You could've invented *tmf*.

Aaron Mazel-Gee

University of California, Berkeley

aaron@math.berkeley.edu

April 21, 2013

Overview

Overview

1. The finite stable homotopy category

Overview

1. The finite stable homotopy category
2. Chromatic homotopy theory

Overview

1. The finite stable homotopy category
2. Chromatic homotopy theory
3. Topological modular forms

Overview

1. The finite stable homotopy category
2. Chromatic homotopy theory
3. Topological modular forms
4. Fun with *tmf*

1. The finite stable homotopy category

The category $\mathbf{SHC}^{fin}$

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology?

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

homotopy category of topological spaces.

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

homotopy category of topological spaces.

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

homotopy category of ~~topological spaces~~.
CW complexes

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

homotopy category of ~~topological spaces~~.
CW complexes

The category **SHC**^{fin}

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

homotopy category of ~~topological spaces~~.
finite CW complexes

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

homotopy category of ~~topological spaces~~.
finite CW complexes

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

stable homotopy category of ~~topological spaces~~.
finite CW complexes

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

stable homotopy category of ~~topological spaces~~.
finite CW complexes

The category $\mathbf{SHC}^{fin}$

Let's do some topology!

So, what is topology? Topology is the study of topological spaces.

But this is *hard*.

Let's make things a bit easier for ourselves, and study the

stable homotopy category of ~~topological spaces~~.
finite CW complexes

This is called the *finite stable homotopy category*, denoted $\mathbf{SHC}^{fin}$.

What does it mean to *stabilize*?

What does it mean to *stabilize*?

Recall that the *suspension* of a space X is given by

$$\Sigma X = [0,1] \times X \Big/ \begin{array}{c} (0,x_1) \sim (0,x_2) \\ (1,x_1) \sim (1,x_2). \end{array}$$

What does it mean to *stabilize*?

Recall that the *suspension* of a space X is given by

$$\Sigma X = [0,1] \times X \Big/ \begin{array}{c} (0,x_1) \sim (0,x_2) \\ (1,x_1) \sim (1,x_2). \end{array}$$

Given a map $X \to Y$, we can suspend it to a map $\Sigma X \to \Sigma Y$.

What does it mean to *stabilize*?

Recall that the *suspension* of a space X is given by

$$\Sigma X = [0,1] \times X \Big/ \begin{array}{c} (0, x_1) \sim (0, x_2) \\ (1, x_1) \sim (1, x_2). \end{array}$$

Given a map $X \to Y$, we can suspend it to a map $\Sigma X \to \Sigma Y$.

This gives us a system

$$[X, Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to [\Sigma^3 X, \Sigma^3 Y] \to \cdots .$$

What does it mean to *stabilize*?

Recall that the *suspension* of a space X is given by

$$\Sigma X = [0,1] \times X \Big/ \begin{array}{c} (0,x_1) \sim (0,x_2) \\ (1,x_1) \sim (1,x_2). \end{array}$$

Given a map $X \to Y$, we can suspend it to a map $\Sigma X \to \Sigma Y$.

This gives us a system

$$[X,Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to [\Sigma^3 X, \Sigma^3 Y] \to \cdots .$$

The *Freudenthal suspension theorem* tells us that this system always *stabilizes*.

What does it mean to *stabilize*?

Recall that the *suspension* of a space X is given by

$$\Sigma X = [0,1] \times X \Big/ \begin{array}{c} (0,x_1) \sim (0,x_2) \\ (1,x_1) \sim (1,x_2). \end{array}$$

Given a map $X \to Y$, we can suspend it to a map $\Sigma X \to \Sigma Y$.

This gives us a system

$$[X,Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to [\Sigma^3 X, \Sigma^3 Y] \to \cdots .$$

The *Freudenthal suspension theorem* tells us that this system always *stabilizes*. So for two finite CW complexes X and Y, we define

$$\mathrm{Hom}_{\mathbf{SHC}^{fin}}(X,Y) = \lim_{n\to\infty} [\Sigma^n X, \Sigma^n Y].$$

The set

$$\mathrm{Hom}_{\mathbf{SHC}^{fin}}(X, Y) = \lim_{n \to \infty} [\Sigma^n X, \Sigma^n Y]$$

is in fact an *abelian group*.

The set

$$\mathrm{Hom}_{\mathbf{SHC}^{fin}}(X, Y) = \lim_{n \to \infty} [\Sigma^n X, \Sigma^n Y]$$

is in fact an *abelian group*.

(Recall that $[\Sigma X, Z]$ is always a group (for the same reason that π_1 is a group), and that $[\Sigma^n X, Z]$ is always an abelian group for $n \geq 2$ (for the same reason that $\pi_{\geq 2}$ is an abelian group).)

And as long as we only need to be able to define maps between arbitrarily high suspensions of our finite CW complexes, we may as well allow *desuspensions* too.

And as long as we only need to be able to define maps between arbitrarily high suspensions of our finite CW complexes, we may as well allow *desuspensions* too.

Then, for example,

$$\mathrm{Hom}_{\mathbf{SHC}^{fin}}(\Sigma^{-i}X, \Sigma^{-j}Y) = \lim_{n\to\infty} [\Sigma^{n-i}X, \Sigma^{n-j}Y].$$

And as long as we only need to be able to define maps between arbitrarily high suspensions of our finite CW complexes, we may as well allow *desuspensions* too.

Then, for example,

$$\mathrm{Hom}_{\mathbf{SHC}^{fin}}(\Sigma^{-i}X, \Sigma^{-j}Y) = \lim_{n\to\infty} [\Sigma^{n-i}X, \Sigma^{n-j}Y].$$

So, the objects of $\mathbf{SHC}^{fin}$ are the finite CW complexes and their formal desuspensions.

The geometry of $\mathbf{SHC}^{fin}$

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category.

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category. What structure does it carry?

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category. What structure does it carry?

- A subcategory is called *thick* if it is closed under mapping cones, retracts, and weak equivalences. So, the "kernel" of any co/homology theory is thick.

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category. What structure does it carry?

- A subcategory is called *thick* if it is closed under mapping cones, retracts, and weak equivalences. So, the “kernel” of any co/homology theory is thick.
- We can use the smash product to define *ideal* subcategories and *prime ideal* subcategories, exactly as in ring theory.

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category. What structure does it carry?

- A subcategory is called *thick* if it is closed under mapping cones, retracts, and weak equivalences. So, the "kernel" of any co/homology theory is thick.
- We can use the smash product to define *ideal* subcategories and *prime ideal* subcategories, exactly as in ring theory.

Given a category $\mathcal{C}$ with this structure, we can define a space Spec($\mathcal{C}$) in much the same way an algebraic geometer defines the *prime spectrum* of a ring:

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category. What structure does it carry?

- A subcategory is called *thick* if it is closed under mapping cones, retracts, and weak equivalences. So, the "kernel" of any co/homology theory is thick.
- We can use the smash product to define *ideal* subcategories and *prime ideal* subcategories, exactly as in ring theory.

Given a category $\mathcal{C}$ with this structure, we can define a space $\mathrm{Spec}(\mathcal{C})$ in much the same way an algebraic geometer defines the *prime spectrum* of a ring:

$$\mathrm{Spec}(\mathcal{C}) = \{\mathscr{P} \subset \mathcal{C} : \mathscr{P} \text{ a thick prime ideal subcategory}\}.$$

The geometry of $\mathbf{SHC}^{fin}$

We would like to study the *global structure* of the finite stable homotopy category. What structure does it carry?

- A subcategory is called *thick* if it is closed under mapping cones, retracts, and weak equivalences. So, the "kernel" of any co/homology theory is thick.
- We can use the smash product to define *ideal* subcategories and *prime ideal* subcategories, exactly as in ring theory.

Given a category $\mathcal{C}$ with this structure, we can define a space $\mathrm{Spec}(\mathcal{C})$ in much the same way an algebraic geometer defines the *prime spectrum* of a ring:

$$\mathrm{Spec}(\mathcal{C}) = \{\mathscr{P} \subset \mathcal{C} : \mathscr{P} \text{ a thick prime ideal subcategory}\}.$$

This packages a lot of information really cleanly.

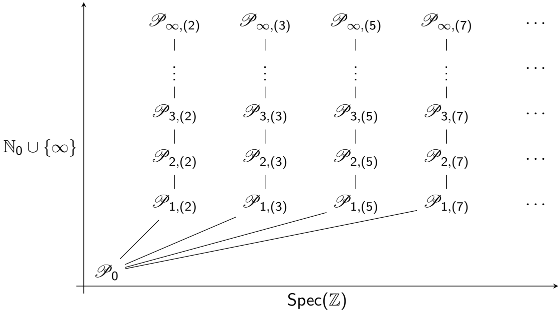
So, what is Spec($\mathbf{SHC}^{fin}$)?

So, what is $\mathrm{Spec}(\mathbf{SHC}^{fin})$?

There is a beautiful answer, given by the *nilpotence theorem* of Devinatz–Hopkins–Smith.

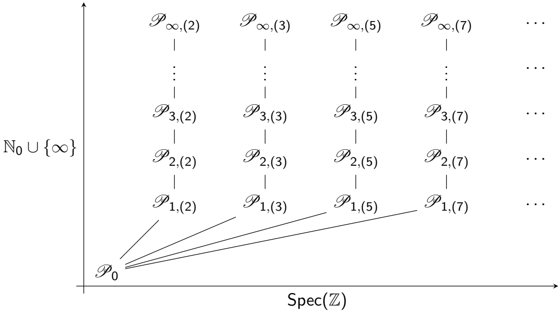
So, what is $\mathrm{Spec}(\mathbf{SHC}^{fin})$?

There is a beautiful answer, given by the *nilpotence theorem* of Devinatz–Hopkins–Smith.



So, what is $\mathrm{Spec}(\mathbf{SHC}^{fin})$?

There is a beautiful answer, given by the *nilpotence theorem* of Devinatz–Hopkins–Smith.



This is exciting! But to explain what the subcategories $\mathscr{P}_{n,(p)}$ are, we'll have to talk about...

2. Chromatic homotopy theory

Formal group laws in topology

Formal group laws in topology

The story of chromatic homotopy theory begins with formal group laws, and the story of formal group laws begins with $\mathbb{CP}^\infty$.

Formal group laws in topology

The story of chromatic homotopy theory begins with formal group laws, and the story of formal group laws begins with $\mathbb{CP}^\infty$.

Recall that $\mathbb{CP}^\infty$ is a *classifying space* for complex line bundles;

Formal group laws in topology

The story of chromatic homotopy theory begins with formal group laws, and the story of formal group laws begins with $\mathbb{CP}^\infty$.

Recall that $\mathbb{CP}^\infty$ is a *classifying space* for complex line bundles; that is, it carries a *universal* line bundle $\mathscr{L}_{univ} \downarrow \mathbb{CP}^\infty$,

Formal group laws in topology

The story of chromatic homotopy theory begins with formal group laws, and the story of formal group laws begins with $\mathbb{CP}^\infty$.

Recall that $\mathbb{CP}^\infty$ is a *classifying space* for complex line bundles; that is, it carries a *universal* line bundle $\mathscr{L}_{univ} \downarrow \mathbb{CP}^\infty$, and there is a natural isomorphism

$$\begin{array}{ccc} \{\text{line bundles over } X\} & \cong & [X, \mathbb{CP}^\infty]. \\ f^* \mathscr{L}_{univ} & \leftrightarrow & f \end{array}$$

By *Yoneda's lemma*, the natural operation of *tensor product* of two line bundles is classified by a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$.

