CHERN-WEIL THEORY
AND
SOME RESULTS ON CLASSIC GENERA

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Abstract. In this survey article, we first review the Chern-Weil theory of characteristic classes of vector bundles over smooth manifolds. Then some well-known characteristic classes appearing in many places in geometry and topology as well as some interesting results on them related to elliptic genera which involve some of my joint work in [HZ1, 2], are briefly introduced.

1. Chern-Weil theory for characteristic classes

The purpose of this section is to give a brief introduction to geometric aspects of the theory of characteristic classes, which was developed by Shiing-shen Chern and André Weil. This section is organized as follows: in a), we briefly review the de Rham cohomology theory; in b), we introduce the theory of connection and curvature on vector bundles over smooth manifolds; in c), the central theorem, Chern-Weil theorem, is introduced in a modern form; Chern classes and Pontrjagin classes are constructed in d); at last in e), we list the main properties of Chern classes and introduce the Chern root algorithm. We will basically follow [Z3] in a)-d).

a). Review of the de Rham Cohomology Theory

Let M be a smooth closed manifold. Let TM and T^*M denote the tangent bundle and cotangent bundle of M respectively. Denote $\Omega^*(M) := \Gamma(\Lambda^*(T^*M))$ as the space of smooth sections of $\Lambda^*(T^*M)$, where $\Lambda^*(T^*M)$ is the complex exterior algebra bundle of T^*M.

Let $d : \Omega^*(M) \to \Omega^*(M)$ be the God-given exterior differential operator on $\Omega^*(M)$, which satisfies $d^2 = 0$. Then one finds that for any integer p such that $0 \le p \le \dim M$,

$$d\Omega^p(M) \subset \ker d\,|_{\Omega^{p+1}(M)},$$

which leads the definition of de Rham complex as well as its associated cohomology: de Rham cohomology.

Definition 1.1. The **de Rham complex** $(\Omega^*(M), d)$ is the complex defined by

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\mathrm{dimM}}(M) \to 0.$$

Definition 1.2. For any integer p such that $0 \le p \le \dim M$, the p-th **de Rham cohomology** of M with complex coefficients is defined by

$$H^p_{dR}(M, \mathbb{C}) := \frac{\ker d\,|_{\Omega^p(M)}}{d\Omega^{p-1}(M)}.$$

The total **de Rham cohomology** of M is defined as

$$H^*_{dR}(M, \mathbb{C}) := \bigoplus_{p=0}^{\dim M} H^p_{dR}(M, \mathbb{C}).$$

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It's easy to check that de Rham cohomology carries a natural ring structure with product induced from wedge product in $\Omega^*(M)$.

The importance of the de Rham cohomology lies in the de Rham theorem which state as follows.

Theorem 1.1. *If M is a smooth closed orientable manifold, then for any integer p such that $0 \le p \le \dim M$,*
(1) $\dim H^p_{dR}(M, \mathbb{C}) < +\infty$;
(2) $H^p_{dR}(M, \mathbb{C})$ is canonically isomorphic to H^p_{Sing}, the p-th singular cohomology of M.

We refer the readers to the book [BoT] for a proof.

b). Connections on Vector Bundles and Their Curvature

Let $E \to M$ be a smooth complex vector bundle over a smooth compact manifold M. We denote by $\Omega^*(M,E)$ the space of smooth sections of tensor product vector bundle $\Lambda^*(T*M) \otimes E$ obtained from $\Lambda^*(T*M)$ and E,

$$\Omega^*(M,E) := \Gamma(\Lambda^*(T*M) \otimes E).$$

A connection on E may be thought of, in some sense, as an extension of exterior differential operator d to include the coefficient E.

Definition 1.3. A **connection**∇^E on E is a $\mathbb{C}$-linear operator $\nabla^E : \Gamma(E) \to \Omega^1(M,E)$ such that for any $f \in C^\infty(M), X \in \Gamma(E)$, the following Leibniz rule holds,

$$\nabla^E(fX) = (df)X + f\nabla^E X.$$

If $X \in \Gamma(TM)$ is a smooth sectionmof TM, then via contraction, ∇^E induces canonically a map $\nabla^E_X : \Gamma(E) \to \Gamma(E)$, which is called the **covariant derivative** of ∇^E along X.

The existence of a connection on a vector bundle can be proved easily by using the method of partitions of unity. All connections on it form a infinite dimensional affine space.

One can canonically extend ∇^E to a map

$$\nabla^E : \Omega^*(M,E) \to \Omega^{*+1}(M,E)$$

such that for any $\omega \in \Omega^*(M), X \in \Gamma(E)$,

$$\nabla^E : \omega X \mapsto (d\omega)X + (-1)^{\deg\omega}\omega \wedge \nabla^E X.$$

The importance of the concept of a connection lies in its curvature.

Definition 1.4. The **curvature** R^E of a connection ∇^E is defined by

$$R^E = \nabla^E \circ \nabla^E : \Gamma(E) \to \Omega^2(M,E),$$

which, for brevity, we will write $R^E = (\nabla^E)^2$.

It's not hard to check that R^E is a $C^\infty(M)$-linear map, i.e., for any $f \in C^\infty(M)$ and $X \in \Gamma(E)$, one has $R^E(fX) = fR^E X$.

Let $End(E)$ denote the vector bundle over M formed by the fibrewise endomorphisms of E. So R^E may be thought of as an element of $\Gamma(End(E))$ with coefficients in $\Omega^2(M)$. In other words, $R^E \in \Omega^2(M, End(M))$.

To give a precise formula, if $X, Y \in \Gamma(TM)$ are two smooth sections of TM, then $R^E(X,Y)$ is the element in $\Gamma(End(E))$ given by

$$R^E(X,Y) = \nabla^E_X\nabla^E_Y - \nabla^E_Y\nabla^E_X - \nabla^E_{[X,Y]}.$$

Finally, using the composition of endomorphisms, one sees that for any integer $k \geq 0$,

$$(R^E)^k = \overbrace{R^E \circ \cdots \circ R^E}^{k} : \Gamma(E) \to \Omega^{2k}(M, E)$$

is a well-defined element lying in $\Omega^{2k}(M, End(E))$.

c). Chern-Weil Theorem

For any smooth section A of the bundle $End(E)$, the fiberwise trace of A forms a smooth function on M. We denote this function by $\mathrm{tr}[A]$. This further induces the map $\mathrm{tr} : \Omega^*(M, End(E)) \to \Omega^*(M)$ such that for any $\omega \in \Omega^*(M)$ and $A \in \Gamma(End(E))$, $\mathrm{tr}[\omega A] = \omega \mathrm{tr}[A]$.

We also extend the Lie brackets operation on $End(E)$ to $\Omega^*(M, End(E))$ as follows: if $\omega, \eta \in \Omega^*(M)$ and $A, B \in \Gamma(End(E))$, then we use the convention that

$$[\omega A, \eta B] = (\omega A)(\eta B) - (-1)^{(\deg\omega)(\deg\eta)}(\eta B)(\omega A). \tag{1.1}$$

Then we have the following vanishing result:

Lemma 1.1. For any $A, B \in \Omega^*(M, End(E))$, the trace of $[A, B]$ vanishes.

