the

Chern-Simons action gives

$$\begin{aligned} S_{M,P} &\mapsto \langle \alpha_{\mathcal{A}} + \delta\nu_{\mathcal{A}}, m\rangle, \\ &= S_{M,P} + \langle \nu_{\mathcal{A}}, \sigma\rangle, \end{aligned}$$

where in the last line we used the bilinearity of $\langle\cdot,\cdot\rangle$ along with the property $\langle \delta B, C\rangle = \langle B, \partial C\rangle$ and setting $\partial M = \Sigma$. Note, when M is closed, $\partial M = 0$, the second term drops out, implying that, in general, the Chern-Simons action is gauge invariant when defined on closed manifolds. While, for compact manifolds, the action is gauge invariant up to boundary terms. Thus, as we saw before with trivial bundles (see section 7.3.12), although the action is not invariant under a gauge transformation, it does change in a controlled way. Indeed, the previous transformation of the action will result in changing the path integral $\int e^{2\pi i S_{M,P}}$ by a phase, namely $e^{2\pi i \langle \nu_{\mathcal{A}}, \sigma\rangle}$, which can be absorbed into the wave functions representing the initial and final states[15].

(2) From [18] one can see that our definition is insensitive to homotopy transformations that leave γ fixed at the boundary of M. However, we still have to address the question of how the action responds to gauge transformations. To begin, let γ_0 and γ_1 be two homotopy equivalent classifying maps which induce the connections $\mathcal{A}$ and $\mathcal{A}'$ on M, here $\mathcal{A}'$ is the gauge transform of $\mathcal{A}$. Then, under this transformation, the action changes by

$$S_{M,P} \mapsto S_{M,P} + \langle \alpha_{\mathcal{A}}, \sigma \times l\rangle + \int_B \gamma^*(\langle \mathcal{F}_u\rangle), \tag{7.47}$$

where $\langle \mathcal{F}_u \in\rangle I^2(G)$ is an invariant polynomial in $\mathcal{F}_u$, $B = M \times I$ and $\sigma \times l \in C_3(\Sigma \times I)$ represents $[\Sigma \times I]$. Once again, as was the case for trivial bundles (see lemma 7.3.14 and the preceding paragraph), one can show that these extra pieces only depend upon the boundary information Σ. This shows that the action on the boundary piece should really be considered as a section of a line bundle over the space of connections modulo gauge transformations. Indeed, since the variation of $S_{M,P}$ only depends on the connection at the boundary and the gauge transformation, the path integral will transform with exactly the same phase as $e^{2\pi i S_{M,P}}$. Consequently, the path integral $Z(M)$, which is a function on the space of connections $\mathcal{A}(\Sigma)$ at the boundary Σ, cannot be considered as a function on the space of connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$. Rather, one should think of $Z(M)$ as a section of a line bundle over $\mathcal{A}/\mathcal{G}$ [18].

So, finally, we have a definition of the Chern-Simons action which holds for any type of principal G-bundle over a manifold M, where G is compact!

Recapping what we have learned so far. In a (classical) d-dimensional field theory we can allow actions which are of the form

$$e^{2\pi i S_{M,P}(\cdot)} : \mathcal{C}_M \to \mathbb{C}, \tag{7.48}$$

[15] When the boundary of M is non-empty, the path-integral on M represents a transition amplitude among initial and final states on the boundary [18].

where $S_{M,P}(\cdot)$ may not even be defined (here M is a d-dimensional closed manifold). Furthermore, we can extend these ideas of fields and actions to closed $(d-1)$-manifolds Σ,

$$L_{\Sigma,\cdot} : \mathcal{C}_\Sigma \to \mathcal{L}, \tag{7.49}$$

where $\mathcal{L}$ is the category of all finite dimensional Hilbert spaces. Finally, if M is a d-dimensional compact manifold with boundary, the exponentiated action (7.48) has a generalization which is explained via the following diagram of line bundles:

$$\begin{tikzcd} r^*(L_{\partial M}) \arrow[r] \arrow[d, "\pi"'] & L_{\partial M} \arrow[d, "\bar{\pi}"] \\ \mathcal{C}_M \arrow[r, "r"] & \mathcal{C}_{\partial M}, \end{tikzcd} \tag{7.50}$$

where $r : M \to \partial M$ is the restriction map, which restricts a field to the boundary, and $r^*(L_{\partial M})$ is the pullback bundle of $L_{\partial M}$ along r^*. The line bundle $L_{\partial M} \xrightarrow{\bar{\pi}} \mathcal{C}_{\partial M}$ is the extended action given in (7.49). We think of the action $e^{2\pi i S_{M,P}(\cdot)}$ is a section of the line bundle $r^*(L_{\partial M}) \xrightarrow{\pi} \mathcal{C}_M$. The next step will be to quantize our, previously constructed, classical theory.

7.6 Quantization: A Primer

In the previous section we defined the Chern-Simons action for any general principal G-bundle over some manifold M. However, everything was purely on a classical level. Now we need to quantize the classical theory given by (7.39). To do this one usually calls upon the technique of quantizing a classical theory by either "defining" a path integral (the quotation marks will become clear in a moment) or by canonical quantization. In this section we would like to lay out these two basic paths (pun intended) which a theory can go through beginning as a classical theory and ending up a quantum theory. In the next chapter we will explicitly work out the details of quantizing the Dijkgraaf-Witten theories.

In order to define a classical theory, one usually starts by defining a *spacetime*. So far, for us spacetime has been the base manifold M. Next, one considers a space of *fields*, which are functions of some kind on the spacetime manifold - or more generally, sections of bundles over M. On the space of fields, is where the *Lagrangian* is defined. For us, the Lagrangian is given by the Chern-Simons form. Additionally, there exists a functional which acts on the Lagrangian, known as the *action* and usually denoted by the letter S (plus a few other indices - we have been denoting our action by $S_{M,P}(\omega)$). From the previous arguments, we know that the action, in our theory, is given by

$$S_{M,P}(\partial\omega') = \frac{1}{n}\left(\int_B \frac{k}{8\pi^2}\langle \mathcal{F}' \wedge \mathcal{F}' \rangle - \langle \gamma^*(\varpi), b\rangle\right) \qquad (\text{mod } 1). \tag{7.51}$$

Note, when M is closed, our action, now viewed as $e^{2\pi i S_{M,P}}$ to be technically correct, gives a complex number of unit, while if M has boundary the action gives an element of unit norm in an abstract metrized line which depends on the restriction of the field to ∂M; that is, the classical action is given by a section of the line bundle $\mathcal{L}_{\partial M} \to \mathcal{C}_{\partial M}$. Viewing ∂M as the space of our theory and M as the spacetime, one can say that the Chern-Simons action assigns a complex metrized line to each space and an element in the line associated to ∂M for each spacetime.

In classical field theory we are done after stating the action. This is all there is to define the theory. From the action one usually then obtains field equations via Hamilton's principle of least action and solves these field equations to gain further insight into the physics described by the action. For the case of Chern-Simons theory, classical solutions are given by flat connections. Even though flat connections imply zero curvature, classical solutions of Chern-Simons theory are far from trivial. Indeed, the fact that the action is not a real number, but is only defined up to integers, leads to interesting geometries.

To go further and define a quantum theory there are usually two routes taken - *canonical quantization* or *path integral quantization.* In canonical quantization one usually starts by promoting the classical fields and their canonical momenta to operators (collections of creation and annihilation operators whose coefficients carry the properties of the fields) which act on a certain Hilbert space, known as a Fock space, which labels each type of particle in each state. For example, the easiest type of quantum field theory is the scalar field theory described by the Klein-Gordon equation of motion,

$$(\Box + m)\,\psi(x) = 0.$$

From this one can show that the field operators and canonical momentum operators become

$$\psi(x) = \int \frac{d^3k}{2\pi^3}\left(a(k)e^{-ik\cdot x} + a^{\dagger}(k)e^{ik\cdot x}\right),$$

$$\Pi(x) = -\frac{i}{2}\int \frac{d^3k}{2\pi^3}\left(a(k)e^{-ik\cdot x} - a^{\dagger}(k)e^{ik\cdot x}\right),$$

where $a^{\dagger}(k)$ acts on the Fock space by adding a particle of momentum k, while $a(k)$ destroys a particle of momentum k. Further restrictions are placed on the operators, such as obeying the principle of causality, Poincaré invariance, and so on. Furthermore, starting from the vacuum Fock space (i.e., the state with zero particles) one can construct all others. For other field theories (such as vector fields, spinors etc.), the procedure is similar. Although, one must be careful when quantizing gauge theories, so as to make sure everything is consistent. In terms of logical flow, it is the author's opinion that the method of canonical quantization prevails. However, when asking questions such as particle scattering there is another method which is far better suited, that of path integral quantization.

Roughly speaking, quantizing a system via path integrals amounts to "defining" an object, from the classical action, known as the *partition function*

$$Z = \int_{\mathcal{C}} \mathcal{D}\psi e^{2\pi i S_{M,P}}. \tag{7.52}$$

Here $\mathcal{C}$ is the space of fields ψ and the integral measure $\mathcal{D}\psi$ is taken over the space of fields. Hence, this is why we have the quotation marks around define - it is usually impossible to define such a measure. Typically one replaces the space $\mathcal{C}$ by the quotient (or moduli) space $\mathcal{C}/\mathcal{S}$, where $\mathcal{S}$ is the space of symmetries of the fields. The reason for doing this is to restrict the space of integration to only those fields which are physically distinct and not related to other fields via a symmetry $s \in \mathcal{S}$. Although this does reduce the size of space to integrate over, the quotient spaces are still usually infinite dimensional. In Chern-Simons theory this is no different. The space of fields is taken to be the space of connections modulo gauge transformations $\mathcal{C} = \mathcal{A}(P)/\mathcal{G}(P)$, which is typically infinite dimensional. However, in the case where G is finite the space of connections modulo gauge transformations becomes finite too. Thus there is no problem in defining the path integral. In the next chapter we consider precisely this case. Finally, from the partition function one can calculate all other properties relevant to the quantum theory.

Chapter 8

Topological Quantum Field Theories (TQFTs)

The main goal of this chapter is to quantize the classical Chern-Simons action. We will restrict to the case where G is a finite group, which we denote by Γ; the so-called Dijkgraaf-Witten theories [9]. By restricting to a finite group G, our path integral becomes a finite sum, thus alleviating the pain caused by the inability to define a measure on the infinite dimensional moduli space $\mathcal{A}/\mathcal{G}$. We will begin from the very beginning: we first define the Lagrangian and the action functional. Then we define the 'path integral' (or rather path sum). Finally, we will show that the quantum field theory arising from the Dijkgraaf-Witten action does, in fact, satisfy the axioms of a *Topological Quantum Field Theory* (see below).

8.1 Atiyah's Axiomatic Definition of a TQFT

We begin this section by reviewing the axioms which any topological quantum field theory should obey. These axioms were first written out by Atiyah [2] in an attempt to mimic the work done by Segal in the area of conformal field theories.

Definition 8.1.1. A n-dimensional **topological quantum field theory** (**TQFT** for short) is an assignment:

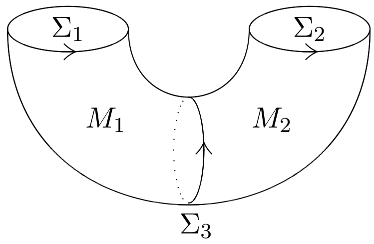
- To each oriented closed $(n-1)$-dimensional manifold Σ, we assign a vector space V_Σ over some fixed field $\Bbbk$.
- To each oriented n-dimensional manifold M, we assign an element (or vector) $Z(M) \in V_{\partial M}$ to the vector space associated to the boundary of M.

Furthermore, these assignments obey the following axioms:

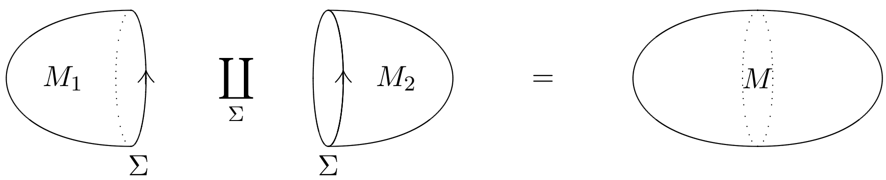
(a) The assignments are functorial; i.e, any orientation preserving diffeomorphism $f : \Sigma_1 \to \Sigma_2$ induces an isomorphism $Z(f) : V_{\Sigma_1} \xrightarrow{\cong} V_{\Sigma_2}$, along with $Z(gf) = Z(g)Z(f)$ where $g : \Sigma_2 \to \Sigma_3$. Additionally, if f can be extended to a diffeomorphism $f : M_1 \to M_2$, where $\partial M_i = \Sigma_i$, then $Z(f)$ maps the vector $Z(M_1) \in V_{\Sigma_1}$ to $Z(M_2) \in V_{\Sigma_2}$.

(b) The $(n-1)$-dimensional closed manifold with reversed orientation, which we denote by $\bar{\Sigma}$, is assigned the dual vector space of that which is assigned to Σ; that is, $\bar{\Sigma} \mapsto V_\Sigma^*$, where V_Σ^* is the dual vector space to V_Σ.

(c) The assignment is multiplicative with respect to disjoint unions. To be more precise, let $\Sigma = \sqcup_i \Sigma_i$ then we have that $\Sigma \mapsto V_{\Sigma_1} \otimes V_{\Sigma_2} \otimes V_{\Sigma_3} \cdots (\equiv \bigotimes_i V_{\Sigma_i})$, while if $\partial M = \sqcup_j \Sigma'_j$ then $Z(M) \in \bigotimes_j V_{\Sigma'_j}$. Moreover, if $\partial M_1 = \Sigma_1 \sqcup \Sigma_3$, $\partial M_2 = \bar{\Sigma_3} \sqcup \Sigma_2$, and $M = M_1 \coprod_{\Sigma_3} M_2$ is the manifold obtained by gluing M_1 to M_2 along Σ_3:



we then require $Z(M) = \langle Z(M_1), Z(M_2)\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing (from the duality map) $V_{\Sigma_1} \otimes V_{\Sigma_3} \otimes V_{\Sigma_3}^* \otimes V_{\Sigma_2} \to V_{\Sigma_1} \otimes V_{\Sigma_2}$. Note, this implies that one can compute $Z(M)$ by cutting M into 'pieces', which is a very powerful property of a TQFT. Indeed, suppose M_1 and M_2 are compact, oriented n-dimensional manifolds with $\partial M_1 = \partial M_2 = \Sigma$. Then, we can glue M_1 and M_2 along Σ (here reversing the orientation of $\partial M_2 = \Sigma$) to obtain a closed n-dimensional manifold $M = M_1 \coprod_\Sigma M_2$:



Furthermore, since $\partial M_1 = \partial M_2 = \Sigma$, we see that the natural pairing $V_\Sigma \otimes V_\Sigma^* \to \Bbbk$ gives an element, $Z(M) \in \Bbbk$, in the ground field; hence $Z(M)$ is an invariant of M. Thus, all invariants of closed n-dimensional manifolds can be obtained by cutting the closed manifold into two pieces and the invariants can be obtained via the previous gluing procedure. However, this is not the end of the story. As we will see later on, one can further cut the resulting two manifolds to give manifolds of higher codimensions, all the way down to points, and the invariant associated to the starting manifold M can be calculated, as before, from these maximal codimension manifolds (i.e. points)[1].

(d) The $(n-1)$-dimensional empty set $\emptyset$ is mapped to the ground field, $\emptyset \mapsto \Bbbk$, while the n-dimensional empty set $\emptyset$ is mapped to the identity element in $\Bbbk$, $Z(\emptyset) = 1_\Bbbk$.

[1]This implies that an *extended TQFT* (i.e. a TQFT which is defined not only on n-dimensional and $(n-1)$-dimensional manifolds, but all the way down to points) is determined by its action on a point, see [31].

Using the language of categories, we can compactly bundle Atiyah's axioms in the following way.

Definition 8.1.2. A n-dimensional **TQFT** is a symmetric monoidal functor (see chapter 2) from the category of n-dimensional cobordisms (whose monoidal structure is given by the disjoint union, $\sqcup$) to the category of vector spaces $\mathscr{V}_1$ over a fixed field $\Bbbk$ (whose monoidal structure is given by the tensor product, $\otimes$)

$$Z : (\mathrm{Cob}_n, \sqcup) \longrightarrow (\mathscr{V}_1, \otimes).$$

Hence (see below), we can view a TQFT as a (unitary) representation of the cobordism category!

Indeed, we can get a functor from Atiyah's axioms as follows. First, note that the objects in Cob_n are $(n-1)$-dimensional, closed, oriented manifolds while the objects in $\mathscr{V}_1$ are vector spaces (over some fixed field $\Bbbk$). Thus, according to the axioms, we already have objects mapping to objects and so, we only need to show that the vector $Z(M)$ assigned to some n-dimensional object (or cobordism) M really is just a linear transformation between the vector spaces assigned to ∂M. To do this, let $M : \Sigma_1 \to \Sigma_2$ be a cobordism, then $\partial M = \bar{\Sigma}_1 \sqcup \Sigma_2$. Therefore, from the axioms, we see that

$$Z(M) \in V_{\Sigma_1}^* \otimes V_{\Sigma_2}.$$

However, we also have the isomorphism $V_{\Sigma_1}^* \otimes V_{\Sigma_2} \cong \mathrm{Hom}_{\mathscr{V}_1}(V_{\Sigma_1}, V_{\Sigma_2})$. Hence, $Z(M)$ is a morphism $Z(M) : V_{\Sigma_1} \to V_{\Sigma_2}$ and thus, we get a (symmetric monoidal) functor from Atiyah's axioms. So, we can and will switch back and forth between the two notions of a TQFT - the axiomatic approach and the functorial approach. Note, when expressed as a cobordism, Atiyah's multiplication axiom (c) shows that, for the manifold $\Sigma \times I$, the linear map $Z(\Sigma \times I) \in \mathrm{End}(V_\Sigma)$ is indempotent - its square gives the identity - and, more generally, it acts as the identity on the subspace of V_Σ spanned by all of the elements $Z(M)$, where $\partial M = \Sigma$. Furthermore, if one replaces V_Σ by its image under the endomorphism $Z(\Sigma \times I)$ they see that all of the previous axioms of a TQFT are still satisfied. Consequently, it is reasonable to assume that $Z(\Sigma \times I) = 1_{\Bbbk}$.

Aside 8.1.3. **(1) Homology vs. TQFT**

Recall, from section 3.1, that a homology theory assigns to a topological space T an abelian group, and to each map between two topological spaces T and T' a mapping between the two groups associated to each space. Hence, we can think of a homology theory is a functor from the category of topological spaces Top to the category of abelian groups Ab,

$$H : \mathrm{Top} \longrightarrow \mathrm{Ab}. \tag{8.1}$$

Now, we can further introduce structure into Top and Ab via the monoidal product $\otimes$ (see chapter 2): for Top the monoidal product is given by disjoint union, while for Ab the monoidal product is the direct sum, $\oplus$. Furthermore, we see that both monoidal structures are symmetric; that is, there exists a braiding[2] in each category. We mention that the

[2]For any pair of objects x and y, there exists a natural isomorphism $B_{x,y} : x \otimes y \longrightarrow y \otimes x$, which satisfies several axioms, in particular the symmetry equation, $B_{x,y}B_{y,x} = 1_{x\otimes y}$ (see section 2.3).

homology functor H must also satisfy the Mayer-Vietoris sequence, along with several others (see [40]). Hence, the homology functor is symmetric and monodial. Additionally, note that the homology functor is additive under disjoint union. Finally, one could refine the codomain category Ab by replacing it with the symmetric, monoidal category of Vector spaces over some fixed field $\Bbbk$, here the monoidal structure is given by the direct sum of vector spaces. So, recapping, we can think of the homology functor H as a symmetric monoidal functor

$$H : (\mathrm{Top}, \sqcup) \longrightarrow (\mathscr{V}_1, \oplus), \tag{8.2}$$

which is additive under disjoint unions (neglecting other axioms of homology of course).

On the other side of the coin we have a TQFT functor, which, as we have seen, is a symmetric, monoidal functor from the category of n-dimensional cobordisms (if our TQFT is of dimension n) - here the monoidal structure is given by the disjoint union - to the category of Vector spaces over a fixed $\Bbbk$ with the tensor product of vector spaces giving the monoidal structure,

$$Z : (\mathrm{Cob}_n, \sqcup) \longrightarrow (\mathscr{V}_1, \otimes).$$

As one can see, the big difference between a TQFT functor and a homology functor is that instead of taking the direct sum of vector spaces as the monoidal product in $\mathscr{V}_1$, we take the tensor product. Hence, under disjoint unions, the TQFT functor Z is multiplicative whereas the homology functor H is additive. Finally, note that one could replace the category $\mathscr{V}_1$ by *any* other symmetric monoidal category and/or impose further constraints on Cob_n. For example, one could define the TQFT functor as a symmetric monoidal functor between the category of cobordisms with spin structure and the category of R-modules over some commutative ring R,

$$Z : (\mathrm{SpCob}_n, \sqcup) \longrightarrow (\text{R-mod}, \otimes).$$

(2) TQFT as a Representation of Cob_n

Thinking of a group G as the category, denoted Grp_G, consisting of one object and morphisms 'labeled' by the elements of G, we see that a representation of G is nothing more than a functor from Grp_G to $\mathscr{V}_1$

$$\rho : \mathrm{Grp}_G \longrightarrow \mathscr{V}_1.$$

Thus, extrapolating, we can think of a TQFT as a (unitary) 'representation' of the category of n-dimensional cobordisms.

8.2 Invariant Section Construction of a TQFT

We now refine the notion of the *invariant section construction* (see section 7.4.1), or *inverse limit construction*, to taylor more to our needs[3]. Recall, the invariant section construction is a procedure which allows one to "glue together" several related objects, the precise manner of the gluing process being specified by morphisms between the objects. This has already come in handy when we took the collection of Chern-Simons lines in

[3]In what follows, we outline the discussion found in [28].

the last section and glued them together to construct a bundle over the moduli space of connections. Further, in the construction of TQFTs on manifolds it is sometimes helpful to assume that each particular manifold has some 'additional' data; e.g., triangulations. On the other hand, the theory will then only produce an invariant of a manifold endowed with this extra structure, and hence, the TQFT is not really a topological invariant. However, we are saved by the invariant section construction, because we can construct a TQFT with additional data and then use the invariant section construction to eliminate the added data, thus giving a true topological invariant. Recall, we have already seen that invariant sections are invariant under the additional data of gauge symmetries (see section 7.4).

To begin, recall that, in its purest form, a TQFT is a symmetric monoidal functor Z from the category of n-dimensional cobordisms to the category of vector spaces over a fixed field $\Bbbk$. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are two domain categories - hence $\mathcal{M}$ and $\mathcal{M}'$ are two categories of n-cobordisms possibly with different added structures, for e.g. we could take $\mathcal{M} = \mathrm{Cob}_n$ and $\mathcal{M}' = \mathrm{SpCob}_n$ - and that there exists a functor $r : \mathcal{M} \to \mathcal{M}'$. Furthermore, let $Z' : \mathcal{M}' \to \mathscr{V}_1$ be a TQFT over $\mathcal{M}'$ (suppressing the monoidal structure symbols for clarity). Now, we want to construct a TQFT on $\mathcal{M}$ in such a way that the following diagram commutes

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "Z'"] \arrow[d, "r"'] & \mathscr{V}_1 \\ \mathcal{M} \arrow[ur, dashed, "Z"'] & \end{tikzcd}$$

Think of the objects in $\mathcal{M}'$ as being objects in $\mathcal{M}$ endowed with extra structure, such as a triangulation or a metric. Thus, we write the objects in $\mathcal{M}'$ as a pair (M, α) where $M \in \mathrm{Obj}(\mathcal{M})$ and α is the additional data. Hence, $r(M, \alpha) = M$ and $Z'(M, \alpha) \in V_{\partial M, \partial\alpha}$, where $\partial\alpha$ is the restriction of the additional data to the boundary ∂M.

Assuming that $Z'(M, \alpha) = Z'(M, \beta)$ for any data α and β that satisfy $\partial\alpha = \partial\beta$, our goal is, for each $(n-1)$-dimensional manifold Σ, to identity (or glue) the vector spaces $Z'(\Sigma, \alpha)$ and then to define a particular vector, in this vector space, associated with each M that obeys $\partial M = \Sigma$. To proceed, fix Σ and assume that the possible extra data on Σ, $r^{-1}(\Sigma)$, forms a category, which we will call $\mathscr{C}$, and that there exists a functor $F : \mathscr{C} \to \mathscr{V}_1$ such that $F(\Sigma, \alpha) = Z'(\Sigma, \alpha)$. Stated differently, we are assuming that for each morphism $f : \alpha \to \beta$ in $\mathscr{C}$, there is defined a linear transformation $F(f) : Z'(\Sigma, \alpha) \to Z'(\Sigma, \beta)$ which is compatible with composition. Now, we can use this functor F to define a new vector space $Z(\Sigma)$ which does not depend on the extra data α. We do so by defining

$$Z(\Sigma) := \Big\{ v : \alpha \mapsto v(\alpha) \mid v(\alpha) \in Z'(\Sigma, \alpha),\ \forall\alpha;\ F(f)\big(v(\alpha)\big) = v(\beta),\ \forall f : \alpha \to \beta \in \mathscr{C} \Big\}.$$

That is, $Z(\Sigma)$ is the space of all flat sections. Thus, an element of $Z(\Sigma)$ is a choice, for each possible additional structure α on Σ, of an element of $Z'(\Sigma, \alpha)$ in such a way that they are compatible with respect to F.

Now, suppose that $\mathscr{C}$ is connected; i.e., any two objects in $\mathscr{C}$ are connected by a morphism. Furthermore, suppose that F has no 'holonomy', that is, $F(f)$ depends only on the source α and target β of f, and so may be denoted $F(\alpha \to \beta)$. Recall, this is equivalent to requiring $F(f) = id$ when f is an automorphism. Indeed, consider the following proposition:

Proposition 8.2.1. *Suppose that $\mathscr{C}$ is connected and that F has no holonomy. Then, there exists an isomorphism*

$$Z(\Sigma) \cong \Big\{ v \in Z'(\Sigma, \alpha_0) \mid F(f)\big(v\big) = v, \ \forall f : \alpha \to \alpha \Big\}.$$

for each possible additional structure α on Σ.

Proof. See [28]. □

Finally, to define Z on objects $M \in \mathrm{Obj}(\mathcal{M})$, pick a representative of $r^{-1}(M)$, say $(M, \theta) \in \mathrm{Obj}(\mathcal{M})'$, and define $Z(M)$ to be the choice

$$\alpha \mapsto F(\partial\theta \to \alpha)\big(Z'(M, \theta)\big).$$

8.3 Dijkgraaf-Witten Theory (Chern-Simons Theory with Finite Group)

N.B. 8.3.1. Typically in a course on TQFTs one usually starts with some kind of algebra (usually a Hopf algebra), and from it they then construct a topological quantum field theory [39]. However, here we go the other (more physically appealing) way. That is, we define a field theory - classical action, path integral, etc. - and from its solutions we derive these algebras. We believe that (at least to the majority of the physics community) this approach is more appealing.

We now restrict the material of the previous chapter to the case where G is a finite (or discrete) group, which, to adhere to conformity, we write as $G \equiv \Gamma$. Additionally, we allow for the base space manifolds M to have dimension d (not necessarily $d = 3$). There are several observations to be made when one restricts the structure group G to be finite. First, when our structure group Γ is finite, we are no longer dealing with principal bundles in the sense one is used to; in the case where the structure group is finite, our bundles become regular covering spaces[4]. Furthermore, we have an isomorphism between the set of connections (which are always flat for finite G) on $\Gamma \hookrightarrow P \xrightarrow{\pi} M$ and the set of homomorphisms from $\pi_1(M; x)$ to Γ, i.e.

$$\mathcal{A}(P) \cong \mathrm{Hom}\big(\pi_1(M; x), \Gamma\big). \tag{8.3}$$

Additionally, we can extend this isomorphism to the set of all connections on P modulo gauge transformations by taking the quotient of the group of homomorphism with Γ,

$$\mathcal{A}(P)/\mathcal{G}(P) \cong \mathrm{Hom}\big(\pi_1(M; x), \Gamma\big) \Big/ \ \Gamma. \tag{8.4}$$

[4]The reader is assumed to know about covering spaces, regular coverings and deck transformations. If these concepts are unfamiliar, the reader may want to consult [25], or they may simply think of covering spaces, regular coverings and deck transformations as the discrete G version of fibre bundles, principal bundles and gauge transformations, respectively.