By *Yoneda's lemma*, the natural operation of *tensor product* of two line bundles is classified by a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$.

(A pair of line bundles $\mathscr{L}_1, \mathscr{L}_2 \downarrow X$ is classified by a map

$$X \xrightarrow{f} \mathbb{CP}^\infty \times \mathbb{CP}^\infty$$

such that $\mathscr{L}_1 \cong f^* \mathscr{L}_{univ,1}$ and $\mathscr{L}_2 \cong f^* \mathscr{L}_{univ,2}$.

By *Yoneda's lemma*, the natural operation of *tensor product* of two line bundles is classified by a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$.

(A pair of line bundles $\mathscr{L}_1, \mathscr{L}_2 \downarrow X$ is classified by a map

$$X \xrightarrow{f} \mathbb{CP}^\infty \times \mathbb{CP}^\infty$$

such that $\mathscr{L}_1 \cong f^* \mathscr{L}_{univ,1}$ and $\mathscr{L}_2 \cong f^* \mathscr{L}_{univ,2}$. By assumption,

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$$

classifies $\mathscr{L}_{univ,1} \otimes \mathscr{L}_{univ,2} \cong \mu^* \mathscr{L}_{univ}$,

By *Yoneda's lemma*, the natural operation of *tensor product* of two line bundles is classified by a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$.

(A pair of line bundles $\mathscr{L}_1, \mathscr{L}_2 \downarrow X$ is classified by a map

$$X \xrightarrow{f} \mathbb{CP}^\infty \times \mathbb{CP}^\infty$$

such that $\mathscr{L}_1 \cong f^* \mathscr{L}_{univ,1}$ and $\mathscr{L}_2 \cong f^* \mathscr{L}_{univ,2}$. By assumption,

$$\mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$$

classifies $\mathscr{L}_{univ,1} \otimes \mathscr{L}_{univ,2} \cong \mu^* \mathscr{L}_{univ}$, so the composite

$$X \xrightarrow{f} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{\mu} \mathbb{CP}^\infty$$

classifies $\mathscr{L}_1 \otimes \mathscr{L}_2 \cong (\mu f)^* \mathscr{L}_{univ}$.)

What does this map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ buy us?

What does this map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ buy us?

Let's see what happens when we apply $H^*(-;\mathbb{Z})$.

What does this map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ buy us?

Let's see what happens when we apply $H^*(-;\mathbb{Z})$.

Recall that $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[[t]]$; by the Künneth formula, $H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}[[x,y]]$.

What does this map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ buy us?

Let's see what happens when we apply $H^*(-;\mathbb{Z})$.

Recall that $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[[t]]$; by the Künneth formula, $H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}[[x,y]]$. So, the ring homomorphism

$$H^*(\mathbb{CP}^\infty) \to H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$

is entirely described by

$$t \mapsto F(x,y) \in \mathbb{Z}[[x,y]].$$

What does this map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ buy us?

Let's see what happens when we apply $H^*(-;\mathbb{Z})$.

Recall that $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[[t]]$; by the Künneth formula, $H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z}[[x, y]]$. So, the ring homomorphism

$$H^*(\mathbb{CP}^\infty) \to H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$

is entirely described by

$$t \mapsto F(x, y) \in \mathbb{Z}[[x, y]].$$

What is $F(x, y)$?

To determine the map

$$\begin{array}{rcl} H^*(\mathbb{CP}^\infty) & \to & H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \\ t & \mapsto & F(x,y), \end{array}$$

we need to remember that the element t also goes by another name: the *first Chern class*.

To determine the map

$$\begin{array}{rcl} H^*(\mathbb{CP}^\infty) & \to & H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \\ t & \mapsto & F(x,y), \end{array}$$

we need to remember that the element t also goes by another name: the *first Chern class*. We also need to remember that for any two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$,

$$c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) = c_1(\mathscr{L}_1) + c_1(\mathscr{L}_2).$$

To determine the map

$$\begin{array}{rcl} H^*(\mathbb{CP}^\infty) & \to & H^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \\ t & \mapsto & F(x,y), \end{array}$$

we need to remember that the element t also goes by another name: the *first Chern class*. We also need to remember that for any two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$,

$$c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) = c_1(\mathscr{L}_1) + c_1(\mathscr{L}_2).$$

Since $F(x,y)$ just encodes how the first Chern class behaves under tensor product, it follows that $F(x,y) = x + y$.

Let's generalize this a bit.

Let's generalize this a bit.

A ring-valued cohomology theory E^* is called *complex-oriented* if we have an isomorphism

$$E^*(\mathbb{CP}^\infty) \cong (E^*(\mathrm{pt}))[[t]] = E_*[[t]];$$

we call the element t a *generalized first Chern class*.

Let's generalize this a bit.

A ring-valued cohomology theory E^* is called *complex-oriented* if we have an isomorphism

$$E^*(\mathbb{CP}^\infty) \cong (E^*(\mathrm{pt}))[[t]] = E_*[[t]];$$

we call the element t a *generalized first Chern class*.

We'll always have $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E_*[[x, y]]$, and so once again the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces a map $E_*[[t]] \to E_*[[x, y]]$, and once again this is determined by $t \mapsto F_E(x, y) \in E_*[[x, y]]$.

Let's generalize this a bit.

A ring-valued cohomology theory E^* is called *complex-oriented* if we have an isomorphism

$$E^*(\mathbb{CP}^\infty) \cong (E^*(\mathrm{pt}))[[t]] = E_*[[t]];$$

we call the element t a *generalized first Chern class*.

We'll always have $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E_*[[x, y]]$, and so once again the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces a map $E_*[[t]] \to E_*[[x, y]]$, and once again this is determined by $t \mapsto F_E(x, y) \in E_*[[x, y]]$.

What can we say about $F_E(x, y)$?

The power series $F_E(x, y) \in E_*[[x, y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

The power series $F_E(x, y) \in E_*[[x, y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

$$\mathscr{L} \otimes \underline{\mathbb{C}} \cong \mathscr{L} \quad \Rightarrow \quad F_E(x, 0) = x \quad \text{(unitality)}$$

The power series $F_E(x, y) \in E_*[[x, y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

$$\mathscr{L} \otimes \underline{\mathbb{C}} \cong \mathscr{L} \quad \Rightarrow \quad F_E(x, 0) = x \quad \text{(unitality)}$$

$$\mathscr{L}_1 \otimes \mathscr{L}_2 \cong \mathscr{L}_2 \otimes \mathscr{L}_1 \quad \Rightarrow \quad F_E(x, y) = F_E(y, x) \quad \text{(commutativity)}$$

The power series $F_E(x,y) \in E_*[[x,y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

$$\begin{aligned}
\mathscr{L} \otimes \underline{\mathbb{C}} \cong \mathscr{L} \quad &\Rightarrow \quad F_E(x,0) = x \quad \text{(unitality)} \\
\mathscr{L}_1 \otimes \mathscr{L}_2 \cong \mathscr{L}_2 \otimes \mathscr{L}_1 \quad &\Rightarrow \quad F_E(x,y) = F_E(y,x) \quad \text{(commutativity)} \\
(\mathscr{L}_1 \otimes \mathscr{L}_2) \otimes \mathscr{L}_3 \cong & \\
\mathscr{L}_1 \otimes (\mathscr{L}_2 \otimes \mathscr{L}_3) \quad &\Rightarrow \quad F_E(F_E(x,y),z) \\
& \qquad = F_E(x, F_E(y,z)) \quad \text{(associativity)}
\end{aligned}$$

The power series $F_E(x, y) \in E_*[[x, y]]$ enjoys certain properties coming from analogous properties of the tensor product of line bundles.

$$\begin{aligned}
\mathscr{L} \otimes \underline{\mathbb{C}} \cong \mathscr{L} \quad &\Rightarrow \quad F_E(x, 0) = x \quad \text{(unitality)} \\
\mathscr{L}_1 \otimes \mathscr{L}_2 \cong \mathscr{L}_2 \otimes \mathscr{L}_1 \quad &\Rightarrow \quad F_E(x, y) = F_E(y, x) \quad \text{(commutativity)} \\
(\mathscr{L}_1 \otimes \mathscr{L}_2) \otimes \mathscr{L}_3 \cong & \\
\mathscr{L}_1 \otimes (\mathscr{L}_2 \otimes \mathscr{L}_3) \quad &\Rightarrow \quad F_E(F_E(x, y), z) \\
& \qquad = F_E(x, F_E(y, z)) \quad \text{(associativity)}
\end{aligned}$$

These are the three defining properties for F_E to be a (1-dimensional commutative) *formal group law* over the ring E_*.

One can obtain *formal group laws* as germs of *algebraic groups*, much as one can obtain Lie algebras as germs of Lie groups.

One can obtain *formal group laws* as germs of *algebraic groups*, much as one can obtain Lie algebras as germs of Lie groups.

The formal group law $F_{H\mathbb{Z}}(x, y) = x + y$ associated to singular cohomology is called the *additive formal group law*, denoted $\widehat{\mathbb{G}}_a$, which is the germ of the *additive group*, denoted $\mathbb{G}_a$.

Another example: complex K-theory.

Another example: complex K-theory.

Recall that $KU^0(X)$ is the group-completion of the monoid $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus)$, with multiplication given by $\otimes$.

Another example: complex K-theory.

Recall that $KU^0(X)$ is the group-completion of the monoid $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus)$, with multiplication given by $\otimes$.

K-theory is complex-oriented: $KU^*(\mathbb{CP}^\infty) \cong KU_*[[t]]$, where $t = [\mathscr{L}_{univ}] - [\underline{\mathbb{C}}] = [\mathscr{L}_{univ}] - 1$.

Another example: complex K-theory.

Recall that $KU^0(X)$ is the group-completion of the monoid $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus)$, with multiplication given by $\otimes$.

K-theory is complex-oriented: $KU^*(\mathbb{CP}^\infty) \cong KU_*[[t]]$, where $t = [\mathscr{L}_{univ}] - [\underline{\mathbb{C}}] = [\mathscr{L}_{univ}] - 1$. So with $x = [\mathscr{L}_1] - 1$ and $y = [\mathscr{L}_2] - 1$, since $\mu^* \mathscr{L}_{univ} = \mathscr{L}_1 \otimes \mathscr{L}_2$ we compute that

$$F_{KU}(x, y) = [\mathscr{L}_1 \otimes \mathscr{L}_2] - 1$$

Another example: complex K-theory.

Recall that $KU^0(X)$ is the group-completion of the monoid $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus)$, with multiplication given by $\otimes$.