Lemma 1.2. If ∇^E is a connection on E, then for any $A \in \Omega^*(M, End(E))$, one has

$$d\mathrm{tr}[A] = \mathrm{tr}\left[[\nabla^E, A]\right]. \tag{1.2}$$

Proof. First of all, if $\widetilde{\nabla^E}$ is another connection on E, then from the Leibniz rule in the definition of the connection, one verifies that

$$\nabla^E - \widetilde{\nabla^E} \in \Omega^1(M, End(E)).$$

Thus by Lemma 1.1 one has

$$\mathrm{tr}\left[[\nabla^E - \widetilde{\nabla^E}, A]\right] = 0.$$

That is to say, the right hand side of (2.2) does not depend on the choice of ∇^E.

On the other hand, it is clear that the operations in the right hand side of (1.2) are local. Thus for any $x \in M$, one can choose a sufficiently small open neighborhood U_x of x such that $E|_{U_x}$ is a trivial vector bundle. Then one can take a trivial connection on $E|_{U_x}$ for which (2.2) holds automatically.

By combining the above independence and local properties, one sees directly that (2.2) holds on the whole manifold M. □

Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

be a power series in one variable.

Let R^E be the curvature of a connection ∇^E on E.

The trace of

$$f(R^E) = a_0 + a_1 R^E + \cdots + a_n (R^E)^n + \cdots$$

is an element in $\Omega^*(M)$.

We can now state a form of **Chern-Weil theorem** (cf. [Che]) as follows.

Theorem 1.2. *(i) The form* $\mathrm{tr}\left[f(R^E)\right]$ *is closed. That is,*

$$d\mathrm{tr}\left[f(R^E)\right] = 0;$$

(ii) If $\widetilde{\nabla}^E$ is another connection on E and $\widetilde{R}^E$ its curvature, the there is a differential form $\omega \in \Omega^*(M)$ such that

$$\operatorname{tr}\left[f(R^E)\right] - \operatorname{tr}\left[f(\widetilde{R}^E)\right] = d\omega. \tag{1.3}$$

Proof. (i) From Lemma 1.2 one verifies directly that

$$d\operatorname{tr}\left[f(R^E)\right] = \operatorname{tr}\left[[\nabla^E, f(R^E)]\right]$$
$$= \operatorname{tr}\left[a_1[\nabla^E, R^E] + \cdots + a_n[\nabla^E, (R^E)^n] + \cdots\right] = 0,$$

as for any integer $k \geq 0$ one has the obvious **Bianchi identity**

$$\left[\nabla^E, (R^E)^k\right] = \left[\nabla^E, (\nabla^E)^{2k}\right] = 0. \tag{1.4}$$

(ii) For any $t \in [0,1]$, let ∇_t^E be the deformed connection on E given by

$$\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}^E.$$

Then ∇_t^E is a connection on E such that $\nabla_0^E = \nabla^E$ and $\nabla_1^E = \widetilde{\nabla}^E$. Moreover,

$$\frac{d\nabla_t^E}{dt} = \widetilde{\nabla}^E - \nabla^E \in \Omega^1(M, End(E)).$$

Let $R_t^E, t \in [0,1]$, denote the curvature of ∇_t^E. We study the change of $\operatorname{tr}\left[f(R_t^E)\right]$ when t changes in $[0,1]$.

Let $f'(x)$ be the power series obtained from the derivative of $f(x)$ with respect to x.

We deduce that

$$\frac{d}{dt}\operatorname{tr}\left[f(R_t^E)\right] = \operatorname{tr}\left[\frac{dR_t^E}{dt}f'(R_t^E)\right] = \operatorname{tr}\left[\frac{d(\nabla_t^E)^2}{dt}f'(R_t^E)\right]$$
$$\operatorname{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt}\right]f'(R_t^E)\right] = \operatorname{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt}f'(R_t^E)\right]\right],$$

where the last equality follows from the Bianchi identity (1.4).

Combining with Lemma 1.2, one then gets

$$\frac{d}{dt}\operatorname{tr}\left[f(R_t^E)\right] = d\operatorname{tr}\left[\frac{d\nabla_t^E}{dt}f'(R_t^E)\right], \tag{1.5}$$

from which one gets

$$\operatorname{tr}\left[f(R^E)\right] - \operatorname{tr}\left[f(\widetilde{R}^E)\right] = -d\int_0^1 \operatorname{tr}\left[\frac{d\nabla_t^E}{dt}f'(R_t^E)\right]dt. \tag{1.6}$$

This completes the proof of part (ii). □

d). Chern Classes and Pontrjagin Classes

By Theorem 1.2 (i), $\operatorname{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]$ is a closed differential form which determines a cohomology class $\left[\operatorname{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]\right] \in H_{dR}^*(M, \mathbb{C})$. While (1.3) says that this class does not depend on the connection ∇^E.

Definition 1.5. (i) We call the differential form $\operatorname{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]$ the **Characteristic form** of E associated to ∇^E and f, and denote it by $f(E, \nabla^E)$.

(ii) we call the cohomology class $\left[\operatorname{tr}[f(\frac{\sqrt{-1}}{2\pi}R^E)]\right] \in H_{dR}^*(M, \mathbb{C}$. the **characteristic class** of E associated to f, and denote it by $f(E)$.

Thus, a characteristic form is a differential form representative of corresponding characteristic class.

We now assume M is oriented, so that one can integrate differential forms on M.

Let $E_1, \cdots, E_k$ be k complex vector bundles over M, and $\nabla^{E_1}, \cdots, \nabla^{E_k}$ connections on them respectively.

Give k power series $f_1, \cdots, f_k$, one can then form the characteristic form

$$f_1(E_1, \nabla^{E_1}) \cdots, f_k(E_k, \nabla^{E_k}) \in \Omega^*(M).$$

Lemma 1.3. The number defined by

$$\int_M f_1(E_1, \nabla^{E_1}) \cdots, f_k(E_k, \nabla^{E_k}) \tag{1.7}$$

does not depend on the choices of the connections $\nabla^{E_i}, 1 \leq i \leq k$.

The number defined in (1.7) is called the **characteristic number** associated to the characteristic class $f_1(E_1) \cdots, f_k(E_k)$, and is denoted by $\langle f_1(E_1) \cdots, f_k(E_k), [M] \rangle$.

In the following, we will define Chern forms (classes, numbers) and Pontrjagin forms (classes, numbers).

Let ∇^E be a connection on a complex vector bundle E over a smooth manifold M, and R^E be the curvature of ∇^E.

The total **Chern form**, denoted by $c(E, \nabla^E)$, associated to ∇^E is defined by

$$c(E, \nabla^E) = \det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right), \tag{1.8}$$

Where I is the identity endomorphism of E.

Since

$$\det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right) = \exp\left(\operatorname{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right)\right]\right), \tag{1.9}$$

one sees that $c(E, \nabla^E)$ is a characteristic form in the sense of Definition 1.3. The associated characteristic class, denoted by $c(E)$, is called the total Chern class of E.

It is clear that one has the decomposition of the total Chern form

$$c(E, \nabla^E) = 1 + c_1(E, \nabla^E) + \cdots + c_k(E, \nabla^E) + \cdots$$

with each

$$c_i(E, \nabla^E) \in \Omega^{2i}(M).$$

we call $c_i(E, \nabla^E)$ the i-th **Chern form** associated to ∇^E, and its associated cohomology class, denoted by $c_i(E)$, the i-th **Chern class** of E.

Note that

$$\log\left(\det\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right)\right) = \operatorname{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi} R^E\right)\right],$$

and by the taylor expansion of the function $\log(1+x)$, one deduce that for any integer $k \geq 0$, $\operatorname{tr}[(R^E)^k]$ can be written as as linear combination of various products of $c_k(E, \nabla^E)$'s.

