Geometric Models for Noncommutative Algebras

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Contents

Preface

Noncommutative geometry is the study of noncommutative algebras *as if* they were algebras of functions on spaces, like the commutative algebras associated to affine algebraic varieties, differentiable manifolds, topological spaces, and measure spaces. In this book, we discuss several types of geometric objects (in the usual sense of sets with structure) which are closely related to noncommutative algebras.

Central to the discussion are symplectic and Poisson manifolds, which arise when noncommutative algebras are obtained by deforming commutative algebras. We also make a detailed study of groupoids, whose role in noncommutative geometry has been stressed by Connes, as well as of Lie algebroids, the infinitesimal approximations to differentiable groupoids.

These notes are based on a topics course, "Geometric Models for Noncommutative Algebras," which one of us (A.W.) taught at Berkeley in the Spring of 1997.

We would like to express our appreciation to Kevin Hartshorn for his participation in the early stages of the project – producing typed notes for many of the lectures. Henrique Bursztyn, who read preliminary versions of the notes, has provided us with innumerable suggestions of great value. We are also indebted to Johannes Huebschmann, Kirill Mackenzie, Daniel Markiewicz, Elisa Prato and Olga Radko for several useful commentaries or references.

Finally, we would like to dedicate these notes to the memory of four friends and colleagues who, sadly, passed away in 1998: Moshé Flato, K. Guruprasad, André Lichnerowicz, and Stanisław Zakrzewski.

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Introduction

We will emphasize an approach to algebra and geometry based on a *metaphor* (see Lakoff and Nuñez [100]):

An algebra (over $\mathbb{R}$ or $\mathbb{C}$) is the set of ($\mathbb{R}$- or $\mathbb{C}$-valued) functions on a space.

Strictly speaking, this statement only holds for commutative algebras. We would like to pretend that this statement still describes noncommutative algebras.

Furthermore, different restrictions on the functions reveal different structures on the space. Examples of distinct algebras of functions which can be associated to a space are:

- polynomial functions,
- real analytic functions,
- smooth functions,
- C^k, or just continuous (C^0) functions,
- L^∞, or the set of bounded, measurable functions modulo the set of functions vanishing outside a set of measure 0.

So we can actually say,

An algebra (over $\mathbb{R}$ or $\mathbb{C}$) is the set of good ($\mathbb{R}$- or $\mathbb{C}$-valued) functions on a space with structure.

Reciprocally, we would like to be able to recover the space with structure from the given algebra. In algebraic geometry that is achieved by considering homomorphisms from the algebra to a field or integral domain.

Examples.

1. Take the algebra $\mathbb{C}[x]$ of complex polynomials in one complex variable. All homomorphisms from $\mathbb{C}[x]$ to $\mathbb{C}$ are given by evaluation at a complex number. We recover $\mathbb{C}$ as the space of homomorphisms.

2. Take the quotient algebra of $\mathbb{C}[x]$ by the ideal generated by x^{k+1}

$$\mathbb{C}[x] \Big/ \langle x^{k+1} \rangle \;=\; \{a_0 + a_1 x + \ldots + a_k x^k \mid a_i \in \mathbb{C}\} \ .$$

   The coefficients $a_0, \ldots, a_k$ may be thought of as values of a complex-valued function plus its first, second, ..., kth derivatives at the origin. The corresponding "space" is the so-called **kth infinitesimal neighborhood** of the point 0. Each of these "spaces" has just one point: evaluation at 0. The limit as k gets large is the space of power series in x.

3. The algebra $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n variables can be interpreted as the algebra $\mathrm{Pol}(V)$ of "good" (*i.e.* polynomial) functions on an n-dimensional complex vector space V for which $(x_1, \ldots, x_n)$ is a dual basis. If we denote the tensor algebra of the dual vector space V^* by

$$\mathcal{T}(V^*) = \mathbb{C} \oplus V^* \oplus (V^* \otimes V^*) \oplus \ldots \oplus (V^*)^{\otimes k} \oplus \ldots \ ,$$

where $(V^*)^{\otimes k}$ is spanned by $\{x_{i_1} \otimes \ldots \otimes x_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$, then we realize the symmetric algebra $\mathcal{S}(V^*) = \mathrm{Pol}(V)$ as

$$\mathcal{S}(V^*) = \mathcal{T}(V^*)/\mathcal{C} \ ,$$

where $\mathcal{C}$ is the ideal generated by $\{\alpha \otimes \beta - \beta \otimes \alpha \mid \alpha, \beta \in V^*\}$.

There are several ways to recover V and its structure from the algebra $\mathrm{Pol}(V)$:

- Linear homomorphisms from $\mathrm{Pol}(V)$ to $\mathbb{C}$ correspond to points of V. We thus recover the set V.
- Algebra endomorphisms of $\mathrm{Pol}(V)$ correspond to polynomial endomorphisms of V: An algebra endomorphism

$$f : \mathrm{Pol}(V) \longrightarrow \mathrm{Pol}(V)$$

is determined by the $f(x_1), \ldots, f(x_n))$. Since $\mathrm{Pol}(V)$ is freely generated by the x_i's, we can choose *any* $f(x_i) \in \mathrm{Pol}(V)$. For example, if $n = 2$, f could be defined by:

$$\begin{array}{ccc} x_1 & \longmapsto & x_1 \\ x_2 & \longmapsto & x_2 + x_1^2 \end{array}$$

which would even be invertible. We are thus recovering a polynomial structure in V.

- Graded algebra automorphisms of $\mathrm{Pol}(V)$ correspond to linear isomorphisms of V: As a *graded* algebra

$$\mathrm{Pol}(V) = \bigoplus_{k=0}^{\infty} \mathrm{Pol}^k(V) \ ,$$

where $\mathrm{Pol}^k(V)$ is the set of homogeneous polynomials of degree k, *i.e.* symmetric tensors in $(V^*)^{\otimes k}$. A graded automorphism takes each x_i to an element of degree one, that is, a linear homogeneous expression in the x_i's. Hence, by using the graded algebra structure of $\mathrm{Pol}(V)$, we obtain a linear structure in V.

4. For a noncommutative structure, let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and define

$$\Lambda^\bullet(V^*) = \mathcal{T}(V^*)/\mathcal{A} \ ,$$

where $\mathcal{A}$ is the ideal generated by $\{\alpha \otimes \beta + \beta \otimes \alpha \mid \alpha, \beta \in V^*\}$. We can view this as a graded algebra,

$$\Lambda^\bullet(V^*) = \bigoplus_{k=0}^{\infty} \Lambda^k(V^*) \ ,$$

whose automorphisms give us the linear structure on V. Therefore, as a graded algebra, $\Lambda^\bullet(V^*)$ still "represents" the vector space structure in V.

The algebra $\Lambda^\bullet(V^*)$ is *not* commutative, but is instead **super-commutative**, *i.e.* for elements $a \in \Lambda^k(V^*), b \in \Lambda^\ell(V^*)$, we have

$$ab = (-1)^{k\ell} ba \ .$$

Super-commutativity is associated to a $\mathbb{Z}_2$-grading:[1]

$$\Lambda^\bullet(V^*) \quad = \quad \Lambda^{[0]}(V^*) \oplus \Lambda^{[1]}(V^*) \ ,$$

where

$$\Lambda^{[0]}(V^*) \quad = \Lambda^{\mathrm{even}}(V^*) \quad := \bigoplus_{k \text{ even}} \Lambda^k(V^*) \ , \qquad \text{and}$$

$$\Lambda^{[1]}(V^*) \quad = \Lambda^{\mathrm{odd}}(V^*) \quad := \bigoplus_{k \text{ odd}} \Lambda^k(V^*) \ .$$

Therefore, V is not just a vector space, but is called an **odd superspace**; "odd" because all nonzero vectors in V have odd($= 1$) degree. The $\mathbb{Z}_2$-grading allows for more automorphisms, as opposed to the $\mathbb{Z}$-grading. For instance,

$$\begin{array}{lcl} x_1 & \longmapsto & x_1 \\ x_2 & \longmapsto & x_2 + x_1x_2x_3 \\ x_3 & \longmapsto & x_3 \end{array}$$

is legal; this preserves the relations since both objects and images anticommute. Although there is more flexibility, we are still not completely free to map generators, since we need to preserve the $\mathbb{Z}_2$-grading. Homomorphisms of the $\mathbb{Z}_2$-graded algebra $\Lambda^\bullet(V^*)$ correspond to "functions" on the (odd) superspace V. We may view the construction above as a definition: a **superspace** is an object on which the functions form a supercommutative $\mathbb{Z}_2$-graded algebra. Repeated use should convince one of the value of this type of terminology!

5. The algebra $\Omega^\bullet(M)$ of differential forms on a manifold M can be regarded as a $\mathbb{Z}_2$-graded algebra by

$$\Omega^\bullet(M) = \Omega^{\mathrm{even}}(M) \oplus \Omega^{\mathrm{odd}}(M) \ .$$

We may thus think of forms on M as functions on a superspace. Locally, the tangent bundle TM has coordinates $\{x_i\}$ and $\{dx_i\}$, where each x_i commutes with everything and the dx_i anticommute with each other. (The coordinates $\{dx_i\}$ measure the components of tangent vectors.) In this way, $\Omega^\bullet(M)$ is the algebra of functions on the **odd tangent bundle** $\overset{\circ}{T}M$; the $\circ$ indicates that here we regard the fibers of TM as odd superspaces.

The exterior derivative

$$d : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

has the property that for $f, g \in \Omega^\bullet(M)$,

$$d(fg) = (df)g + (-1)^{\deg f} f(dg) \ .$$

Hence, d is a derivation of a superalgebra. It exchanges the subspaces of even and odd degree. We call d an **odd vector field** on $\overset{\circ}{T}M$.

6. Consider the algebra of complex valued functions on a "phase space" $\mathbb{R}^2$, with coordinates (q, p) interpreted as position and momentum for a one-dimensional physical system. We wish to impose the standard equation from quantum mechanics

$$qp - pq = i\hbar \ ,$$

[1]The term "super" is generally used in connection with $\mathbb{Z}_2$-gradings.

which encodes the uncertainty principle. In order to formalize this condition, we take the algebra freely generated by q and p modulo the ideal generated by $qp - pq - i\hbar$. As $\hbar$ approaches 0, we recover the commutative algebra $\mathrm{Pol}(\mathbb{R}^2)$.

Studying examples like this naturally leads us toward the **universal enveloping algebra** of a Lie algebra (here the Lie algebra is the Heisenberg algebra, where $\hbar$ is considered as a variable like q and p), and towards **symplectic geometry** (here we concentrate on the phase space with coordinates q and p).

$\diamondsuit$

Each of these latter aspects will lead us into the study of **Poisson algebras**, and the interplay between Poisson geometry and noncommutative algebras, in particular, connections with representation theory and operator algebras.

In these notes we will be also looking at **groupoids**, Lie groupoids and groupoid algebras. Briefly, a groupoid is similar to a group, but we can only multiply certain pairs of elements. One can think of a groupoid as a category (possibly with more than one object) where all morphisms are invertible, whereas a group is a category with only one object such that all morphisms have inverses. Lie algebroids are the infinitesimal counterparts of Lie groupoids, and are very close to Poisson and symplectic geometry.

Finally, we will discuss Fedosov's work in deformation quantization of arbitrary symplectic manifolds.

All of these topics give nice geometric models for noncommutative algebras!

Of course, we could go on, but we had to stop somewhere. In particular, these notes contain almost no discussion of Poisson Lie groups or symplectic groupoids, both of which are special cases of Poisson groupoids. Ample material on Poisson groups can be found in [25], while symplectic groupoids are discussed in [162] as well as the original sources [34, 89, 181]. The theory of Poisson groupoids [168] is evolving rapidly thanks to new examples found in conjunction with solutions of the classical dynamical Yang-Baxter equation [136].

The time should not be long before a sequel to these notes is due.

Part I
Universal Enveloping Algebras

1 Algebraic Constructions

Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]$. We will assume that $\mathfrak{g}$ is a finite dimensional algebra over $\mathbb{R}$ or $\mathbb{C}$, but much of the following also holds for infinite dimensional Lie algebras, as well as for Lie algebras over arbitrary fields or rings.

1.1 Universal Enveloping Algebras

Regarding $\mathfrak{g}$ just as a vector space, we may form the **tensor algebra**,

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} \ ,$$

which is the free associative algebra over $\mathfrak{g}$. There is a natural inclusion $j : \mathfrak{g} \to \mathcal{T}(\mathfrak{g})$ taking $\mathfrak{g}$ to $\mathfrak{g}^{\otimes 1}$ such that, for any *linear* map $f : \mathfrak{g} \to \mathcal{A}$ to an associative algebra $\mathcal{A}$, the assignment $g(v_1 \otimes \ldots \otimes v_k) \mapsto f(v_1) \ldots f(v_k)$ determines the unique algebra homomorphism g making the following diagram commute.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "j"] \arrow[dr, "f"'] & \mathcal{T}(\mathfrak{g}) \arrow[d, "g"] \\
 & \mathcal{A}
\end{tikzcd}$$

Therefore, there is a natural one-to-one correspondence

$$\mathrm{Hom}_{\mathrm{Linear}}(\mathfrak{g}, \mathrm{Linear}(\mathcal{A})) \simeq \mathrm{Hom}_{\mathrm{Assoc}}(\mathcal{T}(\mathfrak{g}), \mathcal{A}) \ ,$$

where $\mathrm{Linear}(\mathcal{A})$ is the algebra $\mathcal{A}$ viewed just as a vector space, $\mathrm{Hom}_{\mathrm{Linear}}$ denotes linear homomorphisms and $\mathrm{Hom}_{\mathrm{Assoc}}$ denotes homomorphisms of associative algebras.

The **universal enveloping algebra** of $\mathfrak{g}$ is the quotient

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I} \ ,$$

where $\mathcal{I}$ is the (two-sided) ideal generated by the set

$$\{ j(x) \otimes j(y) - j(y) \otimes j(x) - j([x,y]) \mid x, y \in \mathfrak{g} \} \ .$$

If the Lie bracket is trivial, *i.e.* $[\cdot,\cdot] \equiv 0$ on $\mathfrak{g}$, then $\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$ is the **symmetric algebra** on $\mathfrak{g}$, that is, the free commutative associative algebra over $\mathfrak{g}$. (When $\mathfrak{g}$ is finite dimensional, $\mathcal{S}(\mathfrak{g})$ coincides with the algebra of polynomials in $\mathfrak{g}^*$.) $\mathcal{S}(\mathfrak{g})$ is the universal commutative enveloping algebra of $\mathfrak{g}$ because it satisfies the universal property above if we restrict to *commutative* algebras; *i.e.* for any commutative associative algebra $\mathcal{A}$, there is a one-to-one correspondence

$$\mathrm{Hom}_{\mathrm{Linear}}(\mathfrak{g}, \mathrm{Linear}(\mathcal{A})) \simeq \mathrm{Hom}_{\mathrm{Commut}}(\mathcal{S}(\mathfrak{g}), \mathcal{A}) \ .$$

The universal property for $\mathcal{U}(\mathfrak{g})$ is expressed as follows. Let $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the composition of the inclusion $j : \mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g})$ with the natural projection $\mathcal{T}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$. Given any associative algebra $\mathcal{A}$, let $\mathrm{Lie}(\mathcal{A})$ be the algebra $\mathcal{A}$ equipped with the bracket $[a, b]_{\mathcal{A}} = ab - ba$, and hence regarded as a Lie algebra. Then, for any Lie algebra homomorphism $f : \mathfrak{g} \to \mathcal{A}$, there is a unique *associative* algebra homomorphism $g : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ making the following diagram commute.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, "g"] \\ & \mathcal{A} \end{tikzcd}$$

In other words, there is a natural one-to-one correspondence

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(\mathcal{A})) \simeq \mathrm{Hom}_{\mathrm{Assoc}}(\mathcal{U}(\mathfrak{g}), \mathcal{A}) \ .$$

In the language of categories [114] the functor $\mathcal{U}(\cdot)$ from Lie algebras to associative algebras is the left adjoint of the functor $\mathrm{Lie}(\cdot)$.

Exercise 1
What are the adjoint functors of $\mathcal{T}$ and $\mathcal{S}$?

1.2 Lie Algebra Deformations

The Poincaré-Birkhoff-Witt theorem, whose proof we give in Sections 2.5 and 4.2, says roughly that $\mathcal{U}(\mathfrak{g})$ has the same size as $\mathcal{S}(\mathfrak{g})$. For now, we want to check that, even if $\mathfrak{g}$ has non-zero bracket $[\cdot, \cdot]$, then $\mathcal{U}(\mathfrak{g})$ will still be approximately isomorphic to $\mathcal{S}(\mathfrak{g})$. One way to express this approximation is to throw in a parameter ε multiplying the bracket; *i.e.* we look at the Lie algebra deformation $\mathfrak{g}_\varepsilon = (\mathfrak{g}, \varepsilon[\cdot, \cdot])$. As ε tends to 0, $\mathfrak{g}_\varepsilon$ approaches an abelian Lie algebra. The family $\mathfrak{g}_\varepsilon$ describes a path in the space of Lie algebra structures on the vector space $\mathfrak{g}$, passing through the point corresponding to the zero bracket.

From $\mathfrak{g}_\varepsilon$ we obtain a one-parameter family of associative algebras $\mathcal{U}(\mathfrak{g}_\varepsilon)$, passing through $\mathcal{S}(\mathfrak{g})$ at $\varepsilon = 0$. Here we are taking the *quotients* of $\mathcal{T}(\mathfrak{g})$ by a family of ideals generated by

$$\{j(x) \otimes j(y) - j(y) \otimes j(x) - j(\varepsilon[x, y]) \mid x, y \in \mathfrak{g}\} \ ,$$

so there is no obvious isomorphism as vector spaces between the $\mathcal{U}(\mathfrak{g}_\varepsilon)$ for different values of ε. We do have, however:

Claim. $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}_\varepsilon)$ for all $\varepsilon \neq 0$.

Proof. For a homomorphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$, the functoriality of $\mathcal{U}(\cdot)$ and the universality of $\mathcal{U}(\mathfrak{g})$ give the commuting diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "i_{\mathfrak{g}}"'] \arrow[dr, "i_{\mathfrak{h}} \circ f"] & \mathfrak{h} \arrow[d, "i_{\mathfrak{h}}"] \\ \mathcal{U}(\mathfrak{g}) \arrow[r, "\exists! g"] & \mathcal{U}(\mathfrak{h}) \end{tikzcd}$$

In particular, if $\mathfrak{g} \simeq \mathfrak{h}$, then $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{h})$ by universality.

Since we have the Lie algebra isomorphism

$$\mathfrak{g} \underset{m_\varepsilon}{\overset{m_{1/\varepsilon}}{\rightleftarrows}} \mathfrak{g}_\varepsilon \ ,$$

given by multiplication by $\frac{1}{\varepsilon}$ and ε, we conclude that $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}_\varepsilon)$ for $\varepsilon \neq 0$. $\square$

In Section 2.1, we will continue this family of isomorphisms to a vector space isomorphism

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}_0) \simeq \mathcal{S}(\mathfrak{g}) \ .$$

The family $\mathcal{U}(\mathfrak{g}_\varepsilon)$ may then be considered as a path in the space of associative multiplications on $S(\mathfrak{g})$, passing through the subspace of commutative multiplications. The first derivative with respect to ε of the path $\mathcal{U}(\mathfrak{g}_\varepsilon)$ turns out to be an anti-symmetric operation called the *Poisson bracket* (see Section 2.2).

1.3 Symmetrization

Let S_n be the symmetric group in n letters, *i.e.* the group of permutations of $\{1, 2, \ldots, n\}$. The linear map

$$s : x_1 \otimes \ldots \otimes x_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$$

extends to a well-defined **symmetrization** endomorphism $s : \mathcal{T}(\mathfrak{g}) \to \mathcal{T}(\mathfrak{g})$ with the property that $s^2 = s$. The image of s consists of the **symmetric tensors** and is a vector space complement to the ideal $\mathcal{I}$ generated by $\{j(x) \otimes j(y) - j(y) \otimes j(x) \mid x, y \in \mathfrak{g}\}$. We identify the symmetric algebra $\mathcal{S}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I}$ with the symmetric tensors by the quotient map, and hence regard symmetrization as a projection

$$s : \mathcal{T}(\mathfrak{g}) \longrightarrow \mathcal{S}(\mathfrak{g}) \ .$$

The linear section

$$\begin{array}{rrcl} \tau : & \mathcal{S}(\mathfrak{g}) & \longrightarrow & \mathcal{T}(\mathfrak{g}) \\ & x_1 \ldots x_n & \longmapsto & s(x_1 \otimes \ldots \otimes x_n) \end{array}$$

is a linear map, but *not* an algebra homomorphism, as the product of two symmetric tensors is generally not a symmetric tensor.

1.4 The Graded Algebra of $\mathcal{U}(\mathfrak{g})$

Although $\mathcal{U}(\mathfrak{g})$ is not a graded algebra, we can still grade it as a vector space.

We start with the natural grading on $\mathcal{T}(\mathfrak{g})$:

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(\mathfrak{g}) \ , \qquad \text{where} \qquad \mathcal{T}^k(\mathfrak{g}) = \mathfrak{g}^{\otimes k} \ .$$

Unfortunately, projection of $\mathcal{T}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g})$ does *not* induce a grading, since the relations defining $\mathcal{U}(\mathfrak{g})$ are not homogeneous unless $[\cdot, \cdot]_\mathfrak{g} = 0$. (On the other hand, symmetrization $s : \mathcal{T}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g})$ does preserve the grading.)

The grading of $\mathcal{T}(\mathfrak{g})$ has associated filtration

$$\mathcal{T}^{(k)}(\mathfrak{g}) = \bigoplus_{j=0}^{k} \mathcal{T}^j(\mathfrak{g}) \ ,$$

such that

$$\mathcal{T}^{(0)} \subseteq \mathcal{T}^{(1)} \subseteq \mathcal{T}^{(2)} \subseteq \ldots \qquad \text{and} \qquad \mathcal{T}^{(i)} \otimes \mathcal{T}^{(j)} \subseteq \mathcal{T}^{(i+j)} \ .$$

We can recover $\mathcal{T}^k$ by $\mathcal{T}^{(k)}/\mathcal{T}^{(k-1)} \simeq \mathcal{T}^k$.

What happens to this filtration when we project to $\mathcal{U}(\mathfrak{g})$?

Remark. Let $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the natural map (as in Section 1.1). If we take $x, y \in \mathfrak{g}$, then $i(x)i(y)$ and $i(y)i(x)$ each "has length 2," but their difference

$$i(y)i(x) - i(x)i(y) = i([y,x])$$

has length 1. Therefore, exact length is not respected by algebraic operations on $\mathcal{U}(\mathfrak{g})$. ♢

Let $\mathcal{U}^{(k)}(\mathfrak{g})$ be the image of $\mathcal{T}^{(k)}(\mathfrak{g})$ under the projection map.

Exercise 2
Show that $\mathcal{U}^{(k)}(\mathfrak{g})$ is linearly spanned by products of length $\leq k$ of elements of $\mathcal{U}^{(1)}(\mathfrak{g}) = i(\mathfrak{g})$.

We do have the relation

$$\mathcal{U}^{(k)} \cdot \mathcal{U}^{(\ell)} \subseteq \mathcal{U}^{(k+\ell)} \ ,$$

so that the universal enveloping algebra of $\mathfrak{g}$ has a *natural* filtration, natural in the sense that, for any map $\mathfrak{g} \to \mathfrak{h}$, the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{h} \arrow[d] \\ \mathcal{U}(\mathfrak{g}) \arrow[r] & \mathcal{U}(\mathfrak{h}) \end{tikzcd}$$

preserves the filtration.

In order to construct a graded algebra, we define

$$\mathcal{U}^k(\mathfrak{g}) = \mathcal{U}^{(k)}(\mathfrak{g})/\mathcal{U}^{(k-1)}(\mathfrak{g}) \ .$$

There are well-defined product operations

$$\begin{array}{ccc} \mathcal{U}^k(\mathfrak{g}) \otimes \mathcal{U}^\ell(\mathfrak{g}) & \longrightarrow & \mathcal{U}^{k+\ell}(\mathfrak{g}) \\ [\alpha] \otimes [\beta] & \longmapsto & [\alpha\beta] \end{array}$$

forming an associative multiplication on what is called the **graded algebra associated to $\mathcal{U}(\mathfrak{g})$**:

$$\bigoplus_{j=0}^{\infty} \mathcal{U}^k(\mathfrak{g}) =: \mathfrak{Gr}\, \mathcal{U}(\mathfrak{g}) \ .$$

Remark. The constructions above are purely algebraic in nature; we can form $\mathfrak{Gr}\, \mathcal{A}$ for any filtered algebra $\mathcal{A}$. The functor $\mathfrak{Gr}$ will usually simplify the algebra in the sense that multiplication forgets about lower order terms. ♢

2 The Poincaré-Birkhoff-Witt Theorem

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$.

2.1 Almost Commutativity of $\mathcal{U}(\mathfrak{g})$

Claim. $\mathfrak{Gr}\,\mathcal{U}(\mathfrak{g})$ is commutative.

Proof. Since $\mathcal{U}(\mathfrak{g})$ is generated by $\mathcal{U}^{(1)}(\mathfrak{g})$, $\mathfrak{Gr}\,\mathcal{U}(\mathfrak{g})$ is generated by $\mathcal{U}^1(\mathfrak{g})$. Thus it suffices to show that multiplication

$$\mathcal{U}^1(\mathfrak{g}) \otimes \mathcal{U}^1(\mathfrak{g}) \longrightarrow \mathcal{U}^2(\mathfrak{g})$$

is commutative. Because $\mathcal{U}^{(1)}(\mathfrak{g})$ is generated by $i(\mathfrak{g})$, any $\alpha \in \mathcal{U}^1(\mathfrak{g})$ is of the form $\alpha = [i(x)]$ for some $x \in \mathfrak{g}$. Pick any two elements $x, y \in \mathfrak{g}$. Then $[i(x)], [i(y)] \in \mathcal{U}^1(\mathfrak{g})$, and

$$\begin{aligned} [i(x)][i(y)] - [i(y)][i(x)] &= [i(x)i(y) - i(y)i(x)] \\ &= [i([x,y]_{\mathfrak{g}})] \; . \end{aligned}$$

As $i([x,y]_{\mathfrak{g}})$ sits in $\mathcal{U}^{(1)}(\mathfrak{g})$, we see that $[i([x,y]_{\mathfrak{g}})] = 0$ in $\mathcal{U}^2(\mathfrak{g})$. $\square$

When looking at symmetrization $s : \mathcal{T}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g})$ in Section 1.3, we constructed a linear section $\tau : \mathcal{S}(\mathfrak{g}) \hookrightarrow \mathcal{T}(\mathfrak{g})$. We formulate the Poincaré-Birkhoff-Witt theorem using this linear section.

Theorem 2.1 (Poincaré-Birkhoff-Witt) *There is a graded (commutative) algebra isomorphism*

$$\lambda : \mathcal{S}(\mathfrak{g}) \xrightarrow{\simeq} \mathfrak{Gr}\,\mathcal{U}(\mathfrak{g})$$

given by the natural maps:

$$\begin{array}{ccccccc} \mathcal{S}^k(\mathfrak{g}) & \xhookrightarrow{\tau} & \mathcal{T}^k(\mathfrak{g}) & \longrightarrow & \mathcal{U}^{(k)}(\mathfrak{g}) & \twoheadrightarrow & \mathcal{U}^k(\mathfrak{g}) \subset \mathfrak{Gr}\,\mathcal{U}(\mathfrak{g}) \\ v_1 \dots v_k & \longmapsto & \displaystyle\frac{1}{k!}\sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)} & & \longmapsto & & [v_1 \dots v_k] \; . \end{array}$$

For each degree k, we follow the embedding $\tau^k : \mathcal{S}^k(\mathfrak{g}) \hookrightarrow \mathcal{T}^k(\mathfrak{g})$ by a map to $\mathcal{U}^{(k)}(\mathfrak{g})$ and then by the projection onto $\mathcal{U}^k$. Although the composition $\lambda : \mathcal{S}(\mathfrak{g}) \to \mathfrak{Gr}\,\mathcal{U}(\mathfrak{g})$ is a graded algebra homomorphism, the maps $\mathcal{S}(\mathfrak{g}) \to \mathcal{T}(\mathfrak{g})$ and $\mathcal{T}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ are not.

We shall prove Theorem 2.1 (for finite dimensional Lie algebras over $\mathbb{R}$ or $\mathbb{C}$) using Poisson geometry. The sections most relevant to the proof are 2.5 and 4.2. For purely algebraic proofs, see Dixmier [46] or Serre [150], who show that the theorem actually holds for free modules $\mathfrak{g}$ over rings.

2.2 Poisson Bracket on $\mathfrak{Gr}\,\mathcal{U}(\mathfrak{g})$

In this section, we denote $\mathcal{U}(\mathfrak{g})$ simply by $\mathcal{U}$, since the arguments apply to *any* filtered algebra $\mathcal{U}$,

$$\mathcal{U}^{(0)} \subseteq \mathcal{U}^{(1)} \subseteq \mathcal{U}^{(2)} \subseteq \dots \; ,$$

for which the associated graded algebra

$$\mathfrak{Gr}\,\mathcal{U} := \bigoplus_{j=0}^{\infty} \mathcal{U}^j \qquad \text{where} \qquad \mathcal{U}^j = \mathcal{U}^{(j)} \Big/ \mathcal{U}^{(j-1)} \ .$$

is commutative. Such an algebra $\mathcal{U}$ is often called **almost commutative**.

For $x \in \mathcal{U}^{(k)}$ and $y \in \mathcal{U}^{(\ell)}$, define

$$\{[x],[y]\} = [xy - yx] \in \mathcal{U}^{k+\ell-1} = \mathcal{U}^{(k+\ell-1)}/\mathcal{U}^{(k+\ell-2)}$$

so that

$$\{\mathcal{U}^k, \mathcal{U}^\ell\} \subseteq \mathcal{U}^{k+\ell-1} \ .$$

This collection of degree -1 bilinear maps combine to form the **Poisson bracket** on $\mathfrak{Gr}\,\mathcal{U}$. So, besides the associative product on $\mathfrak{Gr}\,\mathcal{U}$ (inherited from the associative product on $\mathcal{U}$; see Section 1.4), we also get a bracket operation $\{\cdot,\cdot\}$ with the following properties:

1. $\{\cdot,\cdot\}$ is anti-commutative (*not* super-commutative) and satisfies the **Jacobi identity**
$$\{\{u,v\},w\} = \{\{u,w\},v\} + \{u,\{v,w\}\} \ .$$
That is, $\{\cdot,\cdot\}$ is a **Lie bracket** and $\mathfrak{Gr}\,\mathcal{U}$ is a **Lie algebra**;

2. the **Leibniz identity** holds:
$$\{uv,w\} = \{u,w\}v + u\{v,w\} \ .$$

Exercise 3
Prove the Jacobi and Leibniz identities for $\{\cdot,\cdot\}$ on $\mathfrak{Gr}\,\mathcal{U}$.