We realize these isomorphisms as follows. To begin, suppose that $\Gamma \hookrightarrow P \xrightarrow{\pi} M$ is a principal covering space over a connected manifold M and fix a basepoint $x \in M$ along with $p \in P_x$, where P_x is the fibre of P over $x \in M$. Then, a principal covering space with flat connection determines a map

$$\pi_1(M; x) \longrightarrow \Gamma,$$

by assigning to a loop in M, starting and ending at x, the holonomy, $g \in \Gamma$, around the loop. That is, the loop at x lifts uniquely to a path in P which starts at $p \in P_x$ and ends at $p' \in P_x$ (see figure 8.1). Defining the *holonomy* to be the unique $g \in \Gamma$ such that $p' = p \cdot g$, we, in fact, construct a map from $\pi_1(M; x)$ to Γ. Finally, it can be shown (see [21]) that: (1)

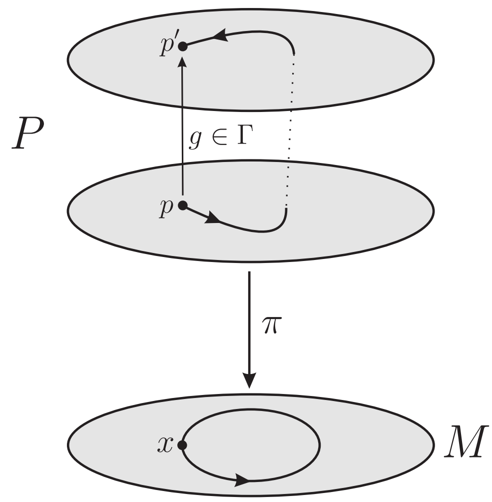


Figure 8.1: Visualization of holonomy.

g only depends on the homotopy class of the loop (i.e. the connections are flat), (2) the map $\pi_1(M; x) \to \Gamma$, which we denote by γ, is in fact a homomorphism and (3) changing $p \in P_x$ and/or $x \in M$ results in a new homomorphism from $\pi_1(M; x)$ to Γ defined by $g \cdot \gamma \cdot g^{-1}$. So, given a connection on P we get a homomorphism from the fundamental group on M to Γ, i.e. we have a map from the set of connections on P modulo gauge transformations to $\mathrm{Hom}\big(\pi_1(M; x), \Gamma\big)/\ \Gamma$. Consequently, the previous properties obeyed by γ imply that $\mathcal{A}(P)/\mathcal{G}(P) \to \mathrm{Hom}\big(\pi_1(M; x), \Gamma\big)/\ \Gamma$ is an isomorphism of sets. Furthermore, since all Γ-bundles are flat when Γ is finite, the topology can only be detected in the possible holonomy around homotpically nontrivial loops on M. Hence, finite Γ-bundles (or covering spaces) are completely determined by homeomorphisms from $\pi_1(M)$ to G. Roughly speaking, *in the case where Γ is finite, the set of connections and the set of Γ-bundles are equivalent.*

Another important observation to be made is that since our set of connection forms on P is isomorphic to $\mathrm{Hom}\big(\pi_1(M; x), \Gamma\big)$ and since M is compact, we see that the set of connections is discrete, or finite. This implies that if we define our partition function as the integral of $\exp(2\pi i S_M(P))$ over the space of connection forms (modulo gauge transformations,

$$Z(M) = \int_{\mathcal{A}(P)/\mathcal{G}(P)} e^{2\pi i S_M(P)} \mathcal{D}\mathcal{A},$$

then, since $\mathcal{A}(P)$ is finite, the path integral reduces to a finite sum; thus, we no longer have to worry about the existence of measures on the space of connections. And so, we will always be able to define a partition function, and hence quantum theory, and we will always be able to evaluate this integral.

Even though restricting to the case of a finite structure group gives a great deal of simplifications, we still do not lose all of the interesting features. For instance, the algebraic and topological structure associated to the theory is virtually unaltered, along with being much clearer when one does not have to worry about all of the analysis required to properly define the integration measures.

8.3.1 Classical Theory

Denote by $B\Gamma$ the classifying space for Γ (see chapter 4). Then, if $\Gamma \hookrightarrow P \xrightarrow{\pi} M$ is a covering space, there exists a covering space map (or classifying map) $\gamma_M : M \to B\Gamma$ and an induced map $\tilde{\gamma}_M : P \to E\Gamma$. We will use the classifying map to pull elements of specific cohomology groups defined on $B\Gamma$ back to M in order to define the action of our theory.

To begin, we define our *spacetime* M as an oriented, compact manifold of dimension d. We denote the boundary of M, or *space*, by $\Sigma \equiv \partial M$. The *fields* $\overline{\mathcal{C}}_M$ in our theory are given by equivalence classes of regular covering spaces.

Remark 8.3.2. Hence, an element in $\overline{\mathcal{C}}_M$ is given by the set of all Γ-covering spaces over M which have bundle maps between them (see section 4.5); i.e., if P and P' are in the same class $[P]$, then this implies that there exists a bundle map $\varphi : P \to P'$, with induced map on M the identity, $\hat{\varphi} = id_M$.

From the preceding arguments, namely since $\mathcal{A}(P)/\mathcal{G}(P) \cong \overline{\mathcal{C}}_M$ for finite Γ-bundles, we see that there is a natural identification

$$\overline{\mathcal{C}}_M \cong \mathrm{Hom}\big(\pi_1(M;x), \Gamma\big) \Big/ \ \Gamma, \tag{8.5}$$

which says that the fields in our theory can be thought of as homomorphisms from $\pi_1(M;x)$ to Γ which are invariant under Γ. Moving along, fix a class $[\alpha] \in H^d(B\Gamma;\mathbb{R}/\mathbb{Z})$ and its representative singular cocycle $\alpha \in Z^d(B\Gamma;\mathbb{R}/\mathbb{Z})$. Additionally, for each covering space $P \to M \in \mathrm{Obj}(\mathcal{C}_M)$ choose a classifying map $\gamma_M : M \to B\Gamma$ (existence of such a classifying map exists by definition of the universal covering space $E\Gamma \to B\Gamma$). Then, we define our *Lagrangian* to be the cohomological[5] pullback of $\alpha \in Z^d(B\Gamma;\mathbb{R}/\mathbb{Z})$ via the classifying map γ_M, $\gamma_M^*(\alpha) \in Z^d(M;\mathbb{R}/\mathbb{Z})$. Finally, when M is <u>closed</u>, we define our action $S_{M,[\alpha]}(P)$ on the field P to be the natural pairing of the pullback of α, via γ_M, with the fundamental class, $[M] \in H_d(M)$, of M; that is

$$S_{M,[\alpha]}(P) := \big\langle \gamma_M^*(\alpha), m \big\rangle, \tag{8.6}$$

where $m \in C_d(M)$ represents the fundamental class of M and $\langle \cdot, \cdot \rangle : C^k(M) \otimes C_k(M) \to \mathbb{Z}$ is the natural pairing[6].

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[5]See chapter 3 for a review of cohomology theory, in particular pullbacks of cochains.

[6]Note, even though we write the action $S_{M,[\alpha]}(P)$ as a function of P, it is really a function of the induced

Remark 8.3.3. Note that, in terms of de Rham cohomology, one can think of (8.6) as the integral of $\gamma_M^*(\alpha)$ over M,

$$S_{M,[\alpha]}(P) = \int_M \gamma_M^*(\alpha).$$

Additionally, since all classifying maps for P are homotopic through Γ maps, the action defined in (8.6) does not depend on the choice of F.

Perhaps it will help the reader if we give a little motivation (sketch a proof) for why (8.6) is the correct expression for the action[7], at least for the case where $\dim(M) = 3$ (in what follows the reader can consult section 7.5 for further details). To begin, recall that, given a bounding 4-manifold B over M and an integer-valued cocycle $\varpi \in Z^4(BG;\mathbb{Z})$ representing the cohomology class $[\varpi] \in H^4(BG,\mathbb{Z})$, the action was defined as

$$S_{M,P} = \frac{1}{n}\left(\int_B \frac{k}{8\pi^2}\langle \mathcal{F}' \wedge \mathcal{F}'\rangle - \langle \gamma_B^*(\varpi), b\rangle\right) \qquad (\text{mod } 1),$$

where $\gamma_B^* : H^4(BG;\mathbb{Z}) \to H^4(B;\mathbb{Z})$ is induced by $\gamma_B : B \to BG$ and $b \in C_4(B)$ represents $[B]$. Furthermore, in the case where G is finite, the class $[\varpi]$ is torsion and hence, due to the isomorphism

$$H^4(BG;\mathbb{Z}) \cong H^3(BG;\mathbb{R}/\mathbb{Z}),$$

(see the proof of proposition 7.5.4) determines a torsion class $[\alpha] \in H^3(BG;\mathbb{R}/\mathbb{Z})$. Additionally, when G is finite, all connections are flat and thus all curvature forms vanish, $\mathcal{F}' = 0$. Consequently, the action, which is now independent of the connection[8], is rewritten as

$$S_{M,[\alpha]}(P) = \langle \gamma_M^*(\alpha), m\rangle,$$

where, in this case, $\gamma_M^* : Z^3(BG;\mathbb{R}/\mathbb{Z}) \to Z^3(M;\mathbb{R}/\mathbb{Z})$ is induced by $\gamma_M : M \to BG$ and α is the 3-cocycle representing the class $[\alpha] \in H^3(BG;\mathbb{R}/\mathbb{Z})$. And so, we recover our original formula for the action by restricting the general action to the case where G is finite. The action (8.6) only depends on the cohomology class $[\alpha] \in H^d(B\Gamma;\mathbb{R}/\mathbb{Z})$ and not on the representative cocycle α. Furthermore, we will often write the action as $e^{2\pi i S_{M,[\alpha]}(P)}$ (in order to avoid any difficulties which might obstruct the action from being (well-)defined) and so, for closed manifolds, the action will take its values in the circle group $\mathbb{T}$ (equivalently $\mathrm{U}(1)$).

When M is not closed we run into the following problem. The d-dimensional action S on a $(d-1)$-manifold Σ,

$$S_{\Sigma,[\alpha]}(Q) = \langle \gamma_\Sigma^*(\alpha), \sigma\rangle,$$

---

classifying map γ_M. However, by definition, any classifying map γ'_M which 'differs' (i.e., not homotopic) from γ_M will belong to a new covering space P'. Therefore, it does not matter if we label the action to be a function of P or γ_M, since they are the same thing (up to homotopy).

[7] Here, just to make things a little more concrete, we will consider the case where M is three dimensional, while the generalizations to (d)-dimensions is completely straightforward.

[8] This is due to the fact that all connections on regular covering spaces are flat.

makes no sense as a natural pairing. Indeed, we are pulling back a cocycle α of degree d to a cocycle of degree d on Σ and then trying to pair it with the fundamental class on Σ, which happens to be of degree $d-1$. Furthermore, what does it mean for Σ to have cocycles of such degree as α (remember $\dim(\Sigma) = d-1$)? The solutions to these problems is to recall that singular homology can have degenerate chains; that is, roughly speaking, singular maps $c : \Delta^n \to M$ (with $\dim(M) \leq n$) which 'pinch' along a face, thus reducing in dimension. In fact, by investigating how singular cochains act on these degeneracies (see proposition 8.3.4), we show that when acting on manifolds of codimension 1, the Dijkgraaf-Witten action defines a one-dimensional metrized line. Hence, it is not correct to think of $S_{\Sigma,[\alpha]}$ not as some integer, but rather as some metrized line - just as was the case when we consider the Chern-Simons action for continuous Lie groups. To begin, let us consider the following proposition about the behavior of singular cocycles and degenerate chains.

Proposition 8.3.4 ([24])**.** *Let Σ be a closed, oriented $(d-1)$-manifold and let $\alpha \in Z^d(\Sigma, \mathbb{R}/\mathbb{Z})$ be a singular cocycle. Then there is a metrized line $I_{\Sigma,\alpha}$ defined. Furthermore, if M is a compact, oriented d-manifold and $\iota : \partial M \hookrightarrow M$ the inclusion map of the boundary, then*

$$e^{2\pi i \langle \alpha, m \rangle}, \tag{8.7}$$

is defined and gives an element of $I_{\partial M, \iota^*(\alpha)}$. Here $m \in C_d(M)$ represents the fundamental class $[M]$ of M.

Proof. The proof uses the invariant section construction and can be found in appendix B of [24]. We give a sketch of the proof. Let $\mathscr{C}_\Sigma$ denote the category whose objects are oriented cycles $\sigma \in Z_{d-1}(\Sigma)$ which represent the fundamental class $[\Sigma] \in H_{d-1}(\Sigma)$ and whose morphisms $a : y \to y'$ are degenerate d-chains, $a \in C_d(Y)$, such that $y' = y + \partial(a)$. Additionally, denote by $\mathscr{L}$ the category consisting of metrized complex lines (i.e., one-dimensional complex inner product spaces) for objects and isometries for morphisms. Then, define a functor $\mathscr{F}_{\Sigma,\alpha} : \mathscr{C}_\Sigma \to \mathscr{L}$ by assigning to each $y \in \mathrm{Obj}(\mathscr{C}_\Sigma)$ a copy of $\mathbb{C}$, which we denote by $\mathbb{C}_y$, and to each $a \in \mathrm{Hom}_{\mathscr{C}_\Sigma}(y, y')$ multiplication by $e^{2\pi i \langle \alpha, a \rangle}$, where $\alpha \in Z^d(\Sigma; \mathbb{R}/\mathbb{Z})$. It is obvious, using the natural metric on $\mathbb{C}$, that $e^{2\pi i \langle \alpha, a \rangle}$ has unit norm. We now want to show that the space of invariant sections of $\mathscr{F}_{\Sigma,\alpha}$ is one-dimensional. We will do this by showing that $\mathscr{F}_{\Sigma,\alpha}$ has no holonomy, which, since $\mathscr{C}_\Sigma$ is connected, implies the invariant section space is one-dimensional. To proceed, let $a : y \to y$ be an automorphism. Hence, $y = y + \partial a$ which implies that $\partial a = 0$. Furthermore, since, $H_d(\Sigma) = 0$, there exists some degenerate $(d+1)$-chain $b \in C_{d+1}(Y)$ with the property $\partial b = a$. Therefore,

$$\begin{aligned} e^{2\pi i \langle \alpha, a \rangle} &= e^{2\pi i \langle \alpha, \partial b \rangle}, \\ &= e^{2\pi i \langle \ \delta\left(\alpha\right), b \rangle}, \\ &= e^{2\pi i \cdot 0}, \\ &= 1. \end{aligned}$$

Thus, $\mathscr{F}_{\Sigma,\alpha}(a) = id_{\mathbb{C}}$ for all automorphisms $a \in \mathrm{Hom}_{\mathscr{C}_\Sigma}(y, y)$; that is, $\mathscr{F}_{\Sigma,\alpha}$ has no holonomy, thus giving us our desired metrized line, $I_{\Sigma,\alpha}$.

Next, for M, fix a chain $m \in Z_d(M)$ which represents the fundamental class $[M] \in H_d(M; \partial M)$ [9]. From familiar arguments, one has, in this case, that $\partial m \in Z_{d-1}(\partial M)$ represents the fundamental class of the boundary $[\partial M] \in H_{d-1}(\partial M)$. Now, consider the section

$$\partial m \longmapsto e^{2\pi i \langle \alpha, m \rangle} \tag{8.8}$$

of the functor $\mathscr{F}_{\partial M, \iota^*(\alpha)}$. This section is (well-)defined since, if m' is some other chain representing $[M]$ such that $\partial m = \partial m'$, then there exists some degenerate chain $c \in H_{d+1}(M)$ with $m' = m + \partial c$. However, then $\langle \alpha, \partial c \rangle = \langle \delta(\alpha), c \rangle$, which vanishes because α is a cocycle (and hence closed), giving us that $e^{2\pi i \langle \alpha, m' \rangle} = e^{2\pi i \langle \alpha, m + \partial c \rangle} = e^{2\pi i \langle \alpha, m \rangle}$; i.e., equation (8.8) is (well-)defined. Identically, under a morphism $m \mapsto m' = m + \partial c$, we see that

$$\begin{aligned} e^{2\pi i \langle \alpha, m' \rangle} &= e^{2\pi i \langle \alpha, m + \partial c \rangle}, \\ &= e^{2\pi i \langle \alpha, m \rangle} e^{2\pi i \langle \alpha, \partial c \rangle}, \\ &= e^{2\pi i \langle \alpha, m \rangle} e^{2\pi i \langle \delta(\alpha), c \rangle}, \\ &= e^{2\pi i \langle \delta(\alpha), c \rangle} e^{2\pi i \langle \alpha, m \rangle}. \end{aligned}$$

And so,

$$\mathscr{F}_{\partial M, \iota^*(\alpha)}(\partial m \xrightarrow{c} \partial m') e^{2\pi i \langle \alpha, m \rangle} = e^{2\pi i \langle \alpha, m' \rangle}. \tag{8.9}$$

Hence, $\partial m \mapsto e^{2\pi i \langle \alpha, m \rangle}$ defines an invariant section of $\mathscr{F}_{\partial M, \iota^*(\alpha)}$, so determines an element of unit norm in $I_{\partial M, \iota^*(\alpha)}$.

□

With the analysis of singular cocycles and degenerate chains fresh at our disposal, we now proceed with the definition of the Dijkgraaf-Witten action on manifolds of codimension 1. To begin, fix a cocycle $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$. Then, denote by $\mathcal{C}_Q$ the category whose objects are induced bundle maps $\tilde{\gamma}_\Sigma : Q \to E\Gamma$ and whose morphisms $\tilde{h} : \tilde{\gamma}_\Sigma \xrightarrow{\sim} \tilde{\gamma}'_\Sigma$ are homotopy classes rel boundary of Γ-homotopies $\tilde{h} : [0,1] \times Q \to E\Gamma$. Note, by definition of the universal covering space, any two classifying maps from the same covering space $Q \to \Sigma$ to $E\Gamma \to B\Gamma$ are homotopy equivalent. Hence, $\mathcal{C}_Q$ is connected. Next, denote by $\mathcal{L}$ the category whose objects are metrized lines and whose morphisms are unitary isomorphisms. Then, we get a functor $\mathcal{F}_Q : \mathcal{C}_Q \to \mathcal{L}$ as follows. For objects $\tilde{\gamma}_\Sigma \in \mathrm{Obj}(\mathcal{C}_Q)$ we assign a metrized line $I_{\Sigma, \gamma^*_\Sigma(\alpha)}$, as in proposition 8.3.4, while to morphisms $\tilde{h} \in \mathrm{Hom}_{\mathcal{C}_Q}(\tilde{\gamma}_\Sigma, \tilde{\gamma}'_\Sigma)$ we assign the isometry

$$e^{2\pi i c_\Sigma(h, \alpha)} : I_{\Sigma, \gamma^*_\Sigma(\alpha)} \longrightarrow I_{\Sigma, \gamma'^*_\Sigma(\alpha)}, \tag{8.10}$$

where $c_\Sigma(h, \alpha)$ is defined by (here we will abuse notation a bit and denote the unit line by $I := [0,1]$)

$$c_\Sigma(h, \alpha) := \langle h^*(\alpha), i \times \sigma \rangle, \tag{8.11}$$

with $h : [0,1] \times \Sigma \to B\Gamma$ the induced map from $\tilde{h} : [0,1] \times Q \to E\Gamma$ and $i \times \sigma \in C_d(I \times \Sigma)$ represents the fundamental class $[I \times \Sigma]$. The fact that $\mathcal{F}_Q$ is a functor and that (8.11) only

[9] Note, when dealing with manifolds which have a non-empty boundary, one must use relative homology to define the fundamental class of the aforementioned manifold. Since, in this case, it is the top relative homology group which is infinite cyclic $H_d(M; \partial M) \cong \mathbb{Z}$ (see chapter 3).

depends on the homotopy class of h follows from proposition $B.1$ of [24]. The functor $\mathcal{F}_Q$ having no holonomy follows from the fact that for any singular cocycle $\alpha \in Z^{d+1}(W; \mathbb{R}/\mathbb{Z})$ defined on a compact oriented $(d+2)$-dimensional manifold W, one has (*Stokes' Theorem*)

$$e^{2\pi i\langle\alpha,\partial w\rangle} = 1, \tag{8.12}$$

where $w \in C_{d+1}(W)$ represents the fundamental class $[W]$. This follows by noting that α is a cocycle and hence closed - i.e., $\langle\alpha, \partial w\rangle = \langle\delta\alpha, w\rangle = 0$. For a detailed proof that $\mathcal{F}_Q$ has no holonomy, the reader can consult page 6 of [24]. Consequently, the invariant sections of $\mathcal{F}_Q$ form a one-dimensional metrized space, or metrized line, which we will denote by $L_{\Sigma,\gamma^*_\Sigma(\alpha)}(Q) \equiv L_\Sigma(Q)$. Also, note that the invariant section construction picks out spaces which are invariant under the automorphisms of Q. Hence, elements in the space of invariant sections $L_\Sigma(Q)$ are invariant under the gauge transformations of Q, $\mathrm{Aut}(Q)$. Thus, our classical Dijkgraaf-Witten action (for some $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$) associates to manifolds of codimension 1 a metrized line.

Next, suppose $\Gamma \hookrightarrow P \xrightarrow{\pi} M$ is a regular Γ-covering space over a compact, oriented d-dimensional manifold M. Then, for each classifying map $\tilde{\gamma}_M : P \to E\Gamma$ define

$$e^{2\pi i S_{M,[\alpha]}(P)} := e^{2\pi i\langle\gamma^*_M(\alpha),m\rangle}, \tag{8.13}$$

where $\gamma_M : M \to B\Gamma$ is the induced classifying map and $m \in C_d(M)$ represents $[M]$. In order for our theory to have any chance of being a topological field theory, we must show that this expression for $e^{2\pi i S_{M,[\alpha]}(P)}$ gives an element of $L_{\partial M,\partial\gamma^*_M(\alpha)}(\partial P) \equiv L_{\partial M}(\partial P)$. Recall that, in order to show $e^{2\pi i S_{M,[\alpha]}(P)} \in L_{\partial M}(\partial P)$ we must show that under a morphism (which in our case is a homotopy $\tilde{k} : [0,1] \times P \to E\Gamma$ between two induced classifying maps $\tilde{\gamma}_M$ and $\tilde{\gamma}'_M$ such that $\tilde{k}|_{\{0\}\times P} = \tilde{\gamma}_M$, while $\tilde{k}|_{\{1\}\times P} = \tilde{\gamma}'_M$), the action transforms as

$$\mathcal{F}_{\partial P,\partial\gamma^*_M(\alpha)}(\partial\tilde{\gamma}_M \xrightarrow{\partial k} \partial\tilde{\gamma}'_M)e^{2\pi i S_{M,[\alpha]}(P,\gamma_M)} = e^{2\pi i S_{M,[\alpha]}(P,\gamma'_M)},$$

where by the symbol ∂ we mean the restriction of maps to the boundary; i.e., ∂k means the induced map between $\partial\tilde{\gamma}_M$ and $\partial\tilde{\gamma}'_M$, which, in turn, are the restrictions of the classifying maps $\tilde{\gamma}_M$ and $\tilde{\gamma}'_M$ to the boundary ∂P. Denoting the unit line as $I = [0,1]$, there exists a product class $[I] \times [M] \in H_{d+1}(I \times M, \partial(I \times M))$ such that

$$\begin{aligned}\partial([I] \times [M]) &= \partial[I] \times M \cup (-1)^{\dim(I)}[I] \times \partial[M],\\ &= \{1\} \times [M] \cup -\{0\} \times [M] \cup -[I] \times [\partial M].\end{aligned}$$

Now, in order to see how our action transforms under a morphism we must see how the pairing $\langle\cdot,\cdot\rangle$ acts on $\partial[I \times M]$. First, letting i and m be cycles representing $[I]$ and $[M]$ respectively, we see (since α is a cocycle; i.e., closed) that $\langle\gamma^*_M(\alpha), \partial(i \times m)\rangle = \langle\delta\gamma^*_M(\alpha), i \times m\rangle = 0$. Consequently, we have

$$0 = \langle\gamma'^*_M(\alpha), m\rangle - \langle\gamma^*_M(\alpha), m\rangle - \langle k^*(\alpha), i \times \partial m\rangle.$$

Hence, by applying the exponent $e^{2\pi i}$, we have

$$\mathcal{F}_{\partial P,\partial\gamma^*_M(\alpha)}(\partial\gamma_M \xrightarrow{\partial k} \partial\gamma'_M)e^{2\pi i S_{M,[\alpha]}(P,\gamma_M)} = e^{2\pi i S_{M,[\alpha]}(P,\gamma'_M)}. \tag{8.14}$$

Consequently, $e^{2\pi i S_{M,[\alpha]}(P,\gamma_M)}$ determines an invariant section of $\mathcal{F}_{\partial P,\partial\gamma^*_M(\alpha)}$, thus giving us

$$e^{2\pi i S_{M,[\alpha]}(P)} \in L_{\partial M}(\partial P). \tag{8.15}$$

So, to recap, the n-dimensional Dijkgraaf-Witten action assigns a metrized line to each codimension 1 manifold and to a manifold with boundary an element of the metrized line assigned to the boundary. Furthermore, our theory obeys several axioms (a discrete version of theorem 7.4.2):

Theorem 8.3.5. *Let Γ be a finite Lie group and fix a cocycle $\alpha \in H^d(B\Gamma;\mathbb{R}/\mathbb{Z})$. Then, the assignments*

$$\begin{aligned} Q &\longmapsto L_\Sigma(Q), \\ P &\longmapsto e^{2\pi i S_{M,P}} \in L_{\partial M}(\partial P), \end{aligned} \tag{8.16}$$

for $Q \in \mathcal{C}_\Sigma$ and $P \in \mathcal{C}_M$ [10]*, previously defined for closed $(d-1)$-manifolds Σ and compact oriented d-manifolds M satisfy:*

(*a*) ***Functoriality:*** *Let $\Gamma \hookrightarrow Q \xrightarrow{\pi} \Sigma$ and $\Gamma \hookrightarrow Q' \xrightarrow{\pi'} \Sigma'$ be two regular covering spaces and let $\psi : Q' \longrightarrow Q$ be a bundle map such that the induced map $\hat{\psi} : \Sigma' \to \Sigma$ is orientation preserving. Then, there is an isometry*

$$\psi^* : L_\Sigma(Q) \to L_{\Sigma'}(Q'). \tag{8.17}$$

Additionally, let $\varphi : P' \to P$ be a bundle map with induced map $\hat{\varphi} : M' \to M$ orientation preserving, then

$$\varphi^*\Big|_{\partial P'} \left(e^{2\pi i S_{M,[\alpha]}(P)}\right) = e^{2\pi i S_{M',[\alpha]}(P')}, \tag{8.18}$$

where $\varphi^*|_{\partial P'}$ is the restriction of $\varphi : P' \to P$ to the boundary.

(*b*) ***Orientation:*** *Denote by $\bar{A}$ the manifold A with opposite orientation. There is a natural isometry*

$$L_{\bar{\Sigma}}(Q) \cong L^*_\Sigma(Q), \tag{8.19}$$

where by $L^*_\Sigma(Q) \equiv L^*_{Q,\Sigma,\tilde{\gamma}^*_\Sigma(\alpha)}$ we mean the dual vector space to $L_\Sigma(Q)$. Furthermore, we have that the action defined on $\bar{M}$ is the complex conjugate of the action defined on M,

$$e^{2\pi i S_{\bar{M},[\alpha]}(P)} = \overline{e^{2\pi i S_{M,[\alpha]}(P)}}. \tag{8.20}$$

(*c*) ***Multiplicativity:*** *If $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_n$ with Q_i denoting covering spaces over Σ_i, then*

$$L_{\Sigma_1\sqcup\cdots\sqcup\Sigma_n}(Q_1 \sqcup \cdots \sqcup Q_n) \cong L_{\Sigma_1}(Q_1) \otimes \cdots \otimes L_{\Sigma_n}(Q_n). \tag{8.21}$$

While, if M decomposes as the disjoint union $M = \sqcup_{i=1}^n M_i$ and P_i are coverings over M_i, then

$$e^{2\pi i S_{\sqcup_i M_i,[\alpha]}(\sqcup_i P_i)} = e^{2\pi i S_{M_1,[\alpha]}(P_1)} \otimes \cdots \otimes e^{2\pi i S_{M_n,[\alpha]}(P_n)}. \tag{8.22}$$

[10] Here and in what follows, we will usually abuse notation and write $P \in \mathcal{C}_M$ rather than $P \in \mathrm{Obj}(\mathcal{C}_M)$.