K-theory is complex-oriented: $KU^*(\mathbb{CP}^\infty) \cong KU_*[[t]]$, where $t = [\mathscr{L}_{univ}] - [\underline{\mathbb{C}}] = [\mathscr{L}_{univ}] - 1$. So with $x = [\mathscr{L}_1] - 1$ and $y = [\mathscr{L}_2] - 1$, since $\mu^* \mathscr{L}_{univ} = \mathscr{L}_1 \otimes \mathscr{L}_2$ we compute that

$$\begin{aligned} F_{KU}(x,y) &= [\mathscr{L}_1 \otimes \mathscr{L}_2] - 1 \\ &= [\mathscr{L}_1] \cdot [\mathscr{L}_2] - 1 \\ &= ([\mathscr{L}_1] \cdot [\mathscr{L}_2] - [\mathscr{L}_1] - [\mathscr{L}_2] + 1) + [\mathscr{L}_1] - 1 + [\mathscr{L}_2] - 1 \\ &= ([\mathscr{L}_1] - 1) \cdot ([\mathscr{L}_2] - 1) + ([\mathscr{L}_1] - 1) + ([\mathscr{L}_2] - 1) \\ &= xy + x + y. \end{aligned}$$

Another example: complex K-theory.

Recall that $KU^0(X)$ is the group-completion of the monoid $(\mathrm{Vect}_{\mathbb{C}}(X), \oplus)$, with multiplication given by $\otimes$.

K-theory is complex-oriented: $KU^*(\mathbb{CP}^\infty) \cong KU_*[[t]]$, where $t = [\mathscr{L}_{univ}] - [\underline{\mathbb{C}}] = [\mathscr{L}_{univ}] - 1$. So with $x = [\mathscr{L}_1] - 1$ and $y = [\mathscr{L}_2] - 1$, since $\mu^*\mathscr{L}_{univ} = \mathscr{L}_1 \otimes \mathscr{L}_2$ we compute that

$$\begin{aligned} F_{KU}(x,y) &= [\mathscr{L}_1 \otimes \mathscr{L}_2] - 1 \\ &= [\mathscr{L}_1] \cdot [\mathscr{L}_2] - 1 \\ &= ([\mathscr{L}_1] \cdot [\mathscr{L}_2] - [\mathscr{L}_1] - [\mathscr{L}_2] + 1) + [\mathscr{L}_1] - 1 + [\mathscr{L}_2] - 1 \\ &= ([\mathscr{L}_1] - 1) \cdot ([\mathscr{L}_2] - 1) + ([\mathscr{L}_1] - 1) + ([\mathscr{L}_2] - 1) \\ &= xy + x + y. \end{aligned}$$

This is called the *multiplicative formal group law*, denoted $\widehat{\mathbb{G}}_m$, which is the germ of the *multiplicative group*, denoted $\mathbb{G}_m$. We'll come back to this.

Returning to the general theory, we have a functor

$$\left\{ \begin{array}{c} \text{complex-oriented} \\ \text{cohomology theories} \end{array} \right\} \xrightarrow{E \mapsto F_E(x,y) \in E_*[[x,y]]} \{\text{formal group laws}\}.$$

Returning to the general theory, we have a functor

$$E \mapsto F_E(x, y) \in E_*[[x, y]]$$

$$\left\{ \begin{array}{c} \text{complex-oriented} \\ \text{cohomology theories} \end{array} \right\} \rightleftarrows \{\text{formal group laws}\}.$$

In fact, there is a partial inverse (i.e. it's not defined on all formal group laws) given by the *Landweber exact functor theorem*.

The Morava K-theories

The Morava K-theories

To define the Morava K-theories, we first need to define the *height* of a formal group law.

The Morava K-theories

To define the Morava K-theories, we first need to define the *height* of a formal group law.

Given a formal group law $F(x,y) \in R[[x,y]]$, we define its *n-series* $[n]_F(x) \in R[[x]]$ inductively by

$$\begin{aligned} [1]_F(x) &= x \\ [n]_F(x) &= F(x, [n-1]_F(x)). \end{aligned}$$

The Morava K-theories

To define the Morava K-theories, we first need to define the *height* of a formal group law.

Given a formal group law $F(x, y) \in R[[x, y]]$, we define its *n-series* $[n]_F(x) \in R[[x]]$ inductively by

$$\begin{aligned} [1]_F(x) &= x \\ [n]_F(x) &= F(x, [n-1]_F(x)). \end{aligned}$$

This classifies "n-fold addition". From the axioms, we know that $F(x, y) = x + y + \cdots$, and so $[n]_F(x) = nx + \cdots$.

The Morava K-theories

To define the Morava K-theories, we first need to define the *height* of a formal group law.

Given a formal group law $F(x, y) \in R[[x, y]]$, we define its *n-series* $[n]_F(x) \in R[[x]]$ inductively by

$$\begin{aligned} [1]_F(x) &= x \\ [n]_F(x) &= F(x, [n-1]_F(x)). \end{aligned}$$

This classifies "n-fold addition". From the axioms, we know that $F(x, y) = x + y + \cdots$, and so $[n]_F(x) = nx + \cdots$.

If $R = k$ is a field of characteristic p, then the first term of $[p]_F(x)$ vanishes.

The Morava K-theories

To define the Morava K-theories, we first need to define the *height* of a formal group law.

Given a formal group law $F(x,y) \in R[[x,y]]$, we define its *n-series* $[n]_F(x) \in R[[x]]$ inductively by

$$\begin{aligned} [1]_F(x) &= x \\ [n]_F(x) &= F(x, [n-1]_F(x)). \end{aligned}$$

This classifies "n-fold addition". From the axioms, we know that $F(x,y) = x + y + \cdots$, and so $[n]_F(x) = nx + \cdots$.

If $R = k$ is a field of characteristic p, then the first term of $[p]_F(x)$ vanishes. In fact, we'll always have $[p]_F(x) = ux^{p^h} + \cdots$ for $u \in k^\times$ and $h \geq 1$, and this integer h is called the *height* of F.

Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the n^{th} *Honda formal group law*, denoted $H_{n,p}$, with p-series $[p]_{H_{n,p}}(x) = x^{p^n}$.

Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the n^{th} *Honda formal group law*, denoted $H_{n,p}$, with p-series $[p]_{H_{n,p}}(x) = x^{p^n}$.

This is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava K-theory*, denoted $K(n, p)$.

Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the n^{th} *Honda formal group law*, denoted $H_{n,p}$, with p-series $[p]_{H_{n,p}}(x) = x^{p^n}$.

This is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava K-theory*, denoted $K(n, p)$.

Here are a few edge cases.

Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the n^{th} *Honda formal group law*, denoted $H_{n,p}$, with p-series $[p]_{H_{n,p}}(x) = x^{p^n}$.

This is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava K-theory*, denoted $K(n,p)$.

Here are a few edge cases.

- $H_{1,p} = (\widehat{\mathbb{G}}_m)_{\mathbb{F}_p}$, and so $K(1,p) = KU/p$ (mod p complex K-theory).

Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the n^{th} *Honda formal group law*, denoted $H_{n,p}$, with p-series $[p]_{H_{n,p}}(x) = x^{p^n}$.

This is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava K-theory*, denoted $K(n,p)$.

Here are a few edge cases.

- $H_{1,p} = (\widehat{\mathbb{G}}_m)_{\mathbb{F}_p}$, and so $K(1,p) = KU/p$ (mod p complex K-theory).
- $H_{\infty,p} = (\widehat{\mathbb{G}}_a)_{\mathbb{F}_p}$, and so $K(\infty,p) = H\mathbb{F}_p$ (mod p singular cohomology).

Over $\mathbb{F}_p$ itself, for each height $n \in [1, \infty]$ we have the n^{th} *Honda formal group law*, denoted $H_{n,p}$, with p-series $[p]_{H_{n,p}}(x) = x^{p^n}$.

This is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava K-theory*, denoted $K(n, p)$.

Here are a few edge cases.

- $H_{1,p} = (\widehat{\mathbb{G}}_m)_{\mathbb{F}_p}$, and so $K(1, p) = KU/p$ (mod p complex K-theory).
- $H_{\infty,p} = (\widehat{\mathbb{G}}_a)_{\mathbb{F}_p}$, and so $K(\infty, p) = H\mathbb{F}_p$ (mod p singular cohomology).
- Even though there's no $H_{0,p}$, it turns out that we can reasonably define $K(0, p) = H\mathbb{Q}$ (rational singular cohomology) for any prime p.

The Morava K-theories represent essentially all† of the homology theories that have Künneth *isomorphisms* (instead of just a Künneth short exact sequence, or worse).

The Morava K-theories represent essentially all[†] of the homology theories that have Künneth *isomorphisms* (instead of just a Künneth short exact sequence, or worse).

That is, they are essentially all the homology theories whose kernel is not just a *thick* subcategory, but is also a *prime ideal* subcategory.

The Morava K-theories represent essentially all† of the homology theories that have Künneth *isomorphisms* (instead of just a Künneth short exact sequence, or worse).

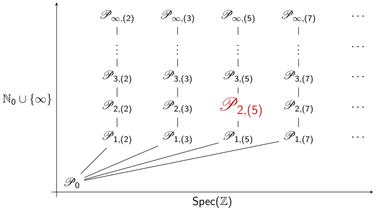
That is, they are essentially all the homology theories whose kernel is not just a *thick* subcategory, but is also a *prime ideal* subcategory.

†The set $\{K(n,p)\}_{n,p}$ plays the same role for ring-valued cohomology theories as the set $\{\mathbb{Q}\} \cup \{\mathbb{F}_p\}_p$ plays for ordinary rings: it is the set of *prime fields*.

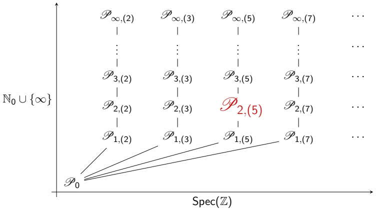
Recall that the points of the space $\text{Spec}(\mathbf{SHC}^{fin})$ are the *thick prime ideal* subcategories of $\mathbf{SHC}^{fin}$.

Recall that the points of the space $\mathrm{Spec}(\mathbf{SHC}^{fin})$ are the *thick prime ideal* subcategories of $\mathbf{SHC}^{fin}$. In fact, the point $\mathscr{P}_{n,(p)}$ is nothing more or less than the kernel of $K(n,p)_*$!

Recall that the points of the space Spec($\mathbf{SHC}^{fin}$) are the *thick prime ideal* subcategories of $\mathbf{SHC}^{fin}$. In fact, the point $\mathscr{P}_{n,(p)}$ is nothing more or less than the kernel of $K(n,p)_*$!



Recall that the points of the space Spec($\mathbf{SHC}^{fin}$) are the *thick prime ideal* subcategories of $\mathbf{SHC}^{fin}$. In fact, the point $\mathscr{P}_{n,(p)}$ is nothing more or less than the kernel of $K(n,p)_*$!



However, as fantastic as the Morava K-theories are for giving us information at the various points of Spec($\mathbf{SHC}^{fin}$), *they do not tell us how to stitch that information back together*.

The Morava E-theories

The Morava E-theories

To define the Morava E-theories, we first need to define a *deformation* of a formal group law.

The Morava E-theories

To define the Morava E-theories, we first need to define a *deformation* of a formal group law.

Let k be a perfect field of characteristic p, let F be a formal group law over k, and let $(A, \mathfrak{m})$ be a complete local ring with projection $A \xrightarrow{\pi} A/\mathfrak{m}$ to its residue field.

The Morava E-theories

To define the Morava E-theories, we first need to define a *deformation* of a formal group law.