This establishes the fundamental importance of Chern classes in the theory of characteristic classes of complex vector bundles.

Let now E be a real vector bundle over M, and ∇^E be a connection on E, and R^E be the curvature of of ∇^E.

One sees easily that one can proceed in exactly the same way as before for real vector bundles with connections. Moreover, the Chern-Weil theorem can be formulated and proved in exactly the same way as in Theorem 1.2.

Now similar to Chern forms for complex vector bundles, we define the total **Pontrjagin form** associated to ∇^E by

$$p(E,\nabla^E) = \det\left(\left(I - \left(\frac{R^E}{2\pi}\right)^2\right)^{1/2}\right). \tag{1.10}$$

Thw associated characteristic class $p(E)$ is called the (total) **Pontrjagin class**.

Clearly, $p(E,\nabla^E)$ admits a decomposition

$$p(E,\nabla^E) = 1 + p_1(E,\nabla^E) + \cdots + p_k(E,\nabla^E) + \cdots$$

with each

$$p_i(E,\nabla^E) \in \Omega^{4k}(M).$$

we call $p_i(E,\nabla^E)$ the i-th **Pontrjagin form** associated to ∇^E, and call the associated class $p_i(E)$ the i-th **Pontrjagin class** of E.

If we denote by $E \otimes \mathbb{C}$ the complexification of E, then one has the following intimate relation between the Pontrjagin classes of E and the Chern classes of $E \otimes \mathbb{C}$, which is usually taken as the definition of Pontrjagin classes: for any $i \geq 0$,

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}). \tag{1.11}$$

e). Main Properties of Chern Classes and Chern Root Algorithm

In this subsection, we first collect some basic properties of the Chern classes without proof. The readers can infer these results easily from Chern-Weil theory. We also introduce the **Chern root algorithm** as well as some references for other aspects of the beautiful theory of characteristic classes. The main references for this subsection are [Bo], [BHi], [Hi], [MS] and [Yu].

1). (Naturality) *If f is a map from Y to X and E is a complex vector bundle over X, then $c(f^{-1}E) = f^*c(E)$, where X and Y are two smooth manifolds.*

2). (Whitney Product Formula) *If E_1 and E_2 are two vector bundles over X, then $c(E_1 \oplus E_2) = c(E_1)c(E_2)$.*

3). *If E has rankn as a complex vector bundle, then $c_i(E) = 0$ for $i > n$.*

4). *If E has a nonvanishing section, then the top Chern class $c_n(E)$ is zero.*

5).The top Chern class of a complex vector bundle E is Euler class of its realization: $c_n(E) = e(E_{\mathbb{R}})$, where $n = \mathrm{rank} E$.

In the following, we will introduce Chern root algorithm.

Let E be a complex vector bundle of rank k over M with an Hermitian metric $\langle , \rangle$ and a compatible connection ∇^E. Then locally the curvature $R^E = (\nabla^E)^2$ is an anti-Hermitian matrix with two-forms as entries. We now **assume** there exist $u_i(\nabla^E)$ such that R^E can be written as

$$R^E = \begin{pmatrix} -2\pi\sqrt{-1}u_1(\nabla^E) & & \\ & \ddots & \\ & & -2\pi\sqrt{-1}u_k(\nabla^E) \end{pmatrix}.$$

Then from (1.8), one has

$$c(E,\nabla^E)=\prod_{i=1}^{n}(1+u_i(\nabla^E)). \tag{1.12}$$

Here $u_i(\nabla^E)$ are called the **new Chern roots**. Consequently, we see

$$\begin{gathered}c_1(E,\nabla^E)=u_1(\nabla^E)+u_2(\nabla^E)+\cdots+u_k(\nabla^E),\\ c_2(E,\nabla^E)=u_1(\nabla^E)u_2(\nabla^E)+u_1(\nabla^E)u_3(\nabla^E)+\cdots+u_{k-1}(\nabla^E)u_k(\nabla^E),\\ \cdots,\\ c_k=u_1(\nabla^E)\cdots u_k(\nabla^E)\end{gathered}$$

are elementary symmetric polynomials of $u_1,\cdots,u_k$.

Remark 1.1. Of course, these u_k's may not exist in $\Omega^2(M)$. We call (1.12) a **formal factorisation** of $c(E,\nabla^E)$. But one actually has the rigorous theory for Chern root algorithm. Please refer to [BHi] and [Yu] for details.

If E is a real vector bundle of rank $2k$ over M with an Euclidean metric $\langle,\rangle$ and a compatible connection also denoted as ∇^E. Then locally the curvature $R^E=(\nabla^E)^2$ is a anti-symmetric matrix with two-forms as entries. Similarly as above, we **assume** that there exist $u_i(\nabla^E)$ such that

$$R^E=\begin{pmatrix}0 & 2\pi u_1(\nabla^E) & & & \\ -2\pi u_1(\nabla^E) & 0 & & & \\ & & \ddots & & \\ & & & 0 & 2\pi u_k(\nabla^E)\\ & & & -2\pi u_k(\nabla^E) & 0\end{pmatrix}.$$

Then by (1.10), one has

$$p(E,\nabla^E)=\prod_{i=1}^{k}(1+u_i(\nabla^E)^2). \tag{1.13}$$

From this formal factorisation, one can easily find again that

$$p_i(E)=(-1)^i c_{2i}(E\otimes\mathbb{C}).$$

The following theorem shows how to calculate Chern classes (forms)of the bundles $E^*, E_1\oplus E_2, E_1\otimes E_2, \Lambda^p E$ from those of E_1 and E_2.

Theorem 1.3. *(cf. [Hi]) Let E_1 be a rank m bundle and E_2 be a rank n bundle over M. Consider formal factorisations,*

$$c(E_1)=\prod_{i=1}^{m}(1+\gamma_i),\qquad c(E_2)=\prod_{i=1}^{n}(1+\delta_i).$$

Then,

(i) $c(E_1^*)=\prod_{i=1}^m(1-\gamma_i)$, i.e. $c_i(E_1)=(-1)^i c_i(E)$.

(ii) $c(E_1\oplus E_2)=\prod_{j=1}^m(1+\gamma_j)\prod_{k=1}^n(1+\delta_k)$.

(iii) $c(E_1\otimes E_2)=\prod_{j,k}(1+\gamma_j+\delta_k), 1\le j\le m, 1\le k\le n$.

(iv) $c(\Lambda^p E_1)=\prod(1+(\gamma_{j_1}+\cdots+\gamma_{j_p}))$, where the product is over all C_m^p combinations with $1\le j_1\le<\cdots<j_p\le m$.

2. SOME INTERESTING RESULTS ON CLASSICAL GENERA

Let M be an even-dimensional Riemanniann manifold. Let ∇^{TM} be the associated Levi-Civita connection and $R^{TM}=\nabla^{TM,2}$ the curvature of ∇^{TM} .