Remark. The Leibniz identity says that $\{\cdot,w\}$ is a derivation of the associative algebra structure; it is a compatibility property between the Lie algebra and the associative algebra structures. Similarly, the Jacobi identity says that $\{\cdot,w\}$ is a derivation of the Lie algebra structure. $\diamondsuit$

A commutative associative algebra with a Lie algebra structure satisfying the Leibniz identity is called a **Poisson algebra**. As we will see (Chapters 3, 4 and 5), the existence of such a structure on the algebra corresponds to the existence of a certain differential-geometric structure on an underlying space.

Remark. Given a Lie algebra $\mathfrak{g}$, we may define new Lie algebras $\mathfrak{g}_\varepsilon$ where the bracket operation is $[\cdot,\cdot]_{\mathfrak{g}_\varepsilon} = \varepsilon[\cdot,\cdot]_{\mathfrak{g}}$. For each ε, the Poincaré-Birkhoff-Witt theorem will give a vector space isomorphism

$$\mathcal{U}(\mathfrak{g}_\varepsilon) \simeq \mathcal{S}(\mathfrak{g}) \ .$$

Multiplication on $\mathcal{U}(\mathfrak{g}_\varepsilon)$ induces a *family* of multiplications on $\mathcal{S}(\mathfrak{g})$, denoted $*_\varepsilon$, which satisfy

$$f *_\varepsilon g = fg + \frac{1}{2}\varepsilon\{f,g\} + \sum_{k\geq 2} \varepsilon^k B_k(f,g) + \dots$$

for some bilinear operators B_k. This family is called a **deformation quantization** of $\mathrm{Pol}(\mathfrak{g}^*)$ in the direction of the Poisson bracket; see Chapters 20 and 21. $\diamondsuit$

2.3 The Role of the Jacobi Identity

Choose a basis $v_1, \ldots, v_n$ for $\mathfrak{g}$. Let $j : \mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g})$ be the inclusion map. The algebra $\mathcal{T}(\mathfrak{g})$ is linearly generated by all monomials

$$j(v_{\alpha_1}) \otimes \ldots \otimes j(v_{\alpha_k}) \ .$$

If $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the natural map (as in Section 1.1), it is easy to see, via the relation $i(x) \otimes i(y) - i(y) \otimes i(x) = i([x, y])$ in $\mathcal{U}(\mathfrak{g})$, that the universal enveloping algebra is generated by monomials of the form

$$i(v_{\alpha_1}) \otimes \ldots \otimes i(v_{\alpha_k}) \ , \qquad \alpha_1 \leq \ldots \leq \alpha_k \ .$$

However, it is not as trivial to show that there are no linear relations between these generating monomials. Any proof of the independence of these generators must use the Jacobi identity. The Jacobi identity is crucial since $\mathcal{U}(\mathfrak{g})$ was defined to be an universal object relative to the category of Lie algebras.

Forget for a moment about the Jacobi identity. We define an **almost Lie algebra** $\mathfrak{g}$ to be the same as a Lie algebra except that the bracket operation does not necessarily satisfy the Jacobi identity. It is not difficult to see that the constructions for the universal enveloping algebra still hold true in this category. We will test the independence of the generating monomials of $\mathcal{U}(\mathfrak{g})$ in this case. Let $x, y, z \in \mathfrak{g}$ for some almost Lie algebra $\mathfrak{g}$. The **jacobiator** is the trilinear map $J : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$J(x, y, z) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \ .$$

Clearly, on a Lie algebra, the jacobiator vanishes; in general, it measures the obstruction to the Jacobi identity. Since J is antisymmetric in the three entries, we can view it as a map $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$, which we will still denote by J.

Claim. $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ vanishes on the image of J.

This implies that we need $J \equiv 0$ for i to be an injection and the Poincaré-Birkhoff-Witt theorem to hold.

Proof. Take $x, y, z \in \mathfrak{g}$, and look at

$$i\left(J(x, y, z)\right) = i\left([[x, y,], z] + c.p.\right) \ .$$

Here, *c.p.* indicates that the succeeding terms are given by applying circular permutations to the x, y, z of the first term. Because i is linear and commutes with the bracket operation, we see that

$$i\left(J(x, y, z)\right) = [[i(x), i(y)]_{\mathcal{U}(\mathfrak{g})}, i(z)]_{\mathcal{U}(\mathfrak{g})} + c.p. \ .$$

But the bracket in the associative algebra always satisfies the Jacobi identity, and so $i(J) \equiv 0$. □

Exercise 4

1. Is the image of J the entire kernel of i?
2. Is the image of J an ideal in $\mathfrak{g}$? If this is true, then we can form the "maximal Lie algebra" quotient by forming $\mathfrak{g}/\mathrm{Im}(J)$. This would then lead to a refinement of Poincaré-Birkhoff-Witt to almost Lie algebras.

Remark. The answers to the exercise above (which we do not know!) should involve the calculus of multilinear operators. There are two versions of this theory:

- skew-symmetric operators – from the work of Frölicher and Nijenhuis [61];
- arbitrary multilinear operators – looking at the associativity of algebras, as in the work of Gerstenhaber [67, 68].

$\diamondsuit$

2.4 Actions of Lie Algebras

Much of this section traces back to the work of Lie around the end of the 19th century on the existence of a Lie group G whose Lie algebra is a given Lie algebra $\mathfrak{g}$.

Our proof of the Poincaré-Birkhoff-Witt theorem will only require local existence of G – a neighborhood of the identity element in the group. What we shall construct is a manifold M with a Lie algebra homomorphism from $\mathfrak{g}$ to vector fields on M, $\rho : \mathfrak{g} \to \chi(M)$, such that a basis of vectors on $\mathfrak{g}$ goes to a pointwise linearly independent set of vector fields on M. Such a map ρ is called a **pointwise faithful representation**, or **free action** of $\mathfrak{g}$ on M.

Example. Let $M = G$ be a Lie group with Lie algebra $\mathfrak{g}$. Then the map taking elements of $\mathfrak{g}$ to left invariant vector fields on G (the generators of the right translations) is a free action. $\diamondsuit$

The Lie algebra homomorphism $\rho : \mathfrak{g} \to \chi(M)$ is called a **right action** of the Lie algebra $\mathfrak{g}$ on M. (For left actions, ρ would have to be an *anti*-homomorphism.) Such actions ρ can be obtained by differentiating right actions of the Lie group G. One of Lie's theorems shows that any homomorphism ρ can be integrated to a local action of the group G on M.

Let $v_1, \ldots, v_n$ be a basis of $\mathfrak{g}$, and $V_1 = \rho(v_1), \ldots, V_n = \rho(v_n)$ the corresponding vector fields on M. Assume that the V_j are pointwise linearly independent. Since ρ is a Lie algebra homomorphism, we have relations

$$[V_i, V_j] = \sum_k c_{ijk} V_k \ ,$$

where the constants c_{ijk} are the **structure constants** of the Lie algebra, defined by the relations $[v_i, v_j] = \sum c_{ijk} v_k$. In other words, $\{V_1, \ldots, V_n\}$ is a set of vector fields on M whose bracket has the same relations as the bracket on $\mathfrak{g}$. These relations show in particular that the span of $V_1, \ldots, V_n$ is an involutive subbundle of TM. By the Frobenius theorem, we can integrate it. Let $N \subseteq M$ be a leaf of the corresponding foliation. There is a map $\rho_N : \mathfrak{g} \to \chi(N)$ such that the $V_j = \rho_N(v_j)$'s form a pointwise *basis* of vector fields on N.

Although we will not need this fact for the Poincaré-Birkhoff-Witt theorem, we note that the leaf N is, in a sense, locally the Lie group with Lie algebra $\mathfrak{g}$: Pick some point in N and label it e. There is a unique local group structure on a neighborhood of e such that e is the identity element and $V_1, \ldots, V_n$ are left invariant vector fields. The group structure comes from defining the flows of the vector fields to be right translations. The hard part of this construction is showing that the multiplication defined in this way is associative.

All of this is part of Lie's third theorem that any Lie algebra is the Lie algebra of a local Lie group. Existence of a global Lie group was proven by Cartan in [23].

Claim. The injectivity of any single action $\rho : \mathfrak{g} \to \chi(M)$ of the Lie algebra $\mathfrak{g}$ on a manifold M is enough to imply that $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.

Proof. Look at the algebraic embedding of vector fields into all vector space endomorphisms of $C^\infty(M)$:

$$\chi(M) \subset \mathrm{End}_{\mathrm{Vect}}(C^\infty(M)) \ .$$

The bracket on $\chi(M)$ is the commutator bracket of vector fields. If we consider $\chi(M)$ and $\mathrm{End}_{\mathrm{Vect}}(C^\infty(M))$ as purely algebraic objects (using the topology of M only to define $C^\infty(M)$), then we use the universality of $\mathcal{U}(\mathfrak{g})$ to see

$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook, "\rho"] \arrow[d, "i"'] & \chi(M) \arrow[r, hook] & \mathrm{End}_{\mathrm{Vect}}(C^\infty(M)) \\ \mathcal{U}(\mathfrak{g}) \arrow[urr, "\exists!\tilde{\rho}"] & & \end{tikzcd}$$

Thus, if ρ is injective for some manifold M, then i must also be an injection. $\square$

The next section shows that, in fact, any pointwise faithful ρ gives rise to a faithful representation $\tilde{\rho}$ of $\mathcal{U}(\mathfrak{g})$ as differential operators on $C^\infty(M)$.

2.5 Proof of the Poincaré-Birkhoff-Witt Theorem

In Section 4.2, we shall actually find a manifold M with a free action $\rho : \mathfrak{g} \to \chi(M)$. Assume now that we have $\mathfrak{g}, \rho, M, N$ and $\tilde{\rho} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathrm{Vect}}(C^\infty(M))$ as described in the previous section.

Choose coordinates $x_1, \ldots, x_n$ centered at the "identity" $e \in N$ such that the images of the basis elements $v_1, \ldots, v_n$ of $\mathfrak{g}$ are the vector fields

$$V_i = \frac{\partial}{\partial x_i} + O(x) \ .$$

The term $O(x)$ is some vector field vanishing at e which we can write as

$$O(x) = \sum_{j,k} x_j a_{ijk}(x) \frac{\partial}{\partial x_k} \ .$$

We regard the vector fields $V_1, \ldots, V_n$ as a set of linearly independent first-order differential operators via the embedding $\chi(M) \subset \mathrm{End}_{\mathrm{Vect}}(C^\infty(M))$.

Lemma 2.2 *The monomials $V_{i_1} \cdots V_{i_k}$ with $i_1 \leq \ldots \leq i_k$ are linearly independent differential operators.*

This will show that the monomials $i(v_{i_1}) \cdots i(v_{i_k})$ must be linearly independent in $\mathcal{U}(\mathfrak{g})$ since $\tilde{\rho}(i(v_{i_1}) \cdots i(v_{i_k})) = V_{i_1} \cdots V_{i_k}$, which would conclude the proof of the Poincaré-Birkhoff-Witt theorem.

Proof. We show linear independence by testing the monomials against certain functions. Given $i_1 \leq \ldots \leq i_k$ and $j_1 \leq \ldots \leq j_\ell$, we define numbers K_i^j as follows:

$$\begin{aligned} K_i^j &:= (V_{i_1} \cdots V_{i_k})\,(x_{j_1} \cdots x_{j_\ell})\,(e) \\ &= \left(\frac{\partial}{\partial x_{i_1}} + O(x)\right) \cdots \left(\frac{\partial}{\partial x_{i_k}} + O(x)\right)(x_{j_1} \cdots x_{j_\ell})\,(e) \end{aligned}$$

1. If $k < \ell$, then any term in the expression will take only k derivatives. But $x_{j_1} \cdots x_{j_\ell}$ vanishes to order ℓ at e, and hence $K_i^j = 0$.
2. If $k = \ell$, then there is only one way to get a non-zero result, namely when the j's match with the i's. In this case, we get
$$K_i^j = \begin{cases} 0 & i \neq j \\ c_i^j > 0 & i = j\ . \end{cases}$$
3. If $k > \ell$, then the computation is rather complicated, but fortunately this case is not relevant.

Assume that we had a dependence relation on the V_i's of the form

$$R = \sum_{\substack{i_1,\ldots,i_k \\ k \leq r}} b^{i_1,\ldots,i_k} V_{i_1} \cdots V_{i_k} = 0\ .$$

Apply R to the functions of the form $x_{j_1} \cdots x_{j_r}$ and evaluate at e. All the terms of R with degree less than r will contribute nothing, and there will be at most one monomial $V_{i_1} \cdots V_{i_r}$ of R which is non-zero on $x_{j_1} \cdots x_{j_r}$. We see that $b^{i_1,\ldots,i_r} = 0$ for each multi-index $i_1, \ldots, i_r$ of order r. By induction on the order of the multi-indices, we conclude that all $b^i = 0$. $\square$

To complete the proof of Theorem 2.1, it remains to find a pointwise faithful representation ρ for $\mathfrak{g}$. To construct the appropriate manifold M, we turn to Poisson geometry.

Part II
Poisson Geometry

3 Poisson Structures

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$. In Section 2.2, we defined a Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{Gr}\,\mathcal{U}(\mathfrak{g})$ using the commutator bracket in $\mathcal{U}(\mathfrak{g})$ and noted that $\{\cdot,\cdot\}$ satisfies the Leibniz identity. The Poincaré-Birkhoff-Witt theorem (in Section 2.1) states that $\mathfrak{Gr}\,\mathcal{U}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}) = \mathrm{Pol}(\mathfrak{g}^*)$. This isomorphism induces a Poisson bracket on $\mathrm{Pol}(\mathfrak{g}^*)$.

In this chapter, we will construct a Poisson bracket directly on all of $C^\infty(\mathfrak{g}^*)$, restricting to the previous bracket on polynomial functions, and we will discuss general facts about Poisson brackets which will be used in Section 4.2 to conclude the proof of the Poincaré-Birkhoff-Witt theorem.

3.1 Lie-Poisson Bracket

Given functions $f, g \in C^\infty(\mathfrak{g}^*)$, the 1-forms df, dg may be interpreted as maps $Df, Dg : \mathfrak{g}^* \to \mathfrak{g}^{**}$. When $\mathfrak{g}$ is finite dimensional, we have $\mathfrak{g}^{**} \simeq \mathfrak{g}$, so that Df and Dg take values in $\mathfrak{g}$. Each $\mu \in \mathfrak{g}^*$ is a function on $\mathfrak{g}$. The new function $\{f, g\} \in C^\infty(\mathfrak{g}^*)$ evaluated at μ is

$$\{f,g\}(\mu) = \mu\left([Df(\mu), Dg(\mu)]_{\mathfrak{g}}\right) .$$

Equivalently, we can define this bracket using coordinates. Let $v_1, \dots, v_n$ be a basis for $\mathfrak{g}$ and let $\mu_1, \dots, \mu_n$ be the corresponding coordinate functions on $\mathfrak{g}^*$. Introduce the structure constants c_{ijk} satisfying $[v_i, v_j] = \sum c_{ijk} v_k$. Then set

$$\{f,g\} = \sum_{i,j,k} c_{ijk}\mu_k \frac{\partial f}{\partial \mu_i}\frac{\partial g}{\partial \mu_j} .$$

Exercise 5
Verify that the definitions above are equivalent.

The bracket $\{\cdot,\cdot\}$ is skew-symmetric and takes pairs of smooth functions to smooth functions. Using the product rule for derivatives, one can also check the Leibniz identity: $\{fg, h\} = \{f, h\}g + f\{g, h\}$.

The bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{g}^*)$ is called the **Lie-Poisson bracket**. The pair $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is often called a **Lie-Poisson manifold**. (A good reference for the Lie-Poisson structures is Marsden and Ratiu's book on mechanics [116].)

Remark. The coordinate functions $\mu_1, \dots, \mu_n$ satisfy $\{\mu_i, \mu_j\} = \sum c_{ijk}\mu_k$. This implies that the linear functions on $\mathfrak{g}^*$ are closed under the bracket operation. Furthermore, the bracket $\{\cdot,\cdot\}$ on the linear functions of $\mathfrak{g}^*$ is exactly the same as the Lie bracket $[\cdot,\cdot]$ on the elements of $\mathfrak{g}$. We thus see that there is an embedding of Lie algebras $\mathfrak{g} \hookrightarrow C^\infty(\mathfrak{g}^*)$. $\diamondsuit$

Exercise 6
As a commutative, associative algebra, $\mathrm{Pol}(\mathfrak{g}^*)$ is generated by the linear functions. Using induction on the degree of polynomials, prove that, if the Leibniz identity is satisfied throughout the algebra and if the Jacobi identity holds on the generators, then the Jacobi identity holds on the whole algebra.

In Section 3.3, we show that the bracket on $C^\infty(\mathfrak{g}^*)$ satisfies the Jacobi identity. Knowing that the Jacobi identity holds on $\mathrm{Pol}(\mathfrak{g}^*)$, we could try to extend to $C^\infty(\mathfrak{g}^*)$ by continuity, but instead we shall provide a more geometric argument.

3.2 Almost Poisson Manifolds

A pair $(M, \{\cdot,\cdot\})$ is called an **almost Poisson manifold** when $\{\cdot,\cdot\}$ is an almost Lie algebra structure (defined in Section 2.3) on $C^\infty(M)$ satisfying the Leibniz identity. The bracket $\{\cdot,\cdot\}$ is then called an **almost Poisson structure**.

Thanks to the Leibniz identity, $\{f,g\}$ depends only on the first derivatives of f and g, thus we can write it as

$$\{f,g\} = \Pi(df,dg) \ ,$$

where Π is a field of skew-symmetric bilinear forms on T^*M. We say that $\Pi \in \Gamma((T^*M \wedge T^*M)^*) = \Gamma(TM \wedge TM) = \Gamma(\wedge^2 TM)$ is a **bivector field**.

Conversely, any bivector field Π defines a bilinear antisymmetric multiplication $\{\cdot,\cdot\}_\Pi$ on $C^\infty(M)$ by the formula $\{f,g\}_\Pi = \Pi(df,dg)$. Such a multiplication satisfies the Leibniz identity because each $X_h := \{\cdot,h\}_\Pi$ is a derivation of $C^\infty(M)$. Hence, $\{\cdot,\cdot\}_\Pi$ is an almost Poisson structure on M.

Remark. The differential forms $\Omega^\bullet(M)$ on a manifold M are the sections of

$$\wedge^\bullet T^*M := \oplus \wedge^k T^*M \ .$$

There are two well-known operations on $\Omega^\bullet(M)$: the wedge product $\wedge$ and the differential d.

The analogous structures on sections of

$$\wedge^\bullet TM := \oplus \wedge^k TM$$

are less commonly used in differential geometry: there is a wedge product, and there is a bracket operation dual to the differential on sections of $\wedge^\bullet T^*M$. The sections of $\wedge^k TM$ are called **k-vector fields** (or **multivector fields** for unspecified k) on M. The space of such sections is denoted by $\chi^k(M) = \Gamma(\wedge^k TM)$. There is a natural commutator bracket on the direct sum of $\chi^0(M) = C^\infty(M)$ and $\chi^1(M) = \chi(M)$. In Section 18.3, we shall extend this bracket to an operation on $\chi^k(M)$, called the Schouten-Nijenhuis bracket [116, 162]. ♢

3.3 Poisson Manifolds

An almost Poisson structure $\{\cdot,\cdot\}_\Pi$ on a manifold M is called a **Poisson structure** if it satisfies the Jacobi identity. A **Poisson manifold** $(M, \{\cdot,\cdot\})$ is a manifold M equipped with a Poisson structure $\{\cdot,\cdot\}$. The corresponding bivector field Π is then called a **Poisson tensor**. The name "Poisson structure" sometimes refers to the bracket $\{\cdot,\cdot\}$ and sometimes to the Poisson tensor Π.

Given an almost Poisson structure, we define the **jacobiator** on $C^\infty(M)$ by:

$$J(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} \ .$$

Exercise 7
Show that the jacobiator is

(a) skew-symmetric, and

(b) a derivation in each argument.

By the exercise above, the operator J on $C^\infty(M)$ corresponds to a **trivector field** $\mathcal{J} \in \chi^3(M)$ such that $\mathcal{J}(df, dg, dh) = J(f,g,h)$. In coordinates, we write

$$J(f,g,h) = \sum_{i,j,k} J_{ijk}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k} \ ,$$

where $J_{ijk}(x) = J(x_i, x_j, x_k)$.

Consequently, the Jacobi identity holds on $C^\infty(M)$ if and only if it holds for the coordinate functions.

Example. When $M = \mathfrak{g}^*$ is a Lie-Poisson manifold, the Jacobi identity holds on the coordinate linear functions, because it holds on the Lie algebra $\mathfrak{g}$ (see Section 3.1). Hence, the Jacobi identity holds on $C^\infty(\mathfrak{g}^*)$. ♢

Remark. Up to a constant factor, $\mathcal{J} = [\Pi, \Pi]$, where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket (see Section 18.3 and the last remark of Section 3.2). Therefore, the Jacobi identity for the bracket $\{\cdot,\cdot\}$ is equivalent to the equation $[\Pi,\Pi] = 0$. We will not use this until Section 18.3. ♢

3.4 Structure Functions and Canonical Coordinates

Let Π be the bivector field on an almost Poisson manifold $(M, \{\cdot,\cdot\}_\Pi)$. Choosing local coordinates $x_1, \ldots, x_n$ on M, we find **structure functions**

$$\pi_{ij}(x) = \{x_i, x_j\}_\Pi$$

of the almost Poisson structure. In coordinate notation, the bracket of functions $f, g \in C^\infty(M)$ is

$$\{f,g\}_\Pi = \sum \pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \ .$$

Equivalently, we have

$$\Pi = \frac{1}{2} \sum \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \ .$$

Exercise 8
Write the jacobiator J_{ijk} in terms of the structure functions π_{ij}. It is a homogeneous quadratic expression in the π_{ij}'s and their first partial derivatives.

Examples.

1. When $\pi_{ij}(x) = \sum c_{ijk} x_k$, the Poisson structure is a **linear Poisson structure**. Clearly the Jacobi identity holds if and only if the c_{ijk} are the structure constants of a Lie algebra $\mathfrak{g}$. When this is the case, the $x_1, \ldots, x_n$ are coordinates on $\mathfrak{g}^*$. We had already seen that for the Lie-Poisson structure defined on $\mathfrak{g}^*$, the functions π_{ij} were linear.

2. Suppose that the $\pi_{ij}(x)$ are **constant**. In this case, the Jacobi identity is trivially satisfied – each term in the jacobiator of coordinate functions is zero. By a linear change of coordinates, we can put the constant antisymmetric matrix (π_{ij}) into the normal form:

$$\left(\begin{array}{cc|c} 0 & I_k & \multirow{2}{*}{0} \\ -I_k & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0_\ell \end{array}\right)$$

where I_k is the $k \times k$ identity matrix and 0_ℓ is the $\ell \times \ell$ zero matrix. If we call the new coordinates $q_1, \ldots, q_k, p_1, \ldots, p_k, c_1, \ldots, c_\ell$, the bivector field becomes

$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \ .$$

In terms of the bracket, we can write

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \ ,$$

which is actually the original form due to Poisson in [138]. The c_i's do not enter in the bracket, and hence behave as parameters. The following relations, called **canonical Poisson relations**, hold:

- $\{q_i, p_j\} = \delta_{ij}$
- $\{q_i, q_j\} = \{p_i, p_j\} = 0$
- $\{\alpha, c_i\} = 0$ for any coordinate function α.

The coordinates c_i are said to be in the **center** of the Poisson algebra; such functions are called **Casimir functions**. If $\ell = 0$, *i.e.* if there is no center, then the structure is said to be **non-degenerate** or **symplectic**. In any case, q_i, p_i are called **canonical coordinates**. Theorem 4.2 will show that this example is quite general.

◊

3.5 Hamiltonian Vector Fields

Let $(M, \{\cdot, \cdot\})$ be an almost Poisson manifold. Given $h \in C^\infty(M)$, define the linear map

$$X_h : C^\infty(M) \longrightarrow C^\infty(M) \qquad \text{by} \qquad X_h(f) = \{f, h\} \ .$$

The correspondence $h \mapsto X_h$ resembles an "adjoint representation" of $C^\infty(M)$. By the Leibniz identity, X_h is a derivation and thus corresponds to a vector field, called the **hamiltonian vector field** of the function h.

Lemma 3.1 *On a Poisson manifold, hamiltonian vector fields satisfy*

$$[X_f, X_g] = -X_{\{f,g\}} \ .$$

Proof. We can see this by applying $[X_f, X_g] + X_{\{f,g\}}$ to an arbitrary function $h \in C^\infty(M)$.

$$\begin{aligned}
\Big([X_f, X_g] + X_{\{f,g\}}\Big)h &= X_f X_g h - X_g X_f h + X_{\{f,g\}}h \\
&= X_f\{h,g\} - X_g\{h,f\} + \{h,\{f,g\}\} \\
&= \{\{h,g\},f\} + \{\{f,h\},g\} + \{\{g,f\},h\} \ .
\end{aligned}$$

The statement of the lemma is thus equivalent to the Jacobi identity for the Poisson bracket. □

Historical Remark. This lemma gives another formulation of the integrability condition for Π, which, in fact, was the original version of the identity as formulated by Jacobi around 1838. (See Jacobi's collected works [86].) Poisson [138] had introduced the bracket $\{\cdot,\cdot\}$ in order to simplify calculations in celestial mechanics. He proved around 1808, through long and tedious computations, that

$$\{f,h\} = 0 \quad \text{and} \quad \{g,h\} = 0 \qquad \Longrightarrow \qquad \{\{f,g\},h\} = 0 \ .$$

This means that, if two functions f, g are constant along integral curves of X_h, then one can form a third function also constant along X_h, namely $\{f,g\}$. When Jacobi later stated the identity in Lemma 3.1, he gave a much shorter proof of a yet stronger result. ♢

3.6 Poisson Cohomology

A **Poisson vector field**, is a vector field X on a Poisson manifold (M, Π) such that $\mathcal{L}_X \Pi = 0$, where $\mathcal{L}_X$ is the Lie derivative along X. The Poisson vector fields, also characterized by

$$X\{f,g\} = \{Xf, g\} + \{f, Xg\} \ ,$$

are those whose local flow preserves the bracket operation. These are also the **derivations** (with respect to *both* operations) of the Poisson algebra.

Among the Poisson vector fields, the hamiltonian vector fields $X_h = \{\cdot, h\}$ form the subalgebra of **inner derivations** of $C^\infty(M)$. (Of course, they are "inner" only for the bracket.)

Exercise 9
Show that the hamiltonian vector fields form an *ideal* in the Lie algebra of Poisson vector fields.

Remark. The quotient of the Lie algebra of Poisson vector fields by the ideal of hamiltonian vector fields is a Lie algebra, called the Lie algebra of **outer derivations**. Several questions naturally arise.

- Is there a group corresponding to the Lie algebra of outer derivations?
- What is the group that corresponds to the hamiltonian vector fields?

In Section 18.4 we will describe these "groups" in the context of Lie algebroids.
$\diamondsuit$

We can form the sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\infty(M) & \longrightarrow & \chi(M) & \longrightarrow & \chi^2(M) \\ & & h & \longmapsto & X_h & & \\ & & & & X & \longmapsto & \mathcal{L}_X \Pi \end{array}$$

where the composition of two maps is 0. Hence, we have a complex. At $\chi(M)$, the homology group is

$$H^1_\Pi(M) := \frac{\text{Poisson vector fields}}{\text{hamiltonian vector fields}} \ .$$

This is called the **first Poisson cohomology**.

The homology at $\chi^0(M) = C^\infty(M)$ is called **0-th Poisson cohomology** $H^0_\Pi(M)$, and consists of the **Casimir functions**, *i.e.* the functions f such that $\{f, h\} = 0$, for all $h \in C^\infty(M)$. (For the trivial Poisson structure $\{\cdot, \cdot\} = 0$, this is all of $C^\infty(M)$.)

See Section 5.1 for a geometric description of these cohomology spaces. See Section 4.5 for their interpretation in the symplectic case. Higher Poisson cohomology groups will be defined in Section 18.4.

4 Normal Forms

Throughout this and the next chapter, our goal is to understand what Poisson manifolds look like geometrically.

4.1 Lie's Normal Form

We will prove the following result in Section 4.3.

Theorem 4.1 (Lie [106]) *If Π is a Poisson structure on M whose matrix of structure functions, $\pi_{ij}(x)$, has constant rank, then each point of M is contained in a local coordinate system with respect to which (π_{ij}) is constant.*

Remarks.

1. The assumption above of constant rank was not stated by Lie, although it was used implicitly in his proof.
2. Since Theorem 4.1 is a local result, we only need to require the matrix (π_{ij}) to have *locally* constant rank. This is a reasonable condition to impose, as the structure functions π_{ij} will always have locally constant rank on an open dense set of M. To see this, notice that the set of points in M where (π_{ij}) has maximal rank is open, and then proceed inductively on the complement of the closure of this set (exercise!). Notice that the set of points where the rank of (π_{ij}) is *maximal* is not necessarily dense. For instance, consider $\mathbb{R}^2$ with $\{x_1, x_2\} = \varphi(x_1, x_2)$ given by an arbitrary function φ.
3. Points where (π_{ij}) has locally constant rank are called **regular**. If all points of M are regular, M is called a **regular Poisson manifold**. A Lie-Poisson manifold $\mathfrak{g}^*$ is not regular unless $\mathfrak{g}$ is abelian, though the regular points of $\mathfrak{g}^*$ form, of course, an open dense subset.

$\diamondsuit$

4.2 A Faithful Representation of $\mathfrak{g}$

We will now use Theorem 4.1 to construct the pointwise faithful representation of $\mathfrak{g}$ needed to complete the proof of the Poincaré-Birkhoff-Witt theorem.

On any Poisson manifold M there is a vector bundle morphism $\widetilde{\Pi} : T^*M \to TM$ defined by

$$\alpha(\widetilde{\Pi}(\beta)) = \Pi(\alpha, \beta) \ , \qquad \text{for any } \alpha, \beta \in T^*M \ .$$

We can write hamiltonian vector fields in terms of $\widetilde{\Pi}$ as $X_f = \widetilde{\Pi}(df)$. Notice that $\widetilde{\Pi}$ is an isomorphism exactly when rank $\Pi = \dim M$, *i.e.* when Π defines a symplectic structure. If we express Π by a matrix (π_{ij}) with respect to some basis, then the same matrix (π_{ij}) represents the map $\widetilde{\Pi}$.

Let $M = \mathfrak{g}^*$ have coordinates $\mu_1, \ldots, \mu_n$ and Poisson structure $\{\mu_i, \mu_j\} = \sum c_{ijk}\mu_k$. If $v_1, \ldots, v_n$ is the corresponding basis of vectors on $\mathfrak{g}$, then we find a representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ by mapping

$$v_i \longmapsto -X_{\mu_i} \ .$$

More generally, we can take $v \in \mathfrak{g}$ to $-X_v$ using the identification $\mathfrak{g} = \mathfrak{g}^{**} \subseteq C^\infty(\mathfrak{g}^*)$. However, this homomorphism might be trivial. In fact, it seldom provides the pointwise faithful representation needed to prove the Poincaré-Birkhoff-Witt theorem. Instead, we use the following trick.