(*d*) ***Gluing:*** *Suppose M is a compact, oriented manifold and that $\Sigma \hookrightarrow M$ is a closed oriented codimension one submanifold of M. Let M^{cut} denote the manifold obtained by cutting M along Σ. Then, $\partial M^{cut} = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$. Now, suppose P is a covering space over M, P^{cut} is a covering space over M^{cut} and Q is the restriction of P to Σ. Then*

$$e^{2\pi i S_{M,[\alpha]}(P)} = \left\langle e^{2\pi i S_{M^{cut},[\alpha]}(P^{cut})} \right\rangle_{L_\Sigma(Q)}, \tag{8.23}$$

where $\langle\cdot\rangle_{L_\Sigma(Q)}$ is the natural contraction from $L_{\partial M}(\partial P) \otimes L_\Sigma(Q) \otimes L_\Sigma(Q)^*$ to $L_{\partial M}(\partial P)$,

$$\langle \cdot \rangle_{L_\Sigma(Q)} : L_{\partial M}(\partial P) \otimes L_\Sigma(Q) \otimes L_\Sigma(Q)^* \longrightarrow L_{\partial M}(\partial P).$$

Remark 8.3.6. For technical reasons we set $L_\emptyset = \mathbb{C}$ along with $S_\emptyset = 0$. From (a), we see that $Q \mapsto L_\Sigma(Q)$ defines a functor $\mathcal{C}_\Sigma \to \mathcal{L}_\Sigma$ and that each M determines an invariant section $e^{2\pi i S_{M,[\alpha]}(P)}$ of the composite functor $\mathcal{C}_M \to \mathcal{C}_{\partial M} \to \mathcal{L}_{\partial M}$. Invariance of the action on closed manifolds M (see (8.18)) implies that if $P \cong P'$, then $S_{M,[\alpha]}(P) = S_{M,[\alpha]}(P')$ and thus the action passes to a function

$$S_{M,[\alpha]}([P]) : \overline{\mathcal{C}}_M \longrightarrow \mathbb{R}/\mathbb{Z}. \tag{8.24}$$

Furthermore, bundle morphisms over $(d-1)$-manifolds Σ act on the corresponding lines via (8.17). Hence there is a "line bundle" (really a functor)

$$\mathcal{L}_\Sigma \longrightarrow \mathcal{C}_\Sigma, \tag{8.25}$$

with a lift of morphisms in $\mathcal{C}_\Sigma$ (Note: since $\mathcal{C}_\Sigma$ is a discrete set, $\mathcal{L}_\Sigma$ is a discrete union of lines). Additionally, if M is a compact, oriented d-manifold, then there is an induced "line bundle"

$$\mathcal{L}_M \longrightarrow \mathcal{C}_M, \tag{8.26}$$

obtained by pulling back $\mathcal{L}_{\partial M}$ via $\mathcal{C}_M \to \mathcal{C}_{\partial M}$,

$$\begin{array}{ccc} r^*(\mathcal{L}_{\partial M}) & \longrightarrow & \mathcal{L}_{\partial M} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\pi}} \\ \mathcal{C}_M & \xrightarrow{r} & \mathcal{C}_{\partial M}, \end{array} \tag{8.27}$$

where $r : M \to \partial M$ is the restriction map, which restricts a field to the boundary, and $r^*(L_{\partial M})$ is the pullback bundle of $L_{\partial M}$ along r^*. The action $e^{2\pi i S_{M,[\alpha]}}$ is an invariant section of the bundle $\mathcal{L}_M \to \mathcal{C}_M$; that is, we think of the classical action $e^{2\pi i S_{M,[\alpha]}(\cdot)}$ as a section of the "line bundle " $r^*(\mathcal{L}_{\partial M}) \to \mathcal{C}_M$ which is invariant under the gauge transformations (or field morphisms) in the sense that under a gauge transformation the action obeys

$$e^{2\pi i S_{M,[\alpha]}(P,\gamma'_M)} = \mathcal{F}_{\partial P, \partial\gamma^*_M(\alpha)}(\partial\gamma_M \xrightarrow{\partial k} \partial\gamma'_M) e^{2\pi i S_{M,[\alpha]}(P,\gamma_M)}.$$

In particular, the group of deck transformations $\mathrm{Aut}(P)$ of $P \to M$ acts on the line over $P \in \mathcal{C}_M$. Finally, the previous theorem expresses the fact that $S_{M,[\alpha]}$ is a local functional of local fields defined as the integral of a local expression, along with being invariant under symmetries of the fields and changing sign under orientation reversal (i.e. leading to a unitary theory).

Proof. See [24]. □

Before ending this section we note that the metrized line bundle defined in (8.25) passes to a (possibly degenerate) metrized line bundle

$$\overline{\mathcal{L}}_\Sigma \longrightarrow \overline{\mathcal{C}}_\Sigma, \tag{8.28}$$

over the finite set (or category) of equivalence classes of covering spaces over Σ. The fibre of this metrized line bundle, which we denote by $L_\Sigma([Q])$, is the space of invariant sections of the functor $Q \to L_\Sigma(Q)$ as Q ranges over $[Q]$. Furthermore, if $\mathrm{Aut}(Q)$ acts nontrivially on $L_\Sigma(Q)$, then $\dim(L_\Sigma([Q])) = 0$, otherwise $\dim(L_\Sigma([Q])) = 1$.

8.3.2 Quantum Theory

Before we can proceed with quantization, we must define one more crucial piece of data - a measure on the space of fields. In most applications of quantum field theories in physics - in particular, when the set of fields is continuous - this can only be done formally, since showing the existence of a measure on an infinite dimensional space is by no means trivial. However, in our case the space of fields is a finite collection and so, is discrete. Therefore, we do not face the technical problems of defining a measure that one would if they were working with continuous fields. In fact, the measure for our discrete theories is, as we will see, rather simple and straightforward - we simply count each bundle according to its number of automorphisms.

We now precisely define a measure μ on the space of fields - i.e., on the category $\mathcal{C}_M$. Since $\mathcal{C}_M$ is discrete (recall $\mathcal{C}_M \cong \mathrm{Hom}(\pi_1(M;x),\Gamma)$), we simply define our measure $\mu : \mathcal{C}_M \to \mathbb{R}$ on the collection $\mathcal{C}_M$ of Γ-covering spaces over M as

$$\mu(P) = \frac{1}{\#\mathrm{Aut}(P)}. \tag{8.29}$$

That is, we count each covering space according to the number of its automorphisms. Note, if $P \cong P'$, then it follows that $\mathrm{Aut}(P) \cong \mathrm{Aut}(P')$. Hence, for isomorphic covering spaces P and P', we have that

$$\mu(P) = \mu(P').$$

This implies that our measure μ on $\mathcal{C}_M$ induces a measure on $\overline{\mathcal{C}}_M$, which we denote by $\mu([P])$ where $[P]$ represents a class in $\overline{\mathcal{C}}_M$. Furthermore, for technical reasons, we assign the empty set measure zero, $\mu(\emptyset) = 0$. Using this property for the empty set along with gluing of automorphisms, it is not hard to show that the defined measure (8.29) obeys the axioms of a measure - namely, it is non-negative, assigns the empty set zero measure, and it is countably additive

$$\mu(\bigcup_i P_i) = \sum_i \mu(P_i).$$

Continuing along, suppose M is a compact oriented d-manifold with boundary ∂M and let $Q \in \mathcal{C}_{\partial M}$ be a Γ-covering space over ∂M. Now, denote by $\mathcal{C}_M(Q)$ the category whose objects are pairs (P,θ) consisting of Γ-covering spaces P over M and isomorphisms

$\theta : \partial P \to Q$, while morphisms in $\mathcal{C}_M(Q)$, $\varphi : (P, \theta) \to (P', \theta')$, are isomorphisms $\varphi : P \to P'$ such that

$$\begin{tikzcd} \partial P \arrow[rr, "\partial\varphi"] \arrow[dr, "\theta"'] & & \partial P' \arrow[dl, "\theta'"] \\ & Q & \end{tikzcd}$$

commutes. Note, these morphisms induce an equivalence relation on the category $\mathcal{C}_M(Q)$. Indeed, we say that two elements (P, θ) and (P', θ'), in $\mathcal{C}_M(Q)$, are equivalent iff there exists a morphism between them,

$$\varphi : (P, \theta) \longrightarrow (P', \theta').$$

We denote the set of equivalence classes by $\overline{\mathcal{C}}_M(Q)$. Furthermore, the measure μ, defined in (8.29), passes to a measure on $\mathcal{C}_M(Q)$ (interpreting $\mathrm{Aut}(P, \theta)$ in the sense just described) and on to $\overline{\mathcal{C}}_M(Q)$ [24]. Finally, note that any morphism $\psi : Q \to Q'$ induces a measure preserving map

$$\psi_* : \overline{\mathcal{C}}_M(Q) \longrightarrow \overline{\mathcal{C}}_M(Q'). \tag{8.30}$$

For future reference, we now investigate the behavior of the measures under the operations of cutting and pasting. To begin, let $N \hookrightarrow M$ be an oriented codimension 1 submanifold and let M^{cut} be the manifold obtained by cutting M along N. Then, as we have previously seen, the boundary of M^{cut} becomes

$$\partial M^{\text{cut}} = \partial M \sqcup N \sqcup \bar{N}.$$

Additionally, let $Q \to N$ be a Γ-covering space over N. Let, $\mathcal{C}_{M^{\text{cut}}}(Q \sqcup Q)$ is the category of triples $(P^{\text{cut}}, \theta_1, \theta_2)$, where $P^{\text{cut}} \to M^{\text{cut}}$ is a Γ-covering space over M^{cut} and $\theta_i : P^{\text{cut}}|_N \to Q$ are isomorphisms, one over each copy of N in M^{cut}. Now, consider the gluing map

$$\begin{aligned} g_Q :\ & \overline{\mathcal{C}}_{M^{\text{cut}}}(Q \sqcup Q) \longrightarrow \overline{\mathcal{C}}_M \\ & (P^{\text{cut}}, \theta_1, \theta_2) \longmapsto P^{\text{cut}}/(\theta_1 = \theta_2). \end{aligned} \tag{8.31}$$

Then we have the following theorem.

Theorem 8.3.7. *The gluing map g_Q satisfies:*

(*a*) *g_Q maps onto the set of coverings over M whose restriction to N is isomorphic to Q.*

(*b*) *Let $\phi \in Aut(Q)$ act on $(P^{cut}, \theta_1, \theta_2) \in \mathcal{C}_{M^{cut}}(Q \sqcup Q)$ by*

$$\phi \cdot (P^{cut}, \theta_1, \theta_2) = (P^{cut}, \phi \circ \theta_1, \phi \circ \theta_2). \tag{8.32}$$

Then the stabilizer of this action at $(P^{cut}, \theta_1, \theta_2)$ is the image $Aut(P) \to Aut(Q)$ determined by the θ_i, where $P = g_Q\big((P^{cut}, \theta_1, \theta_2)\big)$.

(*c*) *There is an induced action on equivalence classes $\overline{\mathcal{C}}_{M^{cut}}(Q \sqcup Q)$, and $Aut(Q)$ acts transitively on $g_Q^{-1}([P])$ for any $[P] \in \overline{\mathcal{C}}_M$.*

(*d*) *For all* $[P] \in \overline{\mathcal{C}}_M$ *we have*

$$\mu([P]) = vol(g_Q^{-1}([P])) \cdot \mu(Q). \tag{8.33}$$

Proof. See [24]. □

We are now in a position to quantize the classical Dijkgraaf-Witten theories.

Definition 8.3.8 (**Path Integral for a Closed Manifold**)**.** Fix a class $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ and its representative $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$. Let $P \in \mathcal{C}_M$ be a covering space $\Gamma \hookrightarrow P \xrightarrow{\pi} M$ representing the class $[P] \in \overline{\mathcal{C}}_M$. Then, we define the path integral to be

$$Z(M) := \int_{\overline{\mathcal{C}}_M} e^{2\pi i S_{M,[\alpha]}([P])} d\mu([P]), \tag{8.34}$$

where $S_{M,[\alpha]}([P]) : \overline{\mathcal{C}}_M \to \mathbb{R}/\mathbb{Z}$ is defined in (8.24)[11].

Remark 8.3.9. As advertised, since $\overline{\mathcal{C}}_M$ is finite, our path integral is really of the form

$$Z(M) = \sum_{[P] \in \overline{\mathcal{C}}_M} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]). \tag{8.35}$$

For covering spaces over manifolds M with boundary, we have the following definition.

Definition 8.3.10 (**Path Integral for a Compact Manifold**)**.** Fix $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ and its representative $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$. Let $\Gamma \hookrightarrow P \xrightarrow{\pi} M$ be a covering space over a manifold M with boundary. Furthermore, let $Q \in \mathcal{C}_{\partial M}$ be a covering space over ∂M and denote by $\overline{\mathcal{C}}_M(Q)$ the category of equivalence classes, $[P]$, of bundles P over M such that their restriction to ∂M is Q, up to isomorphisms which are the identity on ∂M (see the paragraph leading up to equation (8.30)). Then, we define the path integral to be

$$Z(M) := \int_{\overline{\mathcal{C}}_M(Q)} e^{2\pi i S_{M,[\alpha]}([P])} d\mu([P]). \tag{8.36}$$

Remark 8.3.11. As was the case before, since $\overline{\mathcal{C}}_M(Q)$ is finite, our path integral reduces to a discrete sum

$$Z(M) = \sum_{[P] \in \overline{\mathcal{C}}_M(Q)} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]). \tag{8.37}$$

Now, having defined the path integrals, we must show that these quantized theories obey Atiyah's axioms for a TQFT (see section 8.1) - that is, we now must construct the functor which to $(n-1)$-dimensional closed manifolds Σ assigns some Hilbert space V_Σ and to n-dimensional compact manifolds M assigns vectors (or morphisms) which lie in the Hilbert space associated to ∂M, $Z_M \in V_{\partial M}$. We will split our task up into two different categories (no pun intended): (1) the untwisted case ($[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ is the trivial class), and (2) the twisted case.

---

[11]Recall, we want to restrict the integration space to the space only consisting of physically different fields. That is, we pick one representative from each class of fields related by the symmetries of our theory - hence, equivalence classes in $\overline{\mathcal{C}}_M$.

8.3.3 Untwisted Theories ($[\alpha] = 0$)

For the untwisted case we take the trivial cohomology class $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ along with a trivial representative $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$, i.e. $[\alpha] = \alpha = 0$. With these choices it is completely obvious that our action becomes

$$e^{2\pi i S_{M,[0]}(P)} = e^{2\pi i \langle \gamma_M^*(0), m \rangle} = 1. \tag{8.38}$$

Hence, our path integral for closed manifolds M reduces to a sum over the different classes $[P] \in \overline{\mathcal{C}}_M$, with each class being weighted by the number of automorphisms

$$Z(M) = \sum_{[P] \in \overline{\mathcal{C}}_M} \mu([P]). \tag{8.39}$$

Whereas, for compact M with boundary ∂M, the path integral reduces to the sum over the different classes in $\overline{\mathcal{C}}_M(Q)$ (with similar weights)

$$Z(M) = \sum_{[P] \in \overline{\mathcal{C}}_M(Q)} \mu([P]), \tag{8.40}$$

where $Q \in \mathcal{C}_{\partial M}$. Now, denote by $L^2(\overline{C}_\Sigma, \mathbb{C}; \mu([Q]))$ the space of all L^2 invariant functions $f : \overline{C}_\Sigma \to \mathbb{C}$ which are measurable (i.e., finite) with respect to the norm defined by

$$||f||_{\mu([Q])} := \left(\sum_{[Q] \in \overline{\mathcal{C}}_\Sigma} |f([Q])|^2 \mu([Q]) \right)^{1/2}. \tag{8.41}$$

It is well-known that $L^2(\overline{C}_\Sigma, \mathbb{C}; \mu([Q]))$ forms a Hilbert space.

In order to construct a topological quantum field theory we associate to each $(d-1)$-manifold Σ the Hilbert space $Z(\Sigma) \equiv V_\Sigma := L^2(\overline{C}_\Sigma, \mathbb{C}; \mu([Q]))$ and to each compact oriented manifold M we associate $Z(M)$. Indeed, we have the following theorem.

Theorem 8.3.12. *Let Γ be a finite group and let $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ with representative $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ be trivial, $[\alpha] = \alpha = 0$. Then the assignments*

$$\begin{aligned} \Sigma &\longmapsto Z(\Sigma) \equiv V_\Sigma := L^2(\overline{C}_\Sigma, \mathbb{C}; \mu([Q])) \\ M &\longmapsto Z(M) = \sum_{[P] \in \overline{\mathcal{C}}_M(Q)} \mu([P]), \end{aligned} \tag{8.42}$$

defined above for $(d-1)$-dimensional closed oriented manifolds Σ and compact oriented manifolds M, satisfy the Atiyah axioms for a TQFT (see section 8.1).

Proof. We first need to check that $Z(M)$ does indeed give an element in

$$V_{\partial M} := L^2(\overline{C}_{\partial M}(Q), \mathbb{C}; \mu([\partial P])).$$

That is, we need to make sure that, under a morphism, $Z(M)$ transforms as an invariant section. To proceed, let $\psi : Q' \to Q$ be a morphism in $\mathcal{C}_M(Q)$. Then, this induces a measure preserving map

$$\psi_* : \overline{\mathcal{C}}_M(Q') \longrightarrow \overline{\mathcal{C}}_M(Q).$$

Now, since ψ_* is measure preserving, under this morphism $Z(M)$ only depends on the equivalence class $[Q] \in \overline{\mathcal{C}}_{\partial M}(Q)$ of Q. Thus giving us

$$Z(M) \in V_{\partial M}, \tag{8.43}$$

as desired. Furthermore, for M closed it is clear that $Z(M) \in \mathbb{C}$. We now, (re)state and prove each axiom:

(a) **Functoriality:** Suppose $f : \Sigma \to \Sigma'$ is an orientation preserving diffeomorphism. Then there is an induced isometry

$$Z(f) : V_{\Sigma'} \longrightarrow V_{\Sigma}, \tag{8.44}$$

Furthermore, if $f, g : \Sigma \to \Sigma'$ are any two such orientation preserving mappings, then

$$Z(gf) = Z(g)Z(f). \tag{8.45}$$

In addition, if $F : M \to M'$ is an orientation preserving diffeomorphism then

$$Z(\partial F)\big(Z(M)\big) = Z(M'), \tag{8.46}$$

where $Z(\partial F) : V_{\partial M} \to V_{\partial M'}$ is the isometry coming from the induced map $\partial F : \partial M \to \partial M'$ over the boundaries.

$\diamond$ To prove this, we proceed as follows. First, note that the map $f : \Sigma \to \Sigma'$ induces a measure preserving functor $f^{\#} : \mathcal{C}_{\Sigma'} \to \mathcal{C}_{\Sigma}$, which (via equation (8.17)) lifts to

$$\tilde{f} : L_{\Sigma'} \longrightarrow L_{\Sigma}.$$

Now, we define the isometry $Z(f) : V_{\Sigma} \to V_{\Sigma'}$ to be the pullback of invariant sections induced by $\tilde{f}$. Furthermore, with our definition of $Z(f)$, composition is immediate from properties of the pullback. Finally, the relation given by $Z(\partial F)\big(Z(M)\big) = Z(M')$ follows immediately from the fact that (see theorem 8.3.5) under a diffeomorphism $F : M \to M'$ the Dijkgraaf-Witten action obeys

$$(\partial F)^* \left(e^{2\pi i S_{M,[\alpha]}(P)}\right) = e^{2\pi i S_{M',[\alpha]}(P')}.$$

(b) **Orientation:** There is a natural isometry

$$V_{\bar{\Sigma}} \cong V_{\Sigma}^*, \tag{8.47}$$

where $\bar{\Sigma}$ is the $(d-1)$-manifold Σ with reversed orientation and V_{Σ}^* is the dual vector space to V_{Σ}, along with

$$Z(\bar{M}) = \overline{Z(M)}, \tag{8.48}$$

where by $\overline{Z(M)}$ we mean the complex conjugate of $Z(M)$.

◇ Both of these results follow directly from the fact that (see theorem 8.3.5 part (b))

$$L_{\bar{\Sigma}}(Q) \cong L_{\Sigma}(Q)^*$$

and

$$e^{2\pi i S_{\bar{M},[\alpha]}(P)} = \overline{e^{2\pi i S_{M,[\alpha]}(P)}}.$$

(c) **Multiplicativity and Gluing:**

(Multiplicativity) If $\Sigma = \sqcup_i \Sigma_i$, then there is a natural isometry

$$Z(\Sigma) = Z(\sqcup_i \Sigma_i) = V_{\sqcup_i \Sigma_i} \cong \bigotimes_i V_{\Sigma_i}. \tag{8.49}$$

While if $M = \sqcup_i M_i$, then

$$Z(M) = Z(\sqcup_i M) = \bigotimes_i Z(M_i) \in \bigotimes_i V_{\partial M_i}. \tag{8.50}$$

Remark 8.3.13. As was already mentioned this is where the TQFT differs from homology, it is multiplicative over disjoint unions rather than additive.

◇ This follows from theorem 8.3.5 part (c), along with the fact that [41]

$$\mathcal{C}_{\sqcup_{i=1}^n M_i} = \mathcal{C}_{M_1} \times \mathcal{C}_{M_2} \times \cdots \times \mathcal{C}_{M_n}. \tag{8.51}$$

(Gluing) Suppose $\Sigma \hookrightarrow M$ is a closed oriented submanifold of codimension one and M^{cut} is the manifold obtained by cutting M along Σ (note, $\partial M = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$). Then,

$$Z(M) = \left\langle Z\left(M^{\text{cut}}\right)\right\rangle_{V_\Sigma}, \tag{8.52}$$

where $\langle \cdot \rangle_{V_\Sigma}$ is the contraction

$$\langle \cdot \rangle_{V_\Sigma} : V_{\partial M} \otimes V_\Sigma \otimes V_\Sigma^* \longrightarrow V_{\partial M},$$

using the Hermitian metric on V_Σ.

◇ Let us show this for the case M is closed, $\partial M = 0$, while proving the more general case in the next section. To begin let $\{Q\}$ be a set of representatives of $\overline{\mathcal{C}}_\Sigma$. Now, restricting to $\alpha = 0$, we have

$$\begin{aligned}
Z(M) &= \sum_{[P]\in\overline{\mathcal{C}}_M} \mu([P]), \\
&= \sum_{Q\in\{Q\}} \sum_{[P]\in\overline{\mathcal{C}}_M(Q)} \mu([P]), \\
&= \sum_{Q\in\{Q\}} \left(\sum_{[P^{\text{cut}}]\in\overline{\mathcal{C}}_{M^{\text{cut}}}(Q\sqcup Q)} \mu([P^{\text{cut}}]) \right) \mu(Q), \\
&= \left\langle Z\left(M^{\text{cut}}\right)\right\rangle_{V_\Sigma},
\end{aligned}$$

where in the third line we used the gluing map g_Q and the equation relating the measures (see equation (8.31)).

(d) **Empty set:** We set, for the $(d-1)$-dimensional empty set $\emptyset$, $V_\emptyset \cong \mathbb{C}$, while, since we define $\mu(\emptyset) = 0$, $Z(\emptyset) = 0$ which is the indentity for $\mathbb{C}$ (viewed as a group with respect to addition[12]).

□

Remark 8.3.14. The reader should note that only for the gluing part in the above proof did we rely on the fact that the cohomology class $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ was trivial (i.e., $[\alpha] = 0$).

So, to recap, we have seen that the Dijkgraaf-Witten theory obeys the axioms of a topological quantum field theory put forth by Atiyah, and hence leads to a topological quantum field theory - at least for the untwisted theories ($[\alpha] = 0$). We will now show that the twisted theories also lead to TQFT's.

8.3.4 Twisted Theories ($[\alpha] \neq 0$)

For the twisted case we allow for $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ to be non-trivial and hence, $\alpha \neq 0$. Thus, our action does not reduce to the trivial action $e^{2\pi i 0}$ as before. Therefore, we must take the original expressions for the path integral. To be more precise, we define our theory as follows. For M, a closed oriented manifold of dimension d, we define the path integral to be

$$Z(M) := \sum_{[P] \in \overline{\mathcal{C}}_M} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]). \tag{8.53}$$

While, for a compact oriented manifold of dimension d, we define the path integral to be

$$Z(M) := \sum_{[P] \in \overline{\mathcal{C}}_M(Q)} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]). \tag{8.54}$$

Now for the Quantum Hilbert space assigned to each $(d-1)$-dimensional manifold Σ, V_Σ. For this, recall that to each oriented $(d-1)$-dimensional manifold Σ we have the possibly degenerate metrized line bundle (see (8.28))

$$\overline{\mathcal{L}}_\Sigma \longrightarrow \overline{\mathcal{C}}_\Sigma.$$

With this in mind, we define our Hilbert space to be the space of all L^2 invariant sections of the bundle $\overline{\mathcal{L}}_\Sigma \to \overline{\mathcal{C}}_\Sigma$,

$$V_\Sigma := L^2(\overline{\mathcal{C}}_\Sigma, \overline{\mathcal{L}}_\Sigma; \mu([Q])). \tag{8.55}$$

That is, our quantum Hilbert space V_Σ is the space of all L^2 invariant sections of the functor

$$\begin{aligned} \mathcal{F}_\Sigma : \mathcal{C}_\Sigma &\longrightarrow \mathcal{L}_\Sigma \\ Q &\longmapsto L_\Sigma(Q). \end{aligned}$$

[12]Recall, a TQFT is a functor (with additional structure) from the category of cobordisms to the category of abelian groups. Hence, we must view $\mathbb{C}$ as an abelian group, thus we must take the group action to be addition.

Furthermore, we define the inner product (and hence norm, to which all invariant sections are measurable against) on V_Σ as

$$(v, v')_{V_\Sigma} := \sum_{[Q]\in\overline{\mathcal{C}}_\Sigma} \big(v([Q]), v'([Q])\big)_{L_\Sigma(Q)} \mu([Q]), \tag{8.56}$$

where $(\cdot,\cdot)_{L_\Sigma(Q)}$ is the inner product on $L_\Sigma(Q)$. With these constructions, V_Σ and $Z(M)$ define a topological quantum field theory. Indeed, we have the following theorem.

Theorem 8.3.15. *Let Γ be a finite Lie group and let $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ with representative $\alpha \in Z^d(B\Gamma; \mathbb{R}/\mathbb{Z})$. Then the assignments*

$$\begin{aligned} Z :&\Sigma \longmapsto Z(\Sigma) \equiv V_\Sigma := L^2(\overline{\mathcal{C}}_\Sigma, \overline{\mathcal{L}}_\Sigma; \mu([Q])) \\ Z :&M \longmapsto Z(M) := \sum_{[P]\in\overline{\mathcal{C}}_M(Q)} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]) \in V_{\partial M}, \end{aligned} \tag{8.57}$$

defined above for closed oriented $(d-1)$-manifolds Σ and compact oriented d-manifolds M, satisfy the axioms of a topological quantum field theory:

(*a*) ***Functoriality:*** *Suppose $f : \Sigma \to \Sigma'$ is an orientation preserving diffeomorphism. Then there is an induced isometry*

$$Z(f) : V_\Sigma \longrightarrow V_{\Sigma'}. \tag{8.58}$$

Furthermore, if $f, g : \Sigma \to \Sigma'$ are any two such orientation preserving mappings, then

$$Z(gf) = Z(g)Z(f). \tag{8.59}$$

In addition, if $F : M \to M'$ is an orientation preserving diffeomorphism then

$$Z(\partial F)\big(Z(M)\big) = Z(M'), \tag{8.60}$$

where $Z(\partial F) : V_{\partial M} \to V_{\partial M'}$ is the isometry coming from the induced map $\partial F : \partial M \to \partial M'$ over the boundaries.

(*b*) ***Orientation:*** *There is a natural isometry*

$$V_{\bar\Sigma} \cong V_\Sigma^*, \tag{8.61}$$

where $\bar\Sigma$ is the $(d-1)$-manifold Σ with reversed orientation and V_Σ^* is the dual vector space to V_Σ. Along with

$$Z(\bar M) = \overline{Z(M)}, \tag{8.62}$$

where by $\overline{Z(M)}$ we mean the complex conjugate of $Z(M)$.

(c) ***Multiplicativity and Gluing:***

(Multiplicativity) If $\Sigma = \sqcup_i \Sigma_i$, then there is a natural isometry

$$Z(\Sigma) = Z(\sqcup_i \Sigma_i) = V_{\sqcup_i \Sigma_i} \cong \bigotimes_i V_{\Sigma_i}. \tag{8.63}$$

While if $M = \sqcup_i M_i$, then

$$Z(M) = Z(\sqcup_i M) = \bigotimes_i Z(M_i) \in \bigotimes_i V_{\partial M_i}. \tag{8.64}$$

(Gluing) Suppose $\Sigma \hookrightarrow M$ is a closed oriented submanifold of codimension one and M^{cut} is the manifold obtained by cutting M along Σ (note, $\partial M = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$). Then,

$$Z(M) = \left\langle Z\left(M^{cut}\right)\right\rangle_{V_\Sigma}, \tag{8.65}$$

where $\langle\,\cdot\,\rangle_{V_\Sigma}$ is the contraction

$$\langle\,\cdot\,\rangle_{V_\Sigma} : V_{\partial M} \otimes V_\Sigma \otimes V_\Sigma^* \longrightarrow V_{\partial M},$$

using the Hermitian metric on V_Σ.