Let $\widehat{A}(TM,\nabla^{TM})$, $\widehat{L}(TM,\nabla^{TM})$ be the Hirzebruch characteristic forms defined by

(2.1)
$$\widehat{A}(TM,\nabla^{TM})=\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right),\quad \widehat{L}(TM,\nabla^{TM})=\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right).$$

As a very special case, if $\dim M=4$, one has

$$\{\widehat{L}(TM,\nabla^{TM})\}^{\max}=\frac{1}{3}p_1(TM,\nabla^{TM}). \tag{2.2}$$

The importance of the $L-$class lies in the **Hirzebruch Signature Theorem** (cf. [Hi]) which says that when M is oriented, then the $L-$genus of M, denoted by $L(M)$ and defined by

$$L(M):=\langle L(TM),[M]\rangle=\int_M L(TM,\nabla^{TM}),$$

equals to the **Signature** of M [Hi]. Topologically, for a closed and oriented manifold M, it's signature $L(M)$ is defined as follows: if $4\nmid \dim M, L(M)=0$; if $4\mid \dim M$, define a symmetric quadratic form $Q(\alpha,\beta):=\int_M\alpha\wedge\beta$ on $\Omega^{\frac{\dim M}{2}}(M)$, $L(M)$ is defined as the index of this quadratic form.Obviously, $L(M)$ is an homotopically invariant integer.

When $\dim M$ is 4, we also have

$$\{\widehat{A}(TM,\nabla^{TM})\}^{\max}=-\frac{1}{24}p_1(TM,\nabla^{TM}). \tag{2.3}$$

We define the $\widehat{A}$-genus of M, denoted by $\widehat{A}(M)$, by

$$\widehat{A}(M):=\langle \widehat{A}(TM),[M]\rangle=\int_M \widehat{A}(TM,\nabla^{TM}).$$

From (2.2) and (2.3), one has

$$\{\widehat{L}(TM,\nabla^{TM})\}^{\max}=-8\{\widehat{A}(TM,\nabla^{TM})\}^{\max}. \tag{2.4}$$

Atiyah and Hirzebruch proved in 1959 [AtH] that if N is an $(8k+4)$-dimensional smooth closed oriented spin manifold and E is a real vector bundle over N, then

$$\langle \widehat{A}(TN)\mathrm{ch}(E\otimes\mathbb{C}),[N]\rangle\in 2\mathbb{Z}. \tag{2.5}$$

From (2.4) and the above theorem of Atiyah and Hirzebruch, one has $16\mid \mathrm{Sign}(M)$ for 4-dimensional spin manifold M.

Actually Rokhlin [R1] in 1952 found this very amazing fact that the Signature of a smooth closed oriented spin 4-manifold is divisible by 16. So we can in some sense view the theorem of Atiyah and Hirzebruch as a higher dimensional generalization of the original result of Rokhlin.

For 12-dimensional Riemannian manifold, the physicists Alvarez-Gaumé and Witten [AGW] found a so called "miraculous cancellation" formula which reveals the beautiful relation between the Hirzebruch $\widehat{L}$-form and the Hirzebruch $\widehat{A}$-form as follows,

$$\left\{\widehat{L}(TM,\nabla^{TM})\right\}^{(12)}=\left\{8\widehat{A}(TM,\nabla^{TM})\mathrm{ch}(T_{\mathbb{C}}M,\nabla^{T_{\mathbb{C}}M})-32\widehat{A}(TM,\nabla^{TM})\right\}^{(12)}, \tag{2.6}$$

where $T_{\mathbb{C}}M$ denotes the complexification of TM, and $\nabla^{T_{\mathbb{C}}M}$ is canonically induced from ∇^{TM}. Here if M is closed and spin, one can easily deduce from (2.6) and the Atiyah-Hirzebruch divisibility theorem for spin manifolds that 16 | Sign(M).

In the 4-dimensional case, (2.4) plays an analogous role to which (2.6), the "miraculous cancellation" formula, plays in the 12-dimensional case. So we may in some sense call (2.4) the cancellation formula for dimension 4. It's natural to ask if there exists an analogous cancellation formula for each $(8k+4)$-dimensional case. If the answer is positive, one then may expect it will be used as a bridge between these two divisibilities.

In 1994, Kefeng Liu [Li] established higher dimensional "miraculous cancellation" formulas for $(8k+4)$-dimensional smooth manifolds by developing modular invariance properties of characteristic forms. In [Li], he proved that for each $(8k+4)$-dimensional smooth Riemannian manifold M the following identity holds,

$$\widehat{L}(TM, \nabla^{TM}) = 2^3 \sum_{j=0}^{k} 2^{6k-6j} \widehat{A}(TM, \nabla^{TM}) \mathrm{ch} b_j, \tag{2.7}$$

where the b_j 's are elements in $KO(M) \otimes \mathbb{C}$. Ochanine divisibility [O1] that $16 \mid \mathrm{Sign(M)}$ for $(8k+4)$-dimensional spin closed manifold M can then be easily deduced from the Atiyah-Hirzebruch divisibility by (2.7).

What's the story for non-spin case? Now let M be a closed oriented smooth 4-manifold not necessarily spin. Let B be an orientable characteristic submanifold of M, that is, B is dual to the second Stiefel-Whitney class of TM. In this case, Rokhlin established in [R2] a congruence formula of the type

$$\frac{\mathrm{Sign}(M) - \mathrm{Sign}(B \cdot B)}{8} \equiv \phi(B) \quad \mathrm{mod} 2\mathbb{Z}, \tag{2.8}$$

where $B \cdot B$ is the self-intersection of B in M and $\phi(B)$ is a spin cobordism invariant associated to (M, B).

Clearly, when M is spin and $B = \varnothing$, (2.8) reduces to the original Rokhlin divisibility.

In the case where B might not be orientable, an extension of (2.8) was proved by Guillou and Marin in [GuM].

Ochanine [O1] generalized (2.8) to $(8k+4)$-dimensional closed spin^c manifolds. His formula is also of type (2.8) and thus extends his divisibility result to spin^c manifolds.

Finashin [F] generalized the Ochanine congruence to the case where the characteristic submanifold B is nonorientable. His formula also extends the Guillou-Marin congruence to any $(8k+4)$-dimensional closed oriented manifold.

Now there exist Ochanine and Finashin congruences as the generalizations of Ochanine divisibility from spin case to non-spin case. Naturally one may ask if there exist generalizations of the Atiyah-Hirzebruch divisibility.

Zhang gave us positive answers in [Z1, 2]. His results generalize the Atiyah-Hirzebruch divisibility and, when restricted to 4-dimensional case, also recovers (2.8) as well as the Guillou-Marin extension of it.

In [Z2, Appendix], with the help of the "miraculous cancellation" formula of Alvarez-Gaumé and Witten [AGW] for 12-dimensional manifold, Zhang could give an intrinsic analytic interpretation of Ochnaine's spin cobordism invariant $\phi(B)$. Moreover, it was pointed that a higher dimensional generalization of the "miraculous cancellation" formula of Alvarez-Gaumé and Witten would lead to a better understanding of the general congruence formulas due to Ochanine [O1] and Finashin [F]. Of course, this required higher dimensional "miraculous cancellation" formula is exactly Liu's result mentioned above (cf. (2.7) in the text).

In [LiZ], by combining Liu's cancellation formula with the analytic arguments in [Z2], Liu and Zhang were able to give an intrinsic analytic interpretation of the Ochanine invariant $\phi(B)$ for any $(8k+2)$-dimensional closed spin manifold B (Compare with [O2]), as well as the Finashin invariant [F] for any $8k+2$ dimensional closed pin$^-$ manifold appearing in the

Finashin congruence formula. In particular, this leads to analytic versions, stated in [LiZ, Theorems 4.1, 4.2], of the Finashin and Ochanine congruences.