Let k be a perfect field of characteristic p, let F be a formal group law over k, and let $(A, \mathfrak{m})$ be a complete local ring with projection $A \xrightarrow{\pi} A/\mathfrak{m}$ to its residue field.

A *deformation* of F from k to A is a formal group law $\overline{F}$ over A and a map $k \xrightarrow{i} A/\mathfrak{m}$ such that $\pi^*\overline{F} = i^*F$ over $A/\mathfrak{m}$.

The Morava E-theories

To define the Morava E-theories, we first need to define a *deformation* of a formal group law.

Let k be a perfect field of characteristic p, let F be a formal group law over k, and let $(A, \mathfrak{m})$ be a complete local ring with projection $A \xrightarrow{\pi} A/\mathfrak{m}$ to its residue field.

A *deformation* of F from k to A is a formal group law $\overline{F}$ over A and a map $k \xrightarrow{i} A/\mathfrak{m}$ such that $\pi^* \overline{F} = i^* F$ over $A/\mathfrak{m}$. In algebraic geometry, the picture looks like this:

[picture]

The Morava E-theories

To define the Morava E-theories, we first need to define a *deformation* of a formal group law.

Let k be a perfect field of characteristic p, let F be a formal group law over k, and let $(A, \mathfrak{m})$ be a complete local ring with projection $A \xrightarrow{\pi} A/\mathfrak{m}$ to its residue field.

A *deformation* of F from k to A is a formal group law $\overline{F}$ over A and a map $k \xrightarrow{i} A/\mathfrak{m}$ such that $\pi^*\overline{F} = i^*F$ over $A/\mathfrak{m}$. In algebraic geometry, the picture looks like this:

[picture]

These collect into $\mathbf{Def}_{F/k}(A)$.

In fact, there is a *universal deformation*, which is a formal group law $\widetilde{F}$ living over the *Lubin–Tate ring* $LT_{F/k}$ such that

$$\begin{array}{rcl} \mathbf{Def}_{F/k}(A) & \simeq & \mathrm{Hom}^{cts}(LT_{F/k}, A) \\ f^*\widetilde{F} & \leftarrow\!\shortmid & f \end{array}$$

(taking only *local* ring homomorphisms).

In fact, there is a *universal deformation*, which is a formal group law $\widetilde{F}$ living over the *Lubin–Tate ring* $LT_{F/k}$ such that

$$\begin{array}{rcl} \mathbf{Def}_{F/k}(A) & \simeq & \mathrm{Hom}^{cts}(LT_{F/k}, A) \\ f^*\widetilde{F} & \leftarrow\!\shortmid & f \end{array}$$

(taking only *local* ring homomorphisms).

[extend picture]

Recall that we have the Honda formal group law $H_{n,p}$ over $\mathbb{F}_p$.

Recall that we have the Honda formal group law $H_{n,p}$ over $\mathbb{F}_p$.

For all $n < \infty$, its universal deformation $\widetilde{H}_{n,p}$ is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava E-theory*, denoted $E_{n,p}$.

Recall that we have the Honda formal group law $H_{n,p}$ over $\mathbb{F}_p$.

For all $n < \infty$, its universal deformation $\widetilde{H}_{n,p}$ is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava E-theory*, denoted $E_{n,p}$. This lives over the ring $LT_{n,p} = LT_{H_{n,p}/\mathbb{F}_p} = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]]$.

Recall that we have the Honda formal group law $H_{n,p}$ over $\mathbb{F}_p$.

For all $n < \infty$, its universal deformation $\widetilde{H}_{n,p}$ is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava E-theory*, denoted $E_{n,p}$. This lives over the ring $LT_{n,p} = LT_{H_{n,p}/\mathbb{F}_p} = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]]$.

Here are a few edge cases.

Recall that we have the Honda formal group law $H_{n,p}$ over $\mathbb{F}_p$.

For all $n < \infty$, its universal deformation $\widetilde{H}_{n,p}$ is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava E-theory*, denoted $E_{n,p}$. This lives over the ring $LT_{n,p} = LT_{H_{n,p}/\mathbb{F}_p} = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]]$.

Here are a few edge cases.

- $LT_{1,p} = \mathbb{Z}_p$ and $\widetilde{H}_{1,p} = (\widehat{\mathbb{G}}_m)_{\mathbb{Z}_p}$, and so $E_{1,p} = KU_p^\wedge$ (p-completed complex K-theory).

Recall that we have the Honda formal group law $H_{n,p}$ over $\mathbb{F}_p$.

For all $n < \infty$, its universal deformation $\widetilde{H}_{n,p}$ is realized in topology by a complex-oriented cohomology theory called the n^{th} *Morava E-theory*, denoted $E_{n,p}$. This lives over the ring $LT_{n,p} = LT_{H_{n,p}/\mathbb{F}_p} = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]]$.

Here are a few edge cases.

- $LT_{1,p} = \mathbb{Z}_p$ and $\widetilde{H}_{1,p} = (\widehat{\mathbb{G}}_m)_{\mathbb{Z}_p}$, and so $E_{1,p} = KU_p^\wedge$ (p-completed complex K-theory).
- Again, even though there's no $H_{0,p}$ and hence no $\widetilde{H}_{0,p}$, we can reasonably define $E_{0,p} = H\mathbb{Q}$ (rational singular cohomology) for any prime p.

There's a sense in which a deformation of H_n can have any height up to n.

There's a sense in which a deformation of H_n can have any height up to n.

This is reflected in topology!

There's a sense in which a deformation of H_n can have any height up to n.

This is reflected in topology!

For any space X, $(E_{n,p})_*X = 0$ if and only if $K(i,p)_*X = 0$ for $i \in [0,n]$.

There's a sense in which a deformation of H_n can have any height up to n.

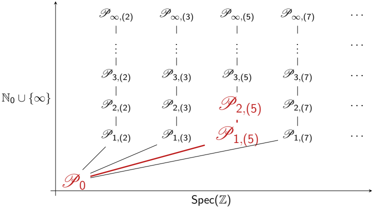
This is reflected in topology!

For any space X, $(E_{n,p})_*X = 0$ if and only if $K(i,p)_*X = 0$ for $i \in [0,n]$. So, the kernel of $E_{n,p}$ is $\{\mathscr{P}_0, \mathscr{P}_{1,(p)}, \ldots, \mathscr{P}_{n,(p)}\}$.

There's a sense in which a deformation of H_n can have any height up to n.

This is reflected in topology!

For any space X, $(E_{n,p})_* X = 0$ if and only if $K(i,p)_* X = 0$ for $i \in [0,n]$. So, the kernel of $E_{n,p}$ is $\{\mathscr{P}_0, \mathscr{P}_{1,(p)}, \ldots, \mathscr{P}_{n,(p)}\}$.



Thus, the Morava E-theories afford us a notion of *chromatic globalization*, i.e. of stitching together information in the chromatic direction.

Thus, the Morava E-theories afford us a notion of *chromatic globalization*, i.e. of stitching together information in the chromatic direction.

But what about *arithmetic* globalization?

3. Topological modular forms

Towards *tmf*

Towards *tmf*

We can see the $E_{n,p}$ as telling us how to globalize in the *chromatic* direction;

Towards *tmf*

We can see the $E_{n,p}$ as telling us how to globalize in the *chromatic* direction; what's much more subtle is the question of how to globalize in the *arithmetic* direction.

Towards *tmf*

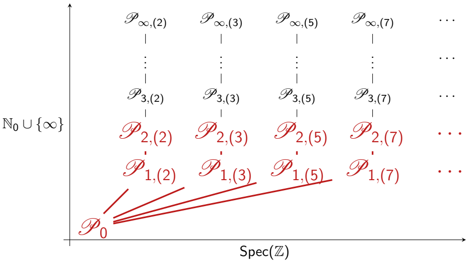
We can see the $E_{n,p}$ as telling us how to globalize in the *chromatic* direction; what's much more subtle is the question of how to globalize in the *arithmetic* direction.

A *global height-n theory* should allow us to recover the $E_{n,p}$ at all primes p.

Towards *tmf*

We can see the $E_{n,p}$ as telling us how to globalize in the *chromatic* direction; what's much more subtle is the question of how to globalize in the *arithmetic* direction.

A *global height-n theory* should allow us to recover the $E_{n,p}$ at all primes p.



So, what are some examples of global height-n theories?

So, what are some examples of global height-n theories?

For all p, $E_{0,p}$ is just $H\mathbb{Q}$, rational cohomology.

So, what are some examples of global height-n theories?

For all p, $E_{0,p}$ is just $H\mathbb{Q}$, rational cohomology. So, obviously we can take $H\mathbb{Q}$ as a global height-0 theory!

So, what are some examples of global height-n theories?

For all p, $E_{0,p}$ is just $H\mathbb{Q}$, rational cohomology. So, obviously we can take $H\mathbb{Q}$ as a global height-0 theory!

Next, $E_{1,p}$ is $KU_p^\wedge$, p-completed complex K-theory.

So, what are some examples of global height-n theories?

For all p, $E_{0,p}$ is just $H\mathbb{Q}$, rational cohomology. So, obviously we can take $H\mathbb{Q}$ as a global height-0 theory!

Next, $E_{1,p}$ is $KU_p^\wedge$, p-completed complex K-theory. What object might allow us to recover the $KU_p^\wedge$ at all primes p?

So, what are some examples of global height-n theories?

For all p, $E_{0,p}$ is just $H\mathbb{Q}$, rational cohomology. So, obviously we can take $H\mathbb{Q}$ as a global height-0 theory!

Next, $E_{1,p}$ is $KU_p^\wedge$, p-completed complex K-theory. What object might allow us to recover the $KU_p^\wedge$ at all primes p?

Well, let's just *not* p-complete the darn thing!

So, what are some examples of global height-n theories?

For all p, $E_{0,p}$ is just $H\mathbb{Q}$, rational cohomology. So, obviously we can take $H\mathbb{Q}$ as a global height-0 theory!

Next, $E_{1,p}$ is $KU_p^\wedge$, p-completed complex K-theory. What object might allow us to recover the $KU_p^\wedge$ at all primes p?

Well, let's just *not* p-complete the darn thing! We can take KU as a global height-1 theory.

In order to go further, let's reimagine this a bit.

In order to go further, let's reimagine this a bit.

We define a *quasicoherent sheaf of cohomology theories* over $X = \text{Spec}(\mathbb{Z})$, which we denote $\mathcal{KU}$:

In order to go further, let's reimagine this a bit.

We define a *quasicoherent sheaf of cohomology theories* over $X = \text{Spec}(\mathbb{Z})$, which we denote $\mathcal{KU}$: we take the global sections to be

$$\mathcal{KU}(X) = KU,$$

In order to go further, let's reimagine this a bit.

We define a *quasicoherent sheaf of cohomology theories* over $X = \mathrm{Spec}(\mathbb{Z})$, which we denote $\mathcal{KU}$: we take the global sections to be

$$\mathcal{KU}(X) = KU,$$

and then by quasicoherence, evaluation on $X_p^\wedge \subset X$ yields

$$\mathcal{KU}(X_p^\wedge) = KU \widehat{\otimes} \mathbb{Z}_p = KU_p^\wedge.$$

(Quasicoherence just means that we have this tensoring-up formula.)

In order to go further, let's reimagine this a bit.