(d) ***Empty set:*** *The $(n-1)$-dimensional empty set $\emptyset$ is mapped to the ground field, $\emptyset \mapsto \mathbb{C}$, while the n-dimensional empty set is mapped to the identity element in $\mathbb{C}$ (viewed as an abelian group), $Z(\emptyset) = 0$.*

Proof. All we need to check is that $Z(M)$, as defined in (8.54), is an element of $V_{\partial M}$ along with generalizing the proof of gluing to nontrivial class $[\alpha] \in H^d(B\Gamma; \mathbb{R}/\mathbb{Z})$ (everything else has already been shown in the proof of theorem 8.3.12).

$\diamond$ The fact that $Z(M)$ is an invariant section of $\mathcal{F}_{\partial M}$ (and hence an element of $V_{\partial M}$) follows immediately from the observation that any morphism $\psi : Q \to Q'$ induces a measure preserving map $\psi_* : \overline{\mathcal{C}}_M(Q) \to \overline{\mathcal{C}}_M(Q')$, along with (8.18).

$\diamond$ We prove the gluing property as follows. To begin, fix a covering space $\tilde{Q} \to \partial M$. Then, from theorem 8.3.5 part (d), for each $Q \to \Sigma$ and $P^{\text{cut}} \in \mathcal{C}_{M^{\text{cut}}}(\tilde{Q} \sqcup Q \sqcup Q)$ we have

$$e^{2\pi i S_{M,[\alpha]}\left(g_Q(P^{\text{cut}})\right)} = \left\langle e^{2\pi i S_{M^{\text{cut}},[\alpha]}(P^{\text{cut}})} \right\rangle_Q, \tag{8.66}$$

where g_Q is the, previously discussed, gluing map. Next, let $\{Q\}$ be a set of representatives of $\overline{\mathcal{C}}_\Sigma$ and let $\overline{\mathcal{C}}_M(\tilde{Q})_Q$ denote the equivalence classes of covering spaces over M whose restriction to ∂M is $\tilde{Q}$ and to Σ is Q. Then, we have that

$$\begin{aligned} Z(M) &= \sum_{[P] \in \overline{\mathcal{C}}_M(\tilde{Q})} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]), \\ &= \sum_{Q \in \{Q\}} \sum_{[P] \in \overline{\mathcal{C}}_M(\tilde{Q})_Q} e^{2\pi i S_{M,[\alpha]}([P])} \mu([P]). \end{aligned}$$

Which, using the gluing map g_Q (defined in equation (8.31)) and the equation relating the measures (8.33) along with (8.66), becomes

$$Z(M) = \sum_{Q\in\{Q\}} \sum_{[P^{\text{cut}}]\in\overline{\mathcal{C}}_M(\tilde{Q}\sqcup Q\sqcup Q)} \left\langle e^{2\pi i S_{M^{\text{cut}},[\alpha]}(P^{\text{cut}})} \right\rangle_Q \mu([P^{\text{cut}}])\mu(Q).$$

Hence, we have that

$$Z(M) = \sum_{Q\in\{Q\}} \left\langle Z\left(M^{\text{cut}}\right)_{\tilde{Q}\sqcup Q\sqcup Q}\right\rangle_Q \mu(Q).$$

Finally, it can be shown via the inner product defined in (8.56) (see [22] for the details) that

$$\sum_{Q\in\{Q\}} \left\langle Z\left(M^{\text{cut}}\right)_{\tilde{Q}\sqcup Q\sqcup Q}\right\rangle_Q \mu(Q) = \left\langle Z\left(M^{\text{cut}}\right)\right\rangle_{V_\Sigma},$$

thus proving that

$$Z(M) = \left\langle Z\left(M^{\text{cut}}\right)\right\rangle_{V_\Sigma},$$

as desired. □

And so, we have just seen that the Dijkgraaf-Witten theories do, in fact, satisfy the axioms of a TQFT put forth by Atiyah - they define a functor from the category of cobordisms to the category of Hilbert spaces, which satisfies certain properties. In the theory developed above, we see that special importance is payed to cutting and gluing manifolds. Indeed, we have seen that given a closed oriented d-manifold X, we can obtain its partition function $Z(M)$ by cutting M along a codimension one submanifold Σ and then using (8.65) to 'combine' both partition functions coming from the two split manifolds which resulted after the cuts. However, what if we further cut each of these codimension one manifolds along some codimension two manifold, could we still then construct the path integral of M? In the next chapter we will see that the answer to this question is yes.

Part III

EXTENDED TOPOLOGICAL FIELD THEORIES

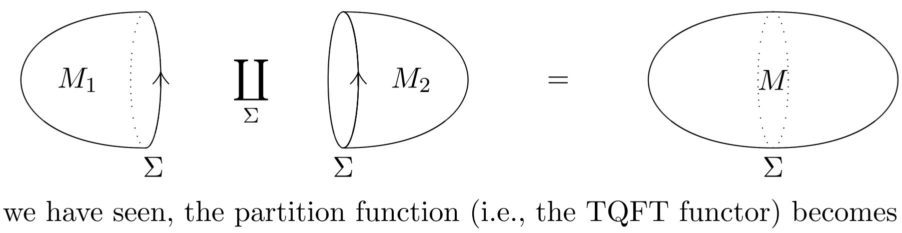
Chapter 9

Extended Dijkgraaf-Witten Theories

We have seen in the previous two chapters what the 3-dimensional Chern-Simons theory attaches to manifolds of dimensions three and two (or, equivalently to manifolds of codimension 0 and 1), by looking at gluing cobordisms along (n-1)-dimensional manifolds. However, what about gluing along manifolds of higher codimension; i.e., we want to answer the question of what does the 3-dimensional Chern-Simons theory assign to 1-dimensional manifolds and points?

Initially, the motivation for the TQFT axioms (see section 8.1) came from the physical ideas of quantum field theory. Here the image under the functor Z of a cobordism, M from Σ_1 to Σ_2, is thought of as the operator which defines propagation, across M, of fields on Σ_1 to those on Σ_2. Hence, we can think of the codimension one manifolds Σ_i as time-slices and M as spacetime. However, in topological theories one does not want to have any directions which are specifically "picked out" (labeled by time) - as above where one would label positive time as the direction which "flowed" via M from Σ_1 to Σ_2, in a TQFT one does not want this extra structure. Furthermore, why should we only consider decompositions of M by codimension one submanifolds?

Consider a d-dimensional TQFT and suppose that M is a closed d-dimensional manifold which has been split into two parts, M_1 and M_2, along a codimension-one submanifold Σ:



Then, as we have seen, the partition function (i.e., the TQFT functor) becomes

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle,$$

where $Z(M_1) \in V_\Sigma, Z(M_2) \in V_\Sigma^*$. Thus, $Z(M)$ can be completely determined by, first being able to split M into two pieces (M_1 and M_2 along Σ) and second, know the value of Z on each piece. However, what if we decide to think outside the box for a moment and further split Σ into two pieces, Σ_1 and Σ_2, along some $(d-2)$-dimensional manifold S? Well, just like the prior case, we would like to be able to express the partition function of Σ as

$$Z(\Sigma) = \langle Z(\Sigma_1), Z(\Sigma_2) \rangle,$$

where $Z(\Sigma_1), Z(\Sigma_2) \in V_S$. After a quick glance though, we realize that the left-hand side of the above equation must be a vector space (from the previous result for $Z(M)$). Thus, how are we supposed to construct an inner product which takes its values in vector spaces? Said differently, what kind of "space" could V_S be, so that one of its elements can be paired with an element of its "dual" to give a vector space? We answer this by employing the use of 2-vector spaces. Power-drunk, we realize that nothing can stop us from splitting S, and so on. TQFTs which can act on manifolds of arbitrary codimension are called extended TQFTs and, as we have just experienced, their behavior relies heavily on the use of higher vector spaces (and hence, higher categories).

9.1 Extended (or Multi-Tiered) TQFTs

As was alluded to in the previous arguments, the axioms of a TQFT where first formulated by considering the properties that would be expected of Feynman integrals, over manifolds with boundary, under various gluing operations. Furthermore, if we want our TQFT to assign objects to manifolds of arbitrary codimension then we must use higher vector spaces. In particular, if objects are to be associated with codimension-one manifolds with boundary, in such a way that they behave reasonably under the gluing axiom, then these objects must be elements of some 2-vector space. In the next section, we give a 'heuristic' definition of an extended TQFT, while in the next we work out the explicit example of the extended Dijkgraaf-Witten theory.

To begin, let M be a manifold of dimension n.

Definition 9.1.1 (ETQFT: Version I). By an **Extended Topological Quantum Field Theory** (ETQFT) of dimension n, we mean an assignment of objects to manifolds of codimension up to n, which behaves 'naturally' under the gluing axioms. This assignment will give:

(a) A k-vector space (or rather k-Hilbert space) V_Σ (for $0 \le k < r$) to closed manifolds Σ of codimension k,

(b) and, to each compact manifold of codimension k, an element $Z(M)$ of the $(k+1)$-vector space $V_{\partial M}$ associated to its boundary ∂M.

Remark 9.1.2. Of course an ETQFT must obey familiar axioms to the regular TQFT. However, we choose to leave those to the explicit definitions given in the next section. Although, we do mention that the only gluing operations which are allowed are those in which all parts of the gluing process are expressed in terms of manifolds of codimension at most n, while those manifolds which appear at codimension n are all closed.

Thus, for $n = 1$, an ETQFT associates complex numbers and Hilbert spaces to closed codimension-0 and codimension-1 manifolds, while associating vectors to compact codimension-1 manifolds. Hence, for $n = 1$, an ETQFT reduces to an ordinary TQFT (as should be expected). Finally, as before, we can think of an n-dimensional ETQFT as a functor between n-categories. Indeed, the collection of n-vector spaces and maps between them defines a n-category, while one can construct the n-category of n-cobordisms (see [31]). Using these, we have the following categorical definition of a n-Extended TQFT.

Definition 9.1.3 (**ETQFT: Version II**). An Extended Topological Quantum Field Theory of dimension n is a symmetric monoidal functor

$$Z : \mathrm{nCob}_n \longrightarrow \mathscr{V}_n$$

between the two n-categories (which we do not define, since it is unnecessary).

Let us now make things more precise.

9.2 Classical Extended Dijkgraaf-Witten Theories

Consider again the d-dimensional field theory discussed in chapter 7 and its finite version discussed in chapter 8. There we were able to define an action on d-manifolds (possibly with boundary) which satisfied several properties, including the gluing axiom (or locality axiom) which keeps our theory local. Furthermore, we asserted that the mapping $f \in \mathcal{C}_\Sigma \mapsto L_{\Sigma,f}$, where f is a field (which takes its appropriate form depending on which type of G one is assuming - i.e., either continuous or finite group), should be considered as an extension of the classical action to closed $(d-1)$-manifolds. Now, we want to go further by defining an action on some $(d-1)$-manifold which has a boundary and then formulate the appropriate gluing law for this case.

A good question to ask is what kind of objects we should expect to run into during this process? To begin, recall that the action on a closed d-manifold gives a complex number while the action on a d-manifold with boundary gives an element in some metrized complex line associated to the boundary. Additionally, the action on a $(d-1)$-dimensional closed manifold gives a 1-dimensional Hilbert space (which we called the Chern-Simons line). So, we should expect that the action on a compact $(d-1)$-dimensional manifold with boundary give some similar objects (as was the case for d-manifolds). It turns out, as we will see shortly, that the action on a $(d-1)$-manifold with boundary gives an element in the category of Hilbert spaces which is associated, by the action, to the boundary of the $(d-1)$-manifold. Wait, we are not done! What is to stop us from going further down in dimension (or, equivalently stated, going up in codimension)? Nothing! Indeed, we can (and will) define the action on a compact $(d-k)$-dimensional manifold with boundary. And here, to the surprise of none, the action of the $(d-k)$-manifold takes its values in the (higher-)category associated which is associated to the boundary. In particular, the 3-dimensional Chern-Simons theory associates to a point a 2-category[1] of 2-Hilbert spaces, to

[1]See chapter 2.

a compact 1-dimensional manifold an element in the 2-category associated to its boundary (the category of 1-Hilbert spaces), to a compact 2-dimensional manifold an element in the category associated to its boundary (a 1-Hilbert space) and finally, to a compact 3-dimensional manifold, the Chern-Simons action associates an element in the Hilbert space associated to its boundary.

Before beginning with the classical case, we first need to introduce some ideas from mathematics - namely, that of torsors and gerbes.

9.2.1 G-Torsors

Although we implicitly used them in the previous sections, we wait until now to use the language of torsors and gerbes; partly because it was unnecessary, but mainly they were left out to not add confusing (to an already confusing treatment). However, for extended TQFTs we need to use higher vector spaces (and higher categories), and so they are of vital importance. Thus, let us begin with an overview of torsors and gerbes.

For a brief (although excellent) introduction to torsors, the reader can consult [10] and [6]. For our purposes we take G to be an abelian group. Roughly speaking, a G-torsor is a manifold X together with some right action of G on X, where the action obeys additional properties. To be more precise:

Definition 9.2.1. A ***G*-torsor** T is a pair (X, ρ) consisting of a manifold X and a simply transitive right G-action, $\rho : X \times G \to X$, of G on X; i.e.,

(a) $\rho_{id_G}(x) \equiv x \cdot id_G = x$ for all $x \in X$,

(b) $\rho_{g_1}\big(\rho_{g_2}(x)\big) = x \cdot (g_1 \cdot g_2)$ (where the $\cdot$ inside the parenthesis represents the action in G),

(c) for ANY two x_1 and x_2 in X, there exists a unique $g \in G$ such that $\rho_g(x_1) \equiv x_1 \cdot g = x_2$.

Remark 9.2.2. From the definition of a G-torsor, in particular item (c), we see that one can talk about the 'ratio' of two elements x_1, x_2 in X. Indeed, the expression $x_1 \cdot g = x_2$ says that the 'ratio' x_2/x_1 is the unique element in G such that $x_1 \cdot (x_2/x_1) = x_2$. Hence, we see that the difference between a group G and a G-torsor is that any two elements in G can be added[2] together to give another element in G, while there is no such notion of adding two elements in a G-torsor. However, we can, add an element of G to an element of X and get another element in X. Additionally, as we have just seen, you can also subtract two elements in a G-torsor, but this gives an element back in the group G, not in the G-torsor itself.

Example 9.2.3. Let us give some quick examples of torsors (see [10]). In Newtonian mechanics, we can only measure energy differences, not energies themselves. This follows from the fact that we can add any real number to our definition of energy without changing any of the physics. Therefore, it does not make much sense to ask what the energy of a system is - we can answer this question only after picking an arbitrary convention about what counts

[2]We will call the group multiplication of an abelian group addition and denote it by $+$.

as "zero energy". What makes more sense is to talk about the difference between the energy of a system in one state and the energy of that system in some other state. We can express this in terms of torsors as follows: energy differences lie in the group of real numbers $\mathbb{R}$, but energies themselves do not - they lie in an $\mathbb{R}$-torsor.

When one calculates the indefinite integral of some function f they get something like $F + C$, where the constant C is any real number. So, there's a whole set of choices: we call any one of these an "antiderivative" of f. This set is an $\mathbb{R}$-torsor. For if we have an antiderivative of f, we can add any number to it and get another antiderivative of f. If we have two antiderivatives of f, we can take their difference and get a real number. But we can not add two antiderivatives of f and get another antiderivative of f.

Let L be any one-dimensional complex space with an inner product, then the set of elements of unit norm gives a $\mathbb{T}$-torsor (here $\mathbb{T} := \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$ is called the circle group; $\mathbb{T}$ is canonically isomorphic to $U(1)$). Note, any $\mathbb{T}$-torsor takes this form for some Hermitian line L. And so, we see that the d-dimensional Dijkgraaf-Witten action gives a $\mathbb{T}$-torsor when acting on closed $(d-1)$-manifolds.

We will now construct the category of G-torsors. This requires us to define what we mean by a morphism between two torsors.

Definition 9.2.4. Let T_1 and T_2 be two G-torsors. Then, we define a **morphism** between them as a map $h : X_1 \to X_2$, between the two manifolds of T_1 and T_2, which commutes with the right action of G,

$$h\big(\rho_g(x)\big) = \rho_g\big(h(x)\big) \equiv h(x) \cdot g, \tag{9.1}$$

where $x \in X_1$ and $g \in G$.

With morphisms between our G-torsors, we are now in a position to define the category of G-torsors, $\mathscr{T}_1$.

Definition 9.2.5. We define the category $\mathscr{T}_1$ to be the category whose objects are G-torsors, while morphisms in $\mathscr{T}_1$ are given by the torsor morphisms previously defined.

Remark 9.2.6. Note, the set of morphisms $\mathrm{Hom}_{\mathscr{T}_1}(T_1, T_2)$ is naturally a G-torsor. Furthermore, every morphism in $\mathscr{T}_1$ has an inverse - hence, $\mathscr{T}_1$ is a groupoid. Finally, the reader should note that the group of automorphisms $\mathrm{Aut}(T)$, where $T \in \mathscr{T}_1$, is isomorphic to G. However, this isomorphism is NOT canonical. Indeed, the group G is itself a G-torsor in an obvious way. And if you hand me any other G-torsor T, we can pretend it's G as soon as we pick one element $t_1 \in X$ and declare it to be the identity element of G. More precisely, we get a map from T to G which sends any element t_2 to the unique g such that

$$t_1 \cdot g = t_2.$$

This map is an isomorphism. However, it depends on an arbitrary choice - hence it is not canonical. So, recapping [6]: Any group G is a G-torsor, and every other G-torsor is isomorphic to G - but not canonically!

As we will see later, one can construct a torsor whose 'abelian group' is given by $\mathscr{T}_1$ itself, which we call a G-*gerbe* (or 2-torsor) and denote by $\mathscr{T}_2$. However, before we can achieve this result, we must first define an 'abelian' structure on $\mathscr{T}_1$. That is, we must define what it means to multiply two G-torsors together, along with defining an inverse G-torsor.

To begin, let's define the product of two G-torsors.

Definition 9.2.7. Let $T_1, T_2 \in \mathscr{T}_1$ be two G-torsors, then we define their product, denoted $T_1 \cdot T_2$, as the quotient space

$$T_1 \cdot T_2 := \big\{(t_1, t_2) \in T_1 \times T_2\big\} \Big/ \sim, \tag{9.2}$$

where the equivalence $\sim$ is given by

$$(t_1 \cdot g, t_2) \sim (t_1, t_2 \cdot g), \tag{9.3}$$

for all $g \in G$. That is, the product of two G-torsors is given by taking their cartesian product and then identifying (or gluing) together the point $(t_1 \cdot g, t_2)$ with $(t_1, t_2 \cdot g)$. Thus, it doesn't matter whether G acts on T_1 or T_2 in the product $T_1 \cdot T_2$ - the actions are equivalent.

Remark 9.2.8. With this in mind, we clearly see that the action of G on $T_1 \cdot T_2$, for any two G-torsors $T_1, T_2 \in \mathscr{T}_1$, is given by

$$\rho_g\big((t_1, t_2)\big) = (t_1 \cdot g, t_2) = (t_1, t_2 \cdot g). \tag{9.4}$$

We next define the inverse of a G-torsor T_1.

Definition 9.2.9. Let $T \in \mathscr{T}_1$ be a G-torsor, we define the **inverse G-torsor**, which is denoted T^{-1}, as the G-torsor consisting of the same underlying set as T but the action of G is now given by

$$\rho_g(t^{-1}) \equiv t^{-1} \cdot g^{-1}. \tag{9.5}$$

Where we have denoted the element in T^{-1} corresponding to $t \in T$ by t^{-1}.

Remark 9.2.10. One should keep in mind that the elements in $\mathscr{T}_1$ CANNOT be declared equal, only isomorphic. Hence, it does NOT make sense to say that $T \cdot T^{-1} = G$, instead we can only say that

$$\begin{aligned} T \cdot T^{-1} &\cong G \\ (t \cdot g, t^{-1}) &\longmapsto g. \end{aligned} \tag{9.6}$$

This isomorphism is part of the data describing $\mathscr{T}_1$ as an abelian group. All other axioms for an abelian group, such as commutative and associativity, must be similarly modified. For example, now the associative law is not an axiom but a piece of the structure - a system of isomorphisms - and these isomorphisms satisfy a higher-order axiom called the *pentagon diagram* [21]. Finally, we also have the following isomorphism

$$T_2 \cdot T_1^{-1} \cong \mathrm{Hom}(T_1, T_2), \tag{9.7}$$

for all $T_1, T_2 \in \mathscr{T}_1$ [11].

So, recapping, given some (abelian) group G and manifold X, we were able to define a G-torsor (X, ρ_G). Additionally, we constructed the category of all G-torsors $\mathscr{T}_1$ - where objects are G-torsors and morphisms are mappings between the manifolds $f : X_1 \to X_2$ which commute with the G action. Finally, we showed that it is possible to define an abelian-like group structure on $\mathscr{T}_1$. Now, let us show how to construct a torsor whose 'abelian group' is given by the category $\mathscr{T}_1$. We call such torsors either G-gerbes or $G2$-torsors.

9.2.2 G-Gerbes ($G2$-Torsors)

Whereas before we defined a G-torsor as a pair (X, ρ_G) consisting of some manifold X which has a simply transitive right G-action, we now define a G-gerbe as some category $\mathscr{G}$ which has a simply transitive right $\mathscr{T}_1$-action. Hence, think of $\mathscr{G}$ as a module over $\mathscr{T}_1$.

Definition 9.2.11. Let $\mathscr{T}_1$ be the category of G-torsors (defined above) with its 'abelian' structure. Then, by a **G-gerbe** (or, equivalently, a **$G2$-torsor**) we mean a pair $(\mathscr{G}, \rho_{\mathscr{T}_1})$, where $\mathscr{G}$ is some arbitrary category which has a simply transitive right $\mathscr{T}_1$ action,

$$\rho_{\mathscr{T}_1} : \mathscr{G} \times \mathscr{T}_1 \longrightarrow \mathscr{G}. \tag{9.8}$$

Here, of course, $\rho_{\mathscr{T}_1}$ is to be understood as a functor between the two categories. We denote such an action by

$$\rho_{\mathscr{T}_1}(Y, T) \equiv Y \cdot T,$$

for all $Y \in \mathscr{G}$ and $T \in \mathscr{T}_1$. Hence, drawing a connection with the usual group action on a manifold, we can think of $Y \in \mathscr{G}$ as a 'point' in the 'manifold' $\mathscr{G}$, and $Y \cdot T$ as the action on Y by the group element T in $\mathscr{T}_1$.

Remark 9.2.12. Note, by $\rho_{\mathscr{T}_1}$ being simply transitive, the functor

$$F : \mathscr{G} \times \mathscr{T}_1 \longrightarrow \mathscr{G} \times \mathscr{G}, \tag{9.9}$$

defined by $(Y, T) \mapsto (Y, Y \cdot T)$, is an equivalence; i.e., there exists another functor $\tilde{F} : \mathscr{G} \times \mathscr{G} \to \mathscr{G} \times \mathscr{T}_1$ and two natural isomorphisms given by

$$\begin{aligned} \epsilon &: F \circ \tilde{F} \longrightarrow 1_{\mathscr{G} \times \mathscr{G}} \\ \eta &: 1_{\mathscr{G} \times \mathscr{T}_1} \longrightarrow \tilde{F} \circ F. \end{aligned} \tag{9.10}$$

Roughly speaking, this says that the categories $\mathscr{G} \times \mathscr{T}_1$ and $\mathscr{G} \times \mathscr{G}$ are essentially the same.

Now, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two G-gerbes. We define a morphism between $\mathscr{G}_1$ and $\mathscr{G}_2$ to be some functor

$$\mathscr{F} : \mathscr{G}_1 \longrightarrow \mathscr{G}_2,$$

which commutes with the right $\mathscr{T}_1$-action. Additionally, we denote the category whose objects are G-gerbes and whose morphisms are functors between the objects which commute with the right $\mathscr{T}_1$-action, as $\mathscr{T}_2$. It should be clear that $\mathscr{T}_2$ actually has the structure of a 2-category; roughly speaking, its objects are categories and its morphisms are functors, while 'functors' from one 2-category to another are natural transformations (of course there is additional structure and axioms to be obeyed, which can be found in chapter 2). Finally, we can, as before, define an abelian-like product structure on $\mathscr{T}_2$, in addition to defining inverses of G-gerbes. Thus, giving $\mathscr{T}_2$ an abelian group-like structure. Therefore, allowing for one to consider $\mathscr{T}_2$-torsors. Not surprisingly, we can proceed further, in the same manner as before, to construct $\mathscr{T}_n$-torsors (for all $n \in \mathbb{N}$). Details for these constructions can be found in the lecture notes by Baez et alii [3, 5, 4].

N.B. 9.2.13. For our purposes we restrict to the case where $G = \mathbb{T}$.

9.2.3 Action of a Group on Torsors and Gerbes

We now discuss symmetries of the previous 'abelian-like' structures. To begin, let A denote a finite group. By saying that A acts on the circle group $\mathbb{T}$ (i.e., $\mathscr{T}_0$) we mean that there exists a homomorphism $\rho : A \to \mathbb{T}$, or a character of A, and then A acts on $\mathbb{T}$ as multiplication by this character. Now, how does A act on a $\mathbb{T}$-torsor $T \in \mathscr{T}_1$? Well, recalling from section 9.2.1 that $\mathrm{Aut}(T) \equiv \mathbb{T}$, we see that the action of A on T is again given by a character. Finally, by group cohomology, the group of homomorphisms $\mathrm{Hom}(A, \mathbb{T})$ (hence the characters, and thus the action of A on $\mathbb{T}$, corresponding to ρ) is classified by $H^1(A; \mathbb{T})$,

$$\mathrm{Hom}(A, \mathbb{T}) \cong H^1(A; \mathbb{T}).$$

Next, by an action of A on the category $\mathscr{T}_1$ we mean that there exists a "homomorphism", $\varrho : A \to \mathscr{T}_1$, giving a T for each element in A which can then act on $\mathscr{T}_1$ via (9.2). That is, for each element $a \in A$ we have a $\mathbb{T}$-torsor T_a and for each $a_1, a_2 \in A$ we have an isomorphism

$$T_{a_1} \cdot T_{a_2} \cong T_{a_1 \cdot a_2}.$$

Note, we must require these isomorphisms to satisfy an associativity constraint. Additionally, note that our collection of torsors T_a gives a central extension $\tilde{A} = \cup_{a \in A} T_a$ of A by $\mathbb{T}$. Hence, $\mathbb{T} \subset Z(\tilde{A})$, where $Z(\tilde{A})$ denotes the center of $\tilde{A}$, and we have the following short exact sequence

$$1 \longrightarrow \mathbb{T} \longrightarrow \tilde{A} \xrightarrow{\pi} A \longrightarrow 1; \tag{9.11}$$

i.e., $A \cong \tilde{A}/\mathbb{T}$. Also, the fibre of $\pi : \tilde{A} \to A$ over a is given by T_a.

Remark 9.2.14. The generators of $\tilde{A}$ which are not present in G are called central charges. These are precisely the generators of $Z(\tilde{A})$. Thus, we see that the generators of $\mathbb{T}$ correspond to central charges.

It is known that the set of isomorphism classes of central extensions of G by $\mathbb{T}$ is in one-to-one correspondence with the cohomology group $H^2(A; \mathbb{T})$. And so, up to an isomorphism, the central extension $\tilde{A} = \cup_{a \in A} T_a$ of A by $\mathbb{T}$ (and hence the action of A on $\mathscr{T}_1$ defined by ϱ) is classified by an element in $H^2(A; \mathbb{T})$.

We mention that it is further possible to define an action of A on $\mathscr{T}_2$, and higher gerbes $\mathscr{T}_n$. Additionally, these actions are classified by representatives of higher cohomology groups.

9.2.4 Classical Theory

Finally, we are now in a position to discuss the classical extended Dijkgraaf-Witten theories. Recall from chapter 8, the d-dimensional (exponentiated) action of fields on a d-manifold takes its values in $\mathbb{T} \equiv \mathscr{T}_0$, while acting on fields on a $(d-1)$-manifold, the action takes its values in the category $\mathscr{T}_1$. This trend continues, as we will see. Indeed, the d-dimensional (exponentiated) action on fields on a $(d-n)$-manifold takes its values in the n-category $\mathscr{T}_n$.