For a regular point in $\mathfrak{g}^*$, Theorem 4.1 states that there is a neighborhood U with canonical coordinates $q_1, \ldots, q_k, p_1, \ldots, p_k, c_1, \ldots, c_\ell$ such that $\Pi = \sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$ (*cf.* Example 2 of Section 3.4). In terms of $\widetilde{\Pi}$, we have

$$\begin{aligned} \widetilde{\Pi}(dq_i) &= -\frac{\partial}{\partial p_i} \\ \widetilde{\Pi}(dp_i) &= \frac{\partial}{\partial q_i} \\ \widetilde{\Pi}(dc_i) &= 0\ . \end{aligned}$$

This implies that the hamiltonian vector field of any function will be a linear combination of the vector fields $\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}$. Unless the structure defined by Π on the regular part of $\mathfrak{g}^*$ is symplectic (that is $l = 0$), the representation of $\mathfrak{g}$ as differential operators on $C^\infty(\mathfrak{g}^*)$ will have a kernel, and hence will not be faithful.

To remedy this, we lift the Lie-Poisson structure to a symplectic structure on a larger manifold. Let $U \times \mathbb{R}^\ell$ have the original coordinates $q_1, \ldots, q_k$, $p_1, \ldots, p_k$, $c_1, \ldots, c_\ell$ lifted from the coordinates on U, plus the coordinates $d_1, \ldots, d_\ell$ lifted from the standard coordinates of $\mathbb{R}^\ell$. We define a Poisson structure $\{\cdot,\cdot\}'$ on $U \times \mathbb{R}^\ell$ by

$$\Pi' = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_i \frac{\partial}{\partial c_i} \wedge \frac{\partial}{\partial d_i}\ .$$

We now take the original coordinate functions μ_i on U and lift them to functions, still denoted μ_i, on $U \times \mathbb{R}^\ell$. Because the μ_i's are independent of the d_j's, we see that $\{\mu_i, \mu_j\}' = \{\mu_i, \mu_j\} = \sum c_{ijk}\mu_k$. Thus the homomorphism $\mathfrak{g} \to C^\infty(U)$, $v_i \mapsto \mu_i$, lifts to a map

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & C^\infty(U \times \mathbb{R}^\ell) & \xrightarrow{-\widetilde{\Pi}' \circ d} & \chi(U \times \mathbb{R}^\ell) \\ v_i & \longrightarrow & \mu_i & \longrightarrow & -\widetilde{\Pi}'(d\mu_i) = -X'_{\mu_i}\ . \end{array}$$

The composed map is a Lie algebra homomorphism. The differentials $d\mu_1, \ldots, d\mu_n$ are pointwise linearly independent on U and thus also on $U \times \mathbb{R}^\ell$. Since $-\widetilde{\Pi}'$ is an isomorphism, the hamiltonian vector fields $-X'_{\mu_1}, \ldots, -X'_{\mu_k}$ are also pointwise linearly independent, and we have the pointwise faithful representation needed to complete the proof of the Poincaré-Birkhoff-Witt theorem.

Remarks.

1. Section 2.4 explains how to go from a pointwise faithful representation to a local Lie group. In practice, it is not easy to find the canonical coordinates in U, nor is it easy to integrate the X'_{μ_i}'s.

2. The integer ℓ is called the **rank** of the Lie algebra, and it equals the dimension of a Cartan subalgebra when $\mathfrak{g}$ is semisimple. This rank should not be confused with the rank of the Poisson structure.

$\diamondsuit$

4.3 The Splitting Theorem

We will prove Theorem 4.1 as a consequence of the following more general result.

Theorem 4.2 (Weinstein [163]) *On a Poisson manifold* (M, Π)*, any point* $O \in M$ *has a coordinate neighborhood with coordinates* $(q_1, \ldots, q_k, p_1, \ldots, p_k, y_1, \ldots, y_\ell)$ *centered at* O*, such that*

$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j} \varphi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \qquad \textit{and} \qquad \varphi_{ij}(0) = 0 \ .$$

The rank of Π at O is $2k$. Since φ depends only on the y_i's, this theorem gives a decomposition of the neighborhood of O as a product of two Poisson manifolds: one with rank $2k$, and the other with rank 0 at O.

Proof. We prove the theorem by induction on $\rho = \text{rank}\ \Pi(O)$.

- If $\rho = 0$, we are done, as we can label all the coordinates y_i.
- If $\rho \neq 0$, then there are functions f, g with $\{f, g\}(O) \neq 0$. Let $p_1 = g$ and look at the operator X_{p_1}. We have $X_{p_1}(f)(O) = \{f, g\}(O) \neq 0$. By the flow box theorem, there are coordinates for which X_{p_1} is one of the coordinate vector fields. Let q_1 be the coordinate function such that $X_{p_1} = \frac{\partial}{\partial q_1}$; hence, $\{q_1, p_1\} = X_{p_1} q_1 = 1$. (In practice, finding q_1 amounts to solving a system of ordinary differential equations.) X_{p_1}, X_{q_1} are linearly independent at O and hence in a neighborhood of O. By the Frobenius theorem, the equation $[X_{q_1}, X_{p_1}] = -X_{\{q_1,p_1\}} = -X_1 = 0$ shows that these vector fields can be integrated to define a two dimensional foliation near O. Hence, we can find functions $y_1, \ldots, y_{n-2}$ such that

  1. $dy_1, \ldots, dy_{n-2}$ are linearly independent;
  2. $X_{p_1}(y_j) = X_{q_1}(y_j) = 0$. That is to say, $y_1, \ldots, y_{n-2}$ are transverse to the foliation. In particular, $\{y_j, q_1\} = 0$ and $\{y_j, p_1\} = 0$.

  Exercise 10
  Show that $dp_1, dq_1, dy_1, \ldots, dy_{n-2}$ are all linearly independent.

  Therefore, we have coordinates such that $X_{q_1} = -\frac{\partial}{\partial p_1}, X_{p_1} = \frac{\partial}{\partial q_1}$, and by Poisson's theorem

  $$\{\{y_i, y_j\}, p_1\} = 0$$
  $$\{\{y_i, y_j\}, q_1\} = 0$$

  We conclude that $\{y_i, y_j\}$ must be a function of the y_i's. Thus, in these coordinates, the Poisson structure is

  $$\Pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{1}{2} \sum_{i,j} \varphi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \ .$$

- If $\rho = 2$, we are done. Otherwise, we apply the argument above to the structure $\frac{1}{2} \sum \varphi_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$.

□

4.4 Special Cases of the Splitting Theorem

1. If the rank is locally constant, then $\varphi_{ij} \equiv 0$ and the splitting theorem recovers Lie's theorem (Theorem 4.1). Hence, by the argument in Section 4.2, our proof of the Poincaré-Birkhoff-Witt theorem is completed.

2. At the origin of a Lie-Poisson manifold, we only have y_i's, and the term $\sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$ does not appear.

3. A **symplectic manifold** is a Poisson manifold (M, Π) where rank $\Pi = \dim M$ everywhere. In this case, Lie's theorem (or the splitting theorem) gives canonical coordinates $q_1, \ldots, q_k$, $p_1, \ldots, p_k$ such that
$$\Pi = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \ .$$
In other words, $\widetilde{\Pi} : T^*M \to TM$ is an isomorphism satisfying
$$\widetilde{\Pi}(dq_i) = -\frac{\partial}{\partial p_i} \qquad \text{and} \qquad \widetilde{\Pi}(dp_i) = \frac{\partial}{\partial q_i} \ .$$
Its inverse $\widetilde{\omega} = \widetilde{\Pi}^{-1} : TM \to T^*M$ defines a 2-form $\omega \in \Omega^2(M)$ by $\omega(u,v) = \widetilde{\omega}(u)(v)$, or equivalently by $\omega = (\widetilde{\Pi}^{-1})^*(\Pi)$. With respect to the canonical coordinates, we have
$$\omega = \sum dq_i \wedge dp_i \ ,$$
which is the content of **Darboux's theorem** for symplectic manifolds. This also gives a quick proof that ω is a closed 2-form. ω is called a **symplectic form**.

4.5 Almost Symplectic Structures

Suppose that (M, Π) is an **almost symplectic manifold**, that is, Π is non-degenerate but may not satisfy the Jacobi identity. Then $\widetilde{\Pi} : T^*M \to TM$ is an isomorphism, and its inverse $\widetilde{\omega} = \widetilde{\Pi}^{-1} : TM \to T^*M$ defines a 2-form $\omega \in \Omega^2(M)$ by $\omega(u,v) = \widetilde{\omega}(u)(v)$.

Conversely, any 2-form $\omega \in \Omega^2(M)$ defines a map
$$\widetilde{\omega} : TM \to T^*M \qquad \text{by} \qquad \widetilde{\omega}(u)(v) = \omega(u,v) \ .$$
We also use the notation $\widetilde{\omega}(v) = i_v(\omega) = v \lrcorner \omega$. Suppose that ω is non-degenerate, meaning that $\widetilde{\omega}$ is invertible. Then for any function $h \in C^\infty(M)$, we define the **hamiltonian vector field** X_h by one of the following equivalent formulations:

- $X_h = \widetilde{\omega}^{-1}(dh)$,
- $X_h \lrcorner \omega = dh$, or
- $\omega(X_h, Y) = Y \cdot h$.

There are also several equivalent definitions for a bracket operation on $C^\infty(M)$, including
$$\{f, g\} = \omega(X_f, X_g) = X_g(f) = -X_f(g) \ .$$
It is easy to check the anti-symmetry property and the Leibniz identity for the bracket. The next section discusses different tests for the Jacobi identity.

4.6 Incarnations of the Jacobi Identity

Theorem 4.3 *The bracket* $\{\cdot,\cdot\}$ *on an almost symplectic manifold (defined in the previous section) satisfies the Jacobi identity if and only if* $d\omega = 0$.

Exercise 11
Prove this theorem. Hints:

- With coordinates, write ω locally as $\omega = \frac{1}{2}\sum \omega_{ij} dx_i \wedge dx_j$. The condition for ω to be closed is then
$$\frac{\partial \omega_{ij}}{\partial x_k} + \frac{\partial \omega_{jk}}{\partial x_i} + \frac{\partial \omega_{ki}}{\partial x_j} = 0 \ .$$
Since $(\omega_{ij})^{-1} = (-\pi_{ij})$, this equation is equivalent to
$$\sum_k \left(\frac{\partial \pi_{ij}}{\partial x_k}\pi_{k\ell} + \frac{\partial \pi_{j\ell}}{\partial x_k}\pi_{ki} + \frac{\partial \pi_{\ell i}}{\partial x_k}\pi_{kj} \right) = 0 \ .$$
Cf. Exercise 8 in Section 3.4.
- Without coordinates, write $d\omega$ in terms of Lie derivatives and Lie brackets as
$$\begin{aligned} d\omega(X,Y,Z) = \quad & \mathcal{L}_X(\omega(Y,Z)) + \mathcal{L}_Y(\omega(Z,X)) + \mathcal{L}_Z(\omega(X,Y)) \\ & -\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y) \ . \end{aligned}$$
At each point, choose functions f, g, h whose hamiltonian vector fields at that point coincide with X, Y, Z. Apply $\mathcal{L}_{X_f}(\omega(X_g, X_h)) = \{\{g,h\},f\}$ and $-\omega([X_f,X_g],X_h) = \{\{f,g\},h\}$.

Remark. For many geometric structures, an integrability condition allows us to drop the "almost" from the description of the structure, and find a standard expression in canonical coordinates. For example, an almost complex structure is complex if it is integrable, in which case we can find complex coordinates where the almost complex structure becomes multiplication by the complex number i. Similarly, an almost Poisson structure Π is integrable if Π satisfies the Jacobi identity, in which case Lie's theorem provides a normal form near points where the rank is locally constant. Finally, an almost symplectic structure ω is symplectic if ω is closed, in which case there exist coordinates where ω has the standard Darboux normal form. $\diamondsuit$

We can reformulate the connection between the Jacobi identity and $d\omega = 0$ in terms of Lie derivatives. Cartan's magic formula states that, for a vector field X and a differential form η,

$$\mathcal{L}_X \eta = d(X \lrcorner \eta) + X \lrcorner d\eta \ .$$

Using this, we compute

$$\begin{aligned} \mathcal{L}_{X_h}\omega &= d(X_h \lrcorner \omega) + X_h \lrcorner d\omega \\ &= d(dh) + X_h \lrcorner d\omega \\ &= X_h \lrcorner d\omega \ . \end{aligned}$$

We conclude that $d\omega = 0$ if and only if $\mathcal{L}_{X_h}\omega = 0$ for each $h \in C^\infty(M)$. (One implication requires the fact that hamiltonian vector fields span the whole tangent bundle, by invertibility of $\widetilde{\omega}$.) It follows that another characterization for ω being closed is ω being invariant under all hamiltonian flows. This is equivalent to saying that hamiltonian flows preserve Poisson brackets, *i.e.* $\mathcal{L}_{X_h}\Pi = 0$ for all h. Ensuring that the symplectic structure be invariant under hamiltonian flows is one of the main reasons for requiring that a symplectic form be closed.

While the Leibniz identity states that all hamiltonian vector fields are derivations of pointwise multiplication of functions, the Jacobi identity states that all hamiltonian vector fields are derivations of the bracket $\{\cdot,\cdot\}$. We will now check directly the relation between the Jacobi identity and the invariance of Π under hamiltonian flows, in the language of hamiltonian vector fields. Recall that the operation of Lie derivative is a derivation on contraction of tensors, and therefore

$$\begin{aligned}
\{\{f,g\},h\} = X_h\{f,g\} &= X_h(\Pi(df,dg)) \\
&= (\mathcal{L}_{X_h}\Pi)(df,dg) + \Pi(\mathcal{L}_{X_h}df,dg) + \Pi(df,\mathcal{L}_{X_h}dg) \\
&= (\mathcal{L}_{X_h}\Pi)(df,dg) + \Pi(d\mathcal{L}_{X_h}f,dg) + \Pi(df,d\mathcal{L}_{X_h}g) \\
&= (\mathcal{L}_{X_h}\Pi)(df,dg) + \Pi(d\{f,h\},dg) + \Pi(df,d\{g,h\}) \\
&= (\mathcal{L}_{X_h}\Pi)(df,dg) + \{X_hf,g\} + \{f,X_hg\} \\
&= (\mathcal{L}_{X_h}\Pi)(df,dg) + \{\{f,h\},g\} + \{f,\{g,h\}\}\ .
\end{aligned}$$

We conclude that the Jacobi identity holds if and only if $(\mathcal{L}_{X_h}\Pi)(df,dg) = 0$ for all $f,g,h \in C^\infty(M)$.

5 Local Poisson Geometry

Roughly speaking, any Poisson manifold is obtained by gluing together symplectic manifolds. The study of Poisson structures involves both local and global concerns: the local structure of symplectic leaves and their *transverse structures*, and the global aspects of how symplectic leaves fit together into a foliation.

5.1 Symplectic Foliation

At a regular point p of a Poisson manifold M, the subspace of T_pM spanned by the hamiltonian vector fields of the canonical coordinates at that point depends only on the Poisson structure. When the Poisson structure is regular (see Section 4.1), the image of $\widetilde{\Pi}$ (formed by the subspaces above) is an involutive subbundle of TM. Hence, there is a natural foliation of M by symplectic manifolds whose dimension is the rank of Π. These are called the **symplectic leaves**, forming the **symplectic foliation**.

It is a remarkable fact that symplectic leaves exist through every point, even on Poisson manifolds $(M, \{\cdot,\cdot\})$ where the Poisson structure is *not* regular. (Their existence was first proved in this context by Kirillov [95].) In general, the symplectic foliation is a singular foliation.

The symplectic leaves are determined locally by the splitting theorem (Section 4.3). For any point O of the Poisson manifold, if (q,p,y) are the normal coordinates as in Theorem 4.2, then the symplectic leaf through O is given locally by the equation $y = 0$.

The Poisson brackets on M can be calculated by restricting to the symplectic leaves and then assembling the results.

Remark. The 0-th Poisson cohomology, H^0_Π, (see Section 3.6) can be interpreted as the set of smooth functions on the space of symplectic leaves. It may be useful to think of H^1_Π as the "vector fields on the space of symplectic leaves" [72]. $\diamondsuit$

Examples.

1. For the zero Poisson structure on M, $H^0_\Pi(M) = \mathbb{C}^\infty(M)$ and $H^1_\Pi(M)$ consists of all the vector fields on M.

2. For a symplectic structure, the first Poisson cohomology coincides with the first de Rham cohomology via the isomorphisms

$$\begin{array}{rcl} \text{Poisson vector fields} & \xrightarrow{\tilde{\omega}} & \text{closed 1-forms} \\ \text{hamiltonian vector fields} & \xrightarrow{\tilde{\omega}} & \text{exact 1-forms} \end{array}$$

$$H^1_\Pi(M) \xrightarrow{\simeq} H^1_{\text{deRham}}(M)\ .$$

   In the symplectic case, the 0-th Poisson cohomology is the set of locally constant functions, $H^0_{\text{deRham}}(M)$. This agrees with the geometric interpretation of Poisson cohomology in terms of the space of symplectic leaves.

   On the other hand, on a symplectic manifold, $H^1_\Pi \simeq H^1_{\text{deRham}}$ gives a finite dimensional space of "vector fields" over the discrete space of connected components

◇

Problem. Is there an interesting and natural way to give a "structure" to the point of the leaf space representing a connected component M of a symplectic manifold in such a way that the infinitesimal automorphisms of this "structure" correspond to elements of $H^1_{\mathrm{deRham}}(M)$? ◇

5.2 Transverse Structure

As we saw in the previous section, on a Poisson manifold (M, Π) there is a natural singular foliation by symplectic leaves. For each point $m \in M$, we can regard M as fibering locally over the symplectic leaf through m. Locally, this leaf has canonical coordinates $q_1, \ldots, q_k, p_1, \ldots, p_k$, where the bracket is given by canonical symplectic relations. While the symplectic leaf is well-defined, each choice of coordinates $y_1, \ldots, y_\ell$ in Theorem 4.2 can give rise to a different last term for Π,

$$\frac{1}{2}\sum_{i,j} \varphi_{ij}(y)\frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \ ,$$

called the **transverse Poisson structure** (of dimension ℓ). Although the transverse structures themselves are not uniquely defined, they are all isomorphic [163]. Going from this local isomorphism of the transverse structures to a structure of "Poisson fiber bundle" on a *neighborhood* of a symplectic leaf seems to be a difficult problem [90].

Example. Suppose that Π is regular. Then the transverse Poisson structure is trivial and the fibration over the leaf is locally trivial. However, the bundle structure can still have holonomy as the leaves passing through a transverse section wind around one another. ◇

Locally, the transverse structure is determined by the structure functions $\pi_{ij}(y) = \{y_i, y_j\}$ which vanish at $y = 0$. Applying a Taylor expansion centered at the origin, we can write

$$\pi_{ij}(y) = \sum_k c_{ijk} y_k + O(y^2)$$

where $O(y^2)$ can be expressed as $\sum d_{ijkl}(y) y_k y_\ell$, though the d_{ijkl} are not unique outside of $y = 0$.

Since the π_{ij} satisfy the Jacobi identity, it is easy to show using the Taylor expansion of the jacobiator that the truncation

$$\pi'_{ij}(y) = \sum_k c_{ijk} y_k$$

also satisfies the Jacobi identity. Thus, the functions π'_{ij} define a Poisson structure, called the **linearized Poisson structure** of π_{ij}.

From Section 3.4 we know that a linear Poisson structure can be identified with a Poisson structure on the dual of a Lie algebra. In this way, for any point $m \in M$, there is an associated Lie algebra, called the **transverse Lie algebra**. We will now show that this transverse Lie algebra can be identified *intrinsically* with the *conormal space* to the symplectic leaf $\mathcal{O}_m$ through m, so that the linearized transverse Poisson structure lives naturally on the normal space to the leaf. When the Poisson structure vanishes at the point m, this normal space is just the tangent space T_mM.

Recall that the normal space to $\mathcal{O}_m$ is the quotient

$$N\mathcal{O}_m = T_mM \Big/ T_m\mathcal{O}_m \ .$$

The **conormal space** is the dual space $(N\mathcal{O}_m)^*$. This dual of this quotient space of T_mM can be identified with the subspace $(T_m\mathcal{O}_m)^\circ$ of cotangent vectors at m which annihilate $T_m\mathcal{O}_m$:

$$(N\mathcal{O}_m)^* \simeq (T_m\mathcal{O}_m)^\circ \subseteq T_m^*M \ .$$

To define the bracket on the conormal space, take two elements $\alpha, \beta \in (T_m\mathcal{O}_m)^\circ$. We can choose functions $f, g \in C^\infty(M)$ such that $df(m) = \alpha, dg(m) = \beta$. In order to simplify computations, we can even choose such f, g which are zero along the symplectic leaf, that is, $f, g|_{\mathcal{O}_m} \equiv 0$. The bracket of α, β is

$$[\alpha, \beta] = d\{f, g\}(m) \ .$$

This is well-defined because

- $f, g|_{\mathcal{O}_m} \equiv 0 \Rightarrow \{f, g\}|_{\mathcal{O}_m} \equiv 0 \Rightarrow d\{f, g\}|_{\mathcal{O}_m} \in (T_m\mathcal{O}_m)^\circ$. That the set of functions vanishing on the symplectic leaf is closed under the bracket operation follows, for instance, from the splitting theorem.
- The Leibniz identity implies that the bracket $\{\cdot, \cdot\}$ only depends on first derivatives. Hence, the value of $[\alpha, \beta]$ is independent of the choice of f and g.

There is then a Lie algebra structure on $(T_m\mathcal{O}_m)^\circ$ and a bundle of duals of Lie algebras over a symplectic leaf. The next natural question is: does this linearized structure determine the Poisson structure on a neighborhood?

5.3 The Linearization Problem

Suppose that we have structure functions

$$\pi_{ij}(y) = \sum_k c_{ijk} y_k + O(y^2) \ .$$

Is there a change of coordinates making the π_{ij} linear? More specifically, given π_{ij}, is there a new coordinate system of the form

$$z_i = y_i + O(y^2)$$

such that $\{z_i, z_j\} = \sum c_{ijk} z_k$?

This question resembles Morse theory where, given a function whose Taylor expansion only has quadratic terms or higher, we ask whether there exist some coordinates for which the higher terms vanish. The answer is yes (without further assumptions on the function) if and only if the quadratic part is non-degenerate.

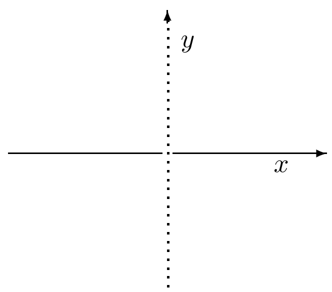
When the answer to the linearization problem is affirmative, we call the structure π_{ij} **linearizable**. Given fixed c_{ijk}, if π_{ij} is linearizable for *all* choices of $O(y^2)$, then we say that the transverse Lie algebra $\mathfrak{g}$ defined by c_{ijk} is **non-degenerate**. Otherwise, it is called **degenerate**.

There are several versions of non-degeneracy, depending on the kind of coordinate change allowed: for example, formal, C^∞ or analytic. Here is a brief summary of some results on the non-degeneracy of Lie algebras.

- It is not hard to see that the zero (or commutative) Lie algebra is degenerate for dimensions ≥ 2. Two examples of non-linearizable structures in dimension 2 demonstrating this degeneracy are

  1. $\{y_1, y_2\} = y_1^2 + y_2^2$,
  2. $\{y_1, y_2\} = y_1 y_2$.

- Arnold [6] showed that the two-dimensional Lie algebra defined by $\{x, y\} = x$ is non-degenerate in all three versions described above. If one decomposes this Lie algebra into symplectic leaves, we see that two leaves are given by the half-planes $\{(x, y)|x < 0\}$ and $\{(x, y)|x > 0\}$. Each of the points $(0, y)$ comprises another symplectic leaf. See the following figure.



- Weinstein [163] showed that, if $\mathfrak{g}$ is semi-simple, then $\mathfrak{g}$ is formally non-degenerate. At the same time he showed that $\mathfrak{sl}(2; \mathbb{R})$ is C^∞ degenerate.

- Conn [27] first showed that if $\mathfrak{g}$ is semi-simple, then $\mathfrak{g}$ is analytically non-degenerate. Later [28], he proved that if $\mathfrak{g}$ is semi-simple of compact type (*i.e.* the corresponding Lie group is compact), then $\mathfrak{g}$ is C^∞ non-degenerate.

- Weinstein [166] showed that if $\mathfrak{g}$ is semi-simple of *non*-compact type and has real rank of at least 2, then $\mathfrak{g}$ is C^∞ degenerate.

- Cahen, Gutt and Rawnsley [22] studied the non-linearizability of some Poisson Lie groups.

Remark. When a Lie algebra is degenerate, there is still the question of whether a change of coordinates can remove higher order terms. Several students of Arnold [6] looked at the 2-dimensional case (*e.g.*: $\{x, y\} = (x^2 + y^2)^p + \ldots$) to investigate which Poisson structures could be reduced in a manner analogous to linearization. Quadratization (*i.e.* equivalence to quadratic structures after a coordinate change) has been established in some situations for structures with sufficiently nice quadratic part by Dufour [49] and Haraki [80]. ♢

We can view Poisson structures near points where they vanish as deformations of their linearizations. If we expand a Poisson structure π_{ij} as

$$\{x_i, x_j\} = \pi_1(x) + \pi_2(x) + \dots ,$$

where $\pi_k(x)$ denotes a homogeneous polynomial of degree k in x, then we can define a deformation by

$$\{x_i, x_j\}_\varepsilon = \pi_1(x) + \varepsilon\pi_2(x) + \dots .$$

This indeed satisfies the Jacobi identity for all ε, and $\{x_i, x_j\}_0 = \pi_1(x)$ is a linear Poisson structure. All the $\{\cdot,\cdot\}_\varepsilon$'s are isomorphic for $\varepsilon \neq 0$.

5.4 The Cases of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2;\mathbb{R})$

We shall compare the degeneracies of $\mathfrak{sl}(2;\mathbb{R})$ and $\mathfrak{su}(2)$, which are both 3-dimensional as vector spaces. First, on $\mathfrak{su}(2)$ with coordinate functions μ_1, μ_2, μ_3, the bracket operation is defined by

$$\begin{array}{rcl} \{\mu_1,\mu_2\} &=& \mu_3 \\ \{\mu_2,\mu_3\} &=& \mu_1 \\ \{\mu_3,\mu_1\} &=& \mu_2 \ . \end{array}$$

The Poisson structure is trivial only at the origin. It is easy to check that the function $\mu_1^2 + \mu_2^2 + \mu_3^2$ is a Casimir function, meaning that it is constant along the symplectic leaves. By rank considerations, we see that the symplectic leaves are exactly the level sets of this function, *i.e.* spheres centered at the origin. This foliation is quite stable. In fact, $\mathfrak{su}(2)$, which is semi-simple of real rank 1, is C^∞ *non*-degenerate.

On the other hand, $\mathfrak{sl}(2;\mathbb{R})$ with coordinate functions μ_1, μ_2, μ_3 has bracket operation defined by

$$\begin{array}{rcl} \{\mu_1,\mu_2\} &=& -\mu_3 \\ \{\mu_2,\mu_3\} &=& \mu_1 \\ \{\mu_3,\mu_1\} &=& \mu_2 \ . \end{array}$$

In this case, $\mu_1^2+\mu_2^2-\mu_3^2$ is a Casimir function, and the symplectic foliation consists of

- the origin,
- two-sheeted hyperboloids $\mu_1^2 + \mu_2^2 - \mu_3^2 = c < 0$,
- the cone $\mu_1^2 + \mu_2^2 - \mu_3^2 = 0$ punctured at the origin, and
- one-sheeted hyperboloids $\mu_1^2 + \mu_2^2 - \mu_3^2 = c > 0$.

There are now non-simply-connected symplectic leaves. Restricting to the horizontal plane $\mu_3 = 0$, the leaves form a set of concentric circles. It is possible to modify the Poisson structure slightly near the origin, so that the tangent plane to each symplectic leaf is tilted, and on the cross section $\mu_3 = 0$, the leaves spiral toward the origin. This process of "breaking the leaves" [163] requires that there be non-simply-connected leaves and that we employ a smooth perturbation whose derivatives all vanish at the origin (in order not to contradict Conn's results listed in the previous section, since such a perturbation cannot be analytic).

Part III
Poisson Category

6 Poisson Maps

Any Poisson manifold has an associated Poisson algebra, namely the algebra of its smooth functions equipped with the Poisson bracket. In this chapter, we will strengthen the analogy between algebras and spaces.

6.1 Characterization of Poisson Maps

Given two Poisson algebras $\mathcal{A}, \mathcal{B}$, an algebra homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ is called a **Poisson-algebra homomorphism** if ψ preserves Poisson brackets:

$$\psi\left(\{f, g\}_{\mathcal{A}}\right) = \{\psi(f), \psi(g)\}_{\mathcal{B}} \ .$$

A smooth map $\varphi : M \to N$ between Poisson manifolds M and N is called a **Poisson map** when

$$\varphi^*\left(\{f, g\}_N\right) = \{\varphi^*(f), \varphi^*(g)\}_M \ ,$$

that is, $\varphi^* : C^\infty(N) \to C^\infty(M)$ is a Poisson-algebra homomorphism. (Every homomorphism $C^\infty(N) \to C^\infty(M)$ of the commutative algebra structures arising from pointwise multiplication is of the form φ^* for a smooth map $\varphi : M \to N$ [1, 16].) A **Poisson automorphism** of a Poisson manifold (M, Π), is a diffeomorphism of M which is a Poisson map.

Remark. The **Poisson automorphisms** of a Poisson manifold (M, Π) form a group. For the trivial Poisson structure, this is the group of all diffeomorphisms. In general, flows of hamiltonian vector fields generate a significant part of the automorphism group. In an informal sense, the "Lie algebra" of the (infinite dimensional) group of Poisson automorphisms consists of the Poisson vector fields (see Section 3.6). $\diamondsuit$

Here are some alternative characterizations of Poisson maps:

- Let $\varphi : M \to N$ be a differentiable map between manifolds. A vector field $X \in \chi(M)$ is **φ-related** to a vector field Y on N when

  $$(T_x\varphi)\, X(x) = Y\left(\varphi(x)\right) \ , \quad \text{for all } x \in M \ .$$

  If the vector fields X and Y are φ-related, then φ takes integral curves of X to integral curves of Y.

  We indicate that X is φ-related to Y by writing

  $$Y = \varphi_* X \ ,$$

  though, in general, φ_* is *not* a map: there may be several vector fields Y on N that are φ-related to a given $X \in \chi(M)$, or there may be none. Thus we understand $Y = \varphi_* X$ as a relation and not as a map.