Now the remained question is whether these analytic versions of the Finashin and Ochanine congruences could be deduced directly from the Rokhlin type congruences in [Z1, 2]. In other words, whether we can find cancellation formulas for non-spin case which play analogous roles as that the “miraculous cancellation” formulas of Alvarez-Gaumé, Witten and Liu play for spin case.

The purpose of the following is to give a positive answer to this question. To be more precise, what we get is a generalization of the “miraculous cancellation” formulas of Alvarez-Gaumé, Witten and Liu to a twisted version where an extra complex line bundle is involved. Modular invariance properties developed in [Li] still play an important role in the proof of such an extended cancellation formula. When applying our formula to spin^c manifolds, we are led directly to an analytic interpretation of the Ochanine invariant $\phi(B)$.

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E=\nabla^{E,2}$ (resp. $R^F=\nabla^{F,2}$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G=E-F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as

$$\mathrm{ch}(G,\nabla^G)=\mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]-\mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^F\right)\right]. \tag{2.9}$$

In the rest of this paper, when there will be no confusion about the Hermitian connection ∇^E on a Hermitian vector bundle E, we will also write simply $\mathrm{ch}(E)$ for the associated Chern character form.

For any complex number t, let

$$\Lambda_t(E)=\mathbb{C}|_M+tE+t^2\Lambda^2(E)+\cdots,\ S_t(E)=\mathbb{C}|_M+tE+t^2S^2(E)+\cdots \tag{2.10}$$

denote respectively the total exterior and symmetric powers of E. They are living in $K(M)[t]$.

We recall the following relations between these two operations (cf. [At, Chap. 3]),

$$S_t(E)=\frac{1}{\Lambda_{-t}(E)},\qquad \Lambda_t(E-F)=\frac{\Lambda_t(E)}{\Lambda_t(F)}. \tag{2.11}$$

Moreover, if $\{\omega_i\}$, $\{{\omega_j}'\}$ be the formal Chern roots for Hermitian vector bundle E, F respectively, then [Hi, Chap. 1]

$$\mathrm{ch}(\Lambda_t(E))=\prod_i(1+e^{\omega_i}t). \tag{2.12}$$

Therefore, we have the following formulas for Chern character forms,

$$\mathrm{ch}(S_t(E))=\frac{1}{\mathrm{ch}(\Lambda_{-t}(E))}=\frac{1}{\prod\limits_i(1-e^{\omega_i}t)}\ , \tag{2.13}$$

$$\mathrm{ch}(\Lambda_t(E-F))=\frac{\mathrm{ch}(\Lambda_t(E))}{\mathrm{ch}(\Lambda_t(F))}=\frac{\prod\limits_i(1+e^{\omega_i}t)}{\prod\limits_j(1+e^{{\omega_j}'}t)}\ . \tag{2.14}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbb{C}}=W\otimes\mathbb{C}$ is a complex vector bundle over M carrying with a canonically induced Hermitian metric from the Euclidean metric of W as well as a Hermitian connection from ∇^W.

Let V be a rank $2l$ real Euclidean vector bundle over M carrying with a Euclidean connection ∇^V.

Let ξ be a rank two real oriented Euclidean vector bundle over M carrying with a Euclidean connection ∇^{ξ}.

If E is a complex vector bundle over M, set $\widetilde{E} = E - \mathbb{C}^{\mathrm{rk}(E)}$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane.

Set

(2.15)

$$\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbb{C}}} - 2\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}),$$

(2.16)

$$\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{V_{\mathbb{C}}} - 2\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbb{C}}}).$$

Clearly,$\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ and $\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ admit formal Fourier expansions in $q^{\frac{1}{2}}$ as

$$\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = A_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + A_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})q^{1/2} + \cdots, \tag{2.17}$$

$$\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = B_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + B_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})q^{1/2} + \cdots, \tag{2.18}$$

where the A_j 's and B_j 's are elements in the semi-group formally generated by Hermitian vector bundles over M. Moreover, they carry canonically induced Hermitian connections.

Let $c = e(\xi, \nabla^{\xi})$ denote the Euler form of ξ canonically associated to (ξ, ∇^{ξ}) (cf. [Z3, Sect. 3.4]). Let $R^V = \nabla^{V,\,2}$ denote the curvature of ∇^V.

If ω is a differential form over M, we denote by $\omega^{(8k+4)}$ its top degree component.

We can now state our main result of Section 2. as follows.

Main Theorem. *([HZ1, 2]) If the equality for the first Pontrjagin forms $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$ holds, then one has the equation for $(8k+4)$-forms,*

$$\left\{ \frac{\widehat{A}(TM, \nabla^{TM})\det{}^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2(\frac{c}{2})} \right\}^{(8k+4)}$$

$$= 2^{l+2k+1}\sum_{r=0}^{k} 2^{-6r}\left\{ \widehat{A}(TM, \nabla^{TM})\mathrm{ch}(b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}))\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)},$$

where each b_r, $0 \le r \le k$, is a canonical integral linear combination of $B_j(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \le j \le r$.

Certainly, when $\xi = \mathbb{R}^2$ and $c = 0$, Theorem 2.1 is exactly Liu's result in [Li, Theorem 1], which in the $(TM, \nabla^{TM}) = (V, \nabla^V)$ and $k = 1$ case, recovers the original "miraculous cancellation" formula of Alvarez-Gaumé and Witten [AGW].

Let M be a closed oriented smooth $8k+4$-manifold not necessarily spin. Let B be an orientable characteristic submanifold of M, that is, B is dual to the second Stiefel-Whitney class of TM. In this case, one finds easily from our main theorem by a simple observation that

(2.19)

$$\phi(B) \equiv \frac{\mathrm{Sign}(M) - \mathrm{Sign}(B\cdot B)}{8} \equiv \langle \widehat{A}(TM, \nabla^{TM})\mathrm{ch}(b_k(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}))\cosh\left(\frac{c}{2}\right), [M]\rangle \mod 2\mathbb{Z},$$

where $B\cdot B$ is the self-intersection of B in M and $\phi(B)$ is a spin cobordism invariant associated to (M, B). This gives the Ochanine invariant an anylatic interpretation.

The rest of this section is to give a detailed proof of our main theorem. This following part is organized as follows. In a.), we present necessary knowledge of theta-functions and modular forms which are crucial to our proof. In b.), two special characteristic forms are defined and explicitly calculated. In c.), by using transformation laws of theta-functions, we verify modular invariance of such two characteristic forms. At last in d.), a number-theoretic theorem is introduced to obtain our main result.

a.) Theta-Functions and Modular Forms

The four Jacobi theta-functions [Ch] defined by infinite multiplications are

$$\theta(v,\tau)=2q^{1/8}\sin\pi v\prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)]\ , \tag{2.20}$$

$$\theta_1(v,\tau)=2q^{1/8}\cos\pi v\prod_{j=1}^{\infty}[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)]\ , \tag{2.21}$$

$$\theta_2(v,\tau)=\prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})]\ , \tag{2.22}$$

$$\theta_3(v,\tau)=\prod_{j=1}^{\infty}[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})]\ , \tag{2.23}$$

where $q=e^{2\pi\sqrt{-1}\tau},\tau\in\mathbb{H}$.

They are all holomorphic functions for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane.

Let $\theta^{'}(0,\tau)=\frac{\partial}{\partial v}\theta(v,\tau)|_{v=0}$, then the following Jacobi identity relates the four theta-functions garcefully.