To begin, fix a finite group Γ. We denote the *fields* in our theory by $\mathcal{C}_M$, which is the category of all regular Γ-covering spaces P over M. Furthermore, there exists symmetries of our fields: A morphism $f : P \to P'$ is a smooth map (which commutes with the right Γ-action), that induces the identity mapping on M (i.e., symmetries are gauge transformations - no surprise here). Now, as before, we define an equivalence relation on $\mathcal{C}_M$ by setting $P \cong P'$ iff there exists a morphism between the two coverings, and we denote the space of equivalence classes of fields by $\overline{\mathcal{C}}_M$. Note, if M is compact, $\overline{\mathcal{C}}_M$ is finite dimensional. Furthermore, if M is connected, there exists an isomorphism[3]

$$\overline{\mathcal{C}}_M \cong \mathrm{Hom}\big(\pi_1(M;x),\Gamma\big)\Big/\ \Gamma, \tag{9.12}$$

for any $x \in M$. Denote by $B\Gamma$ the classifying space for Γ, which we fix along with the universal covering space $E\Gamma \to B\Gamma$. Pick a cohomology class $[\alpha] \in H^d(B\Gamma;\mathbb{R}/\mathbb{Z})$ and its representative singular d-cocycle $\alpha \in Z^d(B\Gamma;\mathbb{R}/\mathbb{Z})$. Then, we define the *Lagrangian* $\mathscr{L}_{M,[\alpha]}(P)$ of our theory to be the pullback of α with respect to the classifying map $\gamma_M : M \to B\Gamma$,

$$\mathscr{L}_{M,[\alpha]}(P) := \gamma_M^*(\alpha) \in Z^d(M;\mathbb{R}/\mathbb{Z}). \tag{9.13}$$

Note, even though we write the Lagrangian $\mathscr{L}_{M,[\alpha]}(P)$ as a function of P, it is really a function of the induced classifying map γ_M. However, by definition, any classifying map γ_M' which 'differs' (i.e., not homotopic) from γ_M will define a new covering space P'. Therefore, it does not matter if we label the Lagrangian to be a function of P or γ_M, since they are the same thing (up to homotopy).

Before we proceed to define the action, we first need to generalize proposition 8.3.4 to include manifolds of any codimension. This will require us to discuss how one goes about integrating d-cocycles over manifolds of dimension $\leq d$.

Integrating singular d-cocycles over manifolds M with dim(M) $\leq$ d

Let $\alpha \in Z^d(M;\mathbb{R}/\mathbb{Z})$ be a singular d-cocycle. As we have seen before, when M is a closed oriented d-manifold, we can define the pairing

$$e^{2\pi i\langle\alpha,m\rangle} \in \mathbb{R}/\mathbb{Z},$$

where $m \in Z_d(M)$ is the d-cycle representing the fundamental class. Furthermore, we saw that this pairing is (well-)defined. Indeed, let $m' \in Z_d(M)$ be another representative

[3] See section 8.3.

of $[M] \in H_d(M)$. Then, by definition of homology, there exists some degenerate chain $w \in C_{d+1}(M)$ such that $m - m' = \partial w$. And so,

$$\begin{aligned} \langle \alpha, m - m' \rangle &= \langle \alpha, m \rangle - \langle \alpha, m' \rangle, \\ &= \langle \alpha, \partial w \rangle, \\ &= \langle \delta(\alpha), w \rangle, \end{aligned}$$

which vanishes since α is a cocycle. Hence, the pairing

$$e^{2\pi i \langle \alpha, m \rangle} \in \mathbb{T} \equiv \mathscr{T}_0$$

is (well-)defined. Let's further remind ourselves of the case where Σ is some closed $(d-1)$-dimensional manifold. In this case we have the (well-)defined pairing

$$I_{\Sigma,\alpha} := e^{2\pi i \langle \alpha, \sigma \rangle} \in \mathscr{T}_1,$$

where $\sigma \in Z_{d-1}(\Sigma)$ represents $[\Sigma] \in H_{d-1}(\Sigma)$, which can be seen as follows. First, denote by $\mathscr{C}_\Sigma$ the category whose objects are cycles $\sigma \in Z_{d-1}(\Sigma)$, which represent $[\Sigma]$, and whose morphisms are degenerate chains $x \in C_d(\Sigma)$ such that $\sigma - \sigma' = \partial x$. Then, define a functor

$$\mathscr{F}_{\Sigma,\alpha} : \mathscr{C}_\Sigma \longrightarrow \mathscr{T}_1,$$

by sending σ to $\mathbb{T}$ and the morphism x to $e^{2\pi i \langle \alpha, x \rangle}$. Then, we define $I_{\Sigma,\alpha}$ as the inverse limit[4] of $\mathscr{F}_{\Sigma,\alpha}$.

Up till now, everything we have mentioned has been previously discussed, in detail, when we looked at regular Dijkgraaf-Witten theories (i.e., not extended theories). However, the goal of this chapter is to define extended Dijkgraaf-Witten theories. And so, we must continue with going down in codimension. So, let us look at the case where S is some closed oriented $(d-2)$-manifold. The claim here is that the pairing

$$I_{S,\alpha} := e^{2\pi i \langle \alpha, s \rangle} \in \mathscr{T}_2,$$

where $s \in Z_{d-2}(S)$ represents $[S]$, is (well-)defined and makes sense as a $\mathbb{T}$-gerbe. Favoring predictability over originality, we will use the invariant section construction to show this. So, to begin, let $\mathscr{C}_S$ denote the category whose objects are oriented cycles $s \in Z_{d-2}(S)$, which represent $[S]$, with morphisms given by degenerate chains $y \in Z_{d-1}(S)$ such that $s - s' = \partial y$. Next, for all $s, s' \in \mathscr{C}_S$, denote by $\mathscr{C}_{s,s'}$ the category whose objects are $(d-1)$-chains y which satisfy $s - s = \partial y$, morphisms between each $(d-1)$-chain are given by degenerate chains $x \in Z_d(S)$ such that $y - y' = \partial x$. Additionally, define a functor

$$\mathscr{F}_{s,s';\alpha} : \mathscr{C}_{s,s'} \longrightarrow \mathscr{T}_1,$$

by sending objects y to $\mathbb{T}$ and morphisms x to $e^{2\pi i \langle \alpha, x \rangle}$. Define the $\mathbb{T}$-torsor $I_{s,s';\alpha}$ to be the space of invariant sections of $\mathscr{F}_{s,s';\alpha}$. Next, define a functor

$$\mathscr{F}_{S,\alpha} : \mathscr{C}_S \longrightarrow \mathscr{T}_2,$$

[4] See section 8.2.

by $\mathscr{F}_{S,\alpha}(s) = \mathscr{T}_1$ for each s and $\mathscr{F}_{S,\alpha}(s \to s')$ acts as multiplication by $I_{s,s';\alpha}$. Then, we define the $\mathbb{T}$-gerbe $I_{S,\alpha}$ to be the space of invariant sections of $\mathscr{F}_{S,\alpha}$, $I_{s,\alpha} \in \mathscr{T}_2$. Note, we can continue cranking out this proceedure to manifolds of codimension d, and in the end we will see that the pairing

$$I_{pt.,\alpha} := e^{2\pi i\langle\alpha,pt.\rangle} \in \mathscr{T}_n,$$

is (well-)defined.

Let us now treat the case of compact manifolds with boundary. As was previously shown in section 8.3.1, when $\alpha \in Z^d(M)$ is a singular d-cocycle on a compact d-manifold M, the pairing of the fundamental class m with α gives an element in the $\mathbb{T}$- torsor associated to ∂M,

$$e^{2\pi i\langle\alpha,m\rangle} \in I_{\partial M,\iota^*(\alpha)}.$$

Here $\iota : \partial M \hookrightarrow M$ is an inclusion. Next, let α be a singular d-cocycle on some compact oriented $(d-1)$-manifold Σ. Then, we claim that the pairing

$$e^{2\pi i\langle\alpha,\sigma\rangle},$$

gives an element of unit norm in the $\mathbb{T}$-gerbe associated to $\partial\Sigma$, $I_{\partial\Sigma,\iota^*(\alpha)}$ (here $\iota : \partial\Sigma \hookrightarrow \Sigma$). The proof of this claim can be found in the paper by Freed [22]; where he proves the most general case, consisting of manifolds in arbitrary codimensions. We combine the previous results into the following lemma.

In what follows, we make the shift $d \mapsto d+1$ in dimension label.

Lemma 9.2.15 (Freed's Lemma). *Let Σ be a closed oriented $(d+1-n)$-dimensional manifold, for some non-zero $n \in \mathbb{N}$, and let $\alpha \in Z^{d+1}(\Sigma;\mathbb{R}/\mathbb{Z})$ be a singular $(d+1)$-cocycle on Σ. Then, there is an element $I_{\Sigma,\alpha} \in \mathscr{T}_n$ defined. Furthermore, if M is a compact oriented $(d+2-n)$-dimensional manifold, $\alpha \in Z^{d+1}(M;\mathbb{R}/\mathbb{Z})$ a singular $(d+1)$-cocycle on M and $\iota : \partial M \hookrightarrow M$ is the inclusion of the boundary, then*

$$e^{2\pi i\langle\alpha,m\rangle} \in I_{\partial M,\iota^*(\alpha)} \tag{9.14}$$

is defined. Additionally, these higher $\mathbb{T}$-torsors and pairings satisfy:

(*a*) ***Functoriality:*** *If $f : \Sigma' \to \Sigma$ is an orientation preserving diffeomorphism, then there is an induced isomorphism*

$$f_* : I_{\Sigma',f^*(\alpha)} \xrightarrow{\cong} I_{\Sigma,\alpha}. \tag{9.15}$$

If $F : M' \to M$ is an orientation preserving diffeomorphism, then there is an induced isomorphism

$$(\partial F)_*\left(e^{2\pi i\langle F^*(\alpha),m'\rangle}\right) \xrightarrow{\cong} e^{2\pi i\langle\alpha,m\rangle}. \tag{9.16}$$

Remark 9.2.16. Note, if $n = 1$ then (9.16) becomes an equality.

(*b*) ***Orientation:*** *There are natural isomorphisms*

$$I_{\bar{\Sigma},\alpha} \xrightarrow{\cong} (I_{\Sigma,\alpha})^{-1}, \tag{9.17}$$

and

$$e^{2\pi i\langle\alpha,-m\rangle} \xrightarrow{\cong} \left(e^{2\pi i\langle\alpha,m\rangle}\right)^{-1}. \tag{9.18}$$

Here, $\bar{\Sigma}$ represents the manifold Σ with reversed orientation while $-m$ represents the fundamental class $[\bar{M}]$.

(*c*) ***Multiplicativity:*** *If $\Sigma = \sqcup_{i=1}^{n}\Sigma_i$, with $\alpha_i \in Z^{d+1}(\Sigma_i;\mathbb{R}/\mathbb{Z})$ a singular $(d+1)$-cocycle on Σ_i, then*

$$I_{\Sigma,\alpha} \cong I_{\Sigma_1,\alpha_1} \cdot I_{\Sigma_2,\alpha_2} \cdot \quad \cdots \quad \cdot I_{\Sigma_n,\alpha_n}. \tag{9.19}$$

If $M = \sqcup_{i=1}^{n}M_i$ and m_i represents the fundamental class $[M_i]$, then

$$e^{2\pi i\langle\alpha,m\rangle} \cong e^{2\pi i\langle\alpha_1,m_1\rangle}e^{2\pi i\langle\alpha_2,m_2\rangle}\cdots e^{2\pi i\langle\alpha_n,m_n\rangle}. \tag{9.20}$$

(*d*) ***Gluing:*** *Suppose $j:\Sigma \hookrightarrow M$ is a closed oriented codimension one submanifold and M^{cut} is the manifold obtained by cutting M along Σ. Then $\partial M^{cut} = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$. Next, suppose $\alpha \in Z^{d+1}(M;\mathbb{R}/\mathbb{Z})$ is a singular $(d+1)$-cocycle on M and $\alpha^{cut} \in Z^{d+1}(M^{cut};\mathbb{R}/\mathbb{Z})$ is the induced singular d-cocycle on M^{cut} and that m^{cut} represents the fundamental class $[M^{cut}]$. Then, there is an natural isomorphism*

$$\left\langle e^{2\pi i\langle\alpha^{cut},m^{cut}\rangle}\right\rangle_{\Sigma,j^*(\alpha)} \xrightarrow{\cong} e^{2\pi i\langle\alpha,m\rangle}, \tag{9.21}$$

where $\langle\cdot\rangle_{\Sigma,j^*(\alpha)}$ is the contraction

$$\langle\cdot\rangle_{\Sigma,j^*(\alpha)} : I_{\partial M^{cut},\alpha^{cut}} \cong I_{\partial M,\iota^*(\alpha)} \otimes I_{\partial M,\iota^*(\alpha)} \otimes I^{-1}_{\partial M,\iota^*(\alpha)} \xrightarrow{\cong} I_{\partial M,\iota^*(\alpha)},$$

with $\iota:\partial M \to M$ as before.

(*d*) ***Stokes' Theorem I:*** *Let $\alpha \in Z^{d+1}(W;\mathbb{R}/\mathbb{Z})$ be a singular cocycle on some compact oriented $(d+3-n)$-manifold W (recall $n \in \mathbb{N}$ and hence $n \neq 0$). Then, there is a natural isomorphism*

$$e^{2\pi i\langle\alpha,\partial w\rangle} \in \mathscr{T}_{n-2}. \tag{9.22}$$

Remark 9.2.17. Note, $\mathscr{T}_{n-2}$ is the identity element in $\mathscr{T}_{n-1}$. Hence, when $n=1$, equation (9.22) reduces to

$$e^{2\pi i\langle\alpha,\partial w\rangle} = 1,$$

since $1 = id_{\mathscr{T}_1}$, which is in agreement with (8.12).

(*f*) ***Stokes' Theorem II:*** *Any singular d-cochain $\beta \in C^d(\Sigma;\mathbb{R}/\mathbb{Z})$ on a closed oriented $(d+1-n)$-manifold Σ, determines a trivialization*

$$I_{\Sigma,\delta\beta} \cong \mathscr{T}_{n-1}. \tag{9.23}$$

Under this isomorphism, any singular d-cochain $\beta \in C^d(M;\mathbb{R}/\mathbb{Z})$ on a compact oriented $(d+2-n)$-manifold M satisfies

$$e^{2\pi i\langle\delta\beta,m\rangle} \cong \mathscr{T}_{n-2}. \tag{9.24}$$

Proof. See [22]. □

Now, back to the classical theory of extended Dijkgraaf-Witten theories in arbitrary codimensions. Recall, for the sake of the readers memory, the fields $\mathcal{C}_M$ in our theory is given by the category of all Γ-covering spaces P over some manifold M and that for each P there exists a classifying map $\tilde{\gamma}_M : P \to E\Gamma$, which carries all of the information of P. Next, fix a cohomology class $[\alpha] \in H^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ and a representative singular $(d+1)$-cocyle $\alpha \in Z^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ [5]. Then, we define the Lagrangian $\mathscr{L}_{M,[\alpha]}(P)$ of our theory to be the pullback of α, from $B\Gamma$ to M, via the classifying map $\gamma_M : M \to B\Gamma$,

$$\mathscr{L}_{M,[\alpha]}(P) := \gamma_M^*(\alpha) \in Z^d(M; \mathbb{R}/\mathbb{Z}).$$

We define our classical action as follows. First, let M be a compact oriented manifold, where $\dim(M) \leq d+1$, and let $P \in \mathcal{C}_M$. Then, if $\tilde{\gamma}_M : P \to E\Gamma$ is a classifying map for P, with induced classifying map $\gamma_M : M \to B\Gamma$, consider the following pairing (which we defined via 9.2.15)

$$e^{2\pi i S_{M,[\alpha]}(P)} := e^{2\pi i \langle \gamma_M^*(\alpha), m \rangle}, \tag{9.25}$$

where m is some cycle representing $[M]$. In order to appeal to the physicists, we could rewrite this more symbolically as

$$e^{2\pi i S_{M,[\alpha]}(P)} \equiv \exp\left(2\pi i \int_M \mathscr{L}_{M,[\alpha]}(P)\right). \tag{9.26}$$

We have already seen, in the previous section, how to define this pairing for both compact and closed manifolds of arbitrary dimension. However, we must now see how we can get rid of the dependence on the classifying map γ_M, since we do not want our theory to depend on which classifying map we pick to define our action. To begin[6], let Σ be some closed oriented $(d+1-n)$-manifold (here, as usual, we take $n \in \mathbb{N}$) and let $Q \in \mathcal{C}_\Sigma$ be a regular Γ-covering space over Σ. Next, we define a category $\mathcal{C}_Q$ whose objects are classifying maps $\tilde{\gamma}_\Sigma : Q \to E\Gamma$ and whose morphisms are homotopies $\tilde{h} : \tilde{\gamma}_\Sigma \simeq \tilde{\gamma}'_\Sigma$. Then, define a functor $\mathcal{F}_{Q;[\alpha]} : \mathcal{C}_Q \to \mathscr{T}_n$ as follows. For objects $\tilde{\gamma}_\Sigma \in \mathrm{Obj}(\mathcal{C}_Q)$ we have

$$\mathcal{F}_{Q;[\alpha]}(\tilde{\gamma}_\Sigma) = e^{2\pi i \langle \gamma_\Sigma^*(\alpha), \sigma \rangle} = I_{\Sigma, \gamma_\Sigma^*(\alpha)}, \tag{9.27}$$

where $\sigma \in Z_{d+1-n}(\Sigma)$ represents $[\Sigma] \in H_{d+1-n}(\Sigma)$, and $I_{\Sigma,\gamma_\Sigma^*(\alpha)}$ is the integration line defined in 9.2.15. For morphisms $\tilde{h} \in \mathrm{Hom}_{\mathcal{C}_Q}(\tilde{\gamma}_\Sigma, \tilde{\gamma}'_\Sigma)$ we set

$$\mathcal{F}_{Q;[\alpha]}(\tilde{h}) = e^{2\pi i \langle h^*(\alpha), i \times \sigma \rangle} : I_{\Sigma, \gamma_\Sigma^*(\alpha)} \longrightarrow I_{\Sigma, \gamma_\Sigma'^*(\alpha)}, \tag{9.28}$$

where $h : [0,1] \times \Sigma \to B\Gamma$ is induced from $\tilde{h} : [0,1] \times Q \to E\Gamma$, and $i \times \sigma \in Z_{d+2-n}([0,1] \times \Sigma)$ represents the fundamental class $[[0,1] \times \Sigma] \in H_{d+2-n}([0,1] \times \Sigma)$. That $\mathcal{F}_{Q;[\alpha]}$ is a functor

[5] Remember, we are using the notations introduced by Freed. And so, we write our cohomology class $[\alpha]$ as degree $d+1$ rather than of degree d.

[6] We only give a brief overview here, since our arguments are just a generalization from what appears in section 8.3.1 for codimensions zero and one.

follows from 9.2.15, see [22]. Furthermore, by Stokes' theorem (see 9.2.15), the morphism $\tilde{h} : \tilde{\gamma}_M \to \tilde{\gamma}_M$ maps to the trivial morphism, via $\mathcal{F}_{Q;[\alpha]}$. Hence, there exists an inverse limit of $\mathcal{F}_{Q;[\alpha]}$ in $\mathscr{T}_n$, which we will denote by $L_\Sigma^{[\alpha]}(Q)$. And so, we define the value of the classical extended $(d+1)$-dimensional Dijkgraaf-Witten action on a covering space Q, over a closed oriented $(d+1-n)$-dimensional manifold Σ, to be $L_\Sigma^{[\alpha]}(Q)$.

Let us now consider compact manifolds. So, suppose M is some compact oriented $(d+2-n)$-dimensional manifold, with boundary ∂M, and let $P \in \mathcal{C}_M$ be a Γ-covering space over M. Next, define a category $\mathcal{C}_P$ whose objects are classifying maps $\tilde{\gamma}_M : P \to E\Gamma$ and whose morphisms are homotopies $\tilde{h} : \tilde{\gamma}_M \simeq \tilde{\gamma}'_M$. Note, by restriction the classifying maps and homotopies to the boundary we get a new category $\mathcal{C}_{\partial P}$. Furthermore, these restrictions induce a functor $\partial : \mathcal{C}_P \to \mathcal{C}_{\partial P}$. Now, any $\tilde{\gamma}_M \in \mathrm{obj}(\mathcal{C}_P)$ induces (via the integration theory previous discussed)

$$e^{2\pi i \langle \gamma_M^*(\alpha), m\rangle} \in I_{\partial M, \partial\gamma_M^*(\alpha)} = \mathcal{F}_{\partial P;\alpha}(\partial\tilde{\gamma}_M). \tag{9.29}$$

Finally, under a morphism (i.e., homotopy $\tilde{h} : \tilde{\gamma}_M \to$), we have that

$$\mathcal{F}_{\partial P;\alpha}\left(\partial\tilde{\gamma}_M \xrightarrow{\partial\tilde{h}} \partial\tilde{\gamma}'_M\right) e^{2\pi i \langle \gamma_M^*(\alpha), m\rangle} = e^{2\pi i \langle \gamma_M'^*(\alpha), m\rangle}. \tag{9.30}$$

The proof of such a statement is completely identical to the case of codimension zero and one, which is discussed with complete details in the paragraph leading up to equation (8.3.1). This leads us to the conclusion that equation (9.30) determines an element

$$e^{2\pi i S_{M,[\alpha]}(P)} \in L_{\partial M}^{[\alpha]}(\partial P). \tag{9.31}$$

Hence, the classical extended Dijkgraaf-Witten action associates the invariant section $L_\Sigma^{[\alpha]} \in \mathscr{T}_n$ to each $(d+1-n)$-dimensional closed manifold Σ, and $e^{2\pi i S_{M,[\alpha]}} \in L_{\partial M}^{[\alpha]} \in \mathscr{T}_n$ to each $(d+2-n)$-dimensional compact manifold M. These assignments satisfy several familiar axioms.

Proposition 9.2.18 (Freed). *Let Γ be a finite group, $\alpha \in Z^{d+1}(B\Gamma;\mathbb{R}/\mathbb{Z})$ a $(d+1)$-cocyle representing the class $[\alpha] \in H^{d+1}(B\Gamma;\mathbb{R}/\mathbb{Z})$ and $Q \in \mathcal{C}_\Sigma$ while $P \in \mathcal{C}_M$. Then, the assignments*

$$\begin{aligned} Q &\longmapsto L_\Sigma^{[\alpha]}(Q) \in \mathscr{T}_n \\ P &\longmapsto e^{2\pi i S_{M,[\alpha]}(P)} \in L_{\partial M}^{[\alpha]}(\partial P), \end{aligned} \tag{9.32}$$

defined above for $(d+1-n)$-manifolds Σ and $(d+2-n)$-manifolds M, satisfy:

(*a*) ***Functoriality:*** *Let $\Gamma \hookrightarrow Q \xrightarrow{\pi} \Sigma$ and $\Gamma \hookrightarrow Q' \xrightarrow{\pi'} \Sigma'$ be two regular covering spaces and let $\psi : Q \to Q'$ be a bundle map such that the induced map $\hat{\psi} : \Sigma \to \Sigma'$ is orientation preserving diffeomorphism. Then, there is an isomorphism*

$$\psi_* : L_\Sigma^{[\alpha]}(Q) \cong L_{\Sigma'}^{[\alpha]}(Q'). \tag{9.33}$$

Additionally, let $\varphi : P \to P'$ be a bundle map with induced map $\hat{\varphi} : M \to M'$ orientation preserving diffeomorphism, then

$$\partial\varphi_* \left(e^{2\pi i S_{M,[\alpha]}(P)}\right) \cong e^{2\pi i S_{M',[\alpha]}(P')}, \tag{9.34}$$

where $\partial\varphi$ is the restriction of $\varphi : P \to P'$ to the boundary.

(*b*) ***Orientation:*** *Denote by $\bar{A}$ the manifold A with opposite orientation. There is a natural isometry*

$$L^{[\alpha]}_{\bar{\Sigma}}(Q) \cong L^{[\alpha]}_{\Sigma}(Q)^{-1}, \tag{9.35}$$

where by $L^{[\alpha]}_{\Sigma}(Q)^{-1}$ we mean the inverse torsor to $L^{[\alpha]}_{\Sigma}(Q)$. Furthermore, we have that the action defined on $\bar{M}$ is isomorphic to the complex conjugate of the action defined on M,

$$e^{2\pi i S_{\bar{M},[\alpha]}(P)} \cong \overline{e^{2\pi i S_{M,[\alpha]}(P)}}. \tag{9.36}$$

(*c*) ***Multiplicativity:*** *If $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \cdots \sqcup \Sigma_n$ with Q_i denoting covering spaces over Σ_i, then*

$$L^{[\alpha]}_{\Sigma}(Q_1 \sqcup Q_2 \sqcup \cdots \sqcup Q_n) \cong L^{[\alpha]}_{\Sigma_1}(Q_1) \cdots L^{[\alpha]}_{\Sigma_n}(Q_n). \tag{9.37}$$

While, if M decomposes as the disjoint union $M = \sqcup_{i=1}^{n} M_i$ and P_i are coverings over M_i, then

$$e^{2\pi i S_{\sqcup_i M_i,[\alpha]}(\sqcup_i P_i)} = e^{2\pi i S_{M_1,[\alpha]}(P_1)} \cdot \cdots \cdot e^{2\pi i S_{M_n,[\alpha]}(P_n)}. \tag{9.38}$$

(*d*) ***Gluing:*** *Suppose M is a compact, oriented manifold and that $\Sigma \hookrightarrow M$ is a closed oriented codimension one submanifold of M. Let M^{cut} denote the manifold obtained by cutting M along Σ. Then, $\partial M^{cut} = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$. Now, suppose P is a covering space over M, P^{cut} is a covering space over M^{cut} and Q is the restriction of P to Σ. Then the is a natural isomorphism*

$$e^{2\pi i S_{M,[\alpha]}(P)} \cong \left\langle e^{2\pi i S_{M^{cut},[\alpha]}(P^{cut})} \right\rangle_{L_\Sigma(Q)}, \tag{9.39}$$

where $\langle \cdot \rangle_{L_\Sigma(Q)}$ is the natural contraction

$$\langle \cdot \rangle_{L_\Sigma(Q)} : L^{[\alpha]}_{\partial M}(\partial P) \cdot L^{[\alpha]}_{\Sigma}(Q_n) \cdot L^{[\alpha]}_{\Sigma}(Q_n)^{-1} \longrightarrow L^{[\alpha]}_{\partial M}(\partial P).$$

Remark 9.2.19. Note, as opposed to the case of codimension zero and one manifolds (see theorem 8.3.5), here we can only, at best, say things are isomorphic, not equal. This has to do with the necessity to use (higher) category theory. For example, if $n = 1$ the above isomorphisms would turn into equalities. Also, the Functoriality axiom (a) means, in particular, that for any $Q \in \mathcal{C}_\Sigma$ there is an action of the finite group $\mathrm{Aut}(Q)$ on $L_\Sigma(Q)$. As explained in section 9.2.3, the isomorphism class of this action is an element in $H^n(\mathrm{Aut}(Q); \mathbb{T})$. Hence, for $n = 2$ the classical action determines a central extension of $\mathrm{Aut}(Q)$ by $\mathbb{T}$.

So, we have shown that the extended Dijkgraaf-Witten theory attributes a complex number to a codimension zero closed manifold, to a codimension one closed manifold it attributes a $\mathbb{T}$-torsor, and so on. While, to a compact codimension zero manifold, the Dijkgraaf-Witten action assigns an element in the $\mathbb{T}$-torsor associated to its boundary and to a compact codimension one manifold, the Dijkgraaf-Witten theory assigns an element in the $\mathbb{T}$-gerbe associated to its boundary, and so. In particular, the three dimensional Dijkgraaf-Witten theory assigns a $\mathbb{T}2$-gerbe to a point, a $\mathbb{T}$-gerbe to a 1-dimensional manifold, a $\mathbb{T}$-torsor to a 2-dimensional manifold manifold, and to a 3-dimensional manifold the extended classical Dijkgraaf-Witten actions assigns an element (of unit norm) in $\mathscr{T}_1$, $e^{2\pi i\langle\gamma_M^*(\alpha),m\rangle}$.

We now proceed to quantize our extended Dijkgraaf-Witten theories. However, as was the case when dealing with the classical theories, we first need to introduce a few more concepts from mathematics.