This definition extends to multivector fields via the induced map on higher wedge powers of the tangent bundle. For $X \in \chi^k(M)$ and $Y \in \chi^k(N)$, we say that X is **φ-related** to Y, writing $Y = \varphi_* X$, if

$$\left(\wedge^k T_x\varphi\right) X(x) = Y\left(\varphi(x)\right) \ , \qquad \text{for all } x \in M \ .$$

Now let $\Pi_M \in \chi^2(M), \Pi_N \in \chi^2(N)$ be bivector fields specifying Poisson structures in M and N. Then φ is a Poisson map if and only if

$$\Pi_N = \varphi_* \Pi_M \ .$$

Exercise 12
Prove that this is an equivalent description of Poisson maps.

- φ being a Poisson map is also equivalent to commutativity of the following diagram for all $x \in M$:

$$\begin{tikzcd} T_x^*M \arrow[r, "\widetilde{\Pi}_M(x)"] & T_xM \arrow[d, "T_x\varphi"] \\ T^*_{\varphi(x)}N \arrow[u, "T_x^*\varphi"] \arrow[r, "\widetilde{\Pi}_N(\varphi(x))"] & T_{\varphi(x)}N \end{tikzcd}$$

That is, φ is a Poisson map if and only if

$$\widetilde{\Pi}_N\left(\varphi(x)\right) = T_x\varphi \circ \widetilde{\Pi}_M(x) \circ T_x^*\varphi \ , \qquad \text{for all } x \in M \ .$$

Since it is enough to check this assertion on differentials of functions, this characterization of Poisson maps translates into $X_{\varphi^* h}$ being φ-related to X_h, for any $h \in C^\infty(N)$:

$$\begin{aligned} X_h(\varphi(x)) = \widetilde{\Pi}_N(\varphi(x))\left(dh\left(\varphi(x)\right)\right) &= (T_x\varphi)\left(\widetilde{\Pi}_M(x)\left(T_x^*\varphi\left(dh\left(\varphi(x)\right)\right)\right)\right) \\ &= (T_x\varphi)\left(\widetilde{\Pi}_M(x)\left(d\left(\varphi^* h(x)\right)\right)\right) \\ &= (T_x\varphi)\left(X_{\varphi^* h}(x)\right) \ , \end{aligned}$$

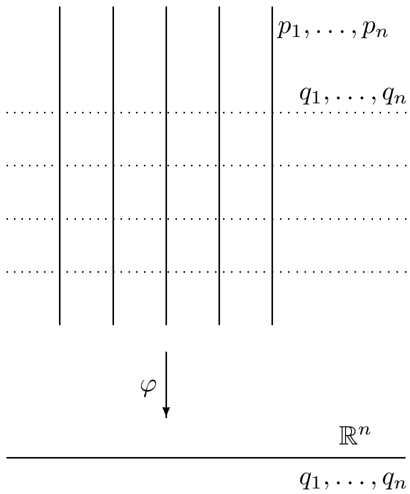
where the first equality is simply the definition of hamiltonian vector field.

The following example shows that $X_{\varphi^* h}$ depends on h itself and not just on the hamiltonian vector field X_h.

Example. Take the space $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ and Poisson structure defined by $\Pi = \sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$. The projection φ onto $\mathbb{R}^n$ with coordinates $(q_1, \ldots, q_n)$ and Poisson tensor 0 is trivially a Poisson map. Any $h \in C^\infty(\mathbb{R}^n)$ has $X_h = 0$, but if we pull h back by φ, we get

$$X_{h\circ\varphi} = -\sum_i \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i} \ .$$

This is a *non*-trivial *vertical* vector field on $\mathbb{R}^{2n}$ (vertical in the sense of being killed by the projection down to $\mathbb{R}^n$) . ♢



6.2 Complete Poisson Maps

Although a Poisson map $\varphi : M \to N$ preserves brackets, the image is not in general a union of symplectic leaves. Here is why: For a point $x \in M$, the image $\varphi(x)$ lies on some symplectic leaf $\mathcal{O}$ in N. We can reach any other point $y \in \mathcal{O}$ from $\varphi(x)$ by following the trajectory of (possibly more than one) hamiltonian vector field X_h. While we can lift X_h to the hamiltonian vector field $X_{\varphi^* h}$ near x, knowing that X_h is complete does not ensure that $X_{\varphi^* h}$ is complete. Consequently, we may not be able to lift the entire trajectory of X_h, so the point y is not necessarily in the image of φ. Still, the image of φ is a union of *open subsets* of symplectic leaves. The following example provides a trivial illustration of this fact.

Example. Let $\varphi : U \hookrightarrow \mathbb{R}^{2n}$ be the inclusion of an open strict subset U of the space $\mathbb{R}^{2n}$ with Poisson structure as in the last example of the previous section. Complete hamiltonian vector fields on $\mathbb{R}^{2n}$ will not lift to complete vector fields on U. ♢

To exclude examples like this we make the following definition.

A Poisson map $\varphi : M \to N$ is **complete** if, for each $h \in C^\infty(N)$, X_h being a complete vector field implies that $X_{\varphi^* h}$ is also complete.

Proposition 6.1 *The image of a complete Poisson map is a union of symplectic leaves.*

Proof. From any image point $\varphi(x)$, we can reach any other point on the same symplectic leaf of N by a chain of integral curves of complete hamiltonian vector fields, X_h's. The definition of completeness was chosen precisely to guarantee that the $X_{\varphi^* h}$'s are also complete. Hence, we can integrate them without restriction, and their flows provide a chain on M. The image of this chain on M has to be the original chain on N since X_h and $X_{\varphi^* h}$ are φ-related. We conclude that any point on the leaf of $\varphi(x)$ is contained in the image of φ. $\square$

Remarks.

1. In the definition of complete map, we can replace completeness of X_h by the condition that X_h has compact support, or even by the condition that h has compact support.

2. A Poisson map does *not* necessarily map symplectic leaves *into* symplectic leaves. Even in the simple example (previous section) of projection $\mathbb{R}^{2n} \to \mathbb{R}^n$, while $\mathbb{R}^{2n}$ has only one leaf, each point of $\mathbb{R}^n$ is a symplectic leaf.

$\diamondsuit$

The example of projecting $\mathbb{R}^{2n}$ to $\mathbb{R}^n$ is important to keep in mind. This projection is a complete Poisson map, as X_h is always trivial (and thus complete) on $\mathbb{R}^n$ and the pull-back $-\sum \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}$ is a complete vector field. However, if we restrict the projection to a subset of $\mathbb{R}^{2n}$, then the map will in general no longer be complete. The subsets of $\mathbb{R}^{2n}$ for which the projection restricts to a complete map are those which are open collections of full vertical p-fibers.

Here is another justification of our terminology.

Proposition 6.2 *Let $\varphi : M \to \mathbb{R}$ be a Poisson map. Then φ is complete if and only if X_φ is complete.*

Proof. First, assume that φ is complete. The hamiltonian vector field X_t for the identity (or coordinate) function $t : \mathbb{R} \to \mathbb{R}$ is trivial, and thus complete. Thus the vector field $X_{\varphi^* t} = X_\varphi$ is complete.

Conversely, assume that X_φ is complete, let $h : \mathbb{R} \to \mathbb{R}$ be any function, and compute

$$\begin{aligned} X_{\varphi^* h} = X_{h\circ\varphi} &= \widetilde{\Pi}_M(d(h\circ\varphi)) \\ &= \widetilde{\Pi}_M(h' \cdot d\varphi) \\ &= h' \cdot \widetilde{\Pi}_M(d\varphi) \\ &= h' \cdot X_\varphi \ . \end{aligned}$$

More precisely, $X_{\varphi^* h}(x) = h'(\varphi(x)) \cdot X_\varphi(x)$. At this point, recall that $X_f \cdot f = \{f, f\} = 0$ for any $f \in C^\infty(M)$ (the law of conservation of energy). Therefore, along any trajectory of X_φ, $h'(\varphi(x))$ is constant, so $X_{\varphi^* h}$, being a *constant* multiple of X_φ, must be also complete. $\square$

6.3 Symplectic Realizations

A Poisson map $\varphi : M \to N$ from a symplectic manifold M is called a **symplectic realization** of the Poisson manifold N.

Examples.

1. A basic example of symplectic realization is the inclusion map of a symplectic leaf into the ambient Poisson manifold.

2. A more significant example is provided by our construction in Section 4.2 of a faithful representation of $\mathfrak{g}$. We took an open subset U of $\mathfrak{g}^*$ with coordinates (q, p, c) and formed the symplectic space $U \times \mathbb{R}^\ell$ with coordinates (q, p, c, d). The map projecting $U \times \mathbb{R}^\ell$ back to $\mathfrak{g}^*$ is a symplectic realization for $\mathfrak{g}^*$. It is certainly not a complete Poisson map. It was constructed to have the property that functions on $\mathfrak{g}^*$ with linearly independent differentials pull back to functions on $U \times \mathbb{R}^\ell$ with linearly independent hamiltonian vector fields.

$\diamondsuit$

If a symplectic realization $\varphi : M \to N$ is a **submersion**, then locally there is a faithful representation of the functions on N (modulo the constants) by vector fields on M, in fact, by hamiltonian vector fields. Example 2 above turns out to be quite general:

Theorem 6.3 (Karasev [89], Weinstein [34]) *Every Poisson manifold has a surjective submersive symplectic realization.*

The proof of this theorem (which is omitted here) relies on finding symplectic realizations of open subsets covering a Poisson manifold and patching them together using a uniqueness property. It is often difficult to find the realization explicitly. We do not know whether completeness can be required in this theorem.

Example. Let $N = \mathbb{R}^2$ with Poisson bracket defined by $\{x, y\} = x$. (This is the dual of the 2-dimensional nontrivial Lie algebra.)

Exercise 13
Study 2-dimensional symplectic realizations of N. Find a surjective realization defined on the union of three copies of $\mathbb{R}^2$. Show that the inverse image of any neighborhood of the origin must have infinite area. Can you find a surjective submersive realization with a connected domain of dimension 2?

We next look for a symplectic realization $\mathbb{R}^4 \to N$. In terms of symplectic coordinates (q_1, p_1, q_2, p_2) on $\mathbb{R}^4$, the two functions

$$f = q_1 \qquad \text{and} \qquad g = p_1 q_1$$

satisfy the same bracket relation as the coordinates on N

$$\{f, g\} = \{q_1, p_1 q_1\} = q_1\{q_1, p_1\} = q_1 = f \ .$$

The map $(f, g) : \mathbb{R}^4 \to N$ is a symplectic realization with a singularity at the origin. To make it a non-singular submersion, simply redefine g to be $p_1 q_1 + q_2$. For this new representation, we compute the hamiltonian vector fields:

$$\begin{aligned} -X_f &= \frac{\partial}{\partial p_1} \\ -X_g &= p_1 \frac{\partial}{\partial p_1} - q_1 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial p_2} \ . \end{aligned}$$

Exercise 14
Is this realization complete? If we can integrate the vector fields X_f and X_g, we have essentially constructed the Lie group with Lie algebra $\mathbb{R}^2$, $[x, y] = x$.

$\diamondsuit$

6.4 Coisotropic Calculus

A submanifold C of a Poisson manifold M is called **coisotropic** if the ideal

$$\mathcal{I}_C = \{f \in C^\infty(M) \mid \ f|_C = 0\}$$

is closed under the bracket $\{\cdot,\cdot\}$. Recalling that $(TC)^\circ$ is the subspace of T^*M which annihilates TC, we can restate the condition above as

$$\widetilde{\Pi}\left((TC)^\circ\right) \subseteq TC \ .$$

Example. Suppose that (M,ω) is symplectic. Then C is coisotropic whenever

$$(TC)^\perp \subseteq TC \ ,$$

where $^\perp$ denotes the symplectic orthogonal space. The term *coisotropic* is linked to the concept of isotropic submanifolds in symplectic geometry. A submanifold C is called **isotropic** if

$$TC \subseteq (TC)^\perp \ .$$

In other words, C is isotropic if $i^*\omega = 0$, where $i : C \hookrightarrow M$ is the inclusion. For more on isotropic submanifolds, see the lecture notes by Bates and Weinstein [11]. $\diamondsuit$

Coisotropic submanifolds play a special role with regard to Poisson maps:

Proposition 6.4 (Weinstein [168]) *A map $f : M_1 \to M_2$ between Poisson manifolds is a Poisson map if and only if its graph is coisotropic in $M_1 \times \overline{M}_2$, where $\overline{M}_2$ has Poisson structure given by minus the Poisson tensor of M_2.*

This suggests defining a **Poisson relation** from M_1 to M_2 to be a coisotropic submanifold $R \subseteq M_1 \times \overline{M}_2$. For relations R and S from M_1 to M_2 and M_2 to M_3, respectively, we can define the composition $S \circ R$ by

$$S \circ R = \{(p_1,p_3) \mid \exists \ p_2 \in M_2, \ (p_1,p_2) \in R, \ (p_2,p_3) \in S\} \ .$$

We can then view Poisson relations as generalized Poisson maps using the following:

Proposition 6.5 *If R and S are Poisson relations as above with* clean composition *[11] in the sense that the composition $S \circ R$ is a smooth submanifold and $T(S \circ R) = TS \circ TR$, then $S \circ R$ is a Poisson relation.*

6.5 Poisson Quotients

Suppose that $\sim$ is an equivalence relation on a Poisson manifold M such that the quotient $M/\sim$ has a C^∞ structure for which the quotient map $\varphi : M \to M/\sim$ is a submersion. Then $\sim$ is called a **regular** equivalence relation. We say that the relation is **compatible** with the Poisson structure if $M/\sim$ has a Poisson structure for which φ is a Poisson map. Equivalently, the relation is compatible when $\varphi^*(C^\infty(M/\sim))$ forms a Poisson subalgebra of $C^\infty(M)$. The manifold $M/\sim$ is called a **Poisson quotient**. Theorem 6.3 implies that all Poisson manifolds can be realized as Poisson quotients of symplectic manifolds.

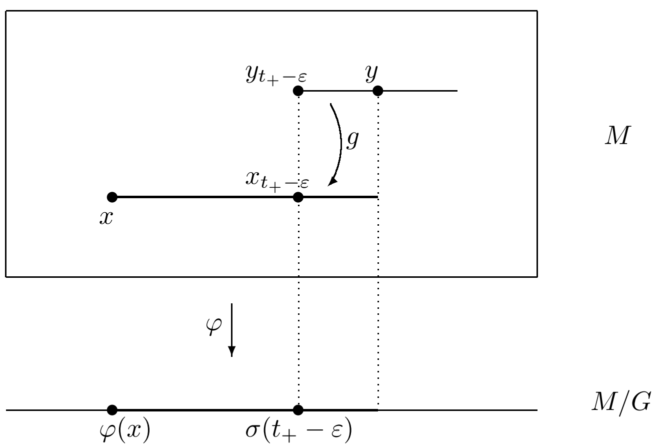
A regular equivalence relation defines a foliation on M. For the relation to be compatible, the set of functions constant along the leaves of this foliation should be closed under the bracket operation. With this notion of compatibility, it makes sense to refer to $\sim$ as compatible even if it is not regular.

Let G be a Lie group acting on a Poisson manifold M by Poisson maps. Then the set of G-invariant functions on M, $C^\infty(M)^G$, is closed under the bracket operation. Hence, if the orbit equivalence relation on M is regular, the orbit space M/G becomes a Poisson manifold, and the quotient map $M \to M/G$ is a Poisson map. When M is symplectic, this gives a symplectic realization of the quotient space. In fact, we have:

Proposition 6.6 *Under the assumptions above, the map $M \to M/G$ is complete.*

Proof. Given a complete function $h \in C^\infty(M/G) \simeq C^\infty(M)^G$ and a point $x \in M$, we need to show that the vector field $X_{h\circ\varphi}$ has a full integral curve through x. We shall suppose that this is not the case and find a contradiction.

Assume that there is a maximal interval (t_-, t_+) of definition for the integral curve through x for which t_+ is finite (the case of t_- finite is essentially the same). If we project down to $\varphi(x)$, then there is no obstruction to extending the integral curve $\sigma = \sigma_h$ of X_h through $\varphi(x)$. At time t_+, the curve σ reaches some point $\sigma(t_+) \in M/G$. Because φ is a projection, there is some $y \in \varphi^{-1}(\sigma(t_+))$. We can lift the integral curve σ to an integral curve of $X_{h\circ\varphi}$ through y and follow the curve back to a lift $y_{t_+-\varepsilon}$ of $\sigma(t_+ - \varepsilon)$. On the integral curve of $X_{h\circ\varphi}$ through x, there is also a lift $x_{t_+-\varepsilon}$ of $\sigma(t_+ - \varepsilon)$, and so there is some element g of G which maps $y_{t_+-\varepsilon}$ to $x_{t_+-\varepsilon}$. Because $X_{h\circ\varphi}$ is G-invariant, we can translate the integral curve through $y_{t_+-\varepsilon}$ by g to extend the curve through x past t_+, giving us a contradiction. Thus t_+ must be ∞.



□

Remark. The proof of Proposition 6.6 shows that any vector field invariant under a regular group action is complete if the projected vector field on the quotient is complete. ♢

For any manifold Q, the cotangent bundle T^*Q has a **canonical symplectic structure**. One way to construct it is to take local coordinates $x_1, \ldots, x_n$ on an open set $U \subseteq Q$. If $\pi : T^*Q \to Q$ is the natural projection, then we can put a corresponding coordinate system $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $T^*Q|_U$ such that $q_i = x_i \circ \pi$ and $p_i = \left\langle \cdot, \frac{\partial}{\partial x_i} \right\rangle$. We define the canonical symplectic structure by $\omega = \sum dq_i \wedge dp_i$ (or by $\Pi = \sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$). This expression for ω is preserved by changes of coordinates on U.

Alternatively, there is a **canonical 1-form** α on T^*Q, defined at any element $v \in T_b(T^*Q)$ by $\alpha(v) = b(\pi_* v)$. The canonical symplectic form is $\omega = -d\alpha$. One can check the equivalence of these two constructions by writing α in coordinates: $\alpha = \sum p_i dq_i$. This shows clearly that ω is independent of the choice of coordinates.

If $\gamma : Q_1 \to Q_2$ is a diffeomorphism, the natural lift of γ to a diffeomorphism $T^*Q_1 \to T^*Q_2$ is a Poisson map.

Example. Let $Q = G$ be a Lie group. It acts on itself by left translations and this action lifts to an action of G on T^*G by Poisson maps. The orbit space T^*G/G is then a Poisson manifold, which can be identified with $T_e^*G \simeq \mathfrak{g}^*$. This gives a Poisson structure on $\mathfrak{g}^*$.

Exercise 15
Show that this Poisson structure on $\mathfrak{g}^*$ is the negative of the one constructed in Section 3.1.

The quotient map $T^*G \to \mathfrak{g}^*$ provides a symplectic realization of $\mathfrak{g}^*$ which is, in general, larger than the one that we found in Section 4.2 (moreover, the symplectic realization here requires the existence of G). $\diamondsuit$

6.6 Poisson Submanifolds

When a Poisson map φ is an **embedding**, we often say that the image of φ is a **Poisson submanifold**, although sometimes the term is applied only when φ is also proper. If $M \subseteq N$ is a closed submanifold, then M is a Poisson submanifold if any of the following equivalent conditions holds:

1. The ideal $I_M \subseteq C^\infty(N)$ defined by
$$I_M = \{f \in C^\infty(N) \mid f|_M = 0\} .$$
is a Poisson ideal. That is, I_M is an ideal under the bracket multiplication as well as the pointwise multiplication of functions. In this case, the inclusion $M \hookrightarrow N$ corresponds to the quotient
$$C^\infty(M) \simeq C^\infty(N)/I_M \longleftarrow C^\infty(N) .$$

2. Every hamiltonian vector field on N is tangent to M.

3. At each point x in M, $\widetilde{\Pi}(T_x^*N) \subseteq T_xM$.

4. At each $x \in M$, $\Pi_x \in \wedge^2 T_xM$, where we consider $\wedge^2 T_xM$ as a subspace of $\wedge^2 T_xN$.

Remark. Symplectic leaves of a Poisson manifold N are minimal Poisson submanifolds, in the sense that they correspond (at least locally) to the maximal Poisson ideals in $C^\infty(N)$. They should be thought of as "points," since each maximal ideal of smooth functions on a manifold is the set of all functions which vanish at a point [16]. ♢

Suppose that M and N are symplectic with Poisson structures induced by the symplectic 2-forms ω_M and ω_N. For a map $\varphi : M \to N$, the symplectic condition $\varphi^*\omega_N = \omega_M$ does *not* make φ a Poisson map, unless φ is a local diffeomorphism. The following two examples illustrate this difference.

Examples.

1. The inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ of symplectic manifolds defined by mapping the coordinates $(q_1, p_1) \mapsto (q_1, p_1, 0, 0)$ *is* a symplectic embedding, but it is *not* a Poisson map, since $\{q_2, p_2\}_{\mathbb{R}^4} = 1$, while the bracket in $\mathbb{R}^2$ of their pull-backs is 0.

2. On the other hand, the projection $\mathbb{R}^4 \to \mathbb{R}^2$ given by mapping

$$(q_1, p_1, q_2, p_2) \longmapsto (q_1, p_1)$$

is a Poisson map, but is *not* symplectic, since $\varphi^*\omega_N = dq_1 \wedge dp_1 \neq \omega_M$.

♢

In general, the condition $\varphi^*\omega_N = \omega_M$ requires the map to be an immersion, while Poisson maps between symplectic manifolds are always submersions.

7 Hamiltonian Actions

A complete Poisson map from a Poisson manifold M to a Lie-Poisson manifold $\mathfrak{g}^*$ gives rise to a left action of the connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$ on M by Poisson automorphisms, as we will now explain and explore.

7.1 Momentum Maps

Each element v of a Lie algebra $\mathfrak{g}$ corresponds to a linear function $h_v \in C^\infty(\mathfrak{g}^*)$ defined by $h_v(\mu) = \mu(v)$. Moreover, this correspondence is a Lie algebra homomorphism: $\{h_v, h_w\} = h_{[v,w]}$; see Section 3.1. Given a Poisson map $J : M \to \mathfrak{g}^*$, the composition

$$\begin{array}{ccccccc} \mathfrak{g} & \xrightarrow{h.} & C^\infty(\mathfrak{g}^*) & \xrightarrow{J^*} & C^\infty(M) & \xrightarrow{X.} & \chi(M) \\ v & \longmapsto & h_v & & h & \longmapsto & X_h \end{array}$$

is a Lie algebra anti-homomorphism $\rho : \mathfrak{g} \to \chi(M)$ (because the last arrow is an *anti*-homomorphism). In other words, J induces a left action of $\mathfrak{g}$ on M by hamiltonian vector fields.

Suppose that J is complete. For each $v \in \mathfrak{g}$, the vector field $X_{h_v} \in \chi(\mathfrak{g}^*)$ is complete. Hence, each $X_{J^*(h_v)}$ is also complete. In this case, the action ρ can be integrated to a left action of the connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$ on M by Poisson automorphisms [134].

Let $J_M : M \to \mathfrak{g}^*$, $J_N : N \to \mathfrak{g}^*$ and $\varphi : M \to N$ be Poisson maps such that the diagram

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "J_M"'] & & N \arrow[dl, "J_N"] \\ & \mathfrak{g}^* & \end{tikzcd}$$

commutes. Then φ will necessarily be compatible with the group actions induced by J_N and J_M.

Example. Let $M = \mathfrak{g}^*$ and let J be the identity map. The induced action of G on $\mathfrak{g}^*$ is just the dual of the adjoint representation, called the **coadjoint action**. In this case, G can be any connected (not necessarily simply connected) Lie group whose Lie algebra is $\mathfrak{g}$. This action of G restricts to a transitive action on each symplectic leaf $\mathcal{O}$ of $\mathfrak{g}^*$; thus, the symplectic leaves are called **coadjoint orbits**. To understand this, consider the inclusion map $\iota : \mathcal{O} \hookrightarrow \mathfrak{g}^*$. The induced commutative diagram

$$\begin{tikzcd} \mathcal{O} \arrow[rr, hook, "\iota"] \arrow[dr, "\iota"'] & & \mathfrak{g}^* \arrow[dl, "\mathrm{id}"] \\ & \mathfrak{g}^* & \end{tikzcd}$$

shows that the G-action on $\mathfrak{g}^*$ restricts to a G-action on $\mathcal{O}$. Furthermore, this action is transitive: at each $\mu \in \mathfrak{g}^*$, the $\{dh_v \mid v \in \mathfrak{g}\}$ span $T^*_\mu \mathfrak{g}^*$, so the corresponding hamiltonian vector fields $\{X_{h_v} \mid v \in \mathfrak{g}\}$ span the tangent space to the symplectic leaf at μ. We conclude that each symplectic leaf $\mathcal{O}$ of $\mathfrak{g}^*$ is a symplectic homogeneous space of G given as an orbit of the coadjoint action. $\diamondsuit$

For a Poisson map $\varphi : M \to \mathfrak{g}^*$, the diagram

$$\begin{tikzcd} M \arrow[rr, "\varphi"] \arrow[dr, "\varphi"'] & & \mathfrak{g}^* \arrow[dl, "\mathrm{id}"] \\ & \mathfrak{g}^* & \end{tikzcd}$$

shows that φ is G-equivariant for the induced action of the corresponding Lie group G on M and the coadjoint action on $\mathfrak{g}^*$.

When $\mathfrak{g} = \mathbb{R}$ and $G = \mathbb{R}$, the induced G-action of a map $J : M \to \mathbb{R}$ is just the hamiltonian flow of J. In general, we say that a complete Poisson map $J : M \to \mathfrak{g}^*$ is a **hamiltonian** or **momentum map** for the resulting action of G on M.

In summary, a complete Poisson map

$$J : M \longrightarrow \mathfrak{g}^*$$

gives rise to a Lie algebra anti-homomorphism

$$\rho : \mathfrak{g} \longrightarrow \chi(M) \ ,$$

which we integrate to a left action of G on M by Poisson automorphisms. The original map J is G-equivariant with respect to this action and the coadjoint action of G on $\mathfrak{g}^*$.

Historical Remark. Much of the construction above is merely a modern formulation of work done by Lie around 1890. Lie even refers to the "dual of the adjoint" (see [106] and [164]). $\diamondsuit$

7.2 First Obstruction for Momentum Maps

Given a Poisson map $J : M \to \mathfrak{g}^*$, we constructed in the previous section an action on M by Poisson automorphisms for which J was the momentum map. Conversely, given an action of a Lie group G by Poisson automorphisms on M, we would like to find a corresponding momentum map.

The sets of Poisson vector fields and of hamiltonian vector fields on M will be denoted $\chi_{\mathrm{Poiss}}(M)$ and $\chi_{\mathrm{Ham}}(M)$.

An action of G on M by Poisson maps can be differentiated to give an anti-homomorphism $\rho : \mathfrak{g} \to \chi_{\mathrm{Poiss}}(M)$. The first step in seeking a momentum map for this G-action is attempting to lift ρ to a linear map $\mathcal{J} : \mathfrak{g} \to C^\infty(M)$ making the following diagram commute:

$$\begin{tikzcd} C^\infty(M) \arrow[r] & \chi_{\mathrm{Ham}}(M) \arrow[r, hook] & \chi_{\mathrm{Poiss}}(M) \\ & \mathfrak{g} \arrow[ul, "\mathcal{J}?"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

The map ρ lifts to $\chi_{\mathrm{Ham}}(M)$ if and only if its image is actually contained in $\chi_{\mathrm{Ham}}(M) \subseteq \chi_{\mathrm{Poiss}}(M)$. Let $H^1_\Pi(M)$ be the first Poisson cohomology of M (defined in Section 3.6). To measure the obstruction, we look at the exact sequence

$$\begin{tikzcd}
\chi_{\mathrm{Ham}}(M) \arrow[r, hook] & \chi_{\mathrm{Poiss}}(M) \arrow[r] & H^1_\Pi(M) \\
\mathfrak{g} \arrow[u, "?"] \arrow[ur, "\rho"] \arrow[urr, "\overline{\rho}"'] & &
\end{tikzcd}$$

which induces a Lie algebra homomorphism $\overline{\rho} : \mathfrak{g} \to H^1_\Pi(M)$ (here we equip $H^1_\Pi(M)$ with the trivial Lie bracket). Clearly, $\overline{\rho} = 0$ if and only if ρ lifts to $\chi_{\mathrm{Ham}}(M)$. This can be interpreted as a first characteristic class for the action of G on a manifold; the vanishing of $\overline{\rho}$ is a necessary and sufficient condition for G to act by hamiltonian vector fields.

Remark. Recall that in the symplectic case $H^1_\Pi(M) = H^1(M; \mathbb{R})$ with trivial bracket, since the bracket of any two Poisson vector fields X_1, X_2 is hamiltonian:

$$[X_1, X_2] = -X_{\omega(X_1, X_2)} \ .$$

Even in this case, $\overline{\rho}$ can of course be non-trivial. ♢

Question: Is the vanishing of $H^1_\Pi(M)$ necessary for *all* group actions to lift to $\chi_{\mathrm{Ham}}(M)$? More generally, are all elements of $H^1_\Pi(M)$ represented by complete Poisson vector fields? (Hint: see [172].)

Metaphorically speaking, $H^1_\Pi(M)$ is the algebra of vector fields on the space of symplectic leaves. It is as if the action of G on M induced an action of G on the space of symplectic leaves, via the algebra homomorphism $\overline{\rho} : \mathfrak{g} \to H^1_\Pi(M)$. The triviality of this action is a necessary and sufficient condition for the lifting to hamiltonian vector fields. The following simple case illustrates that Poisson vector fields are not necessarily tangent to the symplectic leaves.

Example. Take $\mathbb{R}^2$ with bracket $\{x, y\} = x$. The Poisson vector field $\frac{\partial}{\partial y}$ preserves the two open symplectic leaves (the half-planes $\{(x, y) \mid x < 0\}$ and $\{(x, y) \mid x > 0\}$), but it is not tangent to the symplectic leaves on the y-axis (the points $\{(0, y)\}$), and acts non-trivially on them. Thus it does not lift to $\chi_{\mathrm{Ham}}(M)$, and hence this Poisson manifold has $H^1_\Pi \neq 0$. ♢

7.3 Second Obstruction for Momentum Maps

Assume that $\overline{\rho} = 0$, so that there is a lift $\widetilde{\rho} : \mathfrak{g} \to \chi_{\mathrm{Ham}}(M)$.

$$\begin{tikzcd}
C^\infty(M) \arrow[r] & \chi_{\mathrm{Ham}}(M) \arrow[r, hook] & \chi_{\mathrm{Poiss}}(M) \arrow[r] & H^1_\Pi(M) \\
& \mathfrak{g} \arrow[ul, "\mathcal{J}?"'] \arrow[u, "\widetilde{\rho}"] \arrow[ur, "\rho"] \arrow[urr, "\overline{\rho} = 0"'] & &
\end{tikzcd}$$

Because the map $C^\infty(M) \to \chi_{\mathrm{Ham}}(M)$ is surjective, we can lift $\widetilde{\rho}$ to a linear map $\mathcal{J} : \mathfrak{g} \to C^\infty(M)$, but $\mathcal{J}$ is *not* necessarily a Lie algebra homomorphism.