Proposition 2.1. (Jacobi identity, [Ch, Chapter 3]) *The following identity holds,*

$$\theta^{'}(0,\tau)=\pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau). \tag{2.24}$$

Let

$$SL_2(\mathbb{Z}):=\left\{\left.\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\right|a,b,c,d\in\mathbb{Z},\ ad-bc=1\right\}$$

as usual be the famous modular group. Let

$$S=\left(\begin{array}{cc}0&-1\\1&0\end{array}\right),\quad T=\left(\begin{array}{cc}1&1\\0&1\end{array}\right)$$

be the two generators of $SL_2(\mathbb{Z})$. Their actions on $\mathbb{H}$ are given by

$$S:\tau\to-\frac{1}{\tau},\quad T:\tau\to\tau+1.$$

Let

$$\Gamma_0(2)=\left\{\left.\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\in SL_2(\mathbb{Z})\right|c\equiv0\pmod 2\right\},$$

$$\Gamma^0(2)=\left\{\left.\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\in SL_2(\mathbb{Z})\right|b\equiv0\pmod 2\right\}$$

be the two modular subgroup of $SL_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST, while the generators of $\Gamma^0(2)$ are STS, T^2STS (cf. [Ch]).

If we act theta-functions by S and T, up to some complex constants, the following transformation formulas hold (cf. [Ch]),

$$\theta(v,\tau+1)=\theta(v,\tau),\quad \theta(v,-\frac{1}{\tau})=\sqrt{-1}\tau^{1/2}e^{-\tau v^2}\theta(\tau v,\tau)\ ; \tag{2.25}$$

$$\theta_1(v,\tau+1)=\theta_1(v,\tau),\quad \theta_1(v,-\frac{1}{\tau})=\tau^{1/2}e^{-\tau v^2}\theta_2(\tau v,\tau)\ ; \tag{2.26}$$

$$\theta_2(v,\tau+1)=\theta_3(v,\tau),\quad \theta_2(v,-\frac{1}{\tau})=\tau^{1/2}e^{-\tau v^2}\theta_1(\tau v,\tau)\ ; \tag{2.27}$$

$$\theta_3(v,\tau+1)=\theta_2(v,\tau),\quad \theta_3(v,-\frac{1}{\tau})=\tau^{1/2}e^{-\tau v^2}\theta_3(\tau v,\tau)\ . \tag{2.28}$$

Definition 2.1. *A modular form over a modular subgroup Γ is a holomorphic function $f(\tau)$ on $\mathbb{H}$ such that for any*

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma\ ,$$

the following property holds

$$f(g\tau):=f(\frac{a\tau+b}{c\tau+d})=\chi(g)(c\tau+d)^kf(\tau)\ , \tag{2.29}$$

where $\chi:\Gamma\to\mathbb{C}^*$ is a character of Γ and k is called the weight of f.

b.) Calculation of characteristic forms

Set for $\tau\in\mathbb{H}, q=e^{2\pi\sqrt{-1}\tau}$,

(2.30)

$$P_1(\xi,\tau)=\left\{\frac{\widehat{A}(TM,\nabla^{TM})\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2(\frac{c}{2})}\mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}}),\nabla^{\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}})}\right)\right\}^{(8k+4)},$$

$$P_2(\xi,\tau)=\left\{\widehat{A}(TM,\nabla^{TM})\mathrm{ch}\left(\Theta_2(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}}),\nabla^{\Theta_2(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}})}\right)\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)}\ . \tag{2.31}$$

Let $\{\pm2\pi\sqrt{-1}y_v\}$ and $\{\pm2\pi\sqrt{-1}x_j\}$ be the formal Chern roots for $(V_{\mathbb{C}},\nabla^{V_{\mathbb{C}}})$ and $(T_{\mathbb{C}}M,\nabla^{T_{\mathbb{C}}M})$ respectively. We also set $c=2\pi\sqrt{-1}u$. Therefore,

$$\begin{aligned}P_1(\xi,\tau)=&\left\{\prod_{j=1}^{4k+2}\frac{\pi\sqrt{-1}x_j}{\sinh(\pi\sqrt{-1}x_j)}\prod_{v=1}^{l}(e^{\pi\sqrt{-1}y_v}+e^{-\pi\sqrt{-1}y_v})\frac{1}{\cosh^2\left(\frac{c}{2}\right)}\right.\\ &\left.\mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}}),\nabla^{\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}})}\right)\right\}^{(8k+4)}\end{aligned} \tag{2.32}$$

$$=2^l\left\{\left(\prod_{j=1}^{4k+2}\frac{\pi x_j}{\sinh(\pi x_j)}\right)\left(\prod_{v=1}^{l}\cos(\pi y_v)\right)\frac{1}{\cosh^2\left(\frac{c}{2}\right)}\mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}}),\nabla^{\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}})}\right)\right\}^{(8k+4)}.$$

In what follows, in case of no confusion, we will suppress the connection notations in the computation of Chern character forms. By (2.10)-(2.16),

$$\mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}), \nabla^{\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})}\right) \tag{2.33}$$

$$\begin{aligned}
&= \prod_{n=1}^{\infty}\mathrm{ch}(S_{q^n}(\widetilde{T_{\mathbb{C}}M}))\prod_{m=1}^{\infty}\mathrm{ch}(\Lambda_{q^m}(\widetilde{V_{\mathbb{C}}}-2\widetilde{\xi_{\mathbb{C}}}))\prod_{r=1}^{\infty}\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}))\prod_{s=1}^{\infty}\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}))\\
&= \prod_{n=1}^{\infty}\frac{1}{\mathrm{ch}(\Lambda_{-q^n}(\widetilde{T_{\mathbb{C}}M}))}\prod_{m=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{q^m}(\widetilde{V_{\mathbb{C}}}))}{\mathrm{ch}(\Lambda_{q^m}(2\widetilde{\xi_{\mathbb{C}}}))}\prod_{r=1}^{\infty}\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}))\prod_{s=1}^{\infty}\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}))\\
&= \prod_{n=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{-q^n}(\mathbb{C}^{8k+4}))}{\mathrm{ch}(\Lambda_{-q^n}(T_{\mathbb{C}}M))}\prod_{m=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{q^m}(V_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{q^m}(\mathbb{C}^{2l}))}\prod_{t=1}^{\infty}\left(\frac{\mathrm{ch}(\Lambda_{q^t}(\mathbb{C}^2))}{\mathrm{ch}(\Lambda_{q^t}(\xi_{\mathbb{C}}))}\right)^2\\
&\quad \prod_{r=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\xi_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{q^{r-\frac{1}{2}}}(\mathbb{C}^2))}\prod_{s=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\xi_{\mathbb{C}}))}{\mathrm{ch}(\Lambda_{-q^{s-\frac{1}{2}}}(\mathbb{C}^2))}\ ;
\end{aligned}$$

by Proposition 2.1 and (2.20),

$$\prod_{j=1}^{4k+2}\frac{\pi x_j}{\sin(\pi x_j)}\prod_{n=1}^{\infty}\frac{\mathrm{ch}(\Lambda_{-q^n}(\mathbb{C}^{8k+4}))}{\mathrm{ch}(\Lambda_{-q^n}(T_{\mathbb{C}}M))} \tag{2.34}$$

$$\begin{aligned}
&= \prod_{j=1}^{4k+2}\frac{\pi x_j}{\sin(\pi x_j)}\prod_{n=1}^{\infty}\frac{(1-q^n)^{8k+4}}{\prod_{j=1}^{4k+2}(1-e^{2\pi\sqrt{-1}x_j}q^n)(1-e^{-2\pi\sqrt{-1}x_j}q^n)}\\
&= \prod_{j=1}^{4k+2}x_j\frac{\pi[2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2][\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2][\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2]}{2q^{\frac{1}{8}}\sin(\pi x_j)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi\sqrt{-1}x_j}q^n)(1-e^{-2\pi\sqrt{-1}x_j}q^n)}\\
&= \prod_{j=1}^{4k+2}x_j\frac{\pi\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)}{\theta(x_j,\tau)}\\
&= \prod_{j=1}^{4k+2}x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\ ;
\end{aligned}$$