9.3 Quantum Theory

As before when we quantized the (normal) Dijkgraaf-Witten classical theory, in this chapter we will quantize the extended theory by defining a path integral (or really, a path sum). That is, we integrate the classical action over the moduli space of equivalence classes of fields on some manifold. However, recalling that the classical action assigns elements in $\mathscr{T}_n$ to manifolds of codimension n (for example, in the top dimension (codimension 0) it assigns a value in $\mathbb{T} = \mathscr{T}_0$), we quickly see that we cannot add two values assigned by the action - since addition is not defined for torsors. Thus, how do we construct a path sum that adds together the values of an action which, in turn, takes its values in the category of certain types of torsors? We can get around this inability to add by embedding each category of (higher) torsors into the category of (higher) inner product spaces, $\mathscr{T}_n \hookrightarrow \mathscr{V}_n$ (where addition is defined). In particular, for the top dimension (codimension 0) case, we embed $\mathbb{T} \hookrightarrow \mathbb{C}$ so that we can perform the path integral. While in general, we embed the n-category of n-torsors $\mathscr{T}_n$ into the n-category of n-inner product spaces $\mathscr{V}_n$ (which has addition defined) to perform the path integral. Thus, the next step in forming the extended Dijkgraaf-Witten quantum theory is to precisely define (higher) inner product spaces and the categories formed from them.

9.3.1 (Higher) Inner Product Spaces

We first begin by reviewing the basic properties of **(complex) inner product spaces** over $\mathbb{C}$. Recall, a vector space V over $\mathbb{C}$ is a pair $(V, \mathbb{C})$, with V some set, together with maps:

(a) **Commutative Sum:** a map $+ : V \times V \longrightarrow V$,

(b) **Scalar Multiplication:** a map $\cdot : \mathbb{C} \times V \longrightarrow V$,

(c) **Inner Product:** a map $(\cdot,\cdot) : V \times \overline{V} \longrightarrow \mathbb{C}$ (here $\overline{V}$ is the conjugate vector space to V defined below), obeying certain axioms. The canonical 1-dimensional complex inner product space is given by $\mathbb{C}$ itself, where the inner product is $(z_1, z_2) := z_1 \cdot \overline{z}_2$. We define the **conjugate inner product** space $\overline{V}$ as having the same structure/properties as V, except in the case of $\overline{V}$ we replace the usual scalar multiplication with conjugate scalar multiplication and the inner product with the transposed inner product. The **dual space** V^* to V is defined as the complex inner product space

$$V^* := \mathrm{Hom}(V, \mathbb{C}).$$

Note, there is a natural isometry $V^* \cong \overline{V}$ determined by the inner product on V.

Definition 9.3.1. We call the category whose objects are given by complex inner product spaces V and whose morphisms are given by linear maps $L : V_1 \to V_2$ which preserve the inner product, the **category of complex inner products** and denote it by $\mathscr{V}_1$.

Given two complex inner product spaces $V_1, V_2 \in \mathscr{V}_1$ we can form their **direct sum**, $V_1 \oplus V_2$, and their **tensor product**, $V_1 \otimes V_2$, together with the induced inner products

$$(v_1 \oplus v_2, w_1 \oplus w_2) = (v_1, w_1) + (v_2, w_2),$$

$$(v_1 \otimes v_2, w_1 \otimes w_2) = (v_1, w_1)(v_2, w_2).$$

It can be shown that there exists natural isomorphisms $0 \oplus V \cong V$ (here 0 is the 'zero' inner product space) and $\mathbb{C} \otimes V \cong V$. Furthermore, the inner product on V induces an inner product on the space $V^* \otimes V$ (remember we can think of $V^* \otimes V$ as the space $\mathrm{Hom}(V; \mathbb{C}) \cong V^*$ - in general, we have $W^* \otimes V \cong \mathrm{Hom}(V, W)$), given by

$$(F_1, F_2) = \mathrm{Trace}(F_1 F_2^*),$$

where $F_i \in \mathrm{Hom}(V, \mathbb{C})$ and F_2^* is the hermitian adjoint of F_2.

Remark 9.3.2. Note, the direct sum of complex inner product spaces, $V_1 \oplus V_2$, and the tensor product of complex inner product spaces, $V_1 \otimes V_2$, gives the category $\mathscr{V}_1$ a *commutative 'ring-like' structure with involution* (the involution comes from the conjugation or duality).

We now embed $\mathscr{T}_1$ into $\mathscr{V}_1$. To begin, let $T \in \mathscr{T}_1$ be a $\mathbb{T}$-torsor. Then, we construct a complex one dimensional inner product space from T by defining

$$L_T = T \times_{\mathbb{T}} \mathbb{C} := \left\{(t, z) \in T \times \mathbb{C}\right\} \Big/ \sim, \tag{9.40}$$

where the equivalence is given by $(t \cdot \lambda, z) \sim (t, \lambda \cdot z)$ for all $\lambda \in \mathbb{T}$. Hence, it does not matter if the right $\mathbb{T}$ action acts on T or $\mathbb{C}$; i.e., $(t, z) \cdot \lambda = (t \cdot \lambda, z) = (t, \lambda \cdot z)$. We call the one dimensional complex inner product spaces L_T **hermitian lines**. This gives us our embedding of $\mathscr{T}_1$ into $\mathscr{V}_1$. Further, the inner product $(\cdot, \cdot) : L_T \times L_T \to \mathbb{C}$ on L_T is defined by

$$\Big((t, z), (t', w)\Big) := (z, w)_{\mathbb{C}} = z \cdot \bar{w}, \tag{9.41}$$

where $\bar{w}$ is the complex conjugate of w and $(\cdot, \cdot)_{\mathbb{C}}$ is the usual inner product on $\mathbb{C}$. We now list some basic properties of L_T [22]:

(a) **Inner product space of inverse torsors $L_{T^{-1}}$:**

$$L_{T^{-1}} \cong L_T^* \cong \overline{L}_T. \tag{9.42}$$

(b) **Inner product space of product of torsors $L_{T_1 \cdot T_2}$:**

$$L_{T_1 \cdot T_2} \cong L_{T_1} \otimes L_{T_2}. \tag{9.43}$$

(c) **Trivialization $L_{\mathbb{T}}$:**

$$L_{\mathbb{T}} \cong \mathbb{C}. \tag{9.44}$$

Proposition 9.3.3. *The embedding defined in* (9.40) *is a homomorphism, while the image of the embedding is closed under the tensor product.*

Proof. See [22]. □

Finally, we can define an 'inner product' on $\mathscr{V}_1$, $(\cdot,\cdot)_{\mathscr{V}_1}$, thus giving the category $\mathscr{V}_1$ its own inner product-like structure. We do so by setting, for all $V_1, V_2 \in \mathscr{V}_1$,

$$(V_1, V_2)_{\mathscr{V}_1} := V_1 \otimes \overline{V}_2. \tag{9.45}$$

Thus, we can view $\mathscr{V}_1$ not only as a category, but also as an 'inner product space'. Also, this inner product defines a 'norm' on $\mathscr{V}_1$, given by

$$|V|_{\mathscr{V}_1} := (V, V)_{\mathscr{V}_1} = V \otimes \overline{V}. \tag{9.46}$$

Proposition 9.3.4. *The elements of 'unit norm' in $\mathscr{V}_1$, that is of norm $\mathbb{C}$, are precisely the hermitian lines L; i.e., the elements in the image of the embedding defined by* (9.40).

Proof. We have that, for all $T \in \mathscr{T}_1$,

$$|L_T|_{\mathscr{V}_1} = (L_T, L_T)_{\mathscr{V}_1} = L_T \otimes \overline{L}_T.$$

Now, since $\overline{L}_T \cong L_T^* \cong L_{T^{-1}}$ and since $L_{T_1} \otimes L_{T_2} = L_{T_1 \cdot T_2}$ (see (9.42) and (9.43), respectively), we conclude

$$|L_T|_{\mathscr{V}_1} = L_T \otimes \overline{L}_T \cong L_T \otimes L_{T^{-1}} \cong L_{T \cdot T^{-1}}.$$

Finally, the result follows from recalling that $T \cdot T^{-1} \cong \mathbb{T}$ (see (9.6), replacing G with $\mathbb{T}$), along with $L_{\mathbb{T}} \cong \mathbb{C}$ (see (9.44)),

$$|L_T|_{\mathscr{V}_1} \cong L_{T \cdot T^{-1}} \cong L_{\mathbb{T}} \cong \mathbb{C}.$$

□

So, recapping, starting with $\mathbb{C}$ we have shown how to construct a commutative ring-like category $\mathscr{V}_1$ (with involution) consisting of all inner product spaces over $\mathbb{C}$. Furthermore, we showed how to embed $\mathscr{T}_1$ into $\mathscr{V}_1$, and that the elements in the image of this embedding have unit norm.

We now iterate this proceedure and consider inner product spaces over the ring-like category $\mathscr{V}_1$. We call such an inner product space, a 2-*inner product space* and denote it by $\mathcal{W}$.

We want to embed the 2-category of $\mathbb{T}$-gerbes $\mathscr{T}_2$ (or, equivalently, the 2-category of $2\mathbb{T}$-torsors) into the 2-category of 2-inner product spaces $\mathscr{V}_2$, in order to perform the sums in the path integral. However, before writing down the embedding, let us first review some basic properties of 2-inner product spaces.

Recall, an inner product space was a set V with a commutative sum, inner product and scalar multiplication by $\mathbb{C}$. We generalize this to 2-inner product spaces by changing V from a set to a category, which we denote by $\mathcal{W}$, and changing $\mathbb{C}$ to the category $\mathscr{V}_1$, which possesses a ring-like structure. Thus, roughly speaking, a complex 2-inner product space is a module $\mathcal{W}$ over $\mathscr{V}_1$. To be more precise, consider the following definition:

Definition 9.3.5. By a **(complex) 2-inner product space** $\mathcal{W}$ we mean a pair of categories $(\mathcal{W}, \mathscr{V}_1)$, together with functors:

(a) **Abelian Group Law:** a map $+ : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{W}$,

(b) **Scalar Multiplication:** a map $\cdot : \mathscr{V}_1 \times \mathcal{W} \longrightarrow \mathcal{W}$,

(c) **Inner Product:** a map $(\cdot, \cdot) : \mathcal{W} \times \overline{\mathcal{W}} \longrightarrow \mathscr{V}_1$.

We can construct conjugate and dual 2-inner product spaces as well as direct sums and tensor products, just as before with regular inner product spaces. Note, there exists a zero 2-inner product space 0 such that $0 \oplus \mathcal{W} \cong \mathcal{W}$ and $\mathscr{V}_1 \otimes \mathcal{W} \cong \mathcal{W}$ (here the equivalence is in the sense of categories). Also, whereas before with regular inner product spaces $\mathbb{C}$ is an inner product space which is an identity for the tensor product, now $\mathscr{V}_1$ takes that role.

Since $\mathcal{W}$ is a category, it has an extra layer of structure. In particular, for all $W_1, W_2 \in \mathcal{W}$ there is a map

$$(W_2, W_1) \cdot W_1 \longrightarrow W_2.$$

Further, we assume that there is an isometry

$$(W_1, W_2) \longrightarrow \overline{(W_2, W_1)},$$

whose square is the identity. This implies that (W, W) has a 'real' structure for all $W \in \mathcal{W}$, since there exists an isometry $(W, W) \to \overline{(W, W)}$. We also assume the existence of compatible maps

$$\mathbb{C} \longrightarrow (W, W) \longrightarrow \mathbb{C}.$$

Whose composition is then multiplication by a real number, which we denote $\dim(W)$. Finally, we briefly mention that it is possible to define linear independence and bases for $\mathcal{W}$.

Definition 9.3.6. A **2-linear map** $L : \mathcal{W}_1 \to \mathcal{W}_2$, between two 2-inner product spaces, is a functor which preserves the 'addition' and scalar multiplication structure.

Note, the space of all 2-linear maps is the 2-inner product space $\mathrm{Hom}(\mathcal{W}_1, \mathcal{W}_2) \cong \mathcal{W}_2 \otimes \mathcal{W}_1^*$.

Definition 9.3.7. We call the 2-category whose objects are given by complex 2-inner product spaces $\mathcal{W}$ and whose morphisms are given by 2-linear maps $L : \mathcal{W}_1 \to \mathcal{W}_2$ which preserve the inner product, the **2-category of complex 2-inner products** and denote it by $\mathscr{V}_2$.

We can give $\mathscr{V}_2$ an inner product-like structure by defining the "inner product" $(\cdot,\cdot)_{\mathscr{V}_2} : \mathscr{V}_2 \times \overline{\mathscr{V}}_2 \to \mathscr{V}_2$ as

$$(\mathcal{W}_1, \mathcal{W}_2)_{\mathscr{V}_2} := \mathcal{W}_1 \otimes \overline{\mathcal{W}}_2, \tag{9.47}$$

for all $\mathcal{W}_1, \mathcal{W}_2 \in \mathscr{V}_2$.

Let us now embed the 2-category $\mathscr{T}_2$ into $\mathscr{V}_2$. We do this by defining, for all $\mathscr{G} \in \mathscr{T}_2$,

$$\mathcal{W}_{\mathscr{G}} := \mathscr{G} \times_{\mathscr{T}_1} \mathscr{V}_1 = \left\{ (G, V) \in \mathscr{G} \times \mathscr{V}_1 \right\} \Big/ \sim, \tag{9.48}$$

where the equivalence is given by $(G \cdot T, V) \sim (G, L_T \otimes V)$ for all $\mathbb{T}$-torsors T. Note, $\mathcal{W}_{\mathscr{T}_1} \cong \mathscr{V}_1$ and that the image of the embedding has 'unit' norm, $|\mathcal{W}_{\mathscr{G}}|_{\mathscr{V}_2} \cong \mathscr{V}_1$. We can show this last property in exactly the same way as before.

Example 9.3.8. Consider the following example of a 2-inner product space. To begin, let A be a finite group and let $(\mathscr{V}_1)^A$ denote the category of finite dimensional unitary representations of A. That is, each object in $(\mathscr{V}_1)^A$ is a representation of A, W, and each morphism is a map $\varphi : W \to W'$ such that

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & W' \\ {\scriptstyle \rho_W(a)}\downarrow & & \downarrow{\scriptstyle \rho_{W'}(a)} \\ W & \xrightarrow{\varphi} & W', \end{array}$$

commutes for all $a \in A$. Now, we define the 'scalar' multiplication $\cdot : \mathscr{V}_1 \times (\mathscr{V}_1)^A \to (\mathscr{V}_1)^A$ as $V \cdot W := V \otimes W$ for all $V \in \mathscr{V}_1$ and $W \in (\mathscr{V}_1)^A$. This is the tensor product representation in the usual sense. The direct sum $+ : (\mathscr{V}_1)^A \times (\mathscr{V}_1)^A \to (\mathscr{V}_1)^A$ is the usual direct sum of representations $W_1 \oplus W_2$, while we define the 'inner product' on $(\mathscr{V}_1)^A$ as

$$(W_1, W_2) := (W_1 \otimes \overline{W}_2)^A.$$

Which is thought of as first taking the tensor product representation $W_1 \otimes \overline{W}_2$ and then restricting to the subspace of $W_1 \otimes \overline{W}_2$ which is invariant under the action of A. We can generalize this example to the following situation (which will prove useful when quantizing the classical action).

Let $\mathscr{G}$ be a $\mathbb{T}$-gerbe with a nontrivial action of A on it, $\rho_{\mathscr{G}}$. Then, for any $G \in \mathscr{G}$, define the one dimensional complex 2-inner product space

$$L_G := \{(G, \mathbb{C})\} \in \mathcal{W}_{\mathscr{G}}. \tag{9.49}$$

Note, for any hermitian line $L \in \mathscr{V}_1$ the element $\{(G, L)\}$ is equivalent to $L_{G'}$ for some $G' \in \mathscr{G}$. Hence, any element of $\mathcal{W}_{\mathscr{G}}$ is isomorphic to a finite sum $L_{G_1} \oplus \cdots \oplus L_{G_n}$. Now, let A act on $\mathcal{W}_{\mathscr{G}}$ by

$$a \cdot L_G = L_{a \cdot G}, \tag{9.50}$$

where $a \cdot G = \rho_{\mathscr{G}}(a)\big(G\big)$. Then, we can define a complex 2-inner product which is invariant under A as

$$(\mathcal{W}_{\mathscr{G}})^{A,\rho} := \operatorname{span}\big\{W = L_{G_1} \oplus \cdots \oplus L_{G_n} \mid W \text{ is invariant under the } A \text{ action}\big\}, \tag{9.51}$$

where we have denoted $\rho_{\mathscr{G}}$ by ρ for simplicity.

Remark 9.3.9. Note, to make a good sense of "invariant" we must identify certain canonically isomorphic elements. For example, we the different permutations of the sum $L_{G_1} \oplus \cdots \oplus L_{G_n}$. Also, note that the dimension of the invariants is larger than the dimension of $\mathcal{W}_{\mathscr{G}}$ [22].

Thus, we have now embedded $\mathscr{T}_1$ and $\mathscr{T}_2$ into the inner product spaces $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively. Hence, we can now perform the path integral for codimension one and two manifolds. For higher codimensions we simply embed the higher categories $\mathscr{T}_n$ into $\mathscr{V}_n$ and perform the sums. For instance, as we have just seen, the 2-category $\mathscr{V}_2$ has a ring-like structure. Then, we define 3-inner product spaces as 2-categories with abelian group law, inner product and scalar multiplication by $\mathscr{V}_2$; i.e., a module over $\mathscr{V}_2$. The category of all 3-inner product spaces is a 3-category which we denote by $\mathscr{V}_3$. We can iterate this process as needed to perform the path integral for manifolds arbitrary codimension.

Now that we know how to add the elements assigned by the classical action, let us proceed with the quantization.

9.3.2 Quantization of the Extended Dijkgraaf-Witten Theory

We are now in a position to carry out the quantization of our $(d+1)$-dimensional [7] *extended Dijkgraaf-Witten classical theory* on any compact oriented manifold of dimension less than or equal to $d+1$. We do this, as before, by 'integrating' the classical action over the space of fields. However, we first must embed the values of the classical action (which, recall, are given by (higher) $\mathbb{T}$-torsors) into (higher) inner product spaces, so that we may perform the integral. Furthermore, since our fields have *symmetries* - Γ-bundle maps - we will only integrate over the *equivalence classes of fields*, or *moduli space of fields*. We must also take the *residual symmetries* - automorphisms - of the fields into account. It will be shown that, for a closed oriented $(d+1-n)$-manifold Σ, $n \in \mathbb{N}$, the resulting *quantum invariant*, which is obtained by evaluating the path integral, is a complex n-inner product space, $V_\Sigma \in \mathscr{V}_n$. While, if Σ is the empty set, $\Sigma = \emptyset$, the associated quantum invariant $V_\emptyset \in \mathscr{V}_{n-1}$ is the trivial space. For instance, if $n = 1$ we have $V_\emptyset = \mathbb{C}$. Additionally, we will see that, to a any compact oriented $(d+2-n)$-manifold M, possibly with boundary, the quantum invariant

[7] Recall, we have switched to saying $(d+1)$-dimensions rather than d-dimensions, simply for aesthetic appeal.

will be an element $Z(M)$ in the complex n-inner product space associated to the boundary, $Z(M) \in V_{\partial M}$. Note, for $n = 1$ we recover everything that was established in the previous sections. Alternatively, when $n = 2$ the quantum invariant of a closed $(d-1)$-dimensional manifold S is a 2-inner product space $V_S \in \mathscr{V}_2$, and the quantum invariant of a compact oriented d-dimensional manifold Σ is an object $Z(\Sigma)$ in the 2-category $V_{\partial\Sigma} \in \mathscr{V}_2$.

Since the gauge group Γ of our theory is finite, the moduli space of fields (i.e., the set of equivalence classes of the fields) on a compact manifold is a finite set. Hence, all we must do in order to perform the path integral (sum) is to define a measure μ on this finite set. Further, after looking at the path integral, which is something of the form

$$Z \sim \sum_{[P]\in\overline{\mathcal{C}}_M} \underbrace{\mu([P])}_{\in\mathbb{R}^+} \cdot \underbrace{\tilde{S}([P])}_{\in\mathcal{W}_n},$$

we see that we must define the product of a positive real number μ (the measure) by an element $\mathcal{W}_n \in \mathscr{V}_n$ (the embedded action $\tilde{S}$). We do this my taking $\mathcal{W}_n$ as before, except now we multiply the inner product on $\mathcal{W}_n$ by μ. That is, $\mu \cdot \mathcal{W}_n$ and $\mathcal{W}_n$ are equivalent as complex n-inner product spaces except that the inner product on $\mu \cdot \mathcal{W}_n$ is μ times the inner product on $\mathcal{W}_n$. Let us now precisely define a measure on our moduli spaces, and then perform the integration (and hence, quantization).

Definition 9.3.10. We define the **measure** $\mu : \mathcal{C}_M \to \mathbb{R}$ on the category of Γ-covering spaces $\mathcal{C}_M$ over any manifold M exactly as before for the non-extended theories. To be precise, for any $P \in \mathcal{C}_M$, set

$$\mu(P) = \frac{1}{|\mathrm{Aut}(P)|}, \tag{9.52}$$

where $|\mathrm{Aut}(P)|$ denotes the order of $\mathrm{Aut}(P)$.

Remark 9.3.11. Note, if P is equivalent to P' [8], $P \cong P'$, then $\mu(P) = \mu(P')$. Hence, the measure μ on $\mathcal{C}_M$ determines a measure on the moduli space $\overline{\mathcal{C}}_M$ (see remark 8.3.2), which we denote as $\mu([P])$ for any $[P] \in \overline{\mathcal{C}}_M$. Furthermore, since $\mu(P) = \mu(P')$ for equivalent coverings $P \cong P'$, we see that the measure is, in fact, invariant under the symmetries (bundle maps) of the fields. Indeed, $P \cong P'$ implies that there exists a bundle (or covering space) map $\varphi : P \to P'$. Thus, if $\mu(P) = \mu(P')$ then it is invariant under this map φ. As a corollary to this statement, note that the measure is therefore gauge invariant.

Continuing along, suppose M is a compact oriented d-manifold with boundary ∂M and let $Q \in \mathcal{C}_{\partial M}$ be a Γ-covering space over ∂M. Denote by $\mathcal{C}_M(Q)$ the category whose objects are pairs (P, θ) consisting of Γ-covering spaces P over M and isomorphisms $\theta : \partial P \to Q$, while morphisms in $\mathcal{C}_M(Q)$, $\varphi : (P, \theta) \to (P', \theta')$, are isomorphisms $\varphi : P \to P'$ such that

$$\begin{tikzcd} \partial P \arrow[rr, "\partial\varphi"] \arrow[dr, "\theta"'] & & \partial P' \arrow[dl, "\theta'"] \\ & Q & \end{tikzcd}$$

[8] Recall, we say that a Γ-covering space P over M is equivalent to a Γ-covering space P' over M if and only if there exists a Γ-covering space map $\varphi : P \to P'$ (see section 4.5), from P to P', with induced map on M equal to the identity, $\hat{\varphi} = id_M$.

commutes. Observe that these morphisms induce an equivalence relation on the category $\mathcal{C}_M(Q)$. Indeed, we say that two elements (P, θ) and (P', θ') in $\mathcal{C}_M(Q)$ are equivalent iff there exists a morphism between them,

$$\varphi : (P, \theta) \longrightarrow (P', \theta').$$

We denote the set of equivalence classes by $\overline{\mathcal{C}}_M(Q)$. Furthermore, the measure μ, defined in (8.29), passes to a measure on $\mathcal{C}_M(Q)$ (interpreting $\mathrm{Aut}(P, \theta)$ in the sense just described) and on to $\overline{\mathcal{C}}_M(Q)$ [24]. Finally, note that any morphism $\psi : Q \to Q'$ induces a measure preserving map

$$\psi_* : \overline{\mathcal{C}}_M(Q) \longrightarrow \overline{\mathcal{C}}_M(Q'). \tag{9.53}$$

For future reference, we now investigate the behavior of our measures under the operations of cutting and pasting. To begin, let $N \hookrightarrow M$ be an oriented codimension 1 submanifold and let M^{cut} be the manifold obtained by cutting M along N. Then, as we have previously seen, the boundary of M^{cut} becomes

$$\partial M^{\text{cut}} = \partial M \sqcup N \sqcup \bar{N}.$$

Additionally, let $Q \to N$ be a Γ-covering space over N and let $Q' \to \partial M$. Then, $\mathcal{C}_{M^{\text{cut}}}(Q \sqcup Q \sqcup Q')$ is the category of quadruples $(P^{\text{cut}}, \theta_1, \theta_2, \theta)$, where $P^{\text{cut}} \to M^{\text{cut}}$ is a Γ-covering space over M^{cut}, $\theta_i : P^{\text{cut}}|_N \to Q$ are isomorphisms, one over each copy of N in M^{cut}, and $\theta : \partial P \to Q'$ is an isomorphism. Now, consider the gluing map

$$\begin{aligned} g_Q :\ & \overline{\mathcal{C}}_{M^{\text{cut}}}(Q \sqcup Q \sqcup Q') \longrightarrow \overline{\mathcal{C}}_M(Q') \\ & (P^{\text{cut}}, \theta_1, \theta_2, \theta) \longmapsto (P^{\text{cut}}/(\theta_1 = \theta_2), \theta), \end{aligned} \tag{9.54}$$

then we have the following theorem.

Theorem 9.3.12. *The gluing map g_Q satisfies:*

(a) *g_Q maps onto the set of coverings over M whose restriction to N is isomorphic to Q.*

(b) *Let $\phi \in Aut(Q)$ act on $(P^{cut}, \theta_1, \theta_2, \theta) \in \mathcal{C}_{M^{cut}}(Q \sqcup Q \sqcup Q')$ by*

$$\phi \cdot (P^{cut}, \theta_1, \theta_2, \theta) = (P^{cut}, \phi \circ \theta_1, \phi \circ \theta_2, \theta). \tag{9.55}$$

Then the stabilizer of this action at $(P^{cut}, \theta_1, \theta_2, \theta)$ is the image $Aut(P) \to Aut(Q)$ determined by the θ_i, where $P = g_Q\big((P^{cut}, \theta_1, \theta_2, \theta)\big)$.

(c) *There is an induced action on equivalence classes $\overline{\mathcal{C}}_{M^{cut}}(Q \sqcup Q)$, and $Aut(Q)$ acts transitively on $g_Q^{-1}([P])$ for any $[P] \in \overline{\mathcal{C}}_M$.*

(d) *For all $[P] \in \overline{\mathcal{C}}_M$ we have*

$$\mu([P]) = vol(g_Q^{-1}([P])) \cdot \mu(Q). \tag{9.56}$$

Proof. See [24]. □

With the measure defined, we are now ready to perform the quantization. To begin, fix a class $[\alpha] \in H^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ and let Σ be a closed oriented $(d+1-n)$-dimensional manifold, $n \in \mathbb{N}$. Recall, in this case, the classical action defines a map,

$$L_{\Sigma}^{[\alpha]} : \mathcal{C}_{\Sigma} \longrightarrow \mathscr{T}_n, \tag{9.57}$$

from the category of Γ-covering spaces of Σ to the n-category of $n\mathbb{T}$-torsors (see conjecture 9.2.18). Note, we can think of (9.57) as a bundle of $n\mathbb{T}$-torsors over $\mathcal{C}_{\Sigma}$, $L_{\Sigma}^{[\alpha]} \to \mathcal{C}_{\Sigma}$. For example, in the case of a continuous group G, we saw that the classical action on a codimension-one manifold determines a line bundle over the space of connections (see proposition 7.4.1) $L_{\Sigma}(Q) \to \mathcal{A}(Q)$. Additionally, part (a) of conjecture 9.2.18 says that for each $Q \in \mathcal{C}_{\Sigma}$ there is an action ρ_Q of $\mathrm{Aut}(Q)$ on $L_{\Sigma}^{[\alpha]}(Q)$. Now, we use the embedding, previously defined, to embed $\mathscr{T}_n$ into $\mathscr{V}_n$, which replaces each $L_{\Sigma}(Q)^{[\alpha]} \equiv L_Q$ with the one dimensional complex n-inner product space $\mathcal{W}_{L_{\Sigma}(Q)}^{[\alpha]} \equiv \mathcal{W}_Q$.