In any case, define the smooth map $J : M \to \mathfrak{g}^*$ by

$$\langle J(x), v\rangle = \mathcal{J}(v)(x)$$

for all $x \in M, v \in \mathfrak{g}$. This is called a **momentum map** for the G action by Kostant [99], Smale [152] and Souriau [153] (though our definition above is more restrictive). The map J is Poisson if and only if $\mathcal{J}$ is a Lie algebra homomorphism. In that case, J is G-equivariant if G is connected. Conversely, we have the following proposition:

Proposition 7.1 *A G-equivariant momentum map is a Poisson map.*

Proof. If $J : M \to \mathfrak{g}^*$ is G-equivariant, then for any $v \in \mathfrak{g}$, the hamiltonian flow of $\mathcal{J}(v)$ on M is mapped by J to the hamiltonian flow of h_v on $\mathfrak{g}^*$, since $\mathcal{J}(v) = J^*(h_v)$. By the last characterization in Section 6.1, the map J is Poisson if, for all functions $f \in C^\infty(\mathfrak{g}^*)$, J maps the hamiltonian flow of $J^*(f)$ to the hamiltonian flow of f. But it actually suffices to check this condition for the h_v's because the collection $\{dh_v\}$ spans the cotangent spaces of $\mathfrak{g}^*$. We conclude that J is a Poisson map. □

What is the obstruction to constructing a lift $\mathcal{J} : \mathfrak{g} \to C^\infty(M)$ which *is* a Lie algebra homomorphism? Here is a test to see whether a given $\mathcal{J}$ preserves the Poisson bracket. For any $v, w \in \mathfrak{g}$, define

$$\overline{\Theta}_J(v, w) = \{\mathcal{J}(v), \mathcal{J}(w)\} - \mathcal{J}([v, w]) \ .$$

We would like to have $\overline{\Theta}_J(v, w) = 0$ for any choice of v, w. Let $\beta : C^\infty(M) \to \chi_{\mathrm{Ham}}(M)$ be the map $\beta(f) = X_f$. Noting that both β and $\widetilde{\rho} = \beta \circ \mathcal{J}$ are anti-homomorphisms, we compute

$$\begin{aligned}
\beta(\overline{\Theta}_J(v, w)) &= \beta\{\mathcal{J}(v), \mathcal{J}(w)\} - \beta\mathcal{J}([v, w]) \\
&= -[\beta(\mathcal{J}(v)), \beta(\mathcal{J}(w))] - \widetilde{\rho}([v, w]) \\
&= -[\widetilde{\rho}(v), \widetilde{\rho}(w)] + [\widetilde{\rho}(v), \widetilde{\rho}(w)] \\
&= 0 \ .
\end{aligned}$$

So $\overline{\Theta}_J$ takes values in

$$\ker\Big(\beta : C^\infty(M) \to \chi_{\mathrm{Ham}}(M)\Big) = H^0_\Pi(M) \ .$$

Since $\overline{\Theta}_J$ is anti-symmetric, we regard it as a map

$$\Theta_J : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow H^0_\Pi(M) \ ,$$

whose vanishing is equivalent to J being G-equivariant, as long as G is connected.

7.4 Killing the Second Obstruction

For a fixed ρ, the definition of Θ_J above does depend on $\mathcal{J}$. As the lift $\mathcal{J}$ varies by elements of $H^0_\Pi(M)$, the corresponding Θ_J's can change. The question becomes: if Θ_J is non-trivial, can we kill it by a different choice of lifting $\mathcal{J}$?

To answer this question, we start by evaluating

$$\Theta_J(u, [v, w]) = \big\{\mathcal{J}(u), \{\mathcal{J}(v), \mathcal{J}(w)\} - \Theta_J(v, w)\big\} - \mathcal{J}([u, [v, w]]) \ .$$

The cyclic sum

$$\delta\Theta_J(u,v,w) = \Theta_J(u,[v,w]) + \Theta_J(v,[w,u]) + \Theta_J(w,[u,v])$$

is called the **coboundary**, $\delta\Theta_J$, of Θ_J.

Exercise 16
Prove that $\delta\Theta_J(u,v,w)$ is 0. You should use the Jacobi identity and the fact that $\Theta_J(v,w)$ is a Casimir function.

Since $\delta\Theta_J(u,v,w) = 0$, Θ_J is called a **2-cocycle** on $\mathfrak{g}$ with values in $H^0_\Pi(M)$.

Suppose that we replace $\mathcal{J}$ with $\mathcal{J}+\mathcal{K}$, where $\mathcal{K} : \mathfrak{g} \to H^0_\Pi(M)$ is a linear map. The momentum map $K : M \to \mathfrak{g}^*$ associated to $\mathcal{K}$ is constant on symplectic leaves. Such a map $\mathcal{K}$ is called a **1-cochain** on $\mathfrak{g}$ with values in $H^0_\Pi(M)$. The 2-cocycle Θ_{J+K} corresponding to $\mathcal{J}+\mathcal{K}$ satisfies

$$\Theta_{J+K}(u,v) = \Theta_J(v,w) - \mathcal{K}([v,w]) \ .$$

We define $\delta\mathcal{K}(v,w) = -\mathcal{K}([v,w])$.

Exercise 17
Using the previous definition of δ for 2-cochains on $\mathfrak{g}$ with values in $H^0_\Pi(M)$, show that $\delta^2\mathcal{K} = 0$.

Let $H^2(\mathfrak{g}; H^0_\Pi(M))$ be the second Lie algebra cohomology of $\mathfrak{g}$ with coefficients in $H^0_\Pi(M)$. We then conclude that the cohomology class

$$[\Theta_J] \in H^2(\mathfrak{g}; H^0_\Pi(M))$$

is independent of the choice of $\mathcal{J}$ and depends only on ρ. Furthermore, $[\Theta_J]$ vanishes if and only if a lift $\mathcal{J}$ exists which is a Lie algebra homomorphism.

7.5 Obstructions Summarized

Given an action of a Lie group G on a Poisson manifold M, there is an induced map $\rho : \mathfrak{g} \to \chi_{\mathrm{Poiss}}(M)$. The first obstruction to lifting ρ to a Lie algebra homomorphism $\mathcal{J} : \mathfrak{g} \to C^\infty(M)$ is the map $\overline{\rho} : \mathfrak{g} \to H^1_\Pi(M)$. If $H^1_\Pi(M)$ is abelian, as for instance in the symplectic case, then $\overline{\rho}$ is actually an element of $H^1(\mathfrak{g}; H^1_\Pi(M))$. We think of $\overline{\rho}$ as an action of $\mathfrak{g}$ on the leaf space of M which needs to be trivial in order to lift ρ.

When $\overline{\rho} = 0$ there is a second obstruction in $H^2(\mathfrak{g}; H^0_\Pi(M))$.

Exercise 18
Check that

$$H^2(\mathfrak{g}; H^0_\Pi(M)) \simeq H^2(\mathfrak{g}) \otimes H^0_\Pi(M) \ .$$

Interpreting $H^0_\Pi(M)$ as the set of functions on the leaf space, we can view this second obstruction as lying on "functions on the leaf space with values in $H^2(\mathfrak{g})$".

Questions: Is there a variant for $[\Theta_J]$ that makes sense even when $\overline{\rho} \neq 0$? Is it possible that the two objects $\overline{\rho}$ and $[\Theta_J]$ be considered as parts of some single geometric object related to the "action of G on the leaf space"? Can we integrate cocycles on the Lie algebra into cocycles on the group? Perhaps some sense can be made of these questions in the realms of Lie algebroid cohomology or equivariant Poisson cohomology.

There is some terminology commonly used in these constructions. An action of G by automorphisms of a Poisson manifold (M, Π) is called **weakly hamiltonian** if there exists a momentum map J. If there is an equivariant momentum map J, then the action is called **hamiltonian**. In some of the literature, weakly hamiltonian actions are simply referred to as *hamiltonian* while hamiltonian actions as we have defined them are called *strongly hamiltonian*.

Remark. For a weakly hamiltonian action of a connected group G on a connected symplectic manifold M, there is a modified Poisson structure on $\mathfrak{g}^*$ for which the momentum map $J : M \to \mathfrak{g}^*$ is a Poisson map. Consider the map

$$\Theta_J : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow H^0_\Pi(M)$$

as an element of $(\mathfrak{g}^* \wedge \mathfrak{g}^*) \otimes H^0_\Pi(M)$, *i.e.* as a bivector field on $\mathfrak{g}^*$ with values in $H^0_\Pi(M)$. Because M is symplectic and connected, $H^0_\Pi(M) \simeq \mathbb{R}$, and thus Θ_J is simply a bivector field. We then add Θ_J to the Poisson tensor $\Pi_{\mathfrak{g}^*}$, defining a new tensor $\Pi'_{\mathfrak{g}^*} = \Pi_{\mathfrak{g}^*} + \Theta_J$, with respect to which J is a Poisson map.

Exercise 19
Show that $\delta\Theta_J = 0$ implies that $\Pi'_{\mathfrak{g}^*}$ is again a Poisson tensor and that with this Poisson structure on $\mathfrak{g}^*$ the map J is Poisson.

$\diamondsuit$

7.6 Flat Connections for Poisson Maps with Symplectic Target

We will classify complete Poisson maps $\varphi : M \to S$, where M is a Poisson manifold and S is a connected symplectic manifold. The structure of these maps turns out to be remarkably simple and rigid.

Claim. Any Poisson map $\varphi : M \to S$ is a submersion.

Proof. If not, then $(T_x\varphi)(T_xM)$ is a proper subspace V of $T_{\varphi(x)}S$, and $(T_x\varphi)(\Pi(x)) \subseteq V \wedge V$, contradicting the fact that the image of Π under $T_x\varphi$ is symplectic. $\square$

We can say even more if we assume that φ is complete:

Claim. Any complete Poisson map $\varphi : M \to S$ is surjective.

Exercise 20
Prove this claim.

Example. Let F be any Poisson manifold and let $p_1 : S \times F \to S$ be the projection onto the first factor. This is clearly a complete Poisson map. $\diamondsuit$

Inspired by this example, the claims above indicate that a complete Poisson map should be a kind of fibration over S. To formalize this idea, we define a **flat connection** for any submersion $\varphi : M \to S$ between manifolds to be a subbundle $E \subseteq TM$ such that

1. $TM = E \oplus \ker T\varphi$,

2. $[E, E] \subseteq E$ (that is, sections of E are closed under $[\cdot, \cdot]$, and so by the Frobenius theorem E is integrable),

3. every path in S has a horizontal lift through each lift of one of its points.

A subbundle $E \subseteq TM$ satisfying conditions 1 and 3, or sometimes just 1, is called an **Ehresmann connection** [52]. Conditions 1 and 3 imply that $\varphi : M \to S$ is a locally trivial fibration. Condition 2 is the **flatness** property, which implies that the fibration has a discrete structure group.

Theorem 7.2 *A complete Poisson map $\varphi : M \to S$ to a symplectic manifold has a natural flat connection.*

Proof. Let $s = \varphi(x)$ for some $x \in M$ and choose a $v \in T_s S$. We want to lift v to $T_x M$ in a canonical way. Because S is symplectic, $\widetilde{\Pi}_S^{-1}(v)$ is a well-defined covector at s. Define a horizontal lift

$$H_x(v) = \widetilde{\Pi}_M \left((T_x\varphi)^* \widetilde{\Pi}_S^{-1}(v) \right) \in T_x M \ .$$

The fact that φ is a Poisson map implies that $(T_x\varphi)(H_x(v)) = v$ (see Section 6.1). We need to check that the bracket of two horizontal lifts is again horizontal. On S, choose canonical coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, and lift their hamiltonian vector fields

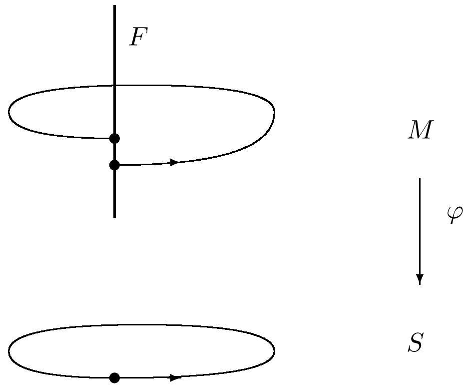
$$-\frac{\partial}{\partial p_1}, \ldots, -\frac{\partial}{\partial p_n}, \frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n} \ .$$

The lifts are closed under commutators, hence span an integrable subbundle. Multiplying these vector fields on S by compactly supported functions if necessary to make them complete, we obtain a local trivialization of φ, because φ is complete. Any path on S lifts to M because any path lifts locally. $\square$

In particular, if S is simply connected, then there is a Poisson manifold F such that M and $S \times F$ are diffeomorphic as Poisson manifolds. In general, φ is determined up to isomorphism by its **holonomy**

$$\pi_1(S) \longrightarrow \mathrm{Aut}(F)$$

on a typical fiber F of the map.



We thus found a functor from the category of complete Poisson maps $M \to S$ to the category of actions of $\pi_1(S)$ by Poisson automorphisms on Poisson manifolds F.

We also have a functor going in the other direction. Let $\widetilde{S}$ be the universal cover of the symplectic manifold S; $\widetilde{S}$ is a symplectic manifold. Let F be a Poisson manifold with a $\pi_1(S)$-action by Poisson automorphisms. On the product $\widetilde{S} \times F$ there is an induced diagonal action

$$\gamma \cdot (\tilde{s}, f) = (\gamma \cdot \tilde{s}, \gamma \cdot f) \qquad \text{for } \gamma \in \pi_1(S) \ .$$

If we form the quotient by this action, we still get a projection

$$\frac{\widetilde{S} \times F}{\pi_1(S)} \xrightarrow{\varphi} S \ .$$

This is a complete Poisson map with fiber F.

Exercise 21
Show that this actually defines a *functor* from the category of actions of $\pi_1(S)$ by Poisson automorphisms on Poisson manifolds to the category of complete Poisson maps from Poisson manifolds to S.

Remark. Comparing the results of this section with the theory of hamiltonian group actions, it is tempting to think of any symplectic manifold S as the "dual of the Lie algebra of $\pi_1(S)$"! $\diamondsuit$

Part IV
Dual Pairs

8 Operator Algebras

In this chapter, we introduce terminology and quote results leading to the double commutant theorem (Theorem 8.3) proved by von Neumann [127]. Chapter 9 will be devoted to analogous results in Poisson geometry.

In the following discussion, we denote the algebra of bounded operators on a complex Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$. There are several topologies worth considering on $B(\mathcal{H})$.

8.1 Norm Topology and C^*-Algebras

The norm of a bounded operator $L \in B(\mathcal{H})$ is by definition

$$||L|| = \sup_{u \in \mathcal{H}\setminus\{0\}} \frac{||Lu||_{\mathcal{H}}}{||u||_{\mathcal{H}}} .$$

Exercise 22
Check that $||\cdot||$ satisfies the axioms for a norm:

(a) $||\lambda \cdot L|| = |\lambda| \cdot ||L||, \quad \lambda \in \mathbb{C}$,

(b) $||L + M|| \leq ||L|| + ||M||, \quad$ and

(c) $||L|| > 0$ if $L \neq 0$.

This induces a (complete) metric

$$d(M, L) = ||L - M||$$

and thus a topology on $B(\mathcal{H})$, called the **norm topology**.

On $B(\mathcal{H})$ there is an **adjoint operation** $*$ defined uniquely by

$$\langle L^* u, v\rangle = \langle u, Lv\rangle$$

which has the properties

- $L^{**} = L$,
- $(LM)^* = M^* L^*$, and
- $||LL^*|| = ||L||^2$.

We say that $B(\mathcal{H})$ equipped with this *-operation is a *C^*-algebra*. In general, a **C^*-algebra** is an algebra with a norm such that the algebra is complete with respect to the topology induced by the norm and possesses a *-operation satisfying the properties above. As general references on C^*-algebras, we recommend [7, 36, 45].

Any norm-closed *-subalgebra of $B(\mathcal{H})$ inherits the properties above and thus is a C^*-algebra. If $\mathcal{A} \subseteq B(\mathcal{H})$ is any *-subalgebra, its norm-closure $\overline{\mathcal{A}}$ is a C^*-algebra. Conversely, we have the following theorem:

Theorem 8.1 (Gel'fand-Naimark [64]) *Any C^*-algebra is isomorphic as a normed *-algebra to a norm-closed subalgebra of $B(\mathcal{H})$.*

Example. The collection of all finite rank operators is a *-subalgebra; its closure is the C^*-subalgebra of **compact** operators on $\mathcal{H}$ – that is, operators L such that L applied to a bounded subset has compact closure. The identity operator I is *not* compact if $\mathcal{H}$ is infinite dimensional, as the closed unit ball in $\mathcal{H}$ is bounded but *not* compact. (For instance, the sequence $a_i = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is in the ith slot, has no convergent subsequences.) For diagonalizable operators, compactness amounts to convergence of the eigenvalues to 0. ♢

Let X be any compact Hausdorff topological space, and let $C(X)$ be the algebra of complex-valued continuous functions on X equipped with the sup norm. Then pointwise addition and multiplication together with the *-operation defined by $f^*(x) = \overline{f(x)}$ give $C(X)$ the structure of a C^*-algebra. The following theorem demonstrates how general this example is:

Theorem 8.2 [63, 65, 64] *Any commutative C^*-algebra $\mathcal{A}$ with identity is isometrically *-isomorphic to $C(X)$ for some compact Hausdorff space X. One can take X to be the space of non-zero *-homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. (X is then called the* **spectrum** *of $\mathcal{A}$.)*

Recalling Theorem 8.1, how can $C(X)$ be regarded as an algebra of operators on a Hilbert space? Because X is compact, we can find a Borel measure on X which is positive on any non-empty open set. $C(X)$ is then realized as an algebra of multiplication operators on $L^2(X)$. For any function $u \in C(X)$, define the multiplication operator m_u by $m_u(g) = ug$ for $g \in L^2(X)$.

Exercise 23
Show that

$$||m_u||_{B(L^2(X))} = ||u||_{C(X)} .$$

8.2 Strong and Weak Topologies

A second topology on $B(\mathcal{H})$ is the **topology of pointwise convergence**, or the **strong topology**. For each $u \in \mathcal{H}$, define a *semi-norm*

$$||L||_u = ||Lu||_{\mathcal{H}} .$$

A semi-norm is essentially the same as a norm except for the positivity requirement: non-zero elements may have 0 semi-norm. We define the strong topology on $B(\mathcal{H})$ by declaring a sequence $\{L_i\}$ to converge if and only if the sequence converges in the semi-norms $|| \cdot ||_u$ for all choices of $u \in \mathcal{H}$.

Example. The sequence of operators L_i on $L^2(\mathbb{N}) =: l^2$ defined by

$$L_i(a_0, a_1, a_2, \ldots) = (0, \ldots, 0, a_i, 0, \ldots)$$

converges to 0 in the strong topology, though each L_i has norm 1. ♢

Example. Let M_i be the operator on $L^2(\mathbb{N})$

$$M_i(a_0, a_1, a_2, \ldots) = (0, \ldots, 0, a_0, 0, \ldots) \ ,$$

where the a_0 on the right is the ith entry. The sequence of the M_i's does *not* converge in the strong topology, yet its adjoint

$$M_i^*(a_0, a_1, a_2, \ldots) = (a_i, 0, 0, \ldots)$$

does converge strongly (exercise). ♢

Another topology on $B(\mathcal{H})$ is the **weak topology**, or the **topology of convergence of matrix elements**. For $u, v \in \mathcal{H}$, define a semi-norm

$$||L||_{u,v} = |\,\langle Lu, v\rangle\,| \ .$$

We say that a sequence $\{L_i\}_{i\in I}$ converges in the weak topology, if $||L_i||_{u,v}$ converges for each choice of u, v.

The sequences L_i, M_i and M_i^* in the examples above converge in the weak topology. In general, any strongly convergent sequence is weakly convergent, and any norm convergent sequence is strongly convergent, so the weak topology is in fact weaker than the strong topology, which is still weaker than the norm topology.

By Exercise 23, the inclusion $C(X) \hookrightarrow B(L^2(X))$ given by $u \mapsto m_u$ is an isometry; this implies that $C(X)$ is norm-closed when considered as a subalgebra of $B(L^2(X))$, which illustrates Theorem 8.1. However, if we use a weaker topology (say the strong or weak topology), then $C(X)$ is no longer closed.

Exercise 24
Construct a sequence of functions in $C(X)$ converging to (multiplication by) a step function in the strong (or weak) topology. Show that this sequence does not converge in the norm topology.

The weak (or strong) closure of $C(X)$ in $B(L^2(X))$ is, in fact, $L^\infty(X)$. Keep in mind that elements of $L^\infty(X)$ cannot be strictly considered as functions on X, since two functions which differ on a set of measure 0 on X correspond to the same element of $B(L^2(X))$.

8.3 Commutants

A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is called **unital** if it contains the identity operator of $B(\mathcal{H})$. For a subset $\mathcal{A} \subseteq B(\mathcal{H})$ closed under the *-operator, we define the **commutant** of $\mathcal{A}$ to be

$$\mathcal{A}' = \{L \in B(\mathcal{H}) \mid \forall a \in \mathcal{A}, La = aL\} \ .$$

Exercise 25
Show that $\mathcal{A}'$ is a weakly closed *-subalgebra.

A weakly closed unital *-subalgebra of $B(\mathcal{H})$ is called a **von Neumann algebra**. [47, 74, 156, 157, 158] are general references on von Neumann algebras. There is a remarkable connection between algebraic and topological properties of these algebras, as shown by the following theorem.

Theorem 8.3 (von Neumann [127]) *For a unital *-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$, the following are equivalent:*

1. $\mathcal{A}'' = \mathcal{A}$,

2. *$\mathcal{A}$ is weakly closed,*

3. *$\mathcal{A}$ is strongly closed.*

Corollary 8.4 *If $\mathcal{A}$ is any subset of $B(\mathcal{H})$, then $\mathcal{A}''' = \mathcal{A}'$.*

For an arbitrary unital *-subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$, the **double commutant** $\mathcal{A}''$ coincides with the weak closure of $\mathcal{A}$.

The **center** of $\mathcal{A}$ is

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'.$$

If $\mathcal{A}$ is a von Neumann algebra with $Z(\mathcal{A}) = \mathbb{C} \cdot 1$, then $\mathcal{A}$ is called a **factor**. These are the building blocks for von Neumann algebras. Von Neumann showed that every von Neumann algebra is a direct integral (generalized direct sum) of factors [129, 130].

Example. We have already seen some classes of von Neumann algebras:

- $B(\mathcal{H})$, which is a factor.

- $L^\infty(X)$ (with respect to a given measure class on X), which is a generalized direct integral of copies of $\mathbb{C}$ (which are factors):

$$L^\infty(X) = \int_X^{\oplus} \text{a copy of } \mathbb{C} \text{ for each point of } X \ .$$

- The commutant of any subset of $B(\mathcal{H})$, for instance the collection of operators which commute with the action of a group on $\mathcal{H}$.

$\diamondsuit$

8.4 Dual Pairs

A **dual pair** $(\mathcal{A}, \mathcal{A}')$ is a pair of unital *-subalgebras $\mathcal{A}$ and $\mathcal{A}'$ of $B(\mathcal{H})$ which are the commutants of one another. By Theorem 8.3, $\mathcal{A}$ and $\mathcal{A}'$ are then von Neumann algebras.

If $\mathcal{A}$ is a von Neumann subalgebra of $B(\mathcal{H})$, then there are inclusions

$$\begin{tikzcd} & B(\mathcal{H}) & \\ \mathcal{A} \arrow[ur] & & \mathcal{A}' \arrow[ul] \end{tikzcd}$$

which form a dual pair. The centers of $\mathcal{A}$ and $\mathcal{A}'$ coincide:

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}' = \mathcal{A}' \cap \mathcal{A}'' = Z(\mathcal{A}') \ ,$$

so that $\mathcal{A}$ is a factor if and only if $\mathcal{A}'$ is.

We next turn to geometric counterparts of dual pairs in the context of Poisson geometry.

9 Dual Pairs in Poisson Geometry

We will discuss a geometric version of dual pairs for Poisson algebras associated to Poisson manifolds.

9.1 Commutants in Poisson Geometry

We have seen that a Poisson manifold $(M, \{\cdot,\cdot\})$ determines a Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$ and that a Poisson map $\varphi : M \to N$ induces a Poisson-algebra homomorphism $\varphi^* : C^\infty(N) \to C^\infty(M)$.

Suppose that N is a Poisson quotient of M. Then there is a map $C^\infty(N) \to C^\infty(M)$ identifying $C^\infty(N)$ as a Poisson subalgebra of $C^\infty(M)$ consisting of functions constant along the equivalence classes of M determined by the quotient map. In the converse direction, we might choose an arbitrary Poisson subalgebra of $C^\infty(M)$ and search for a corresponding quotient map. In general this is *not* possible. To understand the tie between Poisson quotients and Poisson subalgebras, we examine examples of commutants in $(C^\infty(M), \{\cdot,\cdot\})$.

Example. Let $M = \mathbb{R}^{2n}$, with the standard Poisson structure $\Pi = \sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$. The Poisson subalgebra, $\mathrm{Pol}(\mathbb{R}^{2n})$, of polynomial functions does *not* correspond to any Poisson quotient manifold. Since $\mathrm{Pol}(\mathbb{R}^{2n})$ separates any two points of $\mathbb{R}^{2n}$, the "quotient map" would have to be the identity map on $\mathbb{R}^{2n}$.

On the other hand, the Poisson subalgebra

$$\pi^* C^\infty(\mathbb{R}^n_{q_1,\ldots,q_n}) \subseteq C^\infty(\mathbb{R}^{2n}_{q_1,\ldots,q_n,p_1,\ldots,p_n})$$

does correspond to the quotient $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$. ♢

The different behavior of the subalgebras $\mathrm{Pol}(\mathbb{R}^{2n})$ and $C^\infty(\mathbb{R}^n_{q_1,\ldots,q_n})$ of $C^\infty(\mathbb{R}^{2n})$ can be interpreted in the following manner.

Let $\mathcal{A}$ be a Poisson algebra and $\mathcal{B} \subseteq \mathcal{A}$ a Poisson subalgebra. Define the **commutant** of $\mathcal{B}$ in $\mathcal{A}$ to be

$$\mathcal{B}' = \{f \in \mathcal{A} \mid \{f, \mathcal{B}\} = 0\} \ .$$

Example. For $\mathcal{A} = C^\infty(\mathbb{R}^{2n})$ we have:

$$\begin{aligned}
\mathrm{Pol}(\mathbb{R}^{2n})' &= \text{constant functions} \\
(\text{constant functions})' &= C^\infty(\mathbb{R}^{2n}) \\
C^\infty(\mathbb{R}^n_{q_1,\ldots,q_n})' &= C^\infty(\mathbb{R}^n_{q_1,\ldots,q_n}) \\
C^\infty(\mathbb{R}_{q_1})' &= C^\infty(\mathbb{R}^{2n-1}_{q_1,\ldots,q_n,p_2,\ldots,p_n}) \ .
\end{aligned}$$

The double commutants of these subalgebras are:

$$\begin{aligned}
\mathrm{Pol}(\mathbb{R}^{2n})'' &= C^\infty(\mathbb{R}^{2n}) \\
C^\infty(\mathbb{R}^n_{q_1,\ldots,q_n})'' &= C^\infty(\mathbb{R}^n_{q_1,\ldots,q_n}) \\
C^\infty(\mathbb{R}_{q_1})'' &= C^\infty(\mathbb{R}_{q_1}) \ .
\end{aligned}$$

Since $\mathrm{Pol}(\mathbb{R}^{2n})$ does not correspond to a Poisson quotient while the other two subalgebras do, this seems to indicate that the Poisson subalgebras that correspond to quotient maps are those which are their own double commutants. $\diamondsuit$

Question. (R. Conti) Is $\mathcal{A}' = \mathcal{A}'''$ for every subset of a Poisson algebra? (See Corollary 8.4.)

9.2 Pairs of Symplectically Complete Foliations

Suppose that M and N are Poisson manifolds and that $J : M \to N$ is a Poisson map with a dense image in N. Then the pull-back J^* is an injection. The commutant of the Poisson subalgebra

$$\mathcal{A} = J^*(C^\infty(N)) \subseteq C^\infty(M)$$

is

$$\begin{aligned} \mathcal{A}' &= \{f \in C^\infty(M) \mid \{f, \mathcal{A}\} = 0\} \\ &= \{f \in C^\infty(M) \mid \forall g \in \mathcal{A},\ X_g f = 0\} \\ &= \{f \in C^\infty(M) \mid \forall g \in \mathcal{A},\ df \text{ annihilates } \widetilde{\Pi}(dg)\} \ . \end{aligned}$$

At a point x of M, we have

$$\begin{aligned} \{\text{values of hamiltonian vector fields } \widetilde{\Pi}(dg) \text{ at } x \mid g \in \mathcal{A}\} &= \widetilde{\Pi}(\text{image } T_x^* J) \\ &= \widetilde{\Pi}((\ker T_x J)^\circ) \ , \end{aligned}$$

where $(\ker T_x J)^\circ$ is the subspace of covectors that annihilate $\ker T_x J \subseteq T_x M$. When M happens to be *symplectic*,

$$\widetilde{\Pi}((\ker T_x J)^\circ) = (\ker T_x J)^\perp \ ,$$

where $W^\perp$ is the symplectic orthogonal to the subspace W inside the tangent space. (In the symplectic case, taking orthogonals twice returns the same subspace: $(W^\perp)^\perp = W$.) In the symplectic case, we have

$$\{\text{values of hamiltonian vector fields } \widetilde{\Pi}(dg) \text{ at } x \mid g \in \mathcal{A}\} = (\ker T_x J)^\perp \ .$$

Exercise 26
Show that

$$\{\text{values of hamiltonian vector fields } \widetilde{\Pi}(dg) \text{ at } x \mid g \in \mathcal{A}'\} = \ker T_x J \ .$$

Suppose now that $J : M \to N$ is a constant-rank map from a symplectic manifold M to a Poisson manifold N. The kernel

$$\ker TJ$$

forms an integrable subbundle of TM, defining a foliation of M. The symplectic orthogonal distribution

$$(\ker TJ)^\perp$$

is generated by a family of vector fields closed under the bracket operation, since they are lifts of hamiltonian vector fields on N. Hence, it is an integrable distribution which defines another foliation. This is a particular instance of the following lemma.

Lemma 9.1 *Let M be a symplectic manifold and $F \subseteq TM$ an integrable subbundle. Then $F^{\perp}$ is integrable if and only if the set of functions on open sets of M annihilated by vectors in F is closed under the Poisson bracket.*

A foliation $\mathcal{F}$ defined by a subbundle $F \subseteq TM$ as in this lemma (*i.e.* integrable, with the set of functions on open sets of M annihilated by vectors in F closed under the Poisson bracket) is called a **symplectically complete foliation** [104]. Symplectically complete foliations come in orthogonal pairs, since $(F^{\perp})^{\perp} = F$.