By similar computations for other terms, we obtain the following:

Lemma 2.1. *The following two identities hold,*

$$P_1(\xi,\tau) = 2^l\left\{\prod_{j=1}^{4k+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\left(\prod_{v=1}^{l}\frac{\theta_1(y_v,\tau)}{\theta_1(0,\tau)}\right)\frac{\theta_1^2(0,\tau)}{\theta_1^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_2(u,\tau)}{\theta_2(0,\tau)}\right\}^{(8k+4)}, \tag{2.35}$$

By similar computation as above, one has

$$P_2(\xi,\tau) = \left\{\prod_{j=1}^{4k+2}\left(x_j\frac{\theta'(0,\tau)}{\theta(x_j,\tau)}\right)\left(\prod_{v=1}^{l}\frac{\theta_2(y_v,\tau)}{\theta_2(0,\tau)}\right)\frac{\theta_2^2(0,\tau)}{\theta_2^2(u,\tau)}\frac{\theta_3(u,\tau)}{\theta_3(0,\tau)}\frac{\theta_1(u,\tau)}{\theta_1(0,\tau)}\right\}^{(8k+4)}. \tag{2.36}$$

c.) Modular invariance of $P_1(\xi,\tau)$ and $P_2(\xi,\tau)$

We will verify directly from the transformation laws of theta-functions that $P_1(\xi,\tau)$ and $P_2(\xi,\tau)$ are modular forms of weight $4k+2$ over $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively.

From (2.25), we deduce that

$$\theta'(0,\tau+1)=\theta'(0,\tau),\quad \theta'(0,-\frac{1}{\tau})=\sqrt{-1}\tau^{3/2}\theta'(0,\tau)\ . \tag{2.37}$$

By transformation laws (2.25) $\sim$ (2.28), (2.37), [Hi, chapter 9.3] and the assumption that $p_1(TM,\nabla^{TM})=p_1(V,\nabla^V)$, i.e. $\sum_{v=1}^{l}y_v^2=\sum_{j=1}^{4k+2}x_j^2$, we have

$$P_2(\xi,STS\tau)=(\tau-1)^{4k+2}P_2(\xi,\tau)\ . \tag{2.38}$$

One can also easily verify as above that

$$P_2(\xi,T^2STS\tau)=(\tau-1)^{4k+2}P_2(\xi,\tau). \tag{2.39}$$

Since $STS\tau=\frac{-\tau}{\tau-1}, T^2STS\tau=\frac{\tau-2}{\tau-1}$ and STS, T^2STS are the two generators of $\Gamma^0(2)$, one gets by Definition 2.1 and (2.38), (2.39) that $P_2(\xi,\tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$.

Moreover, by (2.25) $\sim$ (2.28) and (2.37),

$$\begin{aligned} & P_1(\xi,\frac{-1}{\tau}) \\ = \ & 2^l\Bigg\{\left[\prod_{j=1}^{4k+2}x_j\frac{\sqrt{-1}\tau^{3/2}\theta'(0,\tau)}{\sqrt{-1}\tau^{1/2}e^{-\tau x_j^2}\theta(\tau x_j,\tau)}\right]\left[\prod_{v=1}^{l}\frac{\tau^{1/2}e^{-\tau y_v^2}\theta_2(\tau y_v,\tau)}{\tau^{1/2}\theta_2(0,\tau)}\right] \\ & \frac{[\tau^{1/2}\theta_2(0,\tau)]^2}{[\tau^{1/2}e^{-\tau u^2}\theta_2(\tau u,\tau)]^2}\frac{\tau^{1/2}e^{-\tau u^2}\theta_3(\tau u,\tau)}{\tau^{1/2}\theta_3(0,\tau)}\frac{\tau^{1/2}e^{-\tau u^2}\theta_1(\tau u,\tau)}{\tau^{1/2}\theta_1(0,\tau)}\Bigg\}^{(8k+4)} \\ = \ & 2^l\left\{\left[\prod_{j=1}^{4k+2}(\tau x_j)\frac{\theta'(0,\tau)}{\theta(\tau x_j,\tau)}\right]\left[\prod_{v=1}^{l}\frac{\theta_2(\tau y_v,\tau)}{\theta_2(0,\tau)}\right]\frac{\theta_2^2(0,\tau)}{\theta_2^2(\tau u,\tau)}\frac{\theta_3(\tau u,\tau)}{\theta_3(0,\tau)}\frac{\theta_1(\tau u,\tau)}{\theta_1(0,\tau)}\right\}^{(8k+4)} \\ = \ & 2^l\tau^{4k+2}P_2(\xi,\tau)\ . \end{aligned} \tag{2.40}$$

One can easily get,

$$P_1(\xi,\tau+1)=P_1(\xi,\tau)\ . \tag{2.41}$$

Thus by (2.40) and (2.41),

$$\begin{aligned} P_1(\xi,ST^2ST\tau) &= 2^l(T^2ST\tau)^{4k+2}P_2(\xi,T^2ST\tau) \\ &= 2^l(T^2ST\tau)^{4k+2}P_2(\xi,ST\tau) \\ &= 2^l(T^2ST\tau)^{4k+2}2^{-l}(T\tau)^{4k+2}P_1(\xi,T\tau) \\ &= (2\tau+1)^{4k+2}P_1(\xi,\tau)\ . \end{aligned} \tag{2.42}$$

Since $T\tau=\tau+1$, $ST^2ST\tau=-\frac{\tau+1}{2\tau+1}$ and T, ST^2ST are the two generators of $\Gamma_0(2)$, one gets by Definition 2.1, (2.41) and (2.42) that $P_1(\tau)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$.

As a summary of c.), we get

Proposition 2.2. *Assume $p_1(TM,\nabla^{TM})=p_1(V,\nabla^V)$, then*

$$P_1(\xi,\tau)=\left\{\frac{\widehat{A}(TM,\nabla^{TM})\det^{1/2}\left(2\cosh\left(\frac{\sqrt{-1}}{4\pi}R^V\right)\right)}{\cosh^2(\frac{c}{2})}\mathrm{ch}\left(\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}}),\nabla^{\Theta_1(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}})}\right)\right\}^{(8k+4)}$$

is a modular form of weight $4k+2$ over $\Gamma_0(2)$;

$$P_2(\xi,\tau)=\left\{\widehat{A}(TM,\nabla^{TM})\mathrm{ch}\left(\Theta_2(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}}),\nabla^{\Theta_2(T_{\mathbb{C}}M,V_{\mathbb{C}},\xi_{\mathbb{C}})}\right)\cosh\left(\frac{c}{2}\right)\right\}^{(8k+4)}$$

is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Moreover, the following identity holds,

$$P_1(\xi,\frac{-1}{\tau})=2^l\tau^{4k+2}P_2(\xi,\tau). \tag{2.43}$$

d.) A number-theoretic theorem and our main result

Writing simply $\theta_j=\theta_j(0,\tau),\ 1\le j\le 3$, we introduce four explicit modular forms (cf. [Liu]),

$$\delta_1(\tau)=\frac{1}{8}(\theta_2^4+\theta_3^4),\qquad \varepsilon_1(\tau)=\frac{1}{16}\theta_2^4\theta_3^4\ ,$$
$$\delta_2(\tau)=\frac{1}{8}(\theta_1^4+\theta_3^4),\qquad \varepsilon_1(\tau)=\frac{1}{16}\theta_1^4\theta_3^4\ .$$