We define a *quasicoherent sheaf of cohomology theories* over $X = \mathrm{Spec}(\mathbb{Z})$, which we denote $\mathcal{KU}$: we take the global sections to be

$$\mathcal{KU}(X) = KU,$$

and then by quasicoherence, evaluation on $X_p^\wedge \subset X$ yields

$$\mathcal{KU}(X_p^\wedge) = KU \widehat{\otimes} \mathbb{Z}_p = KU_p^\wedge .$$

(Quasicoherence just means that we have this tensoring-up formula.)

This corresponds to the sheaf of formal groups over X determined by $\widehat{\mathbb{G}}_m$, which evaluates as $\widehat{\mathbb{G}}_m(X_p^\wedge) = (\widehat{\mathbb{G}}_m)_{\mathbb{Z}_p} = \widetilde{H}_{1,p}$.

So, to get a *global height-2 theory*, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes p.

So, to get a *global height-2 theory*, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes p.

The previous case was easy, since all the $\widetilde{H}_{1,p}$ came from the same formal group $\widehat{\mathbb{G}}_m$.

So, to get a *global height-2 theory*, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes p.

The previous case was easy, since all the $\widetilde{H}_{1,p}$ came from the same formal group $\widehat{\mathbb{G}}_m$. The $\widetilde{H}_{2,p}$ are much more complicated!

So, to get a *global height-2 theory*, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes p.

The previous case was easy, since all the $\widetilde{H}_{1,p}$ came from the same formal group $\widehat{\mathbb{G}}_m$. The $\widetilde{H}_{2,p}$ are much more complicated!

However, we have a clue:

So, to get a *global height-2 theory*, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes p.

The previous case was easy, since all the $\widetilde{H}_{1,p}$ came from the same formal group $\widehat{\mathbb{G}}_m$. The $\widetilde{H}_{2,p}$ are much more complicated!

However, we have a clue: Recall that we can obtain a formal group as the germ of (1-dimensional commutative) algebraic group.

So, to get a *global height-2 theory*, we should look for some object with a sheaf of formal groups which contains as sections the $\widetilde{H}_{2,p}$ for all primes p.

The previous case was easy, since all the $\widetilde{H}_{1,p}$ came from the same formal group $\widehat{\mathbb{G}}_m$. The $\widetilde{H}_{2,p}$ are much more complicated!

However, we have a clue: Recall that we can obtain a formal group as the germ of (1-dimensional commutative) algebraic group.

Besides $\mathbb{G}_a$ and $\mathbb{G}_m$, what other 1-dimensional commutative algebraic groups *are* there, anyways?

As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

Very luckily, some elliptic curves over $\mathbb{F}_p$ have formal group of height 2!

As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

Very luckily, some elliptic curves over $\mathbb{F}_p$ have formal group of height 2! Such elliptic curves are called *supersingular*.

As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

Very luckily, some elliptic curves over $\mathbb{F}_p$ have formal group of height 2! Such elliptic curves are called *supersingular*. (The rest, which have height 1, are called *ordinary*.)

As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

Very luckily, some elliptic curves over $\mathbb{F}_p$ have formal group of height 2! Such elliptic curves are called *supersingular*. (The rest, which have height 1, are called *ordinary*.)

For each prime p, there is a *moduli of deformations* of supersingular elliptic curves, denoted $\mathcal{M}_{ell,p}^{ss}$, and its canonical sheaf of formal groups does indeed contain $\widetilde{H}_{2,p}$.

As it turns out, there is only one other sort of 1-dimensional commutative algebraic group besides $\mathbb{G}_a$ and $\mathbb{G}_m$: the *elliptic curves*.

Very luckily, some elliptic curves over $\mathbb{F}_p$ have formal group of height 2! Such elliptic curves are called *supersingular*. (The rest, which have height 1, are called *ordinary*.)

For each prime p, there is a *moduli of deformations* of supersingular elliptic curves, denoted $\mathcal{M}^{ss}_{ell,p}$, and its canonical sheaf of formal groups does indeed contain $\widetilde{H}_{2,p}$.

So, we might hope to bring these all together and define a sheaf of cohomology theories over $\coprod_p \mathcal{M}^{ss}_{ell,p}$.

Unfortunately, there is a problem: $\coprod_p \mathcal{M}^{ss}_{ell,p}$ is extremely disconnected.

Unfortunately, there is a problem: $\coprod_p \mathcal{M}_{ell,p}^{ss}$ is extremely disconnected.

(By the sheaf condition, evaluating a sheaf on a disjoint union simply returns the product of the values on the individual components.)

Unfortunately, there is a problem: $\coprod_p \mathcal{M}^{ss}_{ell,p}$ is extremely disconnected.

(By the sheaf condition, evaluating a sheaf on a disjoint union simply returns the product of the values on the individual components.)

So, the question becomes: Is there some *connected* object $\mathcal{M}$ admitting an embedding

$$\coprod_p \mathcal{M}^{ss}_{ell,p} \hookrightarrow \mathcal{M}$$

and with a sheaf of formal groups extending that of $\coprod_p \mathcal{M}^{ss}_{ell,p}$?

And the answer is:

And the answer is: Yes!

And the answer is: Yes!

We can use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \mathcal{M}_{ell}$$

into the moduli of *all* elliptic curves.

And the answer is: Yes!

We can use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \mathcal{M}_{ell}$$

into the moduli of *all* elliptic curves. Actually, following our friends in number theory, we instead use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \overline{\mathcal{M}}_{ell}$$

into its *Deligne–Mumford compactification*.

And the answer is: Yes!

We can use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \mathcal{M}_{ell}$$

into the moduli of *all* elliptic curves. Actually, following our friends in number theory, we instead use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \overline{\mathcal{M}}_{ell}$$

into its *Deligne–Mumford compactification*. The moral is:

And the answer is: Yes!

We can use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \mathcal{M}_{ell}$$

into the moduli of *all* elliptic curves. Actually, following our friends in number theory, we instead use the embedding

$$\coprod_p \mathcal{M}_{ell,p}^{ss} \hookrightarrow \overline{\mathcal{M}}_{ell}$$

into its *Deligne–Mumford compactification*. The moral is:

We use the ordinary locus to interpolate between the supersingular neighborhoods.

So what the heck is this sheaf, anyways?

So what the heck is this sheaf, anyways?

Recall that $\overline{\mathcal{M}}_{ell}$ is the moduli of *generalized* elliptic curves (i.e. of smooth elliptic curves and their nodal degenerations).

So what the heck is this sheaf, anyways?

Recall that $\overline{\mathcal{M}}_{ell}$ is the moduli of *generalized* elliptic curves (i.e. of smooth elliptic curves and their nodal degenerations). We endow $\overline{\mathcal{M}}_{ell}$ with a sheaf $\mathscr{O}^{top}$ of ring-valued cohomology theories.

So what the heck is this sheaf, anyways?

Recall that $\overline{\mathcal{M}}_{ell}$ is the moduli of *generalized* elliptic curves (i.e. of smooth elliptic curves and their nodal degenerations). We endow $\overline{\mathcal{M}}_{ell}$ with a sheaf $\mathscr{O}^{top}$ of ring-valued cohomology theories.

If $\mathrm{Spec}(R) \subset \overline{\mathcal{M}}_{ell}$ carries the generalized elliptic curve C over the ring R, then $E = \mathscr{O}^{top}(\mathrm{Spec}(R))$ is a complex-oriented cohomology theory whose formal group law F_E coincides with $\widehat{C}$.

So what the heck is this sheaf, anyways?

Recall that $\overline{\mathcal{M}}_{ell}$ is the moduli of *generalized* elliptic curves (i.e. of smooth elliptic curves and their nodal degenerations). We endow $\overline{\mathcal{M}}_{ell}$ with a sheaf $\mathscr{O}^{top}$ of ring-valued cohomology theories.

If $\text{Spec}(R) \subset \overline{\mathcal{M}}_{ell}$ carries the generalized elliptic curve C over the ring R, then $E = \mathscr{O}^{top}(\text{Spec}(R))$ is a complex-oriented cohomology theory whose formal group law F_E coincides with $\widehat{C}$.

This is called the *elliptic cohomology theory* associated to C.

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$.

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.)

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{\mathcal{M}}_{ell}$.

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{\mathcal{M}}_{ell}$.

A *modular form* of *weight* n is a global section $f \in \omega^{\otimes n}(\overline{\mathcal{M}}_{ell})$.

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{\mathcal{M}}_{ell}$.

A *modular form* of *weight* n is a global section $f \in \omega^{\otimes n}(\overline{\mathcal{M}}_{ell})$. Taken over all n, these assemble into a graded ring

$$MF_* = \bigoplus_{n \in \mathbb{Z}} MF_n = \bigoplus_{n \in \mathbb{Z}} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell})$$

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{\mathcal{M}}_{ell}$.

A *modular form* of *weight* n is a global section $f \in \omega^{\otimes n}(\overline{\mathcal{M}}_{ell})$. Taken over all n, these assemble into a graded ring

$$MF_* = \bigoplus_{n \in \mathbb{Z}} MF_n = \bigoplus_{n \in \mathbb{Z}} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell}) = \bigoplus_{n \geq 0} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell}).$$

(There are no modular forms of negative weight.)

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{\mathcal{M}}_{ell}$.

A *modular form* of *weight* n is a global section $f \in \omega^{\otimes n}(\overline{\mathcal{M}}_{ell})$. Taken over all n, these assemble into a graded ring

$$MF_* = \bigoplus_{n \in \mathbb{Z}} MF_n = \bigoplus_{n \in \mathbb{Z}} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell}) = \bigoplus_{n \geq 0} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell}).$$

(There are no modular forms of negative weight.)

If E is an elliptic cohomology theory associated to C, then $E_{2n} \cong \omega(C)^{\otimes n}$ and $E_{2n+1} = 0$ for all $n \in \mathbb{Z}$.

Any generalized elliptic curve C has a *cotangent space* at the identity, denoted $\omega(C)$. (This is also the module of invariant 1-forms.) These assemble into a line bundle $\omega \downarrow \overline{\mathcal{M}}_{ell}$.

A *modular form* of *weight* n is a global section $f \in \omega^{\otimes n}(\overline{\mathcal{M}}_{ell})$. Taken over all n, these assemble into a graded ring

$$MF_* = \bigoplus_{n \in \mathbb{Z}} MF_n = \bigoplus_{n \in \mathbb{Z}} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell}) = \bigoplus_{n \geq 0} \omega^{\otimes n}(\overline{\mathcal{M}}_{ell}).$$

(There are no modular forms of negative weight.)

If E is an elliptic cohomology theory associated to C, then $E_{2n} \cong \omega(C)^{\otimes n}$ and $E_{2n+1} = 0$ for all $n \in \mathbb{Z}$.

Thus, it is reasonable to call the global sections of our sheaf

$$Tmf = \mathscr{O}^{top}(\overline{\mathcal{M}}_{ell}),$$

the cohomology theory of *topological modular forms*.

What is the coefficient ring Tmf_*?

What is the coefficient ring Tmf_*?

The operations of taking global sections and taking coefficient rings do not commute.

What is the coefficient ring Tmf_*?

The operations of taking global sections and taking coefficient rings do not commute.