9.3 Symplectic Dual Pairs

Example. Suppose that M is symplectic and $J : M \to \mathfrak{g}^*$ is a constant-rank Poisson map. The symplectic orthogonal to the foliation by the level sets of J is exactly the foliation determined by hamiltonian vector fields generated by functions on $\mathfrak{g}^*$, which is the same as the foliation determined by the hamiltonian vector fields generated by *linear* functions on $\mathfrak{g}^*$ (since the differentials of linear functions span the cotangent spaces of $\mathfrak{g}^*$). The leaves of this foliation are simply the orbits of the induced G-action on M. We could hence consider the "dual" to J to be the projection of M to the orbit space, and write

$$\begin{tikzcd} & M \arrow[dl, "J"'] \arrow[dr, "p"] & \\ \mathfrak{g}^* & & M/G\ . \end{tikzcd}$$

Some conditions are required for this diagram to make sense as a pair of Poisson maps between manifolds, in particular, for M/G to exist as a manifold:

1. J must have constant rank so that the momentum levels form a foliation.
2. The G-orbits must form a fibration (*i.e.* the G-action must be regular).

In this situation, the subalgebras $J^*(C^\infty(\mathfrak{g}^*))$ and $p^*(C^\infty(M/G))$ of $C^\infty(M)$ are commutants of one another, and hence their centers are isomorphic. Furthermore, when J is a submersion, the transverse structures to corresponding leaves in $\mathfrak{g}^*$ and M/G are anti-isomorphic [163]. So the Poisson geometry of the orbit space M/G is "modulo symplectic manifolds" very similar to the Poisson geometry of $\mathfrak{g}^*$. This construction depends on J being surjective or, equivalently, on the G-action being locally free. When J is not surjective, we should simply throw out the part of $\mathfrak{g}^*$ not in the image of J. $\diamondsuit$

In general, given a symplectic manifold M and Poisson manifolds P_1 and P_2, a **symplectic dual pair** is a diagram

$$\begin{tikzcd} & M \arrow[dl, "J_1"'] \arrow[dr, "J_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

of Poisson maps with symplectically orthogonal fibers. Orthogonality implies

$$\{J_1^*(C^\infty(P_1)), J_2^*(C^\infty(P_2))\} = 0 \ .$$

Sometimes, this relation is written as $\{J_1, J_2\} = 0$.

Remark. For a pair of Poisson maps $J_1 : M \to P_1$ and $J_2 : M \to P_2$, imposing $\{J_1, J_2\} = 0$ is equivalent to imposing that the product map

$$M \xrightarrow{J_1 \times J_2} P_1 \times P_2$$

be a Poisson map. ♢

9.4 Morita Equivalence

Let J_1, J_2 be surjective Poisson submersions from a symplectic manifold M to Poisson manifolds P_1, P_2. If

$$J_1^*(C^\infty(P_1)) = J_2^*(C^\infty(P_2))' \quad \text{and} \quad J_2^*(C^\infty(P_2)) = J_1^*(C^\infty(P_1))' \ ,$$

then the J_1-fibers are symplectic orthogonals to the J_2-fibers:

$$\ker TJ_1 = (\ker TJ_2)^\perp.$$

The reverse implication is not true unless we assume that the fibers are connected, essentially because $\ker TJ_1 = (\ker TJ_2)^\perp$ is a local condition while the hypothesis was a global condition. If J_1 and J_2 have connected fibers, then the two conditions above are equivalent. To require that fibers be connected is appropriate because of the following property for such dual pairs.

Proposition 9.2 *Let J_1, J_2 be a pair of complete surjective Poisson submersions*

$$\begin{tikzcd} & M \arrow[dl, "J_1"'] \arrow[dr, "J_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

from a symplectic manifold M. Assume that the J_1-fibers are symplectically orthogonal to the J_2-fibers, and that all fibers are connected. Then there is a one-to-one correspondence between the symplectic leaves of P_1 and the symplectic leaves of P_2.

Proof. Let $F_j \subseteq TM$ be the distribution spanned by the hamiltonian vector fields of functions in $J_j^*(C^\infty(P_j))$. The assumption says that, at each point, the distribution F_1 (respectively F_2) gives the subspace tangent to the fibers of J_2 (respectively J_1); this clearly shows that each of F_1 and F_2 is integrable. To see that $F_1 + F_2$ is also integrable, note that $F_1 + F_2$ is spanned by hamiltonian vector fields, and that the vector fields from J_1 commute with those from J_2. So we can integrate $F_1 + F_2$ to a foliation of M.

A leaf $\mathcal{L}$ of the foliation defined by $F_1 + F_2$ projects by each map J_i to a set $J_i(\mathcal{L})$, which is in fact a symplectic leaf of P_i ($i = 1, 2$) for the following two reasons.

First, by completeness, we can move anywhere within a symplectic leaf of P_i by moving in the F_i direction in $\mathcal{L}$. Secondly, if we move in the F_2 (respectively F_1) direction in $\mathcal{L}$, then nothing happens in the projection to P_1 (respectively P_2).

Therefore, there is a map from the leaf space of $F_1 + F_2$ to the product of the leaf spaces of P_1 and P_2. The image $\mathcal{R}$ of this map gives a relation between the leaf space of P_1 and the leaf space of P_2. Additionally, the projection of $\mathcal{R}$ to either factor of the product is surjective. Because the fibers of J_1, J_2 are connected, it follows that $\mathcal{R}$ is the graph of a bijection. $\square$

We say that two Poisson manifolds P_1, P_2 are **Morita equivalent** [176, 177] if there is a symplectic manifold M and surjective submersions J_1, J_2

$$\begin{tikzcd} & M \arrow[dl, "J_1"'] \arrow[dr, "J_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

satisfying the following conditions:

- J_1 is a Poisson map and J_2 is an anti-Poisson map (*anti* in the sense of being an anti-homomorphism for the bracket).
- each J_i is complete and has constant rank,
- each J_i has connected, simply connected fibers,
- the fibers of J_1, J_2 are symplectically orthogonal to one another. Equivalently, $J_1^*(C^\infty(P_1))$ and $J_2^*(C^\infty(P_2))$ are commutants of one another.

Remark. The map J_2 in the Morita equivalence is sometimes denoted as a Poisson map $J_2 : M \to \overline{P}_2$, where $\overline{P}_2$ is the manifold P_2 with Poisson bracket defined by $\{\cdot,\cdot\}_{\overline{P}_2} = -\{\cdot,\cdot\}_{P_2}$. $\diamondsuit$

Remark. In spite of the name, Morita equivalence is not an equivalence relation, as it fails to be reflexive [176, 177]. $\diamondsuit$

9.5 Representation Equivalence

The Morita equivalence of Poisson manifolds provides a classical analogue to the Morita equivalence of algebras. Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras over a field $\mathbb{K}$. Define an $(\mathcal{A}_1, \mathcal{A}_2)$-bimodule E to be an abelian group E with a left action of $\mathcal{A}_1$ and a right action of $\mathcal{A}_2$ such that for $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, $e \in E$

$$(a_1 e)a_2 = a_1(e a_2) \ .$$

So we have injective maps

$$\begin{array}{rcl} \mathcal{A}_1 & \longrightarrow & \mathrm{End}_{\mathbb{K}}(E) \\ \mathcal{A}_2^{\mathrm{opp}} & \longrightarrow & \mathrm{End}_{\mathbb{K}}(E) \ . \end{array}$$

where $\mathcal{A}_2^{\rm opp}$ denotes $\mathcal{A}_2$ acting on the left by inverses. A **Morita equivalence** from $\mathcal{A}_1$ to $\mathcal{A}_2$ is an $(\mathcal{A}_1, \mathcal{A}_2)$-bimodule E such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are mutual commutants in $\mathrm{End}_{\mathbb{K}}(E)$. Morita introduced this as a weak equivalence between algebras, and he showed that it implies that $\mathcal{A}_1$-modules and $\mathcal{A}_2$-modules are equivalent categories.

Xu [176, 177] showed that we can imitate this construction for symplectic realizations of Poisson manifolds. In particular, if P_1, P_2 are Poisson manifolds, then we say that they are **representation equivalent** if the category of complete Poisson maps to P_1 is equivalent to the category of complete Poisson maps to P_2. Xu then proved the following theorem:

Theorem 9.3 (Xu [176, 177]) *If two Poisson manifolds are Morita equivalent, then they are representation equivalent.*

For a survey of Xu's work and Morita equivalence in general, see the article by Meyer [118]. For a survey of the relation between Poisson geometry and von Neumann algebras, see the article by Shlyakhtenko [151].

9.6 Topological Restrictions

The importance of the condition that the fibers of J_i be simply connected in the definition of Morita equivalence between Poisson manifolds is explained by the following property for the case where P_1 and P_2 are symplectic.

Proposition 9.4 *Let S_1, S_2 be symplectic manifolds. Then S_1 and S_2 are Morita equivalent if and only if they have isomorphic fundamental groups.*

Proof. Suppose that S_1, S_2 are Morita equivalent. Then, from the long exact sequence for homotopy

$$0 = \pi_1(\text{fiber}) \longrightarrow \pi_1(M) \longrightarrow \pi_1(S_j) \longrightarrow \pi_0(\text{fiber}) = 0 \ ,$$

we conclude that

$$\pi_1(S_1) \simeq \pi_1(M) \simeq \pi_1(S_2) \ .$$

Furthermore, the Morita equivalence induces a specific isomorphism via pull-back by the maps from S.

Conversely, suppose that $\pi_1(S_1) \simeq \pi_1(S_2) \simeq \pi$. Let $\widetilde{S_j}$ be the universal cover of S_j, so that $\widetilde{S_j}$ is a principal π-bundle over S_j. Because π acts on $\widetilde{S_1}$ and $\widetilde{\overline{S_2}}$, there is a natural diagonal action of π on $\widetilde{S_1} \times \widetilde{\overline{S_2}}$ which allows us to define the dual pair

$$\begin{tikzcd}[column sep=small] & \dfrac{\widetilde{S}_1 \times \widetilde{\overline{S}}_2}{\pi} \arrow[dl] \arrow[dr] & \\ S_1 = \widetilde{S}_1/\pi & & \overline{S}_2 = \widetilde{\overline{S}}_2/\pi \end{tikzcd}$$

Exercise 27
Check that these maps have simply connected fibers and that this defines a Morita equivalence.

□

Isomorphism of fundamental groups implies isomorphism of first de Rham cohomology groups. For symplectic manifolds, the de Rham cohomology is isomorphic to Poisson cohomology. For general Poisson manifolds, we have the following result.

Theorem 9.5 (Ginzburg-Lu [72]) *If P_1, P_2 are Morita equivalent Poisson manifolds, then $H^1_\Pi(P_1) \simeq H^1_\Pi(P_2)$.*

Since any two simply connected symplectic manifolds are Morita equivalent, we are not able to say anything about the higher Poisson cohomology groups.

10 Examples of Symplectic Realizations

A symplectic realization of a Poisson manifold P is a Poisson map φ from a symplectic manifold M to P.

10.1 Injective Realizations of $\mathbb{T}^3$

Let $\mathbb{R}^3$ have coordinates (x_1, x_2, x_3) and (by an abuse of notation) let $\mathbb{T}^3$ be the 3-torus with coordinates (x_1, x_2, x_3) such that $x_i \sim x_i + 2\pi$. Define a Poisson structure (on $\mathbb{R}^3$ or $\mathbb{T}^3$) by

$$\Pi = \left(\frac{\partial}{\partial x_1} + \alpha_1 \frac{\partial}{\partial x_3}\right) \wedge \left(\frac{\partial}{\partial x_2} + \alpha_2 \frac{\partial}{\partial x_3}\right) .$$

The Poisson bracket relations are:

$$\{x_1, x_2\} = 1 , \qquad \{x_2, x_3\} = -\alpha_1 , \qquad \{x_1, x_3\} = \alpha_2 .$$

On $\mathbb{R}^3$, Π defines a foliation by planes with slope determined by α_1, α_2. If $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$, then Π also defines a foliation on $\mathbb{T}^3$ by planes, each of which is dense in $\mathbb{T}^3$. This is called a **(fully) irrational foliation**. If both α_1 and α_2 are rational, then the foliation of $\mathbb{T}^3$ is by 2-tori, and if exactly two of $1, \alpha_1, \alpha_2$ are linearly dependent over $\mathbb{Q}$, then the foliation is by cylinders.

In the fully irrational case, the algebra $H^0_\Pi(\mathbb{T}^3)$ of Casimir functions is trivial; in fact, the constants are the only L^∞ functions constant on symplectic leaves, since the foliation on $\mathbb{T}^3$ is ergodic. There are no proper Poisson ideals. This structure allows us to regard $\mathbb{T}^3$ as being “almost symplectic”. We will see, however, that its complete symplectic realizations are more interesting than those of a symplectic manifold.

Exercise 28
If Π defines a foliation by cylinders, are there any (nontrivial) Casimir functions?

First we may define a realization J by inclusion of a symplectic leaf,

$$\begin{array}{ccc} \mathbb{R}^2 & & (x_1, x_2) \\ \Big\downarrow J & & \Big\downarrow \\ \mathbb{T}^3 & & (x_1, x_2, \alpha_1 x_1 + \alpha_2 x_2) \pmod{2\pi} \end{array}$$

Although J is not a submersion, it is a complete map. There is such a realization for each symplectic leaf of $\mathbb{T}^3$, defined by

$$J_c : (x_1, x_2) \longmapsto (x_1, x_2, \alpha_1 x_1 + \alpha_2 x_2 + c) ,$$

with $c \in \mathbb{R}$. For any integers n_0, n_1, n_2, substituting $c + 2\pi(n_0 + \alpha_1 n_1 + \alpha_2 n_2)$ for c gives the same leaf. Thus the leaf space of $\mathbb{T}^3$ is parametrized by $c \in \mathbb{R}/2\pi(\mathbb{Z} + \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z})$.

The leaf space is highly singular; there is not even a sensible way to define nonconstant measurable functions. It is better to consider the Poisson manifold $\mathbb{T}^3$

itself as a model for the leaf space, just as one uses noncommutative algebras to model such singular spaces in noncommutative geometry [32].

The map $J : \mathbb{R}^2 \to \mathbb{T}^3$ has a dense image, and thus the induced pull-back on functions, $J^* : C^\infty(\mathbb{T}^3) \to C^\infty(\mathbb{R}^2)$, is injective. The following (periodic or quasi-periodic) functions on $\mathbb{R}^2$,

$$e^{ix_1}, \quad e^{ix_2}, \quad e^{i(\alpha_1 x_1 + \alpha_2 x_2)} ,$$

are in the image of J^*, and generate such a large class of functions that any function in $C^\infty(\mathbb{R}^2)$ can be uniformly C^∞-approximated by them on compact sets. Thus

$$J^*(C^\infty(\mathbb{T}^3))' = \text{ constants} \qquad \text{and} \qquad J^*(C^\infty(\mathbb{T}^3))'' = C^\infty(\mathbb{R}^2) \ .$$

Since the Poisson algebra $J^*(C^\infty(\mathbb{T}^3))$ is not its own double commutant, there can not be another Poisson manifold P which will make

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dl, "J"'] \arrow[dr] & \\ \mathbb{T}^3 & & P \end{tikzcd}$$

into a Morita equivalence. In fact, to form a dual pair, such a "manifold" P would have to be a single point because each fiber of J is a single point and because of orthogonality of fibers. The diagram

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dl, "J"'] \arrow[dr] & \\ \mathbb{T}^3 & & \text{point} \end{tikzcd}$$

satisfies the conditions that the fibers be symplectic orthogonals and that the fibers be all connected and simply connected. However, the function spaces of this pair are not mutual commutants. Of course, the problem here is that J is not a submersion.

10.2 Submersive Realizations of $\mathbb{T}^3$

Noticing that $\mathbb{T}^3$ is a regular Poisson manifold, we can use the construction for proving Lie's theorem (Chapter 4) to form a symplectic realization by adding enough extra dimensions. Specifically, consider the map

$$\begin{tikzcd} \mathbb{R}^4 \arrow[d, "J"'] & (x_1, x_2, x_3, x_4) \arrow[d] \\ \mathbb{T}^3 & (x_1, x_2, x_3) \end{tikzcd}$$

where $\mathbb{R}^4$ has symplectic structure defined by

$$\Pi_{\mathbb{R}^4} = \left(\frac{\partial}{\partial x_1} + \alpha_1 \frac{\partial}{\partial x_3} \right) \wedge \left(\frac{\partial}{\partial x_2} + \alpha_2 \frac{\partial}{\partial x_3} \right) + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \ ,$$

and $\mathbb{T}^3$ has the fully irrational Poisson structure as above:

$$\Pi = \left(\frac{\partial}{\partial x_1} + \alpha_1 \frac{\partial}{\partial x_3}\right) \wedge \left(\frac{\partial}{\partial x_2} + \alpha_2 \frac{\partial}{\partial x_3}\right) .$$

Exercise 29
Check that $\Pi_{\mathbb{R}^4}$ defines a non-degenerate 2-form on $\mathbb{R}^4$ which is equivalent to the standard symplectic structure

$$\Pi_{\rm std} = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} .$$

Show that the symplectic structures induced on $\mathbb{T}^4$ by $\Pi_{\mathbb{R}^4}$ and $\Pi_{\rm std}$ are *not* equivalent, though they both have the same volume element

$$\Pi_{\mathbb{R}^4} \wedge \Pi_{\mathbb{R}^4} = \Pi_{\rm std} \wedge \Pi_{\rm std} = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} .$$

(Consider $\frac{1}{2\pi}$ times the cohomology class of each symplectic structure.)

To find the commutant of $J^*(C^\infty(\mathbb{T}^3))$ in this case, we examine the symplectic orthogonals to the fibers of J. First, we list the Poisson brackets for the symplectic structure on $\mathbb{R}^4$:

$$\begin{array}{ll} \{x_1, x_2\} = 1 & \{x_1, x_4\} = 0 \\ \{x_2, x_3\} = -\alpha_1 & \{x_2, x_4\} = 0 \\ \{x_1, x_3\} = \alpha_2 & \{x_3, x_4\} = 1 \end{array}$$

and the hamiltonian vector fields

$$\begin{aligned} X_{x_1} &= -\frac{\partial}{\partial x_2} - \alpha_2 \frac{\partial}{\partial x_3} , \\ X_{x_2} &= \frac{\partial}{\partial x_1} + \alpha_1 \frac{\partial}{\partial x_3} , \\ X_{x_3} &= \alpha_2 \frac{\partial}{\partial x_1} - \alpha_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4} , \\ X_{x_4} &= \frac{\partial}{\partial x_3} . \end{aligned}$$

The commutant of $J^*(C^\infty(\mathbb{T}^3))$ consists of the functions killed by X_{x_1}, X_{x_2} and X_{x_3}. Since these three vector fields are constant, it suffices to find the *linear* functions $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ killed by these vector fields, *i.e.* solve the system

$$\begin{cases} -c_2 - \alpha_2 c_3 = 0 \\ c_1 + \alpha_1 c_3 = 0 \\ \alpha_2 c_1 - \alpha_1 c_2 - c_4 = 0 \end{cases}$$

The linear solutions are the constant multiples of

$$\alpha_1 x_1 + \alpha_2 x_2 - x_3 ,$$

and $J^*(C^\infty(\mathbb{T}^3))' \subseteq C^\infty(\mathbb{R}^4)$ consists of functions of $\alpha_1 x_1 + \alpha_2 x_2 - x_3$.

Given the commutant, we can geometrically define the other leg of a dual pair to be the map $J_2 : \mathbb{R}^4 \to \mathbb{R}$ given

$$(x_1, x_2, x_3, x_4) \longmapsto \alpha_1 x_1 + \alpha_2 x_2 - x_3 .$$

Thus we have the diagram

$$\begin{tikzcd} & \mathbb{R}^4 \arrow[dl, "J"'] \arrow[dr, "J_2"] & \\ \mathbb{T}^3 & & \mathbb{R} \end{tikzcd}$$

Although $\alpha_1 x_1 + \alpha_2 x_2 - x_3$ is not quasi-periodic, it lies in the closure of $J^*(C^\infty(\mathbb{T}^3))$. One can check that $J_2^*(C^\infty(\mathbb{R}))' \neq J^*(C^\infty(\mathbb{T}^3))$, and so this does not define a Morita equivalence. The obstruction stems from the fact that J does not have connected fibers; a fiber of J is an infinite collection of parallel lines in $\mathbb{R}^4$.

We can factor J through the quotient $\mathbb{R}^4 \to \mathbb{T}^4$, and denote the induced map by $J_{\mathbb{T}^4} : \mathbb{T}^4 \to \mathbb{T}^3$. The commutant of $J_{\mathbb{T}^4}^*(C^\infty(\mathbb{T}^3))$ in $C^\infty(\mathbb{T}^4)$ should be generated by the linear function $\alpha_1 x_1 + \alpha_2 x_2 - x_3$ on $\mathbb{T}^4$, but this is not periodic on $\mathbb{R}^4$ and thus is not defined on $\mathbb{T}^4$. Therefore, the commutant of $J_{\mathbb{T}^4}^*(C^\infty(\mathbb{T}^3))$ in $C^\infty(\mathbb{T}^4)$ is trivial, and the double commutant must be all of $C^\infty(\mathbb{T}^4)$, which again prevents Morita equivalence (moreover, fibers would fail to be simply connected). As before, the other leg of the dual pair would have to be a single point rather than $\mathbb{R}$:

$$\begin{tikzcd} & \mathbb{T}^4 \arrow[dl, "J_{\mathbb{T}^4}"'] \arrow[dr] & \\ \mathbb{T}^3 & & \text{point} \end{tikzcd}$$

The "dual" to $\mathbb{T}^3$ thus depends on the choice of realization, but requiring that the realization have connected fibers seems to imply that the dual is "pointlike".

We close these sections on $\mathbb{T}^3$ by mentioning that there is still much to investigate in the classification of complete realizations. For instance, it would be interesting to be able to classify complete Poisson maps from (connected) symplectic manifolds to

- $\mathbb{T}^3$ with the Poisson tensor $\Pi = \left(\frac{\partial}{\partial x_1} + \alpha_1 \frac{\partial}{\partial x_2}\right) \wedge \left(\frac{\partial}{\partial x_2} + \alpha_2 \frac{\partial}{\partial x_3}\right)$, or
- a given manifold M with the zero Poisson tensor.

10.3 Complex Coordinates in Symplectic Geometry

The symplectic vector space $\mathbb{R}^{2n}$ can be identified with the complex space $\mathbb{C}^n$ by the coordinate change

$$z_j = q_j + i p_j \ .$$

In order to study $\mathbb{C}^n$ as a (real) manifold, it helps to use the *complex* valued functions, vector fields, etc., even though the (real) symplectic form is not holomorphic.

On a general manifold M, the **complexified tangent bundle** is

$$\begin{aligned} T_{\mathbb{C}} M &= TM \otimes \mathbb{C} \\ &= TM \oplus iTM \ , \end{aligned}$$

and the complexified cotangent bundle is

$$\begin{aligned} T^*_{\mathbb{C}}M &= T^*M \otimes \mathbb{C} \\ &= T^*M \oplus iT^*M \\ &= \mathrm{Hom}_{\mathbb{C}}\,(T_{\mathbb{C}}M, \mathbb{C}) \\ &= \mathrm{Hom}_{\mathbb{R}}\,(TM, \mathbb{C}). \end{aligned}$$

Introducing complex conjugate coordinates $\overline{z}_j = q_j - ip_j$, we find $dz_j = dq_j + idp_j$, $d\overline{z}_j = dq_j - idp_j$ as linear functionals on $T_{\mathbb{C}}M$, and

$$\begin{aligned} dz_j \wedge d\overline{z}_j &= (dq_j + idp_j) \wedge (dq_j - idp_j) \\ &= -2i\,(dq_j \wedge dp_j)\ . \end{aligned}$$

Thus the standard symplectic structure on $T^*_{\mathbb{C}}M$ can be written in complex coordinates as

$$\omega = \frac{i}{2} \sum_j dz_j \wedge d\overline{z}_j\ .$$

We linearly extend the Poisson bracket $\{\cdot,\cdot\}$ to complex valued functions and compute

$$\begin{aligned} \{z_k, z_j\} &= 0 \\ \{\overline{z}_k, \overline{z}_j\} &= 0 \\ \{z_k, \overline{z}_j\} &= -2i\delta_{kj}\ . \end{aligned}$$

By these formulas, the Poisson tensor becomes

$$\Pi_{\mathbb{R}^{2n}} = -2i \sum_j \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \overline{z}_j}\ ,$$

where $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}$ form the dual basis to $dz_j, d\overline{z}_j$, and hence satisfy

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial q_j} - i\frac{\partial}{\partial p_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial q_j} + i\frac{\partial}{\partial p_j}\right)\ .$$

10.4 The Harmonic Oscillator

The **harmonic oscillator** is a system of n simple harmonic oscillators without coupling, modeled by $(\mathbb{R}^{2n}, \Pi_{\mathbb{R}^{2n}})$ with hamiltonian function

$$h_\alpha = \frac{1}{2} \sum_j \alpha_j (q_j^2 + p_j^2)\ .$$

The coefficients α_j are the n frequencies of oscillation. Using complex coordinates, we rewrite h_α as

$$h_\alpha = \frac{1}{2} \sum_j \alpha_j z_j \overline{z}_j\ .$$

To compute the flow of h_α, we work out the hamiltonian equations:

$$\frac{dz_k}{dt} = \{z_k, h_\alpha\} = \frac{1}{2} \sum_j \alpha_j \{z_k, z_j \overline{z}_j\} = \frac{1}{2}\alpha_k z_k (-2i) = -i\alpha_k z_k\ ,$$

and similarly, $\frac{d\overline{z}_k}{dt} = i\alpha_k \overline{z}_k$. The solution is thus $z_k(t) = z_k(0)e^{-i\alpha_k t}$.

If $\alpha_k = 1$ for all k, the flow is the standard action of S^1 on $\mathbb{C}^n$, which is free on $\mathbb{C}^n \setminus \{0\}$.

If all the α_k are rationally related, then we can assume after a change of time scale that $\alpha_k \in \mathbb{Z}$ and see that we still have an action of S^1. This action on $\mathbb{C}^n \setminus \{0\}$ will generally not be free, but rather will have discrete stabilizers.

If the α_k are not rationally related, then this defines an $\mathbb{R}$-action, as the typical orbits will not be closed, but will be dense on a torus. From now on, we will concentrate on this case.

To study the orbit space of the $\mathbb{R}$-action, we start by calculating the commutant. Specifically, we want to find the polynomial functions commuting with the hamiltonian.

Exercise 30
For a typical monomial $z^\ell \overline{z}^m = z_1^{\ell_1} \cdots z_n^{\ell_n} \overline{z}_1^{m_1} \cdots \overline{z}_n^{m_n}$, compute:

$$\begin{array}{ll} \text{(a)} & \{z_j^{\ell_j} \overline{z}_j^{m_j}, z_j \overline{z}_j\} = \ell_j z_j^{\ell_j} \overline{z}_j^{m_j} (2i) + \ell_j z_j^{\ell_j} \overline{z}_j^{m_j} (-2i) \ , \\ \text{(b)} & \{z^\ell \overline{z}^m, \frac{1}{2} \sum \alpha_j z_j \overline{z}_j\} = i \sum \alpha_j (m_j - \ell_j) z^\ell \overline{z}^m \ . \end{array}$$

Thus the monomials in z_j and $\overline{z}_j$ are eigenvectors of the hamiltonian vector field of the oscillator h_α. The corresponding eigenvalues are

$$i \sum_j \alpha_j (m_j - \ell_j) \ .$$

The commutant of h_α in $\mathrm{Pol}(\mathbb{C}^{n*})$ is spanned by the monomials $z^\ell \overline{z}^m$ with $\sum \alpha_j (m_j - \ell_j) = 0$.

Example. Suppose that the α_j's are linearly independent over $\mathbb{Q}$. Then the only monomials in the commutant are those with $m_j = \ell_j$ for all j, that is, monomials of the form $z^\ell \overline{z}^\ell = (z\overline{z})^\ell$. In this case, the functions invariant under the hamiltonian action are just polynomials in $I_j = z_j \overline{z}_j = |z_j|^2$. Then we can see this roughly as a pair

$$\begin{tikzcd} & \mathbb{C}^n \arrow[dl, "I_j"'] \arrow[dr, "h_\alpha"] & \\ \mathbb{R}^n & & \mathbb{R} \end{tikzcd}$$

Of course, I_j has a singularity at 0, and its image is only in the positive orthant of $\mathbb{R}^n$. This also could not be a dual pair of symplectic realizations, as the dimensions of the fibers do not match up properly unless we delete the origin. Even so, this example provides some intuition toward our study of dual pairs. $\diamondsuit$

If $\alpha_j \in \mathbb{Z}$ for all j, then calculating the commutant of h_α is equivalent to solving the system of linear equations

$$\sum_j \alpha_j (m_j - \ell_j) = 0$$

over the integers. What makes this problem non-trivial is that we are only interested in non-negative integer solutions for ℓ_j, m_j, in order to study the ring of invariant functions defined on *all* of $\mathbb{C}^n$.

To avoid this difficulty, we look first at the case $\alpha_j = 1$ for all j. Thus our equation reduces to $\sum(m_j - \ell_j) = 0$, or $\sum m_j = \sum \ell_j$. The set of solutions for this system of equations is spanned by the monomials $z_j \overline{z}_k$. In fact, the set $\{z_j \overline{z}_k\}$ forms a basis for the subring of solutions.

Remark. The real part of $z_j \overline{z}_k$ is invariant under the hamiltonian action since it can be expressed as $z_j \overline{z}_k + z_k \overline{z}_j$. Similarly, the imaginary part $z_j \overline{z}_k - z_k \overline{z}_j$ is invariant under the hamiltonian flow. $\diamondsuit$

The most general linear combination of the basis elements (that is, the most general quadratic solution) is

$$h_a = \sum_{j,k} a_{jk} z_j \overline{z}_k \ , \quad a_{jk} \in \mathbb{C} \ ,$$

and any function of this form is invariant under the hamiltonian flow. Furthermore, these are all the quadratic invariants. The invariant functions will not commute with one another, as the basis elements themselves did not commute.

10.5 A Dual Pair from Complex Geometry

To summarize the previous section: on $\mathbb{C}^n$, the hamiltonian $h = \frac{1}{2} \sum z_j \overline{z}_j$ generates a flow, which is just multiplication by unit complex numbers. The invariant functions $h_a = \sum_{j,k} a_{jk} z_j \overline{z}_k$ generate *complex linear* flows (*i.e.* flows by transformations commuting with multiplication by complex constants), which preserve h as well as the symplectic form ω.

Hence, transformations generated by h_a are unitary. The group of all linear transformations leaving h invariant is the unitary group $\mathrm{U}(n)$. We would like to show that the flows of the h_a's give a basis for the unitary Lie algebra $\mathfrak{u}(n)$.