Now part (a) of conjecture 9.2.18 says that any diffeomorphism $\psi : Q \to Q'$ induces an isomorphism $\psi_* : L_Q \cong L_{Q'}$, which (due to the embedding) induces an isomorphism $\psi_* : \mathcal{W}_Q \cong \mathcal{W}_{Q'}$. However, restricting ψ to be an automorphism does not guarantee that $\mathcal{W}_Q$ will transform trivially under the induced ψ_*. Indeed, $\mathrm{Aut}(Q)$ only acts trivially on those subspace of invariants under $\mathrm{Aut}(Q)$, $\mathcal{W}_Q^{\mathrm{Aut}(Q)}$ (see (9.51)). But we want the space $\mathcal{W}_Q$ to be invariant under the automorphisms of Q, in addition to being defined on equivalence classes of fields $[Q]$. We solve this by defining the quotient complex inner product space $\mathcal{W}_{[Q]}$, associated to the equivalence class $[Q] \in \overline{\mathcal{C}}_{\Sigma}$, as the invariant section of the functor $\mathcal{F}_{[Q]} : \mathcal{C}_{[Q]} \to \mathscr{V}_n$ - here $\mathcal{C}_{[Q]}$ is the category of Γ-covering spaces over Σ which are in the same isomorphism class as Q, $[Q]$, and we take $\mathcal{F}_{[Q]}(Q) = \mathcal{W}_Q$. Since then, due to $\mathcal{F}_{[Q]}$ having no holonomy (which must be assumed), the automorphisms of Q will act trivially on $\mathcal{W}_{[Q]}$. To be a bit more precise, let $\mathcal{C}_{[Q]}$ be the category of covering spaces in the same isomorphism class $[Q]$, and let $\mathcal{F}_{[Q]} : \mathcal{C}_{[Q]} \to \mathscr{V}_n$ be the functor whose value at Q is $\mathcal{W}_Q$. Then, we define $\mathcal{W}_{[Q]}$ as the space of invariant sections of $\mathcal{F}_{[Q]}$. Note, as we let $[Q]$ vary over $\overline{\mathcal{C}}_{\Sigma}$ we obtain a map

$$\mathcal{W}_{\Sigma} : \overline{\mathcal{C}}_{\Sigma} \longrightarrow \mathscr{V}_n. \tag{9.58}$$

We next define the quantum invariants of our theory.

Definition 9.3.13. The **quantum space** (or quantum invariant) V_{Σ} is defined to be the integral of $\mathcal{W}_{\Sigma}$ over $\overline{\mathcal{C}}_{\Sigma}$, which in our case reduces to a finite sum:

$$V_{\Sigma} := \int_{\overline{\mathcal{C}}_{\Sigma}} \mathcal{W}_{\Sigma}([Q]) d\mu([Q]) = \bigoplus_{[Q] \in \overline{\mathcal{C}}_{\Sigma}} \mu([Q]) \cdot \mathcal{W}_{[Q]} \in \mathscr{V}_n. \tag{9.59}$$

Remark 9.3.14. Note, if we think of $\mathcal{W}_{\Sigma}$ as a bundle of n-inner product spaces over $\overline{\mathcal{C}}_{\Sigma}$, then V_{Σ} is the space of L^2 sections of that bundle,

$$V_{\Sigma} := \left\{ \text{space of } L^2 \text{ invariant sections of } \mathcal{W}_{\Sigma} \longrightarrow \overline{\mathcal{C}}_{\Sigma} \right\}. \tag{9.60}$$

Next, let M be a compact oriented manifold (possibly with boundary) of dimension $(d + 2 - n)$. As we saw in the last section, the classical action on ∂M gives a bundle of

n-torsors $L_{\partial M} \to \mathcal{C}_{\partial M}$ and the classical action on M, $e^{2\pi i S_{M,[\alpha]}}$, is a section of the pullback bundle $r^*(L_{\partial M})$, where r is the restriction to the boundary, $r : M \to \partial M$:

$$\begin{tikzcd} r^*(L_{\partial M}) \arrow[r] \arrow[d] & L_{\partial M} \arrow[d] \\ \mathcal{C}_M \arrow[r, "r"] & \mathcal{C}_{\partial M} \end{tikzcd}$$

Note, by part (a) of conjecture 9.2.18, we see that the action is invariant under the morphisms in $\mathcal{C}_M$, and hence, the action is invariant under the symmetries of the fields - as is desired. Now, for each $P \in \mathcal{C}_M$ define (analogously to (9.49))

$$\tilde{L}_M(P) = \tilde{L}_{e^{2\pi i S_{M,[\alpha]}(P)}} \in \mathcal{W}_{L_{\partial M}(\partial P)} \equiv \mathcal{W}_{\partial P}. \tag{9.61}$$

Recall that $\mathcal{W}$ is not necessarily invariant under the automorphisms, thus we are not guaranteed that $\tilde{L}_M(P)$ will be invariant under $\mathrm{Aut}(P)$. In fact, it transforms under $\psi \in \mathrm{Aut}(P)$ according to the action of the restricted automorphism $\partial\psi \in \mathrm{Aut}(\partial P)$ on $\mathcal{W}_{\partial P}$. However, we will retain invariance under the automorphisms after we integrate.

To proceed, fix some $Q \in \mathcal{C}_{\partial M}$ and consider $\mathcal{C}_M(Q)$. If $(P, \theta) \in \mathcal{C}_M(Q)$ then, by using θ to identify $L_{\partial M}(\partial P) \cong L_{\partial M}(Q)$, we can define an action $e^{2\pi i S_{M,[\alpha]}(P,\theta)} \in L_{\partial M}(Q)$ and its associated $\tilde{L}_M(P, \theta) \in \mathcal{W}_Q$. Furthermore, if $(P, \theta) \cong (P', \theta')$ then there is an isomorphism the values of the action on these fields as elements of $L_{\partial M}(Q)$. And so, we define $\tilde{L}_M([P, \theta]) \in \mathcal{W}_Q$ via the invariant section construction previously used to defined $\mathcal{W}_{[Q]}$. Define

$$Z_M(Q) := \int_{\overline{\mathcal{C}}_M(Q)} \tilde{L}_M([P, \theta]) d\mu([P, \theta]) = \bigoplus_{[P,\theta] \in \overline{\mathcal{C}}_M(Q)} \mu([P, \theta]) \cdot \tilde{L}_M([P, \theta]) \in \mathcal{W}_Q.$$

We can show that if there exists a diffeomorphism between Q and Q', then $Z_M(Q)$ is isomorphic to $Z_M(Q')$. In particular, $Z_M(Q)$ is invariant under the $\mathrm{Aut}(Q)$ action on $\mathcal{W}_Q$ (see [22]). This, in turn, shows that $\{Z_M(Q) \mid Q \in [Q]\}$ is a collection of elements in $\{\mathcal{W}_Q \mid Q \in [Q]\}$ which are invariant under the symmetries. Hence, it is an element of $\mathcal{W}_{[Q]}$, $Z_M([Q]) \in \mathcal{W}_{[Q]}$.

Finally, we arrive at the following definition:

Definition 9.3.15. The **quantum invariant** $Z(M)$ assigned to M is defined to be

$$Z(M) := \bigoplus_{[Q] \in \overline{\mathcal{C}}_{\partial M}} Z_M([Q]) \in \bigoplus_{[Q] \in \overline{\mathcal{C}}_{\partial M}} \mu_{[Q]} \cdot \mathcal{W}_{[Q]} = V_{\partial M}. \tag{9.62}$$

The preceding constructions have the following properties, analogous to a non-extended TQFT.

Proposition 9.3.16 (Freed). *Let Γ be a finite Lie group and let $[\alpha] \in H^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ with representative cocycle $\alpha \in Z^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$. Then, the assignments*

$$\begin{aligned} Z &: \Sigma \longmapsto Z(\Sigma) \equiv V_\Sigma \in \mathscr{V}_n, \\ Z &: M \longmapsto Z(M) \in V_{\partial M}, \end{aligned} \tag{9.63}$$

defined above for closed oriented $(d+1-n)$-manifolds Σ and compact oriented $(d+2-n)$-manifolds M, satisfy the axioms of a topological quantum field theory:

(a) ***Functoriality:*** *Suppose $f : \Sigma \to \Sigma'$ is an orientation preserving diffeomorphism. Then there is an induced isometry*

$$Z(f) : V_\Sigma \longrightarrow V_{\Sigma'}. \tag{9.64}$$

Furthermore, if $f, g : \Sigma \to \Sigma'$ are any two such orientation preserving mappings, then there is an isometry

$$Z(fg) \cong Z(g)Z(f). \tag{9.65}$$

In addition, if $F : M \to M'$ is an orientation preserving diffeomorphism then there is an induced isometry

$$Z(\partial F)\big(Z(M)\big) \longrightarrow Z(M'), \tag{9.66}$$

where $Z(\partial F) : V_{\partial M} \to V_{\partial M'}$ is the isometry coming from the induced map $\partial F : \partial M \to \partial M'$ over the boundaries. Note, when $n = 1$ the previous two isometries in (9.65) *and* (9.66) *become equalities.*

(b) ***Orientation:*** *There is a natural isometry*

$$V_{\bar{\Sigma}} \cong V_\Sigma^*, \tag{9.67}$$

where $\bar{\Sigma}$ is the $(d+1-n)$-manifold Σ with reversed orientation and V_Σ^* is the dual space to V_Σ. Along with a natural isometry

$$Z(\bar{M}) \cong \overline{Z(M)}, \tag{9.68}$$

where by $\overline{Z(M)}$ we mean the conjugate of $Z(M)$.

(c) ***Multiplicativity and Gluing:***

(*Multiplicativity*) *If $\Sigma = \sqcup_i \Sigma_i$, then there is a natural isometry*

$$V_{\sqcup_i \Sigma_i} \cong \bigotimes_i V_{\Sigma_i}. \tag{9.69}$$

While if $M = \sqcup_i M_i$, then there is a natural isometry

$$Z(M) = Z(\sqcup_i M) \cong \bigotimes_i Z(M_i) \in \bigotimes_i V_{\partial M_i}. \tag{9.70}$$

(*Gluing*) *Suppose $\Sigma \hookrightarrow M$ is a closed oriented submanifold of codimension one and M^{cut} is the manifold obtained by cutting M along Σ (note, $\partial M = \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$). Then, there is a natural isometry*

$$Z(M) \longrightarrow \left\langle Z\left(M^{cut}\right)\right\rangle_{V_\Sigma}, \tag{9.71}$$

where $\langle \cdot \rangle_{V_\Sigma}$ is the contraction

$$\langle \cdot \rangle_{V_\Sigma} : V_{\partial M} \otimes V_\Sigma \otimes V_\Sigma^* \longrightarrow V_{\partial M},$$

using the inner product defined on V_Σ.

(d) ***Empty set:*** *If Σ is the empty set $\emptyset$, then $V_\emptyset \in \mathscr{V}_{n-1}$ is the trivial space. Similarly for M.*

Proof. The proof is extremely tedious but straightforward, except for proving the gluing axiom. However, this can be found in [22]. ☐

We present the following table as a recap of the story so far.

Codimension(s)	Classical Action	Path Integral
0 (M)	$P \longmapsto e^{2\pi i S_{M,[\alpha]}(P)} \in \mathscr{T}_0$ Complex number of unit norm	$Z(M) \in \mathscr{V}_0$ Complex number
1 (Σ)	$Q \longmapsto L_\Sigma(Q) \in \mathscr{T}_1$ Hermitian line ($\mathbb{T}$-torsor)	$V_\Sigma \in \mathscr{V}_1$ Hilbert space
2 (S)	$R \longmapsto L_S(R) \in \mathscr{T}_2$ $2\mathbb{T}$-torsor	$V_S \in \mathscr{V}_2$ 2-Hilbert space
3 (K)	$W \longmapsto L_K(W) \in \mathscr{T}_3$ $3\mathbb{T}$-torsor	$V_K \in \mathscr{V}_3$ 3-Hilbert space
$\vdots$	$\vdots$	$\vdots$

In the next section we carry out some calculations, specifically for the case of $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

9.4 Tying Everything Together: Example with $\Gamma = \mathbb{Z}_2$

We now carry out some explicit calculations to better understand the material presented so far. In particular, we want to explicitly find the quantum invariants associated to manifolds in each codimension from zero up to three. To begin, we take, for our discrete group Γ, the cyclic group of order two $\mathbb{Z}_2$ and consider the $d + 1 = 3$ theory. Let us now define the initial data we require.

Classifying Space

From the definition of $B\Gamma$, namely that it can be realized as the quotient space

$$B\Gamma = E\Gamma \Big/ \Gamma,$$

along with $E\Gamma = S^\infty$, we see that the classifying space for $\mathbb{Z}_2$ takes the form

$$B\mathbb{Z}_2 = S^\infty \Big/ \mathbb{Z}_2 = S^\infty \Big/ x \sim -x. \tag{9.72}$$

Recalling that $\mathbb{R}\mathbf{P}^\infty := S^\infty/x \sim -x$, we conclude

$$B\mathbb{Z}_2 = \mathbb{R}\mathbf{P}^\infty. \tag{9.73}$$

For future references, note that, in terms of Eilenberg-Mac Lane spaces, $\mathbb{R}\mathbf{P}^\infty = K(\mathbb{Z}_2, 1)$, while in terms of Lens spaces $\mathbb{R}\mathbf{P}^\infty = L(\infty, 2)$. These two notifications will come in handy when we try to calculate the cohomology classes of $\mathbb{R}\mathbf{P}^\infty$.

Cohomology of $\mathbb{R}\mathbf{P}^\infty$

Although there are several ways to calculate the degree four cohomology of $\mathbb{R}\mathbf{P}^\infty$ with integer coefficients, the following way is the quickest and easiest (although probably not very satisfying to certain readers). To begin, recall that $\mathbb{R}\mathbf{P}^\infty = K(\mathbb{Z}_2, 1)$. Next, for some positive integer q greater than one, we have the following (see exercise 18.8 page 245 of [13])

$$H^n\Big(K(\mathbb{Z}_q, 1); \mathbb{Z}\Big) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}_q & \text{if } n > 0 \text{ and even}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.74}$$

Consequently, since $\mathbb{R}\mathbf{P}^\infty = K(\mathbb{Z}_2, 1)$, we have that $H^4(\mathbb{R}\mathbf{P}^\infty; \mathbb{Z}) = H^4(K(\mathbb{Z}_2, 1); \mathbb{Z}) \cong \mathbb{Z}_2$. Restating,

$$H^4(\mathbb{R}\mathbf{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}_2. \tag{9.75}$$

Remark 9.4.1. We could have also used the fact that $\mathbb{R}\mathbf{P}^\infty = L(\infty, 2)$ along with the knowledge that $H^4(L(\infty, 2); \mathbb{Z} \cong \mathbb{Z}_2$ to calculate $H^4(\mathbb{R}\mathbf{P}^\infty); \mathbb{Z})$, or that

$$H^\bullet(\mathbb{R}\mathbf{P}^\infty; \mathbb{Z}) \cong Z[\alpha]/(2\alpha),$$

where $\dim(\alpha) = 2$ and (2α) represents the ideal generated by 2α.

So, since our action $e^{2\pi i S_{M,[\alpha]}}$ depends on a particular choice of cocycle $[\alpha] \in H^4(\mathbb{R}\mathbf{P}^\infty; \mathbb{Z})$ and since $H^4(\mathbb{R}\mathbf{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}_2$, we see that there exists two cases: (1) the untwisted theories $[\alpha] = 0$, and (2) the twisted theories $[\alpha] \neq 0$.

The Untwisted Case $[\alpha] = 0$

Recall, the fields in our theory are principal $\mathbb{Z}_2$-bundles (or regular covering spaces) over some manifold of arbitrary codimension, which we denote by M for the moment,

$$\mathcal{C}_M = \Big\{\mathbb{Z}_2 \hookrightarrow P \xrightarrow{\pi} M\Big\}. \tag{9.76}$$

Furthermore, using the notion of bundle morphisms (i.e., symmetries between the fields), we can define the equivalence class of a field and hence the moduli space $\overline{\mathcal{C}}_M$. Here members $[P] \in \overline{\mathcal{C}}_M$ of a class are related to each of the other members of the class by bundle maps

- that is, the class $[P]$ is the set of all $\mathbb{Z}_2$-bundles over M which have a bundle map in common. For example, if P and P' are in $[P]$, then there exists a bundle map $\varphi : P \to P'$ which induces the identity on M. Additionally, since M is compact (which we have been assuming) $\overline{\mathcal{C}}_M$ is finite. While, if M is connected (which we also assume), we have the identification

$$\overline{\mathcal{C}_M} \cong \mathrm{Hom}\big(\pi(M), \mathbb{Z}_2\big)\Big/\mathbb{Z}_2. \tag{9.77}$$

Next, when we take the gauge group G to be finite, the path integral reduces to a finite sum,

$$\int_{\mathcal{A}/\mathcal{G}} e^{2\pi i S} d\mathcal{A} \rightsquigarrow \sum_{[P]\in\overline{\mathcal{C}}_M} e^{2\pi i S}\mu([P]). \tag{9.78}$$

Since the path integral is finite, it is subsequently (well-)defined, and we can use this to define "generalized" path integrals which act on $\mathbb{Z}_2$-bundles over manifolds of varying codimensions. This was how we defined the quantum invariants over d-dimensional manifolds $Z(M)$, $(d-1)$-dimensional manifolds V_Σ, and so on. Recall, with $[\alpha] = 0$, the classical action becomes trivial. Indeed, let $P \to M$ be an arbitrary $\mathbb{Z}_2$-principal bundle over some closed manifold M of top dimension ($\dim(M) = 3$). Then, consider the following:

$$\begin{aligned} e^{2\pi i S_{M,[\alpha=0]}(P)} &= e^{2\pi i \langle \gamma_M^*(0), m\rangle}, \\ &= e^{2\pi i \langle 0, m\rangle}, \\ &= 1. \end{aligned}$$

Hence, $e^{2\pi i S_{M,[0]}([P])} = 1$ for all $[P] \in \overline{\mathcal{C}}_M$. Furthermore, since $[\alpha] = 0$, the extended actions are also trivial. In particular, for any 2-dimensional closed manifold Σ, we have that $L_\Sigma([Q]) = \mathbb{C}$ for all $[Q] \in \overline{\mathcal{C}}_\Sigma$. While, for any 1-dimensional closed manifold S, the extended action assigns, to each equivalence class of bundles over S, the category $\mathscr{T}_1$. Likewise, to any $[W] \in \overline{\mathcal{C}}_{pt.}$, the extended action assigns $\mathscr{T}_2$.

We now explicitly quantize the above results. In the 3-dimensional untwisted Dijkgraaf-Witten theory, the partition function of a 3-dimensional closed manifold M is given by

$$Z(M) := \sum_{[P]\in\overline{\mathcal{C}}_M} \mu([P]).$$

Taking the usual definition of the measure (see (9.52)) gives

$$Z(M) = \sum_{[P]\in\overline{\mathcal{C}}_M} \frac{1}{|\mathrm{Aut}([P])|}, \tag{9.79}$$

where by $|A|$ we mean the order of A.

Remark 9.4.2. Note, it is trivial to show that under a diffeomorphism, $\hat{f} : M \to M'$, the partition function is invariant, $Z(M) = Z(M')$.

In general (hence possibly twisted case), the quantum invariant of a closed 2-manifold Σ is given by the set of all L^2 invariant sections of the Hermitian line bundle $\overline{\mathcal{L}}_\Sigma \to \overline{\mathcal{C}}_\Sigma$,

$$V_\Sigma = \big\{L^2 \text{ invariant sections of } \overline{\mathcal{L}}_\Sigma \longrightarrow \overline{\mathcal{C}}_\Sigma\big\}. \tag{9.80}$$

Recall, this is really shorthand notation for

$$V_\Sigma = \{L^2 \text{ invariant sections of } \mathcal{W}_\Sigma \longrightarrow \overline{\mathcal{C}}_\Sigma\},$$

since we must first embed $\mathcal{L}_\Sigma$ into $\mathcal{W}_\Sigma$ in order to perform the path integral. Thus, if M has a boundary, the partition function $Z(M)$ is a given by an invariant section of the bundle $r^*(\overline{\mathcal{L}}_{\partial M}) \to \overline{\mathcal{C}}_M$. Similarly, we can generalize this concept to manifolds of dimension one and zero. In particular, the quantum invariant of a closed 1-manifold S is given by the set of L^2 invariant sections of the bundle $\mathcal{W}_S \to \overline{\mathcal{C}}_S$. Note, restricting to the untwisted case gives $L_\Sigma([Q]) = \mathbb{C}$ for all $[Q]$. Hence, V_Σ is then given by the set of all L^2 invariant sections of the bundle $\mathbb{C} \to \overline{\mathcal{C}}_\Sigma$.

Quantum Invariants of 3-Manifolds

Let us now further simplify our present situation. The following arguments will hold for any manifold of arbitrary dimension. However, we explicitly work out the details for the 3-manifold M. To begin, let $x \in M$ and consider the space of all bundles over M with a basepoint:

$$\mathcal{C}'_M = \Big\{(P, p_x) \mid P \text{ is a principal bundle and } p_x \in P_x \text{ covers } x \in M \Big\}. \tag{9.81}$$

Then, the objects in $\mathcal{C}'_M$ are *rigid*, and hence, have no nontrivial automorphisms. Additionally, one can use holonomy (as we did before) to give a $1:1$ correspondence between the equivalence classes in $\overline{\mathcal{C}}'_M$ and elements in $\mathrm{Hom}\big(\pi_1(M,x),\Gamma\big) \to \Gamma$. We can act on $\mathcal{C}'_M$ with Γ be setting, for all $g \in \Gamma$, $g \cdot (P, p_x) := (P, p_x \cdot g)$. Furthermore, the action on the quotient space corresponds exactly to the action of Γ on $\mathrm{Hom}\big(\pi_1(M,x),\Gamma\big)$. Hence, we are back to the usual set up, except now our fields have no nontrivial automorphisms; i.e., $|\mathrm{Aut}(P)| = 1$ for all P! Tying it all together, the partition function now takes the much simpler form

$$Z(M) = \frac{|\mathrm{Hom}\big(\pi_1(M),\Gamma\big)|}{|\Gamma|}, \tag{9.82}$$

or, with $\Gamma = \mathbb{Z}_2$,

$$Z(M) = \frac{|\mathrm{Hom}\big(\pi_1(M),\mathbb{Z}_2\big)|}{2}.$$

Using (9.82) it is almost trivial to calculate the 3-dimensional Dijkgraaf-Witten partition function of a closed 3-manifold. In particular, consider the case where $M = S^3$. Here the fundamental group is isomorphic to the trivial group of one element, $\pi_1(S^3) \cong \{1\}$. Thus, $\mathrm{Hom}\big(\pi_1(S^3),\Gamma\big) \cong \mathrm{Hom}\big(\{1\},\Gamma\big)$. Now, noting that homomorphisms must map identity elements to identity elements, we conclude that the order of $\mathrm{Hom}\big(\{1\},\Gamma\big)$ is equal to 1. Therefore,

$$\begin{aligned} Z(S^3) &= \frac{|\mathrm{Hom}\big(\{1\},\Gamma\big)|}{|\Gamma|}, \\ &= \frac{1}{|\Gamma|}. \end{aligned}$$

So, when $\Gamma = \mathbb{Z}_2$, we have

$$Z(S^3) = \frac{1}{2} \in \mathbb{C}.$$

With a quick glance we see that $Z(S^3) \in V_{\partial S^3}$. Indeed, since $\partial S^3 = \emptyset$, we have that $V_{\partial S^3} = \mathbb{C}$.

Remark 9.4.3. We could arrive at this result alternatively as follows. To begin, recall that $\mathrm{Hom}\big(\pi_1(M), \Gamma\big) \cong [M, B\Gamma]$, where by $[M, B\Gamma]$ we mean the collection of homotopy distinct maps $M \to B\Gamma$. Now, since all bundles over S^3 are trivial, there exists only one distinct classifying map $S^3 \to B\Gamma$. Hence, $[M, B\Gamma]$ consists of only one element. Thus, $Z(S^3) = 1/|\Gamma|$.

Proposition 9.4.4. *Let M be a 3-dimensional closed, connected, simply connected manifold. Then, its partition function is given by*

$$Z(M) = \frac{1}{|\Gamma|}. \tag{9.83}$$

Proof. Suppose M is a 3-dimensional closed, connected, simply connected manifold. Then, by the properties of the fundamental group, we have that $\pi_1(M) \cong \{1\}$. Hence,

$$\begin{aligned} Z(M) &= \frac{|\mathrm{Hom}\big(\pi_1(M), \Gamma\big)|}{|\Gamma|}, \\ &= \frac{|\mathrm{Hom}\big(\{1\}, \Gamma\big)|}{|\Gamma|}, \\ &= \frac{1}{|\Gamma|}. \end{aligned}$$

□

As another (non-trivial) example, let us consider the case of $M = \mathbb{R}\mathbf{P}^3$ and $\Gamma = \mathbb{Z}_2$. The fundamental group of $\mathbb{R}\mathbf{P}^3$ is given by the cyclic group of order two,

$$\pi_1\big(\mathbb{R}\mathbf{P}^3\big) \cong \mathbb{Z}_2.$$

Therefore, we need to determine the order of the group of automorphisms on $\mathbb{Z}_2$ (recall, in the formula for the quantum invariant Z, we must determine $|\mathrm{Hom}\big(\pi_1(M), \Gamma\big)|$ and here both $\pi_1(M)$ and Γ are given by $\mathbb{Z}_2$ - hence the term, automorphism). Now, there are exactly two distinct automorphisms of $\mathbb{Z}_2$: $\psi_1 : 0 \to 0,\ 1 \to 0$ and $\psi_2 : 0 \to 0,\ 1 \mapsto 1$. So,

$$\begin{aligned} Z\big(\mathbb{R}\mathbf{P}^3\big) &= \frac{|\mathrm{Hom}\big(\pi_1\big(\mathbb{R}\mathbf{P}^3\big), \mathbb{Z}_2\big)|}{|\mathbb{Z}_2|}, \\ &= \frac{|\mathrm{Hom}\big(\mathbb{Z}_2, \mathbb{Z}_2\big)|}{|\mathbb{Z}_2|}, \\ &= \frac{2}{2} = 1. \end{aligned}$$

Once again, we could determine the correct expression $Z\big(\mathbb{R}\mathbf{P}^3\big)$ in a slightly different way. Indeed, note that, since the fundamental group of $\mathbb{R}\mathbf{P}^3$ has order two, there exists only two different bundles over $\mathbb{R}\mathbf{P}^3$ when $\Gamma = \mathbb{Z}_2$. This implies that there are only two distinct elements of $[\mathbb{R}\mathbf{P}^3, B\mathbb{Z}_2]$. And so, using the fact that $\mathrm{Hom}\big(\pi_1\big(\mathbb{R}\mathbf{P}^3\big), \mathbb{Z}_2\big) \cong [\mathbb{R}\mathbf{P}^3, B\mathbb{Z}_2]$, we conclude that $Z\big(\mathbb{R}\mathbf{P}^3\big) = 2/2 = 1$.

Finally, let $M = S^2 \times S^1$ and let $\Gamma = \mathbb{Z}_2$. For $\pi_1\big(S^2 \times S^1\big)$ we get

$$\begin{aligned}\pi_1\big(S^2 \times S^1\big) &\cong \pi_1\big(S^2\big) * \pi_1\big(S^1\big),\\ &= \{1\} * \mathbb{Z},\\ &\cong \mathbb{Z},\end{aligned}$$

where by $*$ we mean the free product of groups. Furthermore, there are only two distinct homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_2$: (1) the homomorphism which sends all elements in $\mathbb{Z}$ to $0 \in \mathbb{Z}_2$, and (2) the homomorphism which maps the even elements in $\mathbb{Z}$ to $0 \in \mathbb{Z}_2$ and the odd elements $\mathbb{Z}$ to $1 \in \mathbb{Z}_2$. Consequently, we have that

$$\begin{aligned}Z(S^2 \times S^1) &= \frac{|\mathrm{Hom}\big(\pi_1\big(S^2 \times S^1\big), \mathbb{Z}_2\big)|}{|\mathbb{Z}_2|},\\ &= \frac{|\mathrm{Hom}\big(\mathbb{Z}, \mathbb{Z}_2\big)|}{|\mathbb{Z}_2|},\\ &= \frac{2}{2} = 1.\end{aligned}$$

Quantum Hilbert Spaces of 2-Manifolds

We now compute the quantum Hilbert space associated to a 2-dimensional surface. To begin, Let Σ be a closed oriented 2-dimensional manifold. We have seen on several occasions that, neglecting morphisms (bundle maps), the quantum Hilbert space V_Σ is given by the space of all L^2 sections of the Hermitian line bundle $\mathcal{L}_\Sigma \to \mathcal{C}_\Sigma$,

$$V_\Sigma = \left\{L^2 \text{ sections of } \mathcal{L}_\Sigma \longrightarrow \mathcal{C}_\Sigma\right\}.$$

Incorporating the morphisms (i.e., gauge symmetry), our quantum Hilbert space reduces to the space of all L^2 invariant sections of $\overline{\mathcal{L}}_\Sigma \to \overline{\mathcal{C}}_\Sigma$; that is L^2 sections which are invariant under gauge transformations. Therefore, if M is a 3-manifold with nonempty boundary, then, by the properties of an extended TQFT, $Z(M) \in V_{\partial M}$ and hence an invariant section of $r^*(\overline{\mathcal{L}}_{\partial M}) \to \overline{\mathcal{C}}_M$.