Instead, there is a *descent spectral sequence* (essentially a Serre spectral sequence, if you squint hard enough) running

$$H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Rightarrow Tmf_{2t-s}$$

which accounts for their interchange.

Recall that by definition,

$$MF_t = \omega^{\otimes t}(\overline{\mathcal{M}}_{ell}) = H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}).$$

Recall that by definition,

$$MF_t = \omega^{\otimes t}(\overline{\mathcal{M}}_{ell}) = H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}).$$

The theory of spectral sequences tells us that the inclusion

$$H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \hookrightarrow H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Rightarrow Tmf_{2t-s}$$

induces an *edge homomorphism*

$$Tmf_{2*} \to MF_*.$$

Recall that by definition,

$$MF_t = \omega^{\otimes t}(\overline{\mathcal{M}}_{ell}) = H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}).$$

The theory of spectral sequences tells us that the inclusion

$$H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \hookrightarrow H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Rightarrow Tmf_{2t-s}$$

induces an *edge homomorphism*

$$Tmf_{2*} \to MF_*.$$

This ring homomorphism is an isomorphism away from 6 (i.e. its additive kernel and cokernel consist only of 2- and 3-torsion) in positive degrees,

Recall that by definition,

$$MF_t = \omega^{\otimes t}(\overline{\mathcal{M}}_{ell}) = H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}).$$

The theory of spectral sequences tells us that the inclusion

$$H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \hookrightarrow H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Rightarrow Tmf_{2t-s}$$

induces an *edge homomorphism*

$$Tmf_{2*} \to MF_*.$$

This ring homomorphism is an isomorphism away from 6 (i.e. its additive kernel and cokernel consist only of 2- and 3-torsion) in positive degrees, and so we might reasonably call $Tmf_{\geq 0}$ the ring of *derived modular forms*.

Recall that by definition,

$$MF_t = \omega^{\otimes t}(\overline{\mathcal{M}}_{ell}) = H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}).$$

The theory of spectral sequences tells us that the inclusion

$$H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \hookrightarrow H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes t}) \Rightarrow Tmf_{2t-s}$$

induces an *edge homomorphism*

$$Tmf_{2*} \to MF_*.$$

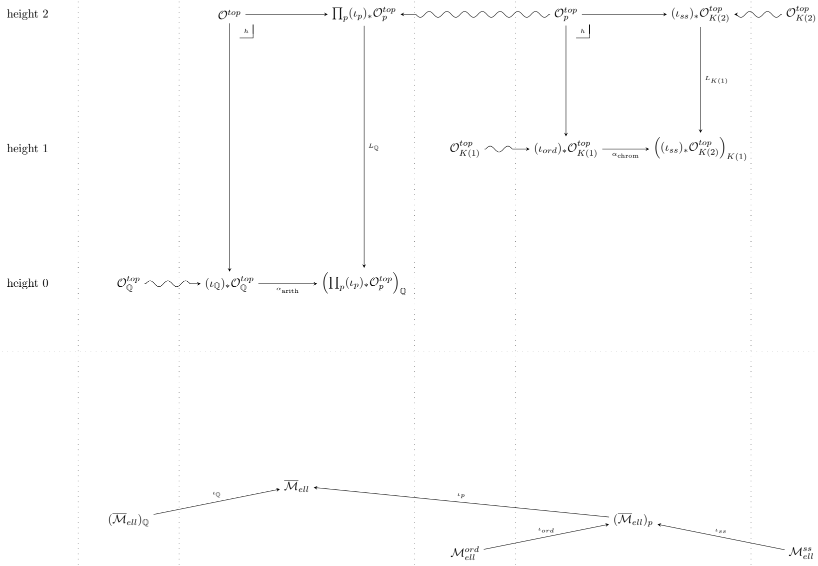
This ring homomorphism is an isomorphism away from 6 (i.e. its additive kernel and cokernel consist only of 2- and 3-torsion) in positive degrees, and so we might reasonably call $Tmf_{\geq 0}$ the ring of *derived modular forms*.

To mimic number theory as closely as possible, we actually usually work with $tmf = \tau_{\geq 0} Tmf$, which is also called *topological modular forms*.

...and how the heck is it constructed?

...and how the heck is it constructed?

Very, very carefully.


height 2
height 1
height 0
$\mathcal{O}^{top}$
$\prod_p (\iota_p)_* \mathcal{O}_p^{top}$
$\mathcal{O}_p^{top}$
$(\iota_{ss})_* \mathcal{O}_{K(2)}^{top}$
$\mathcal{O}_{K(2)}^{top}$
h
$L_{K(1)}$
$L_{\mathbb{Q}}$
$\mathcal{O}_{K(1)}^{top}$
$(\iota_{ord})_* \mathcal{O}_{K(1)}^{top}$
α_{chrom}
$\left((\iota_{ss})_* \mathcal{O}_{K(2)}^{top}\right)_{K(1)}$
$\mathcal{O}_{\mathbb{Q}}^{top}$
$(\iota_{\mathbb{Q}})_* \mathcal{O}_{\mathbb{Q}}^{top}$
α_{arith}
$\left(\prod_p (\iota_p)_* \mathcal{O}_p^{top}\right)_{\mathbb{Q}}$
$\overline{\mathcal{M}}_{ell}$
$(\overline{\mathcal{M}}_{ell})_{\mathbb{Q}}$
$\iota_{\mathbb{Q}}$
ι_p
$(\overline{\mathcal{M}}_{ell})_p$
ι_{ord}
ι_{ss}
$\mathcal{M}_{ell}^{ord}$
$\mathcal{M}_{ell}^{ss}$

Actually, the category of ring-valued cohomology theories is not nearly rigid enough to allow us to glue all of these elliptic cohomology theories into a sheaf.

Actually, the category of ring-valued cohomology theories is not nearly rigid enough to allow us to glue all of these elliptic cohomology theories into a sheaf.

Instead, we use the category of $\mathcal{E}_\infty$-*ring spectra*; these are much more rigid, and give rise to ring-valued cohomology theories with lots of extra structure.

Actually, the category of ring-valued cohomology theories is not nearly rigid enough to allow us to glue all of these elliptic cohomology theories into a sheaf.

Instead, we use the category of $\mathcal{E}_\infty$-*ring spectra*; these are much more rigid, and give rise to ring-valued cohomology theories with lots of extra structure.

$$\begin{array}{ccc} & & \{\mathcal{E}_\infty\text{-ring spectra}\} \\ & \nearrow & \downarrow \\ (\mathbf{Open}(\overline{\mathcal{M}}_{ell}))^{op} & \longrightarrow \left\{\begin{array}{c}\text{ring-valued}\\ \text{cohomology theories}\end{array}\right\} \simeq & \{\text{ring spectra}\} \end{array}$$

We have the presheaf of ring-valued cohomology theories represented by the bottom arrow thanks to the Landweber exact functor theorem.

Actually, the category of ring-valued cohomology theories is not nearly rigid enough to allow us to glue all of these elliptic cohomology theories into a sheaf.

Instead, we use the category of $\mathcal{E}_\infty$-*ring spectra*; these are much more rigid, and give rise to ring-valued cohomology theories with lots of extra structure.

$$\begin{array}{ccc} & & \{\mathcal{E}_\infty\text{-ring spectra}\} \\ & \nearrow & \downarrow \\ (\mathbf{Open}(\overline{\mathcal{M}}_{ell}))^{op} & \longrightarrow \left\{ \begin{array}{c} \text{ring-valued} \\ \text{cohomology theories} \end{array} \right\} \simeq & \{\text{ring spectra}\} \end{array}$$

We have the presheaf of ring-valued cohomology theories represented by the bottom arrow thanks to the Landweber exact functor theorem. We would like to lift this to a presheaf of $\mathcal{E}_\infty$-ring spectra, since there we have a good notion of sheaves and sheafification

The *Goerss–Hopkins–Miller obstruction theory* for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique.

The *Goerss–Hopkins–Miller obstruction theory* for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique. In the immortal words of Lurie:

The *Goerss–Hopkins–Miller obstruction theory* for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique. In the immortal words of Lurie:

"Although it is much harder to write down an $\mathcal{E}_\infty$-ring than a spectrum, it is also much harder to write down a map between $\mathcal{E}_\infty$-rings than a map between spectra.

The *Goerss–Hopkins–Miller obstruction theory* for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique. In the immortal words of Lurie:

"Although it is much harder to write down an $\mathcal{E}_\infty$-ring than a spectrum, it is also much harder to write down a map between $\mathcal{E}_\infty$-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the *wrong* maps between $\mathcal{E}_\infty$-rings and much easier to find the right ones."

The *Goerss–Hopkins–Miller obstruction theory* for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique. In the immortal words of Lurie:

"Although it is much harder to write down an $\mathcal{E}_\infty$-ring than a spectrum, it is also much harder to write down a map between $\mathcal{E}_\infty$-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the *wrong* maps between $\mathcal{E}_\infty$-rings and much easier to find the right ones."

There is also a construction of $\mathscr{O}^{top}$ due to Lurie, which was the original motivation for his theory of *derived algebraic geometry*.

The *Goerss–Hopkins–Miller obstruction theory* for $\mathcal{E}_\infty$-ring spectra guarantees that there is indeed such a lift, and moreover that it is essentially unique. In the immortal words of Lurie:

"Although it is much harder to write down an $\mathcal{E}_\infty$-ring than a spectrum, it is also much harder to write down a map between $\mathcal{E}_\infty$-rings than a map between spectra. The practical effect of this, in our situation, is that it is much harder to write down the *wrong* maps between $\mathcal{E}_\infty$-rings and much easier to find the right ones."

There is also a construction of $\mathscr{O}^{top}$ due to Lurie, which was the original motivation for his theory of *derived algebraic geometry*. This ultimately relies on the Goerss–Hopkins–Miller obstruction theory, too.

Here is a result, modulo some minor loose ends.

Here is a result, modulo some minor loose ends.

Pre-theorem (M-G–Spitzweck)

There is a Goerss–Hopkins–Miller obstruction theory in the setting of motivic homotopy theory.

Here is a result, modulo some minor loose ends.

Pre-theorem (M-G–Spitzweck)

There is a Goerss–Hopkins–Miller obstruction theory in the setting of motivic homotopy theory.

Motivic homotopy theory (a/k/a $\mathbb{A}^1$-*homotopy theory*) is what you get when you mix algebraic geometry with homotopy theory.

Here is a result, modulo some minor loose ends.

Pre-theorem (M-G–Spitzweck)

There is a Goerss–Hopkins–Miller obstruction theory in the setting of motivic homotopy theory.

Motivic homotopy theory (a/k/a $\mathbb{A}^1$*-homotopy theory*) is what you get when you mix algebraic geometry with homotopy theory.

(Schemes represent functors-of-points. So, we just redefine "point" to mean "scheme" and then proceed from there:

Here is a result, modulo some minor loose ends.

Pre-theorem (M-G–Spitzweck)

There is a Goerss–Hopkins–Miller obstruction theory in the setting of motivic homotopy theory.

Motivic homotopy theory (a/k/a $\mathbb{A}^1$-*homotopy theory*) is what you get when you mix algebraic geometry with homotopy theory.

(Schemes represent functors-of-points. So, we just redefine "point" to mean "scheme" and then proceed from there: a simplex is just a fattened-up point, and this suggests the definition for "motivic simplicial complexes".)