Remark. The function h_a is real valued if and only if $a_{jk} = \overline{a_{kj}}$, *i.e.* the matrix (a_{jk}) is hermitian. Thus the set of real valued quadratic solutions corresponds to the set of hermitian matrices.

Recall that the Poisson bracket of two invariant functions is again invariant under the hamiltonian flow. Moreover, the bracket of two quadratics is again quadratic, and thus we can use the correspondence above to define a bracket on the group of hermitian matrices.

Exercise 31
Check that

$$\{h_a, h_b\} = h_{i[a,b]} \ ,$$

where $[a, b]$ is the standard commutator bracket of matrices.

$\diamondsuit$

The algebra $\mathfrak{u}(n)$ is the Lie algebra of *skew*-hermitian matrices. Denoting the space of hermitian matrices by $\mathfrak{h}_n$, we identify

$$\begin{array}{ccc} \mathfrak{h}_n & \longleftrightarrow & \mathfrak{u}(n) \\ a & \stackrel{\lambda}{\longmapsto} & ia \ . \end{array}$$

For $a, b \in \mathfrak{h}_n$, it is easy to check that

$$[\lambda a, \lambda b] = \lambda(i[a, b]) \ ,$$

and thus the bilinear map $\mathfrak{h}_n \times \mathfrak{h}_n \to \mathfrak{h}_n$ taking (a, b) to $i[a, b]$ is the usual commutator bracket on $\mathfrak{u}(n)$ pulled back by λ to $\mathfrak{h}_n$. With this identification of invariant flows as unitary matrices, we see that the map

$$\mathfrak{h}_n \simeq \mathfrak{u}(n) \longrightarrow C^\infty(\mathbb{C}^n)$$

is a Lie algebra homomorphism. From our discussion in Section 7.2, we conclude that there is a complete momentum map $J : \mathbb{C}^n \to \mathfrak{u}(n)^* \simeq \mathfrak{h}_n^*$ corresponding to an action of $\mathrm{U}(n)$ on $\mathbb{C}^n$. This is the standard action of the unitary group on $\mathbb{C}^n$. We may view J as a map $J : z \mapsto z \otimes \overline{z} \simeq (z_j \overline{z}_k)$. The value of the function h_a at $(z_j \overline{z}_k) \in \mathfrak{u}(n)^*$ is the inner product of the matrix (a_{jk}) with the matrix $(z_j \overline{z}_k)$.

Therefore, we have a pair

$$\begin{tikzcd} & \mathbb{C}^n \arrow[dl, "J"'] \arrow[dr, "h"] & \\ \mathfrak{h}_n^* \simeq \mathfrak{u}(n)^* & & \mathbb{R} \simeq \mathfrak{u}(1)^* \end{tikzcd}$$

Removing the origin in $\mathbb{C}^n$, we get a dual pair for which the image of the left leg is the collection of rank-one skew-hermitian, positive semi-definite matrices, and the image of the right leg is $\mathbb{R}^+$. A function which commutes with J is invariant on the concentric spheres centered at 0 and is thus a function of $|z\overline{z}|$ – the square of the radius. On the other hand, even though there is a singularity at $0 \in \mathbb{C}^n$, any function on $\mathbb{C}^n$ commuting with h is in fact a pull-back of a function on $\mathfrak{u}(n)$ by the map J. In general, functions which are pull-backs by the momentum map J are called **collective functions**.

Conjecture 10.1 (Guillemin-Sternberg [76]) *Suppose that a symplectic torus $\mathbb{T}^k$ acts linearly on $\mathbb{C}^n$ with quadratic momentum map $J : \mathbb{C}^n \to (\mathfrak{t}^k)^*$. If the map $\mathbb{C}^n \to \mathbb{C}^n/\mathbb{T}^k$ corresponds to the invariant functions under the torus action, then*

$$\begin{tikzcd} & \mathbb{C}^n \arrow[dl, "J"'] \arrow[dr, "p"] & \\ (\mathfrak{t}^k)^* & & \mathbb{C}^n/\mathbb{T}^k \end{tikzcd}$$

is a dual pair, in the sense that the images of J^* and p^* are mutual commutants in $C^\infty(\mathbb{C}^n)$.

Guillemin and Sternberg [76] almost proved this as stated for tori and conjectured that it held for any compact connected Lie group acting symplectically on $\mathbb{C}^n$. Lerman [103] gave a counterexample and, with Karshon [93], provided a proof of the conjecture for $(\mathfrak{t}^k)^*$ as well as an understanding of when this conjecture does and does not hold for arbitrary compact groups.

Example. Lerman's counterexample for the more general conjecture is the group $\mathrm{SU}(2)$ acting on $\mathbb{C}^2$ (see [93, 103] for more information). As for the case of $\mathfrak{u}(2)$ studied above, the invariant functions corresponding to the collective functions are functions of the square of the radius. The commutator of these functions are pulled back from $\mathfrak{u}(2)^*$, not $\mathfrak{su}(2)^*$. For instance, the function $z_1\overline{z}_1 + z_2\overline{z}_2$ cannot be the pull-back of a smooth function on $\mathfrak{su}(2)^*$, although the function $(z_1\overline{z}_1 + z_2\overline{z}_2)^2$ can be so expressed. Thus the pair of maps

$$\begin{tikzcd} & \mathbb{C}^2 \arrow[dl, "J"'] \arrow[dr] & \\ \mathfrak{su}(2)^* \simeq \mathbb{R}^3 & & \mathbb{R} \end{tikzcd}$$

is not a dual pair.

Exercise 32
What happens when we remove the origin from each space?

$\diamondsuit$

Part V
Generalized Functions

11 Group Algebras

Multiplication on a (locally compact) group G can be coded into a *coproduct* structure on the algebra $C(G)$ of continuous real functions on G, making it into a *commutative Hopf algebra*. Conversely, the algebra $C(G)$ determines the multiplication on G. Noncommutative analogues of $C(G)$ are studied as if they were algebras of functions on so-called *quantum groups*.

11.1 Hopf Algebras

Example. Let G be a finite set, and let $C(G)$ be its algebra of real functions. The tensor product $C(G) \otimes C(G)$ is naturally isomorphic as an algebra to $C(G \times G)$ via the map

$$\varphi \otimes \psi \longmapsto ((g,h) \mapsto \varphi(g)\psi(h)) \ .$$

Now suppose that G is a group. Besides the pointwise *product* of functions,

$$m : C(G) \otimes C(G) \longrightarrow C(G) \ , \qquad m(\varphi \otimes \psi) = \varphi\psi \ ,$$

we can use the group multiplication $G \times G \xrightarrow{m} G$ to define a *coproduct* on $C(G)$

$$m^* : C(G) \longrightarrow C(G \times G) = C(G) \otimes C(G) \ , \qquad m^*(\varphi)(g,h) = \varphi(gh) \ .$$

It is an easy exercise to check that this is a homomorphism with respect to the pointwise products on $C(G)$ and $C(G \times G)$. With this product and coproduct, $C(G)$ becomes a *Hopf algebra*. $\diamondsuit$

In general, a **Hopf algebra** is a vector space $\mathcal{A}$ equipped with the following operations:

1. a **multiplication**
$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \ , \quad \text{also denoted} \quad m(\varphi, \psi) = \varphi \cdot \psi \ ,$$
2. a **comultiplication**
$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \ ,$$
3. a **unit** (or **identity**),
$$i : \mathbb{C} \longrightarrow \mathcal{A} \ ,$$
4. a **co-unit** (or **coidentity**),
$$\varepsilon : \mathcal{A} \longrightarrow \mathbb{C} \ , \quad \text{and}$$
5. an **antipode** map
$$\alpha : \mathcal{A} \longrightarrow \mathcal{A} \ ,$$

satisfying the following axioms:

1. the multiplication is associative, *i.e.*

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "m_{1,2}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m_{2,3}"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "m"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}$$

   commutes, where $m_{1,2}\otimes \mathrm{id} : \varphi\otimes\psi\otimes\rho \mapsto m(\varphi,\psi)\otimes\rho$, and similarly for other indexed maps on tensor product spaces,

2. the comultiplication is **coassociative**, *i.e.*

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \mathcal{A}\otimes\mathcal{A} \arrow[l, "\Delta\otimes \mathrm{id}"'] \\
\mathcal{A}\otimes\mathcal{A} \arrow[u, "\mathrm{id}\otimes\Delta"] & \mathcal{A} \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

   commutes,

3. the comultiplication Δ is a homomorphism of algebras, *i.e.*

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A} & \mathcal{A} \arrow[l, "\Delta"'] \\
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[u, "m_{1,3}\otimes m_{2,4}"] & \mathcal{A}\otimes\mathcal{A} \arrow[l, "\Delta\otimes\Delta"'] \arrow[u, "m"']
\end{tikzcd}$$

   commutes, (that is, $\Delta(\varphi\cdot\psi) = \Delta(\varphi)\cdot\Delta(\psi)$ where the multiplication on the right hand side is $m\otimes m$),

4. the unit is an identity for multiplication, *i.e.*

$$\begin{tikzcd}
\mathcal{A}\otimes\mathbb{C}\simeq\mathbb{C}\otimes\mathcal{A} \arrow[r, "\mathrm{id}\otimes i"] \arrow[d, "i\otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "m"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}$$

   commutes,

5. the co-unit is a co-identity for comultiplication, *i.e.*

$$\begin{tikzcd}
\mathcal{A} & \mathcal{A}\otimes\mathcal{A} \arrow[l, "\mathrm{id}\otimes\varepsilon"'] \\
\mathcal{A}\otimes\mathcal{A} \arrow[u, "\varepsilon\otimes \mathrm{id}"] & \mathcal{A} \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \arrow[ul, "\mathrm{id}"']
\end{tikzcd}$$

commutes,

6. the unit is a homomorphism of coalgebras, *i.e.*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "i"] \arrow[d] & \mathcal{A} \arrow[d, "\Delta"] \\ \mathbb{C}\otimes\mathbb{C} \arrow[r, "i\otimes i"] & \mathcal{A}\otimes\mathcal{A} \end{tikzcd}$$

commutes, where the left arrow is $c \mapsto c \otimes 1$,

7. the co-unit is a homomorphism of algebras, *i.e.*

$$\begin{tikzcd} \mathbb{C} & \mathcal{A} \arrow[l, "\varepsilon"'] \\ \mathbb{C}\otimes\mathbb{C} \arrow[u] & \mathcal{A}\otimes\mathcal{A} \arrow[l, "\varepsilon\otimes\varepsilon"'] \arrow[u, "m"'] \end{tikzcd}$$

commutes, where the left arrow is multiplication of complex numbers, and

8. the antipode is an anti-homomorphism of algebras, *i.e.*

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} \arrow[r, "\alpha\otimes\alpha"] \arrow[d, "m"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\overline{m}"] \\ \mathcal{A} \arrow[r, "\alpha"] & \mathcal{A} \end{tikzcd}$$

commutes, where $\overline{m}(\varphi, \psi) = m(\psi, \varphi)$.

9. the antipode is an anti-homomorphism of coalgebras, *i.e.*

$$\begin{tikzcd} \mathcal{A}\otimes\mathcal{A} & \mathcal{A}\otimes\mathcal{A} \arrow[l, "\alpha\otimes\alpha"'] \\ \mathcal{A} \arrow[u, "\overline{\Delta}"] & \mathcal{A} \arrow[l, "\alpha"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

commutes, where $\overline{\Delta}$ is Δ composed with the map $\varphi \otimes \psi \mapsto \psi \otimes \varphi$.

10. the following diagram involving the antipode commutes[2]

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{C} \arrow[r, "i"] & \mathcal{A} \\ \mathcal{A}\otimes\mathcal{A} \arrow[rr, "\alpha\otimes\mathrm{id}"] & & \mathcal{A}\otimes\mathcal{A} \arrow[u, "m"'] \end{tikzcd}$$

[2]It is not generally true that the square of the antipode equals the identity map.

and the similar diagram with the bottom arrow being $\mathrm{id} \otimes \alpha$ also commutes. This is sometimes regarded as the defining axiom for the antipode.

Hopf came across the structure just described while studying the cohomology rings of topological groups.

11.2 Commutative and Noncommutative Hopf Algebras

When $\mathcal{A} = C(G)$ is the algebra of continuous real functions on a locally compact topological group G, the (pointwise) multiplication of functions extends to a product on a topological completion " $\otimes$ " of the standard algebraic tensor product for which $C(G \times G) \simeq C(G)$"$\otimes$"$C(G)$ (see [159]). The algebra $C(G)$ is a **commutative Hopf algebra** (commutativity here refers to the first multiplication) where

1. the multiplication is pointwise multiplication of functions,

2. the comultiplication is the pull-back m^* of the multiplication on G,

3. the identity is the function identically equal to 1, or, equivalently, the homomorphism $\mathbb{C} \to C(G)$, $c \mapsto \varphi \equiv c$,

4. the coidentity is given by evaluation at the identity of G, and

5. the antipode is the pull-back by the inversion map on G.

Exercise 33
Show that the associativity of group multiplication on G translates to coassociativity on $C(G)$.

Commutative Hopf algebras are closely related to groups: if $\mathcal{A} = C(X)$ is the set of continuous functions on a locally compact Hausdorff space X, then a Hopf algebra structure on $\mathcal{A}$ (with $C(X \times X)$ playing the role of $\mathcal{A} \otimes \mathcal{A}$) defines a (not necessarily commutative) multiplication on X which can be shown to satisfy the group axioms.

A **noncommutative Hopf algebra** is thus to be thought of as "the algebra of functions on a **quantum group**". There is no universally accepted definition of a quantum group. Many people restrict the name to the objects obtained by deforming a Hopf algebra of functions on a Lie group.

Between commutative and noncommutative Hopf algebras lies the category of *Poisson Hopf algebras.* A **Poisson Hopf algebra** $\mathcal{A}$ is a commutative Hopf algebra equipped with a bracket operation making $\mathcal{A}$ into a Poisson algebra. We then require the comultiplication and co-unit to be Poisson algebra homomorphisms, while the antipode is an anti-homomorphism. When $\mathcal{A} = C^\infty(P)$ for some Poisson manifold P, the comultiplication gives P the structure of a **Poisson Lie group**; *i.e.* the multiplication map $P \times P \to P$ is a Poisson map. Poisson Lie groups can be regarded as the transitional objects between groups and quantum groups, or as classical limits of quantum groups. A comprehensive reference on quantum groups and Poisson Lie groups is [25].

11.3 Algebras of Measures on Groups

Let G be a locally compact topological group G, and let $C(G)$ be its algebra of continuous real functions. The dual space $C'(G)$ consists of compactly supported measures on G. (The *Lie* group version of this construction will be presented in Section 11.5).

Denoting by m the multiplication map on G, we described in Section 11.2 a coproduct

$$C(G) \xrightarrow{\Delta = m^*} C(G)\text{“}\otimes\text{”}C(G) \ , \quad m^*(\varphi)(g,h) = \varphi(gh) \ .$$

On $C'(G)$ we obtain a map

$$C'(G) \xleftarrow{\Delta^* = m_*} C'(G)\text{“}\otimes\text{”}C'(G)$$

defined by

$$\Delta^*(\mu)(S) = \mu(m^{-1}S) \ ,$$

where $\mu \in C'(G)\text{“}\otimes\text{”}C'(G) \simeq C'(G \times G)$ is a measure on $G \times G$, S is any measurable subset of G, and $m^{-1}S = \{(g,h) \in G \times G \mid \ gh \in S\}$. The map Δ^* is just the push-forward of measures by the multiplication map.

Composing Δ^* with the natural bilinear map $(\mu,\nu) \mapsto \mu \otimes \nu$ from $C'(G) \times C'(G)$ to the tensor product, we obtain a multiplication of measures on G. By a simple diagram chase through the axioms, we can check that Δ^* is associative. This multiplication is called **convolution**, and we will denote $\Delta^*(\mu \otimes \nu)$ by $\mu * \nu$. The following (abusive) notation is commonly used

$$\int f(x) \ d(\mu * \nu)(x) = \int\int f(yz) \ d\mu(y) \ d\nu(z) \qquad \text{for } f \in C(G) \ .$$

The space $C'(G)$ (or a suitable completion, such as the integrable signed measures) with the convolution operation is known as the **measure group algebra** of G.

The diagonal map in the group

$$\begin{array}{rccc} D: & G & \longrightarrow & G \times G \\ & g & \longmapsto & (g,g) \end{array}$$

induces by push-forward a coproduct on measures

$$C'(G) \xrightarrow{D_*} C'(G)\text{“}\otimes\text{”}C'(G)$$

defined by

$$D_*(\mu)(S) = \mu(D^{-1}S) \ ,$$

where $D^{-1}S = \{g \in G \mid (g,g) \in S\}$.

The space $C'(G)$ becomes a Hopf algebra for the convolution product Δ^* and this coproduct D_*; the unit is the delta measure at the identity e of G (or rather, it is the map $\mathbb{C} \ni c \mapsto c\delta_e$), the co-unit is evaluation of measures on the total set G, and the antipode of a measure is its pushforward by the group inversion map.

In summary, we see that the group structure on G gives rise to:

- a (pull-back of group multiplication) coproduct on $C(G)$, and its dual

- a (convolution) multiplication on $C'(G)$.

Independent of the group structure we have:

- a (pointwise) multiplication on $C(G)$, and its dual
- a (push-forward by the diagonal map) coproduct on $C'(G)$.

Remark. Each element $g \in G$ defines an evaluation functional δ_g on G by

$$\delta_g(f) := f(g) \ .$$

This identification allows us to think of G as sitting inside $C'(G)$. Note that $\delta_{gh} = \delta_g * \delta_h$. Moreover, the push-forward of the diagonal map behaves nicely on $G \subseteq C'(G)$:

$$D_*(\delta_g) = \delta_g \otimes \delta_g \ .$$

An element of $C'(G)$ is called **group(-element)-like** if it satisfies the property above. ♢

11.4 Convolution of Functions

If we choose a reference Borel measure λ on G, we can identify locally integrable functions φ on G with measures by $\varphi \mapsto \varphi\lambda$. The map from compactly supported locally integrable functions to $C'(G)$ is neither surjective (its image is the set of compactly supported measures which are absolutely continuous with respect to λ [146]), nor injective (if two functions differ only on a set of λ-measure 0, then they will map to the same measure). In any case, we can use this rough identification together with convolution of measures to describe a new product on functions on G.

Before we can do this, we need to make a digression through measures on groups. We define a measure λ to be **quasi-invariant** if, for each $g \in G$, the measure $(\ell_g)_*\lambda$ induced by left translation is absolutely continuous with respect to λ; in other words, there is a locally integrable function φ such that $(\ell_g)_*\lambda = \varphi\lambda$. We define λ to be **left-invariant** if $(\ell_g)_*\lambda = \lambda$ for all $g \in G$.

Theorem 11.1 *If G is locally compact, then there exists a left-invariant measure which is unique up to multiplication by positive scalars.*

Such a measure is called a **Haar measure**.

Remarks.

- For Lie groups, this theorem can be proven easily using a left-invariant volume form, which can be identified with a non-zero element of the highest dimensional exterior power $\wedge^{\mathrm{top}}\mathfrak{g}^*$ of $\mathfrak{g}^*$: use left translation to propagate such an element to the entire group.
- For general locally compact groups, this theorem is not trivial [149].

- For some quantum groups, an analogous result holds; the study of Haar measures on quantum groups is still in progress (see [25], Section 13.3B).

◊

Observe that if λ is a left-invariant measure, then $(r_g)_*\lambda$ is again left-invariant for any $g \in G$. Thus by Theorem 11.1, there is a function $\delta : G \to \mathbb{R}^+$ such that

$$(r_g)_*\lambda = \delta(g)\lambda \ .$$

It is easy to check the following lemma.

Lemma 11.2 *$\delta(gh) = \delta(g)\delta(h)$.*

δ is known as the **modular function** or the **modular character** of G. Due to the local compactness of G, we also know that δ is continuous. If G is *compact*, then we see that $\delta \equiv 1$. Any group with $\delta \equiv 1$ is called **unimodular**. Notice that δ is independent of the choice for λ. Also, when G is a Lie group, we can compute

$$\begin{aligned} \delta(g)\lambda &= (\ell_{g^{-1}})_*(r_g)_*\lambda \\ &= (\mathrm{Ad}\ g^{-1})_*\lambda \\ &= |(\det \mathrm{ad}\ g)|\lambda \ . \end{aligned}$$

Thus we interpret the modular function of a Lie group as (the absolute value of) the determinant of the adjoint representation on the Lie algebra.

Exercise 34

1. Compute the modular function for the group of affine transformations, $x \mapsto ax + b$, of the real line.
2. Prove that $\mathrm{GL}(n)$ is unimodular.
3. To check the formula above for $\delta(g)$ on a Lie group, see whether $\delta(g)$ is greater or smaller than 1 when $\mathrm{Ad}\ g^{-1}$ is expanding. Is $\det \mathrm{ad}\ g$ greater or smaller than 1?

Let λ be a Haar measure and $\delta : G \to \mathbb{R}^+$ the modular function. Given functions $\varphi, \psi \in C(G)$, their **convolution** $\varphi *_\lambda \psi$ with respect to λ is

$$(\varphi *_\lambda \psi)\lambda := \varphi\lambda * \psi\lambda \ ,$$

or, equivalently,

$$(\varphi *_\lambda \psi)(g) = \int_{h \in G} \varphi(gh^{-1})\ \psi(h)\ \delta(h)\ d\lambda(h) \ .$$

When G is unimodular, the δ factor drops out, and we recover the familiar formula for convolution of functions. We can rewrite this formula in terms of a kernel for the convolution (*cf.* Sections 14.1 and 14.2):

$$(\varphi *_\lambda \psi)(g) = \iint \varphi(k)\ \psi(\ell)\ K(g,k,\ell)\ d\lambda(k)\ d\lambda(\ell) \ .$$

We think of K as a generalized function, and we interpret the expression $K(g,k,\ell)d\lambda(k)d\lambda(\ell)$ as a measure on $G \times G \times G$ supported on $\{(g,k,\ell) \mid g = k\ell\}$, which is the graph of multiplication.

Remark. There is a *-operation on complex-valued measures which is the push-forward by the inversion map on the group, composed with complex conjugation. To transfer this operation to functions $\varphi \in C(G)$, we need to incorporate the modular function:

$$\varphi^*(x) = \overline{\varphi(x^{-1})}\, \delta(x^{-1}) \ .$$

The map $\varphi \mapsto \varphi^*$ is an anti-isomorphism of $C(G)$. ♢

11.5 Distribution Group Algebras

In Sections 11.2 and 11.3, we realized both $C(G)$ and its dual $C'(G)$ as Hopf algebras, with products and coproducts naturally induced from the group structure of G.

There is a smooth version of this construction. If G is a Lie group, then $\mathcal{D}(G) = C^\infty(G)$ is a Hopf algebra. The product is pointwise multiplication of functions, while the coproduct is again the pull-back of group multiplication. The "tensor" here needs to be a smooth kind of completion, so that $C^\infty(G)$"$\otimes$"$C^\infty(G)$ becomes $C^\infty(G \times G)$.

The dual space $\mathcal{D}'(G)$ of compactly supported distributions [148] is called the **distribution group algebra** of G. The space $\mathcal{D}'(G)$ is larger than the measure group algebra: an example of a distribution that is not a measure is evaluation of a second derivative at a given point. As in the case of $C'(G)$, we can define a product (convolution) and coproduct (push-forward of the diagonal map) on $\mathcal{D}'(G)$ to provide a Hopf algebra structure.

Remark. At the end of Section 11.3, we noted how the group G was contained in $C'(G)$ as evaluation functionals:

$$g \in G \longmapsto \delta_g \in C'(G) \ .$$

Inside $\mathcal{D}'(G)$, the evaluation functionals can be used to define left and right **translation maps**:

$$\begin{array}{rrcl} \delta_g * \cdot : & \mathcal{D}'(G) & \longrightarrow & \mathcal{D}'(G) \\ & \varphi(x) & \longmapsto & (\delta_g * \varphi)(x) = \varphi(g^{-1}x) \end{array}$$

and

$$\begin{array}{rrcl} \cdot * \delta_g : & \mathcal{D}'(G) & \longrightarrow & \mathcal{D}'(G) \\ & \varphi(x) & \longmapsto & (\varphi * \delta_g)(x) = \varphi(xg^{-1}) \end{array}$$

♢

Exercise 35
Show that the algebra of differential forms on a Lie group forms a Hopf algebra. What is its dual?

12 Densities

As we have seen, group algebras, measure group algebras and distribution group algebras encode much, if not all, of the structure of the underlying group. There are counterparts of these algebras for the case of manifolds, as algebras of "generalized functions".

12.1 Densities

To construct spaces of distributions which behave as generalized *functions*, rather than measures, we need the notion of *density* on a manifold.

Let V be a finite dimensional vector space (over $\mathbb{R}$), and let $\mathcal{B}(V)$ be the set of bases of V. An α-**density** on V is a function $\sigma : \mathcal{B}(V) \to \mathbb{C}$ such that, for every $A \in \mathrm{GL}(V)$ and $\beta \in \mathcal{B}(V)$, we have the relation

$$\sigma(\beta \cdot A) = |\det A|^{\alpha}\ \sigma(\beta)\ ,$$

where

$$(\beta \cdot A)_i = \sum_j \beta_j A_{ji}$$

when A is written as $A = (A_{ij})$ and β is the basis $\beta = (\beta_1, \ldots, \beta_n)$.

Remarks.

- When $\alpha = 1$, σ is equal up to signs to the function on bases given by an element of $\wedge^{\mathrm{top}} V^*$. We often denote the space of α-densities on V by
$$\left|\wedge^{\mathrm{top}}\right|^{\alpha} V^*\ .$$
In fact, if θ is an element of $\wedge^{\mathrm{top}} V^* \simeq (\wedge^{\mathrm{top}} V)^*$, there is an α-density $|\theta|^{\alpha}$ defined by
$$|\theta|^{\alpha}(\beta_1, \ldots, \beta_n) = |\theta(\beta_1 \wedge \cdots \wedge \beta_n)|^{\alpha}\ .$$
- A density σ is completely determined by its value on one basis, so $\left|\wedge^{\mathrm{top}}\right|^{\alpha} V^*$ is a one-dimensional vector space.

$\diamondsuit$

Lemma 12.1

1. *For any vector space A and any $\alpha, \beta \in \mathbb{R}$, there is a natural isomorphism*
$$\left|\wedge^{\mathrm{top}}\right|^{\alpha} A \otimes \left|\wedge^{\mathrm{top}}\right|^{\beta} A \simeq \left|\wedge^{\mathrm{top}}\right|^{\alpha+\beta} A\ .$$
2. *For any vector space A and any $\alpha \in \mathbb{R}$, there are natural isomorphisms*
$$\left|\wedge^{\mathrm{top}}\right|^{-\alpha} A \simeq (\left|\wedge^{\mathrm{top}}\right|^{\alpha} A)^* \simeq \left|\wedge^{\mathrm{top}}\right|^{\alpha} A^*\ .$$
3. *Given an exact sequence*
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
of vector spaces, there is a natural isomorphism
$$\left|\wedge^{\mathrm{top}}\right|^{\alpha} A \otimes \left|\wedge^{\mathrm{top}}\right|^{\alpha} C \simeq \left|\wedge^{\mathrm{top}}\right|^{\alpha} B\ .$$

Exercise 36
Prove the lemma above.

Now suppose that E is a vector bundle over a smooth manifold M with fiber V. Letting $\mathcal{B}(E)$ be the bundle of bases of E, a C^k **α-density** on E is a C^k map $\sigma : \mathcal{B}(E) \to \mathbb{C}$ which satisfies

$$\sigma(\beta \cdot A) = |\det A|^\alpha \sigma(\beta) \ .$$

In other words, σ must be $\mathrm{GL}(V)$-equivariant with respect to the natural $\mathrm{GL}(V)$-action on the fibers of $\mathcal{B}(E)$, and the action of $\mathrm{GL}(V)$ on $\mathbb{C}$ where $A \in \mathrm{GL}(V)$ acts by multiplication by $|\det A|^\alpha$. Hence, we can think of an α-density on a vector bundle E as a section of an associated line bundle $\mathcal{B}(E) \times \mathbb{C}/\sim$, where $(\beta \cdot A, z) \sim (\beta, |\det A|^\alpha z)$, for $A \in \mathrm{GL}(V)$. Equivalently, an α-density is a section of the bundle $|\wedge^{\mathrm{top}}|^\alpha E^*$, whose fiber at a point p is $|\wedge^{\mathrm{top}}|^\alpha E_p^*$. Therefore, a density on E is a family of densities on the fibers. When $E = TM$, we write $|\wedge^{\mathrm{top}}|^\alpha M := |\wedge^{\mathrm{top}}|^\alpha T^*M$ and $|\wedge^{\mathrm{top}}| M := |\wedge^{\mathrm{top}}|^1 T^*M$.

Remark. All the bundles $|\wedge^{\mathrm{top}}|^\alpha E^*$ are trivializable. However, they have no *natural* trivialization. ♢

Example. A riemannian manifold carries for each α a natural α-density which assigns the value 1 to every orthonormal basis. The orientation of the basis is not relevant to the density. ♢

12.2 Intrinsic L^p Spaces

Suppose that σ is a compactly supported C^0 1-density on a manifold M. In [42], de Rham referred to such objects as *odd differential forms*. The integral

$$\int_M \sigma$$

can be given a precise meaning (whether or not M is orientable!). To do so, use a partition of unity to express σ as a sum of densities supported in local coordinate systems. Thus we can restrict to the case

$$\sigma = f(x_1, \ldots, x_n)\ |dx_1 \wedge \cdots \wedge dx_n| \ .$$

Expressed in this way, the density can be integrated as

$$\int f(x_1, \ldots, x_n)\ dx_1 \ldots dx_n \ .$$

This integral is well-defined because the jacobian of a coordinate change is the absolute value of the determinant of the transformation.

If λ is a compactly supported $\frac{1}{p}$-density on M, then

$$\left(\int |\lambda|^p \right)^{1/p}$$

is well-defined. Thus there is an intrinsic L^p norm on the space of compactly supported $\frac{1}{p}$-densities on a manifold M. Also, if λ_1, λ_2 are two compactly supported $\frac{1}{2}$-densities, then we can define a hermitian inner product

$$\int_M \lambda_1 \overline{\lambda_2}.$$

Completion with respect to the norm given by the inner product produces an intrinsic Hilbert space $L^2(M)$. The group of diffeomorphisms of M acts on $L^2(M)$ by unitary transformations.

Trivializing $|\wedge^{\text{top}}|M$ amounts to choosing a positive (smooth) density σ_0, or equivalently, to choosing a nowhere vanishing (smooth) measure on M. Given such a trivialization, which also trivializes $|\wedge^{\text{top}}|^\alpha M$ for each α, we can identify functions with densities and hence obtain L^p spaces of *functions*.

Exercise 37
Show that the L^p spaces obtained in this way are the usual L^p spaces of functions with respect to the given measure.