They have the following Fourier expansions in $q^{1/2}$:

$$\delta_1(\tau)=\frac{1}{4}+6q+\cdots,\qquad \varepsilon_1(\tau)=\frac{1}{16}-q+\cdots\ , \tag{2.44}$$

$$\delta_2(\tau)=-\frac{1}{8}-3q^{1/2}+\cdots,\qquad \varepsilon_2(\tau)=q^{1/2}+\cdots\ . \tag{2.45}$$

where the "$\cdots$" terms are the higher degree terms, all of which have integral coefficients. If Γ is a modular subgroup, let $M_{\mathbb{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients.

The following weaker version of [Li, Lemma 2] will be used in our proof.

Proposition 2.3. *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, while $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, and moreover $M_{\mathbb{R}}(\Gamma^0(2))=\mathbb{R}[\delta_2(\tau),\varepsilon_2(\tau)]$.*

We can now proceed to prove our Main Theorem as follows.

Observe that at any point $x\in M$, up to the volume form determined by the metric on T_xM, both $P_i(\xi,\tau), i=1,2$, can be viewed as a power series of $q^{1/2}$ with real Fourier coefficients. Thus, one can combine Proposition 2.2 and Proposition 2.3 to get, at x, that

$$P_2(\xi,\tau)=h_0(8\delta_2)^{2k+1}+h_1(8\delta_2)^{2k-1}\varepsilon_2+\cdots+h_k(8\delta_2)\varepsilon_2^k\ , \tag{2.46}$$

where each h_r, $0\le r\le k$, is a real multiple of the volume form at x.

Now recall that (cf. [L, page 36]) $\delta_i,\varepsilon_i, i=1,2$, verify the transformation laws

$$\delta_2\left(-\frac{1}{\tau}\right)=\tau^2\delta_1(\tau)\qquad,\qquad \varepsilon_2\left(-\frac{1}{\tau}\right)=\tau^4\varepsilon_1(\tau).$$

So by (2.40),

$$P_1(\xi,\tau)=2^l\frac{1}{\tau^{4k+2}}P_2\left(\xi,-\frac{1}{\tau}\right) \tag{2.47}$$

$$
\begin{aligned}
&= 2^l \frac{1}{\tau^{4k+2}} \left[h_0 \left(8\delta_2 \left(-\frac{1}{\tau} \right) \right)^{2k+1} + h_1 \left(8\delta_2 \left(-\frac{1}{\tau} \right) \right)^{2k-1} \varepsilon_2 \left(-\frac{1}{\tau} \right) + \cdots \right. \\
&\quad \left. + h_k \left(8\delta_2 \left(-\frac{1}{\tau} \right) \right) \left(\varepsilon_2 \left(-\frac{1}{\tau} \right) \right)^k \right] \\
&= 2^l [h_0 (8\delta_1)^{2k+1} + h_1 (8\delta_1)^{2k-1} \varepsilon_1 + \cdots + h_k (8\delta_1) \varepsilon_1^k] \,.
\end{aligned}
$$

Set $q = 0$, then $8\delta_1 = 2, \varepsilon_1 = 2^{-4}$, one obtains

$$
\begin{aligned}
&\left\{ \frac{\widehat{A}(TM, \nabla^{TM}) \det^{1/2} \left(2\cosh \left(\frac{\sqrt{-1}}{4\pi} R^V \right) \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(8k+4)} \\
&= 2^l [h_0 2^{2k+1} + h_1 2^{2k-1} \cdot 2^{-4} + \cdots + h_k 2 \cdot 2^{-4k}] \\
&= 2^{l+2k+1} \Sigma_{r=0}^k 2^{-6r} h_r .
\end{aligned}
$$

Now in order to prove our Main Theorem, one need to show that each $h_r, 0 \leq r \leq k$, can be expressed through a canonical integral linear combination of $\{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(B_j, \nabla^{B_j}) \cosh(\frac{c}{2})\}^{(8k+4)}, 0 \leq j \leq r$, with coefficients not depending on $x \in M$.

As in [L], one can use the induction method in the following to prove this fact by comparing the coefficients of $q^{j/2}, j \geq 0$, between the two sides of (2.46).

Formally expand the left side of (2.46) into Fourier series in $q^{1/2}$, then

(2.48)
$$
\begin{aligned}
P_2(\xi, \tau) &= \{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch} B_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) \cosh(\frac{c}{2})\}^{(8k+4)} \\
&\quad + \{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch} B_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) \cosh(\frac{c}{2})\}^{(8k+4)} q^{1/2} \\
&\quad + \{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch} B_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) \cosh(\frac{c}{2})\}^{(8k+4)} q + \cdots \,.
\end{aligned}
$$

Formally expand the right hand side of (2.46) into Fourier series in $q^{1/2}$, then

(2.49)
$$
\begin{aligned}
r.h.s &= h_0(-1 - 24q^{1/2} + \cdots)^{2k+1} + h_1(-1 - 24q^{1/2} + \cdots)^{2k-1}(q^{1/2} + \cdots) \\
&\quad + h_2(-1 - 24q^{1/2} + \cdots)^{2k-3}(q^{1/2} + \cdots)^2 + \cdots \\
&= p_0(h_0) + p_1(h_0, h_1) q^{1/2} + p_2(h_0, h_1, h_2) q + \cdots \\
&= -h_0 + (-h_1 - 24(2k+1)h_0) q^{1/2} + \cdots \,,
\end{aligned}
$$

where each $p_i(h_0, h_1, \cdots, h_i), 0 \leq i \leq k$, is a finite and linear combination of $h_0, h_1, \cdots, h_i$ with integral coefficients.

Fortunately, h_i has coefficient -1 in $p_i(h_0, h_1 \cdots h_i)$. This fact allows us to identify h_i step by step. We only compute h_0 and h_1 here.

(2.50)
$$
h_0 = -\{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch} B_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) \cosh\left(\frac{c}{2}\right)\}^{(8k+4)} \,,
$$

(2.51)
$$
h_1 = \{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}[-B_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + 24(2k+1) B_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})] \cosh\left(\frac{c}{2}\right)\}^{(8k+4)} \,.
$$

Remark 2.1. From (2.31), (2.45), (2.46) and Proposition 2.3, one finds that for any integer $r \geq 0, \{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(B_r, \nabla^{B_r}) \cosh(\frac{c}{2})\}^{(8k+4)}$ can be expressed through a canonical integral linear combination of $\{\widehat{A}(TM, \nabla^{TM}) \mathrm{ch}(B_j, \nabla^{B_j}) \cosh(\frac{c}{2})\}^{(8k+4)}, 0 \leq j \leq k$. This fact is by no means trivial for $r \geq k+1$. It depends heavily on the modular invariance of $P_2(\xi, \tau)$.

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