This result should eventually yield a motivic version of *tmf*, though there are still a number of substantial hurdles to overcome.

This result should eventually yield a motivic version of *tmf*, though there are still a number of substantial hurdles to overcome.

It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of *algebraic K-theory*, the motivic analog of complex K-theory.

This result should eventually yield a motivic version of *tmf*, though there are still a number of substantial hurdles to overcome.

It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of *algebraic K-theory*, the motivic analog of complex K-theory. This would tell us about the *power operations* on algebraic K-theory.

This result should eventually yield a motivic version of *tmf*, though there are still a number of substantial hurdles to overcome.

It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of *algebraic K-theory*, the motivic analog of complex K-theory. This would tell us about the *power operations* on algebraic K-theory.

Power operations are the "extra structure" present on cohomology theories represented by $\mathcal{E}_\infty$-ring spectra referred to earlier.

This result should eventually yield a motivic version of *tmf*, though there are still a number of substantial hurdles to overcome.

It should also tell us about the space of $\mathcal{E}_\infty$-endomorphisms of *algebraic K-theory*, the motivic analog of complex K-theory. This would tell us about the *power operations* on algebraic K-theory.

Power operations are the "extra structure" present on cohomology theories represented by $\mathcal{E}_\infty$-ring spectra referred to earlier. (This refinement is analogous to enriching ordinary cohomology from a graded group to a graded ring.)

Curiously enough, the above result requires no real algebraic geometry!

Curiously enough, the above result requires no real algebraic geometry! (Everything is wrapped up in hard work that's already been done by motivic homotopy theorists.)

Curiously enough, the above result requires no real algebraic geometry! (Everything is wrapped up in hard work that's already been done by motivic homotopy theorists.)

This suggests the following generalization.

Curiously enough, the above result requires no real algebraic geometry! (Everything is wrapped up in hard work that's already been done by motivic homotopy theorists.)

This suggests the following generalization.

Work in progress (M-G)

There is a Goerss–Hopkins–Miller obstruction theory in the setting of ∞*-categories.*

An ∞-categorical obstruction theory would give not just a motivic obstruction theory as a corollary, but also e.g. an obstruction theory for *equivariant* stable homotopy theory.

An ∞-categorical obstruction theory would give not just a motivic obstruction theory as a corollary, but also e.g. an obstruction theory for *equivariant* stable homotopy theory.

It would also re-prove the original obstruction theory in a much cleaner and more streamlined way, although of course it would still rely on the very heavy machinery of ∞-categories.

An ∞-categorical obstruction theory would give not just a motivic obstruction theory as a corollary, but also e.g. an obstruction theory for *equivariant* stable homotopy theory.

It would also re-prove the original obstruction theory in a much cleaner and more streamlined way, although of course it would still rely on the very heavy machinery of ∞-categories.

(The original obstruction theory is the result of six hefty papers, together over 500 pages, which use extremely technical results in the theory of *model categories*.

An ∞-categorical obstruction theory would give not just a motivic obstruction theory as a corollary, but also e.g. an obstruction theory for *equivariant* stable homotopy theory.

It would also re-prove the original obstruction theory in a much cleaner and more streamlined way, although of course it would still rely on the very heavy machinery of ∞-categories.

(The original obstruction theory is the result of six hefty papers, together over 500 pages, which use extremely technical results in the theory of *model categories*. An ∞-category is to a model category as a manifold is to an atlas.)

4. Fun with *tmf*!

The Witten genus and the String orientation

The Witten genus and the String orientation

Using arguments from physics, Witten defined a *genus* for String manifolds, which associates to each String manifold a modular form.

The Witten genus and the String orientation

Using arguments from physics, Witten defined a *genus* for String manifolds, which associates to each String manifold a modular form.

What is a genus?

The Witten genus and the String orientation

Using arguments from physics, Witten defined a *genus* for String manifolds, which associates to each String manifold a modular form.

What is a genus?

Given some structure group G (e.g. unoriented, oriented, Spin, String, etc.), the G *bordism ring* is the graded ring Ω_*^G whose elements are cobordism classes of G manifolds, with addition given by disjoint union and multiplication given by Cartesian product.

The Witten genus and the String orientation

Using arguments from physics, Witten defined a *genus* for String manifolds, which associates to each String manifold a modular form.

What is a genus?

Given some structure group G (e.g. unoriented, oriented, Spin, String, etc.), the *G bordism ring* is the graded ring Ω_*^G whose elements are cobordism classes of G manifolds, with addition given by disjoint union and multiplication given by Cartesian product.

Then, a *genus* is just a homomorphism $\Omega_*^G \to R_*$ of graded rings.

The Witten genus and the String orientation

Using arguments from physics, Witten defined a *genus* for String manifolds, which associates to each String manifold a modular form.

What is a genus?

Given some structure group G (e.g. unoriented, oriented, Spin, String, etc.), the *G bordism ring* is the graded ring Ω_*^G whose elements are cobordism classes of G manifolds, with addition given by disjoint union and multiplication given by Cartesian product.

Then, a *genus* is just a homomorphism $\Omega_*^G \to R_*$ of graded rings. So, the Witten genus is a homomorphism

$$\Omega_*^{\mathrm{String}} \to MF_*.$$

Any structure group G determines a homology theory MG, called *G bordism homology*; this is defined just like singular homology, but using G-manifolds instead of simplicial complexes.

Any structure group G determines a homology theory MG, called *G bordism homology*; this is defined just like singular homology, but using G-manifolds instead of simplicial complexes.

Of course, $MG_* = \Omega_*^G$.

Any structure group G determines a homology theory MG, called *G bordism homology*; this is defined just like singular homology, but using G-manifolds instead of simplicial complexes.

Of course, $MG_* = \Omega_*^G$.

So, we might hope for a *topological* version of the Witten genus...but what should be its target?

Any structure group G determines a homology theory MG, called *G bordism homology*; this is defined just like singular homology, but using G-manifolds instead of simplicial complexes.

Of course, $MG_* = \Omega_*^G$.

So, we might hope for a *topological* version of the Witten genus...but what should be its target?

Its target should be *tmf*!

Any structure group G determines a homology theory MG, called *G bordism homology*; this is defined just like singular homology, but using G-manifolds instead of simplicial complexes.

Of course, $MG_* = \Omega_*^G$.

So, we might hope for a *topological* version of the Witten genus...but what should be its target?

Its target should be *tmf*!

Indeed, Ando–Hopkins–Rezk–Strickland construct the *σ-orientation*

$$M\text{String} \to \mathit{tmf}.$$

The σ-orientation $M\text{String} \to \mathit{tmf}$ is actually part of a much bigger story.

The σ-orientation $M\mathrm{String} \to \mathit{tmf}$ is actually part of a much bigger story.

The *sphere spectrum*, denoted $\mathbb{S}$, is the initial ring spectrum (just as $\mathbb{Z}$ is the initial ring). But $\mathbb{S}$ also gives the bordism homology theory for *framed* manifolds.

The σ-orientation $M\text{String} \to \mathit{tmf}$ is actually part of a much bigger story.

The *sphere spectrum*, denoted $\mathbb{S}$, is the initial ring spectrum (just as $\mathbb{Z}$ is the initial ring). But $\mathbb{S}$ also gives the bordism homology theory for *framed* manifolds. The σ-orientation is a factorization of the unit map

$$\begin{array}{ccc} \mathbb{S} = M\text{Framed} & \longrightarrow & M\text{String} \\ & \searrow & \downarrow \\ & & \mathit{tmf} \end{array}$$

through the natural "forgetful" map from Framed bordism to String bordism.

The σ-orientation $M\mathrm{String} \to \mathit{tmf}$ is actually part of a much bigger story.

The *sphere spectrum*, denoted $\mathbb{S}$, is the initial ring spectrum (just as $\mathbb{Z}$ is the initial ring). But $\mathbb{S}$ also gives the bordism homology theory for *framed* manifolds. The σ-orientation is a factorization of the unit map

$$\begin{array}{ccc} \mathbb{S} = M\mathrm{Framed} & \longrightarrow & M\mathrm{String} \\ & \searrow & \downarrow \\ & & \mathit{tmf} \end{array}$$

through the natural "forgetful" map from Framed bordism to String bordism. (A Framed structure gives a String structure, just like a Spin structure gives an orientation.)

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

$$\begin{array}{ccccccccccc} \mathbb{S} = M\mathrm{Framed} & \xrightarrow{\lim} & \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\ & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & & & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2 \end{array}$$

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

$$\begin{array}{ccccccccccc}
\mathbb{S} = M\mathrm{Framed} & \xrightarrow{\lim} & \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \mathit{tmf} & & \mathit{ko} & & H\mathbb{Z} & & H\mathbb{F}_2
\end{array}$$

These are *topological* versions of various genera. From right to left, they represent:

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

$$\begin{array}{ccccccccc}
\mathbb{S} = M\text{Framed} \xrightarrow{\text{lim}} \cdots & \longrightarrow & M\text{String} & \longrightarrow & M\text{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2
\end{array}$$

These are *topological* versions of various genera. From right to left, they represent:

- the *Stiefel–Whitney* classes;

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

$$\begin{array}{ccccccccccc} \mathbb{S} = M\mathrm{Framed} & \xrightarrow{\lim} & \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\ & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & & & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2 \end{array}$$

These are *topological* versions of various genera. From right to left, they represent:

- the *Stiefel–Whitney* classes;
- the *Pontrjagin* classes;

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

$$\begin{array}{ccccccccc} \mathbb{S} = M\mathrm{Framed} \xrightarrow{\lim} \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2 \end{array}$$

These are *topological* versions of various genera. From right to left, they represent:

- the *Stiefel–Whitney* classes;
- the *Pontrjagin* classes;
- the *ko-Pontrjagin* classes of Atiyah–Bott–Shapiro;

This fits into a whole tower of factorizations of unit maps through various bordism homology theories.

$$\begin{array}{ccccccccccc}
\mathbb{S} = M\mathrm{Framed} & \xrightarrow{\lim} & \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \mathit{tmf} & & \mathit{ko} & & H\mathbb{Z} & & H\mathbb{F}_2
\end{array}$$

These are *topological* versions of various genera. From right to left, they represent:

- the *Stiefel–Whitney* classes;
- the *Pontrjagin* classes;
- the *ko-Pontrjagin* classes of Atiyah–Bott–Shapiro;
- the *σ-orientation* of Ando–Hopkins–Rezk–Strickland.

$$\begin{array}{ccccccccccc}
\mathbb{S} = M\mathrm{Framed} & \xrightarrow{\lim} & \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2
\end{array}$$

What does this tell us when we pass to coefficient rings?

$$
\begin{array}{ccccccccc}
\mathbb{S} = M\text{Framed} \xrightarrow{\lim} \cdots & \longrightarrow & M\text{String} & \longrightarrow & M\text{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2
\end{array}
$$

What does this tell us when we pass to coefficient rings?

It is a classical result that a Spin-manifold is Spin-nullcobordant if (and only if) its *ko*-Pontrjagin and Stiefel–Whitney classes vanish.

$$\begin{array}{ccccccccccc}
\mathbb{S} = M\mathrm{Framed} & \xrightarrow{\lim} & \cdots & \longrightarrow & M\mathrm{String} & \longrightarrow & M\mathrm{Spin} & \longrightarrow & MSO & \longrightarrow & MO \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & tmf & & ko & & H\mathbb{Z} & & H\mathbb{F}_2
\end{array}$$

What does this tell us when we pass to coefficient rings?