Now, since we are considering the untwisted case, $[\alpha] = 0$, the generalized classical action for Σ is trivial; that is,

$$L_\Sigma([Q]) = \mathbb{C}, \qquad \forall \ \ [Q] \in \overline{\mathcal{C}}_\Sigma.$$

Therefore, in the untwisted case, V_Σ reduces to the space of all L^2 invariant $\mathbb{C}$-valued functions on $\overline{\mathcal{C}}_\Sigma$. Hence, using the fact that $\overline{\mathcal{C}}_\Sigma \cong \mathrm{Hom}\big(\pi_1(\Sigma), \Gamma\big)/\Gamma$ (here we must assume that Σ is both compact and connected), we see that V_Σ is given by the space of $\mathbb{C}$-valued functions on $\mathrm{Hom}\big(\pi_1(\Sigma), \Gamma\big)/\Gamma$. Furthermore, employing the same procedure with basepoints as before (see the arguments leading up to (9.82)), we rewrite V_Σ as

$$V_\Sigma \cong \frac{1}{|\Gamma|} \cdot L^2\Big(\mathrm{Hom}\big(\pi_1(\Sigma), \Gamma\big), \Gamma\Big)^\Gamma. \tag{9.84}$$

Here, $1/|\Gamma|$ multiplies the L^2 metric, which, due to the arguments preceding (9.82), weights each section by 1 (hence, this is why it is not appearing in the expression for V_Σ). Also, by $(\cdots)^\Gamma$ we mean that each object in $(\cdots)$ is invariant under the conjugation action of Γ; i.e., objects in $L^2\big(\mathrm{Hom}\big(\pi_1(\Sigma), \Gamma\big), \Gamma\big)^\Gamma$ are L^2 maps (in this case homomorphisms) from $\mathrm{Hom}\big(\pi_1(\Sigma), \Gamma\big)$ to Γ, which are L^2 with respect to $1/|\Gamma|$ times the usual metric on the space, and invariant under the conjugate action of Γ.

As an example, let us calculate the quantum Hilbert space for S^2. In this case, we have that $\pi_1(M) = \pi_1\big(S^2\big) \cong \{1\}$, the trivial group consisting of one element. Thus, $\mathrm{Hom}\big(\pi_1(S^2), \Gamma\big) \cong \mathrm{Hom}\big(\{1\}, \Gamma\big)$. Now, it has been mentioned that there exists only one distinct homomorphism from $\{1\}$ to Γ, for any (finite) group Γ. Hence, denoting this homomorphism by $\star$ we have

$$V_{S^2} \cong \frac{1}{|\Gamma|} \cdot L^2\big(\star, \Gamma\big)^\Gamma.$$

Furthermore, there only exists one morphism from $\star$ to Γ, the one which sends $\star$ to the identity element in Γ. Note, this is incidentally invariant under conjugation by Γ (since it maps to the identity element). So, $V_{S^2} = 1/|\Gamma| \cdot L^2\big(\text{one pt. space}\big)$. Using the fact that L^2 of a one point space is a 1-dimensional Hilbert space and the fact that a 1-dimensional Hilbert space is canonically isomorphic to $\mathbb{C}$, we arrive at the conclusion

$$V_{S^2} \cong \frac{1}{|\Gamma|} \cdot \mathbb{C},$$

where, now, $1/|\Gamma|$ multiplies the usual metric on $\mathbb{C}$. Finally, restricting to $\Gamma = \mathbb{Z}_2$, we have

$$V_{S^2} \cong \frac{1}{2} \cdot \mathbb{C}. \tag{9.85}$$

Quantum Invariants of 1-Manifolds

We now turn to the quantum space associated to a manifold of codimension two - the circle S^1. To begin, recall that the quantum space associated to a bundle R over a closed 1-dimensional manifold S is defined as (neglecting morphisms)

$$V_S := \big\{\text{space of } L^2 \text{ sections of } \mathcal{W}_S \longrightarrow \mathcal{C}_S\big\}, \tag{9.86}$$

or, in the more appealing notation,

$$V_S = \int_{\mathcal{C}_S} \mathcal{W}_S(R)\ d\mu(R).$$

Next, since we are working in the untwisted case, $[\alpha] = 0$, the classical generalized action is trivial,

$$\mathcal{W}_S(R) = \mathscr{V}_1, \tag{9.87}$$

for all $R \in \mathcal{C}_S$. Here, $\mathscr{V}_1$ is the 1-category of Hilbert spaces (see section 9.3.1). Thus, an object of the quantum space V_S is simply a choice of a (finite dimensional) Hilbert space W_R for all $R \in \mathcal{C}_S$. Hence, stated differently, objects of V_S are Hermitian vector bundles over $\mathcal{C}_S$ and morphisms are bundle maps. That is,

$$V_S = \mathrm{Vect}(\mathcal{C}_S), \tag{9.88}$$

where $\mathrm{Vect}(\mathcal{C}_S)$ is the category of Hermitian vector bundles over $\mathcal{C}_S$. Indeed, an element of the quantum space V_S associates to each 'point' on $\mathcal{C}_S$ a vector space. So, we get a vector bundle by viewing the space of fields $\mathcal{C}_S$ as the base manifold and the vector spaces as the fibres (we also assume that we can glue the fibres together in a smooth way, which gives the total space). Furthermore, we can show (see [22]) that V_S comes equipped with 'addition', 'scalar product', and an 'inner product'. Hence, it has the structure of a 2-Hilbert space (which is no surprise from the previous discussions).

Now, let us restrict the case $S = S^1$. To begin, we need to explicitly determine the space of fields $\mathcal{C}_{S^1}$. We will show that the moduli space of fields on the circle $\overline{\mathcal{C}}_{S^1}$ is isomorphic to the set of conjugacy classes in Γ,

$$\overline{\mathcal{C}}_{S^1} \cong \mathrm{Cl}(\Gamma).$$

To proceed, let $x \in S^1$ be a basepoint in S^1, and let $\mathcal{C}'_{S^1}$ denote the category of Γ-bundles R over S^1, together with a point chosen $p \in R_x$ which covers x. That is, an element of $\mathcal{C}'_{S^1}$ is a pair (R, p) such that $R \to S^1$ is a bundle and $p \in R_x$, where R_x is the fibre over $x \in S^1$.

Next, consider sitting at the point $p \in R_x$. Then, if we transverse around the circle (in the direction chosen as positive orientation), we arrive back at some other point $p' \in R_x$ (see figure 8.1). However, by definition of a Γ-bundle (namely the fact that Γ acts on the fibres in a simply transitive fashion), there exists some $g \in \Gamma$ such that $p' = p \cdot g$. This element $g \in \Gamma$ is called the holonomy (see section 8.3). Consequently, we have a map

$$\mathrm{hol} : \mathcal{C}'_{S^1} \longrightarrow \Gamma,$$

which induces an isomorphism

$$\overline{\mathcal{C}'}_{S^1} \longrightarrow \Gamma.$$

Furthermore, under a change of basepoint $p \mapsto p'$, the holonomy g changes by conjugation, $h \cdot g \cdot h^{-1}$, for some $h \in \Gamma$. Thus, we conclude that the conjugacy classes of the holonomy g, $[g] = \{h \cdot g \cdot h^{-1} \mid h \in \Gamma\}$, are blind to changes in the basepoint. Therefore, we have an isomorphism between the moduli space of unpointed bundles $\overline{\mathcal{C}}_{S^1}$ and the conjugacy classes in Γ,

$$\overline{\mathcal{C}}_{S^1} \cong \mathrm{Cl}(\Gamma). \tag{9.89}$$

And so, we see that the moduli space of fields on S^1 can be represented by the conjugacy classes in Γ.

We can view $\mathrm{Cl}(\Gamma)$ as a category $\mathcal{C}_\Gamma$ as follows. The category $\mathcal{C}_\Gamma$ has as objects given by the elements of Γ, $g \in \Gamma$, and a morphism between any two objects $g, h \in \Gamma$ is given by $g \mapsto h \cdot g \cdot h^{-1}$. Thus, replacing $\overline{\mathcal{C}}_{S^1}$ with $\mathcal{C}_\Gamma$ in (9.86) gives the bundle $\mathscr{V}_1 \to \mathcal{C}_\Gamma$. So, neglecting symmetries of the fields (i.e., morphisms in $\mathcal{C}_\Gamma$), we can view V_{S^1} as the space of sections of the functor $\mathscr{V}_1 \to \mathcal{C}_\Gamma$. Equivalently, neglecting symmetries, $V_{S^1} \cong \mathrm{Vect}(\mathcal{C})$. That is, V_{S^1} is viewed as the category of Hermitian vector bundles W over the conjugacy classes in Γ,

$$V_{S^1} = \big\{W \longrightarrow [g] \mid [g] \in \mathrm{Cl}(\Gamma)\big\}.$$

Hence, an element of V_{S^1} is a collection of Hermitian vector spaces W_g indexed by the elements of Γ. We now take into account the symmetries of the fields.

Recall, when we quantized in dimension three we took care of the symmetries by integrating, while in dimension two we took care of the symmetries by only considering invariant sections. Here we take care of the symmetries by restricting V_{S^1} to only include those sections of $\mathscr{V}_1 \to \mathcal{C}_\Gamma$ which are invariant. In this case, since the morphisms in $\mathcal{C}_\Gamma$ are given by conjugation, the invariant sections of $\mathscr{V}_1 \to \mathcal{C}_\Gamma$ are those which are invariant under the lift of this conjugate action. That is, for each morphism $g \mapsto h \cdot g \cdot h^{-1}$ (in the base space) we have an isomorphism

$$A_h : W_g \longrightarrow W_{h\cdot g\cdot h^{-1}}, \tag{9.90}$$

between the fibres. So, including symmetries, we see that an object in V_{S^1} is simply a Hermitian vector bundle $W \to \Gamma$ together with a lift of the conjugate action by Γ on itself. We denote the collection of these *equivariant vector bundles* over Γ as $\mathrm{Vect}_\Gamma(\Gamma)$. Hence,

$$V_{S^1} \cong \frac{1}{|\Gamma|} \cdot \mathrm{Vect}_\Gamma(\Gamma), \tag{9.91}$$

where the factor $1/|\Gamma|$ is a normalization factor [22].

Remark 9.4.5. Note, the rank of the fibre R_g, although constant on each class $[g]$, can vary over Γ. Also, we write an element in $\mathrm{Vect}_\Gamma(\Gamma)$ as

$$R = \bigoplus_{g\in\Gamma} R_g. \tag{9.92}$$

Furthermore, we can give $\mathrm{Vect}_\Gamma(\Gamma)$ the structure of a 2-Hilbert space. For instance, we define the inner product on $\mathrm{Vect}_\Gamma(\Gamma)$ to be

$$(W, W')_{\mathrm{Vect}_\Gamma(\Gamma)} := \frac{1}{|\Gamma|} \cdot \left(\bigoplus_{g\in\Gamma} W_g \otimes W'_g\right)^\Gamma, \tag{9.93}$$

for all $W = \bigoplus_{g\in\Gamma} W_g$ and $W' = \bigoplus_{g\in\Gamma} W'_g$ in $\mathrm{Vect}_\Gamma(\Gamma)$.

Let's now restrict to the case of $\Gamma = Z_2$. To begin, recall that a Γ-equivariant vector bundle over some manifold X is a vector bundle $W \to X$ together with a lift of the Γ-action on X to the fibres. That is, let $\rho : \Gamma \times X \to X$, $x \mapsto \rho_g(x)$ denote the action of Γ on X. Then, by saying that the action ρ lifts to the fibres (hence, vector spaces), we mean that $x \mapsto \rho_g(x)$ induces an isomorphism between fibres $W_x \to W_{\rho_g(x)}$. Now, consider the case of a trivial action, $\rho_g(x) = x$. When this is the case, the Γ-action on X $x \mapsto \rho_g(x) = x$ lifts to an automorphism of the fibre $W_x \to W_x$. Hence, each $g \in \Gamma$ induces a element

End(W_x); i.e., when the Γ-action is trivial on X, a Γ-equivariant vector bundle over X is a vector bundle where each fibre has a representation of Γ.

Now, since $\mathbb{Z}_2$ is abelian, the conjugate action of Γ on itself is trivial. Consequently, any $\mathbb{Z}_2$-equivariant vector bundle over $\mathbb{Z}_2$ (with conjugate action) is a bundle of representations of $\mathbb{Z}_2$; that is, every element of $\mathrm{Vect}_\Gamma(\Gamma)$ is a vector bundle on the two-point space $\mathbb{Z}_2$ (since $|\mathbb{Z}_2| = 2$) and each fibre is a representation of $\mathbb{Z}_2$. Additionally, since all representations of $\mathbb{Z}_2$ are completely reducible, and since there are exactly two irreducible representations of $\mathbb{Z}_2$ (namely, the trivial representation and the sign representation), we see that every $\mathbb{Z}_2$-equivariant vector bundle over $\mathbb{Z}_2$ is constructed out of combinations of these two representations. Indeed, since $\mathrm{Vect}_\Gamma(\Gamma)$ is a monoidal category with tensor products of representations, we see that each object in $\mathrm{Vect}_\Gamma(\Gamma)$, namely a vector bundle of representations of Γ, is obtained from taking tensor products of these two irreducible representations as the representations on the fibres, over the two-point space.

Quantum Invariants of Points

As we have seen, our untwisted extended Dijkgraaf-Witten theory assigns a complex number to each closed 3-manifold, a 1-dimensional Hilbert space to each closed 2-manifold, and the category of Γ-equivariant vector bundles over Γ to each closed 1-manifold. Hence, we have constructed a 1-2-3 theory. Now, we can further extend the Dijkgraaf-Witten theory to a point; that is construct a 0-1-2-3 theory which truncates to the previously defined 1-2-3 theory. We claim that assigning the category of vector bundles over Γ to the point gives a 0-1-2-3 theory. By now the reader may be confused. After all, we have stated on several occasions that the Dijkgraaf-Witten theory assigns a 2-category to a point. However, note that a ring R determines a 1-category, the category of R-modules. Similarly, a monoidal category $\mathcal{R}$ determines a 2-category of its modules. Hence, we can view the quantum space associated to a point, $V_{pt.}$, as the category of $\mathcal{R}$-modules for some monoidal category $\mathcal{R}$. And it just so happens that the category of vector bundles over Γ, $\mathrm{Vect}(\Gamma)$, gives such a category[9]. Furthermore, we can show that the *Drinfeld center* of $\mathrm{Vect}(\Gamma)$, $Z(\mathrm{Vect}(\Gamma))$, corresponds exactly to $\mathrm{Vect}_\Gamma(\Gamma)$, the category of Γ-equivariant vector bundles over Γ (hence, V_{S^1}). Therefore, to get a 0-1-2-3 theory, we assign the quantum space to a point to be the 2-category of $\mathrm{Vect}(\Gamma)$-modules, $V_{pt.} = \mathcal{R}_{\mathrm{Vect}(\Gamma)}$. Hence, $V_{pt.}$ is the 2-category of all the modules over $\mathrm{Vect}(\Gamma)$ and its center (namely V_{S^1}) is the category of all vector bundles over Γ, where each fibre has a representation of Γ (at least when the Γ-action we take on Γ is trivial).

To begin, consider the following definition:

Definition 9.4.6. Let $\mathcal{R}$ be a monoidal category. Its **Drinfeld center** $Z(\mathcal{R})$ is the category whose objects are pairs (X, ϵ) consisting of an object X in $\mathcal{R}$ and a natural transformation $\epsilon(\cdot) : X \otimes \cdot \to \cdot \otimes X$. Furthermore, the natural transformation is compatible with the monoidal structure, in that for all objects $X, Y \in \mathcal{R}$ we require

$$\epsilon(X \otimes Y) = \big(id_X \otimes \epsilon(Y)\big) \circ \big(\epsilon(X) \otimes id_Y\big). \tag{9.94}$$

[9]For the reader who is curious of exactly how one introduces the monoidal structure on $\mathrm{Vect}(\Gamma)$, we refer them to [22].

Now, viewing Vect(Γ) as a monoidal category, again see [22] for the particulars, we see that elements in the Drinfeld center $Z(\text{Vect}(\Gamma))$ correspond precisely to vector bundles over Γ together with a lift of the conjugate action by Γ; i.e., $Z(\text{Vect}(\Gamma))$ corresponds to the Γ-equivariant vector bundles over Γ, $Z(\text{Vect}(\Gamma)) = \text{Vect}_\Gamma(\Gamma)$. Hence, we have a functor

$$\mathcal{F} : Z(V_{pt.}) \longrightarrow V_{S^1}, \tag{9.95}$$

as desired.

So, we get a 0-1-2-3 theory which reduces to the 1-2-3 untwisted extended Dijkgraaf-Witten theory by taking $V_{pt.} = \text{Vect}(\Gamma)$. Therefore, we have constructed an extended topological quantum field theory down to points, which was our goal when we started!

The Twisted Case $[\alpha] \neq 0$

We would like to breifly mention the case of taking a nontrivial cohomology class $[\alpha] \in H^4(B\Gamma; \mathbb{Z})$. When we assume $[\alpha]$ to be nontrivial, we no longer only get trivial classical actions. One can see that this is clearly the case by inspecting the classical action for a codimension zero manifold, M,

$$e^{2\pi i \langle \gamma_M^*(\alpha), m\rangle};$$

i.e., there is nothing to require that $\langle \gamma_M^*(\alpha), m\rangle$ vanish. We must take this into account when we want to construct the quantum invariant associated to a manifold. As an example of this, let us construct the twisted partition function for $M = \mathbb{R}\mathbf{P}^3$, when $\Gamma = \mathbb{Z}_2$. In this case, since $|\pi_1(\mathbb{R}\mathbf{P}^3)| = 2$, we see that there are exactly two Γ-bundles over $\mathbb{R}\mathbf{P}^3$. Hence, there are two elements, or maps, in the set $[\mathbb{R}\mathbf{P}^3, \mathbb{R}\mathbf{P}^\infty]$ (recall, $B\mathbb{Z}_2 = \mathbb{R}\mathbf{P}^\infty$); the trivial classifying map and the, aptly named, nontrivial classifying map. The nontrivial map corresponds exactly to the embedding of $\mathbb{R}\mathbf{P}^3$ into $\mathbb{R}\mathbf{P}^\infty$ [10], which generates the third homology group and is dual to α [18]. If we denote the nontivial map by $\gamma_{1,\mathbb{R}\mathbf{P}^3}$, then we have that $\langle \gamma_{1,\mathbb{R}\mathbf{P}^3}^*(\alpha), [\mathbb{R}\mathbf{P}^3]\rangle = 1/2$ (here we are writing $[\mathbb{R}\mathbf{P}^3]$ for both the fundamental class and its representative). Also, if $\gamma_{0,\mathbb{R}\mathbf{P}^3}$ is the trivial map, then $\gamma_{0,\mathbb{R}\mathbf{P}^3}^*(\alpha) = 0$. And so, we conclude that

$$\begin{aligned} Z(\mathbb{R}\mathbf{P}^3) &= \sum_{\gamma_{i,\mathbb{R}\mathbf{P}^3} \in [\mathbb{R}\mathbf{P}^3, \mathbb{R}\mathbf{P}^\infty]} \frac{e^{2\pi i \langle \gamma_{i,\mathbb{R}\mathbf{P}^3}^*(\alpha), [\mathbb{R}\mathbf{P}^3]\rangle}}{|\mathbb{Z}_2|}, \\ &= \frac{e^{2\pi i \langle \gamma_{0,\mathbb{R}\mathbf{P}^3}^*(\alpha), [\mathbb{R}\mathbf{P}^3]\rangle} + e^{2\pi i \langle \gamma_{1,\mathbb{R}\mathbf{P}^3}^*(\alpha), [\mathbb{R}\mathbf{P}^3]\rangle}}{2}, \\ &= \frac{e^{2\pi i \times 0} + e^{2\pi i \times \frac{1}{2}}}{2} = \frac{1 + (-1)}{2} = 0, \end{aligned}$$

[10] If we think of $\mathbb{R}\mathbf{P}^\infty$ as containing all the $\mathbb{R}\mathbf{P}^n$, then the nontrivial classifying map sends $\mathbb{R}\mathbf{P}^3$ right onto the $\mathbb{R}\mathbf{P}^3$ inside $\mathbb{R}\mathbf{P}^\infty$.

Therefore, we see that when α represents a nontrivial class, it is possible for the partition function to vanish on a particular manifold.

Having a nontrivial classical action will result in a twisting of the quantum space associated to the circle, V_{S^1}. Indeed, through the classical Dijkgraaf-Witten action, the level $[\alpha] \in H^4(B\Gamma;\mathbb{Z})$ produces a central extension of the action groupoid[11] $\Gamma//\Gamma$ of Γ acting on itself by conjugation. That is, for all $g_1, g_2 \in \Gamma$ the classical action defines a Hermitian line L_{g_1,g_2}, while to every triple of elements g_1, g_2, g_3 it assigns an isomorphism

$$L_{g_2 \cdot g_1 \cdot g_2^{-1}, g_3} \otimes L_{g_1,g_2} \longrightarrow L_{g_1, g_3 \cdot g_2},$$

together with some consistency conditions. In this case, the quantum space V_{S^1} is now given by the category of L-twisted Γ-equivariant vector bundles over Γ, L-$\mathrm{Vect}_\Gamma(\Gamma)$. It is known that L-$\mathrm{Vect}_\Gamma(\Gamma)$ gives a concrete model for *twisted K-theory*, where the twisting is defined by L (see the lectures of Dan Freed on twisted K-theory and the Verlinde algebra). Furthermore, L-$\mathrm{Vect}_\Gamma(\Gamma)$ has the structure of a monoidal category (which is no surprise, since $\mathrm{Vect}_\Gamma(\Gamma)$ does as well). Additionally, it is the center of a monoidal tensor category (which will be the quantum invariant assigned to the point). Let us now sketch this proof.

To begin, start by viewing $H^4(B\Gamma;\mathbb{Z})$ as $H^2(B\Gamma;\mathbb{C}\mathbf{P}^\infty)$, this way we can interpret the level $[\alpha] \in H^4(B\Gamma;\mathbb{Z})$ as representing a central extension of Γ by the abelian group-like category of Hermitian lines. Then, a cocycle α which represents $[\alpha]$ is given a Hermitian line K_{g_1,g_2} for every pair $g_1, g_2 \in \Gamma$, a cocycle isomorphism

$$K_{g_1,g_2} \otimes K_{g_1 \cdot g_2, g_3} \longrightarrow K_{g_1, g_2 \cdot g_3} \otimes K_{g_2,g_3},$$

for every triple $g_1, g_2, g_3 \in \Gamma$, together with some consistency conditions. Now, let L-$\mathrm{Vect}(\Gamma)$ be the category of L-twisted complex vector bundles on Γ. We turn this into a monoidal tensor category by defining, for all objects $W = \oplus_{g\in\Gamma} W_g \in \mathrm{Obj}(L\text{-}\mathrm{Vect}(\Gamma))$ and $W' = \oplus_{g\in\Gamma} W'_g \in \mathrm{Obj}(L\text{-}\mathrm{Vect}(\Gamma))$,

$$(W \otimes W')_{g_3} = \bigoplus_{g_1 \cdot g_2 = g_3} K_{g_1,g_2} \otimes W_{g_1} \otimes W'_{g_2}.$$

Furthermore, we can define duals of objects, W^* - namely,

$$(W^*)_{g_1} := K^*_{g_1, g_1^{-1}} \otimes (W^*_{g_1^{-1}})$$

- along with the other properties of inner product spaces. Thus giving L-$\mathrm{Vect}(\Gamma)$ the structure of a (twisted) inner product space. Finally, it is straightforward to check that an object in the center of L-$\mathrm{Vect}(\Gamma)$, $Z\big(L\text{-}\mathrm{Vect}(\Gamma)\big)$, belongs to L-$\mathrm{Vect}_\Gamma(\Gamma)$. That is, it is a vector bundle $W \to \Gamma$ together with isomoprhisms

$$L_{g_1,g_2} \otimes W_{g_1} \longrightarrow W_{g_2 \cdot g_1 \cdot g_2^{-1}},$$

$$L_{g_1,g_2} := K^*_{g_2 \cdot g_1 \cdot g_2^{-1}, g_2} \otimes K_{g_2,g_1}.$$

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[11]Given an action ρ of a group G on a set S, the action groupoid $S//G$ is a bit like the quotient set S/G (the set of G-orbits). But, instead of taking elements of S in the same G-orbit as being equal in S/G, in the action groupoid they are just isomorphic [7].

Hence, we have a functor, which is really an equivalence of braided monoidal categories,

$$\mathcal{F} : Z\big(L\text{-Vect}(\Gamma)\big) \longrightarrow L\text{-Vect}_\Gamma(\Gamma).$$

Consequently, using the fact that (for twisted theories) $V_{S^1} \cong L\text{-Vect}_\Gamma(\Gamma)$, we see that if we take $V_{pt.} \cong L\text{-Vect}(\Gamma)$ we then get a 0-1-2-3 theory!

So, recapping, we have just seen how to construct the Dijkgraaf-Witten theory down to points, in both the untwisted (trivial group cohomology) case and the twisted (nontrivial group cohomology case). Furthermore, we saw that allowing for a twisting gives a concrete example of twisted K-theory, $L\text{-Vect}_\Gamma(\Gamma)$. This concludes our treatment of the extended Dijkgraaf-Witten TQFT.

Chapter 10

Conclusion

The study of topology field theory has been at the forefront of modern research in mathematical physics for the last twenty years. They have been used in (seemingly) diverse areas, such as quantum gravity and knot theory. The best known, concrete example of a topological field theory is that of Chern-Simons theory, or the so-called Dijkgraaf-Witten theories when working with a finite structure group.

These theories obey the axioms of a TQFT put forth by Atiyah, and hence define a topological quantum field theory. Namely, one can consider them to be (symmetric, monoidal) functors from the category of n-dimensional cobordisms to the category of vector spaces over some fixed field $\Bbbk$. Hence, to each closed manifold of codimension one, the theory assigns a vector space (called the quantum space), while to each compact manifold of codimension zero, the theory assigns an element in the vector space associated to the boundary of the aforementioned manifold. When the top dimensional manifold is closed, the theory assigns an element in the trivial vector space $\mathbb{C}$, known as the quantum invariant. Furthermore, there is also a gluing property that allows for one to calculate such quantum invariants by 'dividing' the closed manifold M into two compact manifolds of codimension zero M_1 and M_2, which share boundaries $\partial M_1 = \partial M_2 = \Sigma$. This requires that an inner product be defined on the vector space associated to the boundary $\langle \cdot, \cdot \rangle_{V_\Sigma}$. Then, taking the inner product of the two quantum vectors assigned to M_1 and M_2 will give the quantum invariant,

$$Z(M) = \big\langle Z(M_1), Z(M_2) \big\rangle_{V_\Sigma}.$$

Next, we have asked the question as to wether one could perform the same technique on the closed codimension one manifolds; i.e., can we construct the quantum space associated to a closed codimension one manifold Σ by first chopping it up into two compact codimension one manifolds Σ_1 and Σ_2, which share a boundary $\partial\Sigma_1 = \partial\Sigma_2 = S$, and then take the 'inner product' of the two resulting 'vectors', V_{Σ_1} and V_{Σ_2}, to give V_Σ? As we saw, the answer to this question is yes. However, we must introduce the concept of a 2-Hilbert space - a category with an inner product tthat takes its values in the category of Hilbert spaces, in order to construct such an inner product. Finally, we were able to show that this process of splitting manifolds along codimension n submanifolds can be iterated indefinitely (at least down to points), with the use of higher categories and higher Hilbert spaces. As a specific example, we calculated the quantum invariants and spaces for several manifolds ranging in dimensions, from three down to zero, when our structure group was equal to $\mathbb{Z}_2$.

So far, everything we have done (quantum mechanically speaking) has been based on a finite structure group Γ. We have not mentioned what happens when one replaces the finite structure group Γ with a continuous group G. The reason for sticking with Γ is that when using a finite group their is no real worry about infinities, and hence renormalization techniques. Additionally, when the structure group is finite and the base manifolds are compact and connected, the path integral - which is the prototypical way to quantize a classical theory - reduces to a sum. And hence, one does not need to fight with the heavy analysis required to show that such an integral exists. Furthermore, when dealing with finite theories, it can be shown (as we did) that any class $[\alpha] \in H^{n+1}(B\Gamma;\mathbb{Z})$ determines an n-dimensional TQFT. However, if we take G to only be compact then, at best, one can only say that a class $[\alpha] \in H^{n+1}(BG;\mathbb{Z})$ determines a n-dimensional TQFT for $n = 1, 2, 3$. Nothing can further, at this moment, be said about $n > 3$, although there is a lot of work currently being done is this area [17].

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