It is a classical result that a Spin-manifold is Spin-nullcobordant if (and only if) its *ko*-Pontrjagin and Stiefel–Whitney classes vanish.

Thus, it is natural to hope that *tmf*-characteristic classes allow us to completely detect String-cobordism.

Transchromatic detection

Transchromatic detection

Let's work at a fixed prime p.

Transchromatic detection

Let's work at a fixed prime p.

Recall that elliptic curves can either have height 1, in which case they are called *ordinary*, or height 2, in which case they are called *supersingular*.

Transchromatic detection

Let's work at a fixed prime p.

Recall that elliptic curves can either have height 1, in which case they are called *ordinary*, or height 2, in which case they are called *supersingular*.

Over the moduli $\mathcal{M}^{ord}_{ell,p}$ of *ordinary* elliptic curves over p-complete rings, there is a covering space

$$\mathcal{M}^{ord}_{ell,p}(p^\infty) \downarrow \mathcal{M}^{ord}_{ell,p},$$

which is associated to Katz's ring V of *p-adic modular forms*.

We can think of the covering space

$$\mathcal{M}^{ord}_{ell,p}(p^\infty) \downarrow \mathcal{M}^{ord}_{ell,p}$$

as the *frame bundle*;

We can think of the covering space

$$\mathcal{M}_{ell,p}^{ord}(p^\infty) \downarrow \mathcal{M}_{ell,p}^{ord}$$

as the *frame bundle*; it is the moduli of ordinary elliptic curves C equipped with a *trivialization*, i.e. a chosen isomorphism $\widehat{C} \cong \widehat{\mathbb{G}}_m$.

We can think of the covering space

$$\mathcal{M}^{ord}_{ell,p}(p^\infty) \downarrow \mathcal{M}^{ord}_{ell,p}$$

as the *frame bundle*; it is the moduli of ordinary elliptic curves C equipped with a *trivialization*, i.e. a chosen isomorphism $\widehat{C} \cong \widehat{\mathbb{G}}_m$.

So, the fiber over a point is a copy of $\mathrm{Aut}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times$.

We can think of the covering space

$$\mathcal{M}^{ord}_{ell,p}(p^\infty) \downarrow \mathcal{M}^{ord}_{ell,p}$$

as the *frame bundle*; it is the moduli of ordinary elliptic curves C equipped with a *trivialization*, i.e. a chosen isomorphism $\widehat{C} \cong \widehat{\mathbb{G}}_m$.

So, the fiber over a point is a copy of $\mathrm{Aut}_{\mathbb{Z}_p}(\widehat{\mathbb{G}}_m) \cong \mathbb{Z}_p^\times$.

We should think of this as the group of *deck transformations*.

There is a result from number theory called *Igusa's theorem*, which roughly says two things:

There is a result from number theory called *Igusa's theorem*, which roughly says two things:

- The covering space $\mathcal{M}_{ell,p}^{ord}(p^\infty) \downarrow \mathcal{M}_{ell,p}^{ord}$ is *connected*.

There is a result from number theory called *Igusa's theorem*, which roughly says two things:

- The covering space $\mathcal{M}^{ord}_{ell,p}(p^\infty) \downarrow \mathcal{M}^{ord}_{ell,p}$ is *connected*.
- In any (punctured) neighborhood of a *supersingular* point, i.e. a point in the complement of

$$\mathcal{M}^{ord}_{ell,p} \subset \mathcal{M}_{ell,p},$$

the covering space remains connected.

This is very tantalizing.

This is very tantalizing.

In broad terms, it's somehow saying that *despite living at height 1, the ring of p-adic modular forms "knows about" the points of height 2.*

This is very tantalizing.

In broad terms, it's somehow saying that *despite living at height 1, the ring of p-adic modular forms "knows about" the points of height 2.*

(Connected covering spaces correspond to quotient groups of the fundamental group.

This is very tantalizing.

In broad terms, it's somehow saying that *despite living at height 1, the ring of p-adic modular forms "knows about" the points of height 2.*

(Connected covering spaces correspond to quotient groups of the fundamental group. So, this covering space "sees the missing points".)

How can we interpret this result in topology?

The study of ramified coverings in arithmetic geometry goes by the name of *class field theory*, which gives results analogous to the Riemann–Hurwitz theorem for ramified coverings of Riemann surfaces.

The study of ramified coverings in arithmetic geometry goes by the name of *class field theory*, which gives results analogous to the Riemann–Hurwitz theorem for ramified coverings of Riemann surfaces.

The ring $\mathbb{Z}$ has no connected unramified extensions: every nontrivial extension ramifies at some prime.

The study of ramified coverings in arithmetic geometry goes by the name of *class field theory*, which gives results analogous to the Riemann–Hurwitz theorem for ramified coverings of Riemann surfaces.

The ring $\mathbb{Z}$ has no connected unramified extensions: every nontrivial extension ramifies at some prime.

(For example, $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$ ramifies at $(2) \in \mathrm{Spec}(\mathbb{Z})$, but defines a unramified extension when we localize away from (2), i.e. if we restrict to $\mathrm{Spec}(\mathbb{Z}[2^{-1}]) \subset \mathrm{Spec}(\mathbb{Z})$.

The study of ramified coverings in arithmetic geometry goes by the name of *class field theory*, which gives results analogous to the Riemann–Hurwitz theorem for ramified coverings of Riemann surfaces.

The ring $\mathbb{Z}$ has no connected unramified extensions: every nontrivial extension ramifies at some prime.

(For example, $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$ ramifies at $(2) \in \text{Spec}(\mathbb{Z})$, but defines a unramified extension when we localize away from (2), i.e. if we restrict to $\text{Spec}(\mathbb{Z}[2^{-1}]) \subset \text{Spec}(\mathbb{Z})$. And of course, by the time we pass all the way to $\text{Spec}(\mathbb{Q}) \subset \text{Spec}(\mathbb{Z})$, there are tons of extensions!)

The study of ramified coverings in arithmetic geometry goes by the name of *class field theory*, which gives results analogous to the Riemann–Hurwitz theorem for ramified coverings of Riemann surfaces.

The ring $\mathbb{Z}$ has no connected unramified extensions: every nontrivial extension ramifies at some prime.

(For example, $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$ ramifies at $(2) \in \mathrm{Spec}(\mathbb{Z})$, but defines a unramified extension when we localize away from (2), i.e. if we restrict to $\mathrm{Spec}(\mathbb{Z}[2^{-1}]) \subset \mathrm{Spec}(\mathbb{Z})$. And of course, by the time we pass all the way to $\mathrm{Spec}(\mathbb{Q}) \subset \mathrm{Spec}(\mathbb{Z})$, there are tons of extensions!)

For this reason, we say that $\mathbb{Z}$ is *separably closed*: it has no nontrivial connected finite Galois extensions.

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

However, when we take its *localization* $L_{K(n,p)}\mathbb{S}$ with respect to the Morava K-theory $K(n,p)$, it ceases to be separably closed;

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

However, when we take its *localization* $L_{K(n,p)}\mathbb{S}$ with respect to the Morava K-theory $K(n,p)$, it ceases to be separably closed; in fact, its maximal unramified extension is essentially the Morava E-theory $E_{n,p}$ (at least for $p > 2$, but probably for $p = 2$ too).

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

However, when we take its *localization* $L_{K(n,p)}\mathbb{S}$ with respect to the Morava K-theory $K(n,p)$, it ceases to be separably closed; in fact, its maximal unramified extension is essentially the Morava E-theory $E_{n,p}$ (at least for $p > 2$, but probably for $p = 2$ too).

Since $\mathscr{O}^{top}(\mathcal{M}^{ss}_{ell,p})$ is essentially $E_{2,p}$ and since $K(1,p)$-localization models restriction to the ordinary locus, Igusa's theorem suggests the following.

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

However, when we take its *localization* $L_{K(n,p)}\mathbb{S}$ with respect to the Morava K-theory $K(n,p)$, it ceases to be separably closed; in fact, its maximal unramified extension is essentially the Morava E-theory $E_{n,p}$ (at least for $p > 2$, but probably for $p = 2$ too).

Since $\mathscr{O}^{top}(\mathcal{M}_{ell,p}^{ss})$ is essentially $E_{2,p}$ and since $K(1,p)$-localization models restriction to the ordinary locus, Igusa's theorem suggests the following.

Conjecture

- *The covering space of $L_{K(1,p)}tmf$ coming from the ring V is a connected Galois extension.*

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

However, when we take its *localization* $L_{K(n,p)}\mathbb{S}$ with respect to the Morava K-theory $K(n,p)$, it ceases to be separably closed; in fact, its maximal unramified extension is essentially the Morava E-theory $E_{n,p}$ (at least for $p > 2$, but probably for $p = 2$ too).

Since $\mathscr{O}^{top}(\mathcal{M}^{ss}_{ell,p})$ is essentially $E_{2,p}$ and since $K(1,p)$-localization models restriction to the ordinary locus, Igusa's theorem suggests the following.

Conjecture

- *The covering space of $L_{K(1,p)}tmf$ coming from the ring V is a connected Galois extension.*
- *The restriction of this covering space to $L_{K(1,p)}E_{2,p}$ is also a connected Galois extension.*

As a ring spectrum, the sphere spectrum $\mathbb{S}$ is also separably closed.

However, when we take its *localization* $L_{K(n,p)}\mathbb{S}$ with respect to the Morava K-theory $K(n,p)$, it ceases to be separably closed; in fact, its maximal unramified extension is essentially the Morava E-theory $E_{n,p}$ (at least for $p > 2$, but probably for $p = 2$ too).

Since $\mathscr{O}^{top}(\mathcal{M}^{ss}_{ell,p})$ is essentially $E_{2,p}$ and since $K(1,p)$-localization models restriction to the ordinary locus, Igusa's theorem suggests the following.

Conjecture

- *The covering space of $L_{K(1,p)}tmf$ coming from the ring V is a connected Galois extension.*
- *The restriction of this covering space to $L_{K(1,p)}E_{2,p}$ is also a connected Galois extension. (In particular, $E_{2,p}$ is no longer separably closed after $K(1,p)$-localization.)*

Thank you!

Thank you!

Further reading, in order of appearance.

- M-G, *An introduction to spectra*.[1]
- Peterson, *The geometry of formal varieties in algebraic topology I* and *II*.[2]
- M-G, *Dieudonné modules and the classification of formal groups*.[1]
- Hopkins, *Complex oriented cohomology theories and the language of stacks* (a/k/a *COCTALOS*).[3]
- Lurie, *A survey of elliptic cohomology*.[3]
- M-G, *What are $\mathcal{E}_\infty$-rings?*[4]
- M-G, *Model categories for algebraists, or: What's* really *going on with injective and projective resolutions, anyways?*[1]
- Katz, *p-adic L-functions via moduli of elliptic curves*.

[1] http://math.berkeley.edu/∼aaron/writing/
[2] http://math.berkeley.edu/∼aaron/xkcd/fall2010.html
[3] googleable
[4] math.stackexchange answer; googleable via the string "what are e-infty rings"