12.3 Generalized Sections

Let E be a vector bundle over M. Define E' to be

$$E' := E^* \otimes |\wedge^{\text{top}}|M \ .$$

There is a natural pairing $\langle \sigma, \tau \rangle$ between compactly supported smooth sections of E, $\sigma \in \Gamma_c(E)$, and smooth sections of E', $\tau \in \Gamma(E')$, given by the pairing between E and E^* and by integration of the remaining density.

Sections $\sigma \in \Gamma_c(E)$ define by $\langle \sigma, \cdot \rangle$ continuous linear functionals with respect to the C^∞-topology on $\Gamma(E')$. (Recall that a sequence converges in the C^∞-topology if and only if it converges uniformly with all its derivatives on compact subsets of domains of coordinate charts and bundle trivializations.)

Denoting the space dual to $\Gamma(E')$ by $\mathcal{D}'(M, E)$, we conclude that there is a natural embedding

$$\Gamma_c(E) \subseteq \mathcal{D}'(M, E) \ .$$

For this reason, arbitrary elements of $\mathcal{D}'(M, E)$ are called compactly supported **generalized sections** of E. Occasionally, they are called compactly supported **distributional sections** or (less accurately) compactly supported distribution-valued sections. Similarly, generalized sections of E which are not necessarily compactly supported are defined as the dual space to compactly supported smooth sections of E', $\Gamma_c(E')$. In this case, we have

$$\Gamma(E) \subseteq \Gamma_c(E')' \ .$$

If $E = |\wedge^{\text{top}}|M$, then E' is the trivial line bundle over M, and we recover the usual compactly supported **distributions** on M:

$$\mathcal{D}'(M) := \mathcal{D}'(M, |\wedge^{\text{top}}|M) \supseteq \Gamma_c(|\wedge^{\text{top}}|M) \ .$$

Similarly, if E is the trivial line bundle ε, then $E' = |\wedge^{\text{top}}|M$, and so

$$\mathcal{D}'(M, \varepsilon) \supseteq C_c^\infty(M) \ .$$

Any differential operator D on $\Gamma_c(E)$ has a unique **formal adjoint** D^*, that is, a differential operator on $\Gamma(E')$ such that

$$\langle D\sigma, \tau\rangle = \langle \sigma, D^*\tau\rangle$$

for all $\sigma \in \Gamma_c(E)$ and $\tau \in \Gamma(E')$. These differential operators are continuous with respect to the C^∞-topology, and we can thus extend them to operators on $\mathcal{D}'(M,E)$ by the same formula

$$\langle D\sigma, \tau\rangle = \langle \sigma, D^*\tau\rangle \ ,$$

where σ now lies in $\mathcal{D}'(M,E)$.

Example. It is easy to check that on $\mathbb{R}^n$, the operator $\frac{\partial}{\partial x_i}$ has formal adjoint $-\frac{\partial}{\partial x_i}$. To see how this extends to generalized sections, note that, for instance, $\frac{\partial \delta_0}{\partial x_i}$ is defined by

$$\left\langle \frac{\partial \delta_0}{\partial x_i}, f \right\rangle = \left\langle \delta_0, -\frac{\partial f}{\partial x_i} \right\rangle = -\frac{\partial f}{\partial x_i}(0) \ .$$

$\diamondsuit$

We have shown that we can regard any Lie group G as sitting inside $\mathcal{D}'(G)$ (see Section 11.5). Similarly, on any manifold M, we can view a tangent vector as a **generalized density**, *i.e.* a generalized section of $|\wedge^{\text{top}}| M$. Let $X \in T_m M$ be any tangent vector. Then, for $\varphi \in C^\infty(M)$, the map

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{X\cdot} & \mathbb{R} \\ \varphi & \longmapsto & X \cdot \varphi \end{array}$$

is continuous with respect to the C^∞-topology, and thus X defines an element of $\mathcal{D}'(M)$. In particular, for $M = G$, we see that *both* G and $\mathfrak{g}$ sit in $\mathcal{D}'(G)$.

Alternatively, let $X \in T_m M$ and let $\widetilde{X} \in \Gamma(TM)$ be a vector field on M whose value at m is X. Then $\widetilde{X}$ acts on densities by the Lie derivative. Its formal adjoint can be shown to be the negative of the usual action of $\widetilde{X}$ on functions, in the following manner.

Exercise 38
For a density α, use Stokes' theorem to verify

$$\begin{aligned} \textstyle\int (\mathcal{L}_{\widetilde{X}}\alpha) \circ \varphi &= \textstyle\int \left(\mathcal{L}_{\widetilde{X}}(\alpha\varphi) - \alpha(\widetilde{X} \circ \varphi)\right) \\ &= -\textstyle\int \alpha(\widetilde{X} \circ \varphi) \ . \end{aligned}$$

Let δ_m be the functional of evaluation at m. Then

$$\begin{aligned} \langle -\mathcal{L}_{\widetilde{X}}\delta_m, \varphi\rangle &= \left\langle \delta_m, \widetilde{X}\varphi \right\rangle \\ &= \widetilde{X}(\varphi)(m) \\ &= X\varphi \ , \end{aligned}$$

and thus we again see X as a generalized density. It is known as a *dipole*, since

$$\begin{aligned} X\varphi &= \lim_{\varepsilon \to 0} \frac{\varphi(m_\varepsilon) - \varphi(m)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} \langle \delta_{m_\varepsilon}, \varphi \rangle - \frac{1}{\varepsilon} \langle \delta_m, \varphi \rangle \right] \ , \end{aligned}$$

where $\varepsilon \mapsto m_\varepsilon$ is a path through m with tangent vector X at $\varepsilon = 0$.

If we apply differential operators to δ_m, then the additional distributions obtained are all supported at m; that is, the action of each of the distributions on a test function φ depends on φ only in a neighborhood of m and thus can be obtained by a finite initial segment of the Taylor series of φ at m.

Example. For the case of a Lie group G, inside the distribution group algebra, $\mathcal{D}'(G)$, we have all of the following spaces:

$C'(G)$	– the measure group algebra as the set of measures,
$C(G)$	– the group algebra as the set of continuous functions,
G	– the group itself as evaluation functionals,
$\mathfrak{g}$	– the Lie algebra as vector fields applied to δ_e , and
$\mathcal{U}(\mathfrak{g})$	– the universal enveloping algebra as arbitrary differential operators applied to δ_e .

We will next see how $\mathcal{U}(\mathfrak{g})$ sits in $\mathcal{D}'(G)$; notice already that $\mathfrak{g}$ is *not* closed under the convolution multiplication in $\mathcal{D}'(G)$. ◊

12.4 Poincaré-Birkhoff-Witt Revisited

If two distributions on a Lie group G are supported at the identity e, so is their convolution, and so the distributions supported at e (all derivatives of δ_e) form a subalgebra of $\mathcal{D}'(G)$. For each such distribution σ, the convolution operation $\sigma * \cdot$ is a differential operator on $C^\infty(G)$. These operators commute with all translation operators $\cdot * \delta_g$, hence the distributions supported at the identity realize the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as a subalgebra of $\mathcal{D}'(G)$.

Remark. There is a general theorem that any distribution supported at a point comes from applying a differential operator to the evaluation function at that point [148, p.100]. ◊

The following remarks are due to Berezin and can be found in [13]. Consider the exponential map $\exp : \mathfrak{g} \to G$ on a Lie group G. In general, distributions cannot be pulled back by this map, since it can have singularities. If we are only interested in distributions supported at e, though, then we can use the fact that the exponential map is a diffeomorphism near e to pull back such distributions.

$$\begin{array}{c} \text{generalized densities on } \mathfrak{g} \\ \text{supported at } 0 \end{array} \xleftarrow{\ \exp^*\ } \mathcal{U}(\mathfrak{g}) = \begin{array}{c} \text{generalized densities on } G \\ \text{supported at } e \end{array}$$

The Fourier transform $\mathcal{F}$ maps α-densities on a vector space $\mathfrak{g}$ to $(1-\alpha)$-densities on its dual $\mathfrak{g}^*$. The Fourier transform of a generalized 1-density supported at $0 \in \mathfrak{g}$

will be a polynomial on $\mathfrak{g}^*$:

$$\begin{array}{ccc} \text{generalized densities on } \mathfrak{g} & \xrightarrow{\mathcal{F}} & \mathrm{Pol}(\mathfrak{g}^*) \simeq \mathcal{S}(\mathfrak{g}) \\ \text{supported at } 0 & & \\ \delta_0 & \longmapsto & 1 \\ \dfrac{\partial \delta_0}{\partial x_i} & \longmapsto & v_i \end{array}$$

where $(v_1, \ldots, v_n)$ is a basis of $\mathfrak{g}$, and x_i is the coordinate function on $\mathfrak{g}$ corresponding to v_i.

Theorem 12.2 (Berezin [13]) *The composite map*

$$\mathcal{U}(\mathfrak{g}) \xrightarrow{\mathcal{F} \circ \exp^*} \mathcal{S}(\mathfrak{g})$$

is the symmetrization map (see Section 1.3).

Exercise 39
Prove the theorem. To do so, first prove the theorem for powers of elements of $\mathfrak{g}$ and then extend to all of $\mathcal{U}(\mathfrak{g})$ by "polarization". See [13] and Chapter 2.

To review our construction, if G is a Lie group, then its differential structure provides an algebra $C^\infty(G)$ with pointwise multiplication. On the other hand, diagonal insertion gives rise to a coproduct on the measure group algebra

$$\Delta : \mathcal{D}'(G) \longrightarrow \mathcal{D}'(G) \text{“}\otimes\text{”} \mathcal{D}'(G) \simeq \mathcal{D}'(G \times G) \ .$$

On $\mathcal{U}(\mathfrak{g}) \subseteq \mathcal{D}'(G)$, this restricts to a map where the "tensor" is the usual algebraic tensor product

$$\Delta : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \ .$$

For $X \in \mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$, the map Δ is defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X \ ,$$

and this condition uniquely determines the algebra homomorphism Δ. This coproduct is **co-commutative**, which means that $P \circ \Delta = \Delta$, where

$$P : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

is the permutation linear map defined on elementary tensors by $P(u \otimes v) = v \otimes u$.

Using our isomorphisms of vector spaces $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}_\varepsilon)$ (Section 2.1), we obtain deformed coproducts

$$\Delta_\varepsilon : \mathcal{S}(\mathfrak{g}) \longrightarrow \mathcal{S}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g})$$

satisfying, for $X \in \mathfrak{g} \subset \mathcal{S}(\mathfrak{g})$,

$$\Delta_\varepsilon(X) = X \otimes 1 + 1 \otimes X \ .$$

In general, the map Δ_ε will be an algebra homomorphism with respect to the algebra structure of $\mathcal{U}(\mathfrak{g}_\varepsilon)$, but *not* with respect to the algebra structure of $\mathcal{S}(\mathfrak{g})$. Whenever $\mathfrak{g}$ is not abelian, these two algebra structures are different.

Letting ε approach 0, we ask what Δ_0 should be. It turns out that if we identify $\mathcal{S}(\mathfrak{g})$ with $\mathrm{Pol}(\mathfrak{g}^*)$, then Δ_0 is the coproduct coming from the *addition* operation on $\mathfrak{g}^*$: $\Delta_0(\Sigma \text{ monomials}) = \Sigma\Delta_0(\text{monomials})$. For instance,

$$\begin{aligned}\Delta_0(\mu_1^4\mu_2 + \mu_3) &= (\Delta_0(\mu_1))^4(\Delta_0(\mu_2)) + \Delta_0(\mu_3) \\ &= (\mu_1 \otimes 1 + 1 \otimes \mu_1)^4(\mu_2 \otimes 1 + 1 \otimes \mu_2) \\ &\quad +\mu_3 \otimes 1 + 1 \otimes \mu_3\end{aligned}$$

So the product and coproduct of $\mathcal{U}(\mathfrak{g})$ are deformations of structures on $\mathfrak{g}^*$; thus $\mathcal{U}(\mathfrak{g})$ can be interpreted as the algebra of "functions on" a quantization of $\mathfrak{g}^*$.

In summary, $\mathcal{U}(\mathfrak{g})$ is a non-commutative, co-commutative Hopf algebra, while $\mathcal{S}(\mathfrak{g})$ is a Hopf algebra which is both commutative and co-commutative. Deformations $\mathcal{U}_q(\mathfrak{sl}(2))$ of the Hopf algebra $\mathcal{U}(\mathfrak{sl}(2))$ were among the earliest known (algebras of "functions on") quantum groups (see [25, 88]).

Part VI
Groupoids

13 Groupoids

A groupoid can be thought of as a generalized group in which only certain multiplications are possible.

13.1 Definitions and Notation

A **groupoid** over a set X is a set G together with the following structure maps:

1. A pair of maps
$$\begin{array}{c} G \\ {\scriptstyle\alpha}\downarrow\downarrow{\scriptstyle\beta} \\ X \end{array}$$
The map α is called the **target** while β is called the **source**. [3] An element $g \in G$ is thought of as an arrow from $x = \beta(g)$ to $y = \alpha(g)$ in X:
$$\begin{tikzcd} y = \alpha(g) & x = \beta(g) \arrow[l, bend right, "g"'] \end{tikzcd}$$

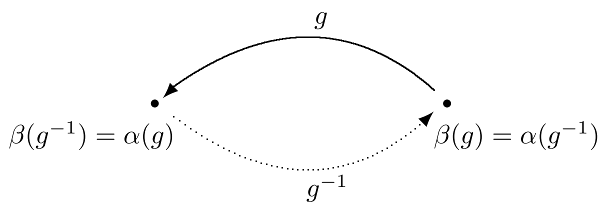
2. A **product** $m : G^{(2)} \to G$, defined on the **set of composable pairs**:
$$G^{(2)} := \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\} \ .$$
We will usually write gh for $m(g, h)$. If h is an arrow from $x = \beta(h)$ to $y = \alpha(h) = \beta(g)$ and g is an arrow from y to $z = \alpha(g)$, then gh is the composite arrow from x to z.
$$\begin{tikzcd} \alpha(g) = \alpha(gh) & \beta(g) = \alpha(h) \arrow[l, bend right, "g"] & \beta(h) = \beta(gh) \arrow[l, bend right, "h"] \arrow[ll, bend right, "gh"'] \end{tikzcd}$$
The multiplication m must have the properties[4]
   - $\alpha(gh) = \alpha(g)$, $\beta(gh) = \beta(h)$, and
   - associativity: $(gh)k = g(hk)$.

3. An embedding $\varepsilon : X \to G$, called the **identity section**, such that $\varepsilon(\alpha(g))g = g = g\varepsilon(\beta(g))$. (In particular, $\alpha \circ \varepsilon = \beta \circ \varepsilon$ is the identity map on X.)

[3] Some authors prefer the opposite convention for α and β.
[4] Whenever we write a product, we are assuming that it is defined.

4. An **inversion** map $\iota : G \to G$, also denoted by $\iota(g) = g^{-1}$, such that for all $g \in G$,

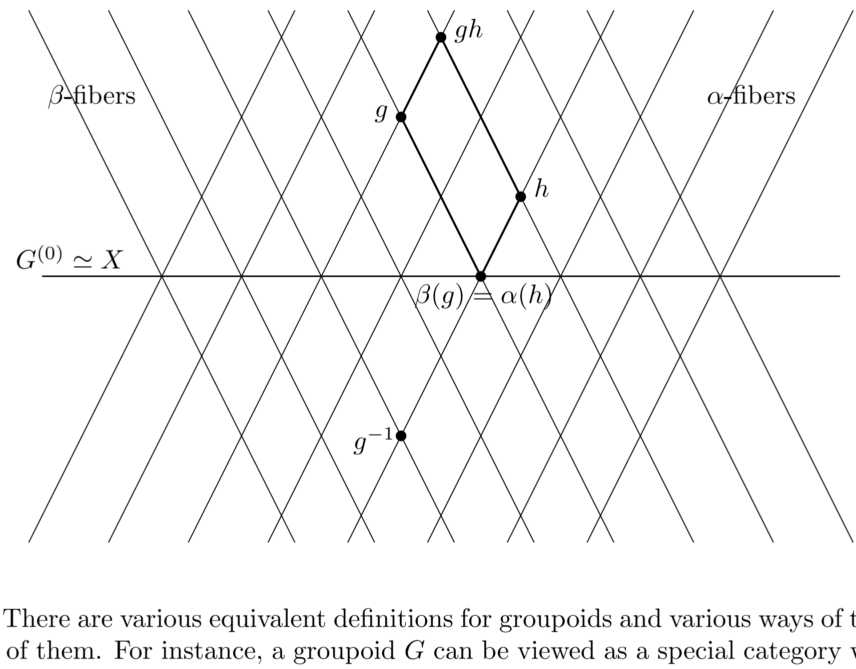
$$\begin{array}{rcl} \iota(g)g & = & \varepsilon(\beta(g)) \\ g\iota(g) & = & \varepsilon(\alpha(g)) \ . \end{array}$$



By an abuse of notation, we shall simply write G to denote the groupoid above. A groupoid G gives rise to a hierarchy of sets:

$$\begin{array}{rcl} G^{(0)} & := & \varepsilon(X) \simeq X \\ G^{(1)} & := & G \\ G^{(2)} & := & \{(g,h) \in G \times G \mid \beta(g) = \alpha(h)\} \\ G^{(3)} & := & \{(g,h,k) \in G \times G \times G \mid \beta(g) = \alpha(h), \beta(h) = \alpha(k)\} \\ \vdots & & \end{array}$$

The following picture can be useful in visualizing groupoids.



There are various equivalent definitions for groupoids and various ways of thinking of them. For instance, a groupoid G can be viewed as a special category whose objects are the elements of the base set X and whose morphisms are all invertible, or as a generalized equivalence relation in which elements of X can be "equivalent in several ways" (see Section 13.2). We refer to Brown [19, 20], as well as [171], for extensive general discussion of groupoids.

Examples.

1. A **group** is a groupoid over a set X with only one element.

2. The **trivial groupoid** over the set X is defined by $G = X$, and $\alpha = \beta =$ identity.

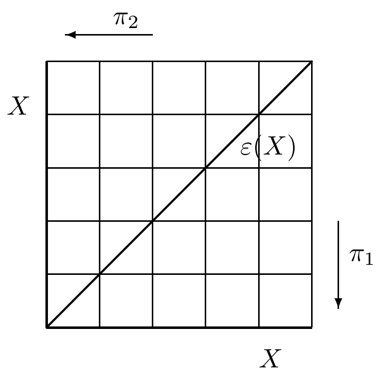
3. Let $G = X \times X$, with the groupoid structure defined by

$$\begin{array}{c} X \times X \\ {\scriptstyle \pi_1} \downarrow\downarrow {\scriptstyle \pi_2} \end{array}$$

$$\alpha(x,y) := \pi_1(x,y) = x \ , \quad \beta(x,y) := \pi_2(x,y) = y \ ,$$

$$\begin{array}{c} (x,y)(y,z) = (x,z) \ , \\ \varepsilon(x) = (x,x) \ , \\ (x,y)^{-1} = (y,x) \ . \end{array}$$

This is often called the **pair groupoid**, or the **coarse groupoid**, or the **Brandt groupoid** after work of Brandt [17], who is generally credited with introducing the groupoid concept.



$\diamondsuit$

Remarks.

1. Given a groupoid G, choose some $\phi \notin G$. The groupoid multiplication on G extends to a multiplication on the set $G \cup \{\phi\}$ by

$$\begin{array}{ll} g\phi = \phi g = \phi & \\ gh = \phi \ , & \quad \text{if } (g,h) \in (G \times G) \setminus G^{(2)} \ . \end{array}$$

The new element ϕ acts as a "receptacle" for any previously undefined product. This endows $G \cup \{\phi\}$ with a **semigroup** structure. A groupoid thus becomes a special kind of semigroup as well.

2. There is a natural way to form the **product of groupoids**:

Exercise 40
If G_i is a groupoid over X_i for $i = 1, 2$, show that there is a naturally defined direct product groupoid $G_1 \times G_2$ over $X_1 \times X_2$.

3. A **disjoint union** of groupoids is a groupoid.

$\diamondsuit$

13.2 Subgroupoids and Orbits

A subset H of a groupoid G over X is called a **subgroupoid** if it is closed under multiplication (when defined) and inversion. Note that

$$h \in H \Rightarrow h^{-1} \in H \Rightarrow \text{ both } \varepsilon(\alpha(h)) \in H \text{ and } \varepsilon(\beta(h)) \in H \ .$$

Therefore, the subgroupoid H is a groupoid over $\alpha(H) = \beta(H)$, which may or may not be all of X. When $\alpha(H) = \beta(H) = X$, H is called a **wide subgroupoid**.

Examples.

1. If $G = X$ is the trivial groupoid, then any subset of G is a subgroupoid, and the only wide subgroupoid is G itself.
2. If X is a one point set, so that G is a group, then the nonempty subgroupoids are the subgroups of G, but the empty set is also a subgroupoid of G.
3. If $G = X \times X$ is the pair groupoid, then a subgroupoid H is a **relation** on X which is symmetric and transitive. A *wide* subgroupoid H is an **equivalence relation**. In general, H is an equivalence relation on the set $\alpha(H) = \beta(H) \subseteq X$.

$\diamondsuit$

Given two groupoids G_1 and G_2 over sets X_1 and X_2 respectively, a **morphism of groupoids** is a pair of maps $G_1 \to G_2$ and $X_1 \to X_2$ which commute with all the structural functions of G_1 and G_2. We depict a morphism by the following diagram.

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\beta_1"] & G_2 \arrow[d, shift right, "\alpha_2"'] \arrow[d, shift left, "\beta_2"] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

If we consider a groupoid as a special type of category, then a morphism between groupoids is simply a covariant functor between the categories.

For any groupoid G over a set X, there is a morphism

$$\begin{tikzcd} G \arrow[r, "{(\alpha,\beta)}"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & X \times X \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ X \arrow[r, phantom, "="] & X \end{tikzcd}$$

from G to the pair groupoid over X. Its image is a wide subgroupoid of $X \times X$, and hence defines an equivalence relation on X. The equivalence classes are called the **orbits** of G in X. In category language, the orbits are the isomorphism classes of the objects of the category. We can also think of a groupoid as an equivalence relation where two elements might be equivalent in different ways, parametrized by the kernel of (α, β). The groupoid further indicates the structure of the set of all ways in which two elements are equivalent.

Inside the groupoid $X \times X$ there is a **diagonal subgroupoid** $\Delta = \{(x,x) \mid x \in X\}$. We call $(\alpha,\beta)^{-1}(\Delta)$ the **isotropy subgroupoid** of G.

$$(\alpha,\beta)^{-1}(\Delta) = \{g \in G \mid \alpha(g) = \beta(g)\} = \bigcup_{x \in X} G_x \ ,$$

where $G_x := \{g \mid \alpha(g) = \beta(g) = x\}$ is the **isotropy subgroup** of x.

If $x, y \in X$ are in the same orbit, then any element g of

$$G_{x,y} := (\alpha,\beta)^{-1}(x,y) = \{g \in G \mid \alpha(g) = x \text{ and } \beta(g) = y\}$$

induces an isomorphism $h \mapsto g^{-1}hg$ from G_x to G_y. On the other hand, the groups G_x and G_y have natural commuting, free transitive actions on $G_{x,y}$, by left and right multiplication, respectively. Consequently, $G_{x,y}$ is isomorphic (as a set) to G_x (and to G_y), but not in a natural way.

A groupoid is called **transitive** if it has just one orbit. The transitive groupoids are the building blocks of groupoids, in the following sense. There is a natural decomposition of the base space of a general groupoid into orbits. Over each orbit there is a transitive groupoid, and the disjoint union of these transitive groupoids is the original groupoid.

Historical Remark. Brandt [17] discovered groupoids while studying quadratic forms over the integers. Groupoids also appeared in Galois theory in the description of relations between subfields of a field K via morphisms of K [108]. The isotropy groups of the constructed groupoid turn out to be the Galois groups. Groupoids occur also as generalizations of equivalence relations in the work of Grothendieck on moduli spaces [75] and in the work of Mackey on ergodic theory [113]. For recent applications in these two areas, see Keel and Mori [94] and Connes [32]. $\diamondsuit$

13.3 Examples of Groupoids

1. Let X be a topological space and let $G = \Pi(X)$ be the collection of homotopy classes of paths in X with all possible fixed endpoints. Specifically, if $\gamma : [0,1] \to X$ is a path from $x = \gamma(0)$ to $y = \gamma(1)$, let $[\gamma]$ denote the homotopy class of γ relative to the points x, y. We can define a groupoid

$$\Pi(X) = \{(x, [\gamma], y) \mid x, y \in X, \gamma \text{ is a path from } x \text{ to } y\} \ ,$$

   where multiplication is concatenation of paths. (According to our convention, if γ is a path from x to y, the target is $\alpha(x,[\gamma],y) = x$ and the source is $\beta(x,[\gamma],y) = y$.) The groupoid $\Pi(X)$ is called the **fundamental groupoid** of X. The orbits of $\Pi(X)$ are just the path components of X. See Brown's text on algebraic topology [20] for more on fundamental groupoids.

There are several advantages of the fundamental groupoid over the fundamental group. First notice that the fundamental group sits within the fundamental groupoid as the isotropy subgroup over a single point. The fundamental groupoid does not require a choice of base point and is better suited to study spaces that are not path connected. Additionally, many of the algebraic properties of the fundamental group generalize to the fundamental groupoid, as illustrated in the following exercise.

Exercise 41
Show that the Seifert-Van Kampen theorem on the fundamental group of a union $U \cup V$ can be generalized to groupoids [20], and that the connectedness condition on $U \cap V$ is then no longer necessary.

2. Let Γ be a group acting on a space X. In the product groupoid $\Gamma \times (X \times X) \simeq X \times \Gamma \times X$ over $\{\text{point}\} \times X \simeq X$, the wide subgroupoid

$$G_\Gamma = \{(x, \gamma, y) \mid x = \gamma \cdot y\}$$

is called the **transformation groupoid** or **action groupoid** of the Γ-action. The orbits and isotropy subgroups of the transformation groupoid are precisely those of the Γ-action.

A groupoid G over X is called **principal** if the morphism $G \xrightarrow{(\alpha,\beta)} X \times X$ is injective. In this case, G is isomorphic to the image $(\alpha, \beta)(G)$, which is an equivalence relation on X. The term “principal” comes from the analogy with bundles over topological spaces.

If Γ acts freely on X, then the transformation groupoid G_Γ is principal, and $(\alpha, \beta)(G_\Gamma)$ is the orbit equivalence relation on X. In passing to the transformation groupoid, we have lost information on the group structure of Γ, as we no longer see *how* Γ acts on the orbits: different free group actions could have the same orbits.

3. Let Γ be a group. There is an interesting ternary operation

$$(x, y, z) \xrightarrow{t} xy^{-1}z \ .$$

It is invariant under left and right translations (check this as an exercise), and it defines 4-tuples $(x, y, z, xy^{-1}z)$ in Γ which play the role of parallelograms. The operation t encodes the affine structure of the group in the sense that, if we know the identity element e, we recover the group operations by setting $x = z = e$ to get the inversion and then $z = e$ to get the multiplication. However, the identity element of Γ cannot be recovered from t.

Denote

$$\begin{aligned} \mathfrak{S}(\Gamma) &= \text{set of subgroups of } \Gamma \\ \mathfrak{B}(\Gamma) &= \text{set of subsets of } \Gamma \text{ closed under } t \ . \end{aligned}$$

Proposition 13.1 *$\mathfrak{B}(\Gamma)$ is the set of* cosets *of elements of* $\mathfrak{S}(\Gamma)$.

The sets of right and of left cosets of subgroups of Γ coincide because $gH = (gHg^{-1})g$, for any $g \in G$ and any subgroup $H \leq G$.

Exercise 42
Prove the proposition above.

We call $\mathfrak{B}(\Gamma)$ the **Baer groupoid** of Γ, since much of its structure was formulated by Baer [10]. We will next see that the Baer groupoid is a groupoid over $\mathfrak{S}(\Gamma)$.

For $D \in \mathfrak{B}(\Gamma)$, let $\alpha(D) = g^{-1}D$ and $\beta(D) = Dg^{-1}$ for some $g \in D$. From basic group theory, we know that α and β are maps into $\mathfrak{S}(\Gamma)$ and are independent of the choice of g. Furthermore, we see that $\beta(D) = g\alpha(D)g^{-1}$ is conjugate to $\alpha(D)$.

$$\begin{array}{c} \mathfrak{B}(\Gamma) \\ {\scriptstyle\alpha} \downarrow\downarrow {\scriptstyle\beta} \\ \mathfrak{S}(\Gamma) \end{array}$$

Exercise 43
Show that if $\beta(D_1) = \alpha(D_2)$, *i.e.* $D_1 g_1^{-1} = g_2^{-1} D_2$ for any $g_1 \in D_1, g_2 \in D_2$, then the product in this groupoid can be defined by

$$D_1 D_2 := g_2 D_1 = g_1 D_2 = \{gh \mid g \in D_1 \ h \in D_2\} \ .$$

Observe that the orbits of $\mathfrak{B}(\Gamma)$ are the conjugacy classes of subgroups of Γ. In particular, over a single conjugacy class of subgroups is a transitive groupoid, and thus we see that the Baer groupoid is a refinement of the conjugacy relation on subgroups.

The isotropy subgroup of a subgroup H of Γ consists of all left cosets of H which are also right cosets of H. Any left coset gH is a right coset $(gHg^{-1})g$ of gHg^{-1}. Thus gH is also a right coset of H exactly when $gHg^{-1} = H$, or, equivalently, when $\beta(gH) = \alpha(gH)$. Thus the isotropy subgroup of H can be identified with $N(H)/H$, where $N(H)$ is the normalizer of H.

4. Let Γ be a compact connected semisimple Lie group. An interesting conjugacy class of subgroups of Γ is

$$\mathcal{T} = \{\text{maximal tori of } \Gamma\} \ ,$$

where a **maximal torus** of Γ is a subgroup

$$\mathbb{T}^k \simeq (S^1)^k = S^1 \oplus \cdots \oplus S^1$$

of Γ which is maximal in the sense that there does not exist an $\ell \geq k$ such that $\mathbb{T}^k < \mathbb{T}^\ell \leq \Gamma$ (here, $S^1 \simeq \mathbb{R}/\mathbb{Z}$ is the circle group). A theorem from Lie group theory (see, for instance, [18]) states that any two maximal tori of a connected Lie group are conjugate, so $\mathcal{T}$ is an orbit of $\mathfrak{B}(\Gamma)$. We call the transitive subgroupoid $\mathfrak{B}(\Gamma)|_{\mathcal{T}} = \mathcal{W}(\Gamma)$ the **Weyl groupoid** of Γ.

Remarks.

- For any maximal torus $\mathbb{T} \in \mathcal{T}$, the quotient $N(\mathbb{T})/\mathbb{T}$ is the classical Weyl group. The relation between the Weyl groupoid and the Weyl group is analogous to the relation between the fundamental groupoid and the fundamental group.

- There should be relevant applications of Weyl groupoids in the representation theory of a group Γ which is acted on by a second group, or in studying the representations of groups that are not connected.

$\diamondsuit$

13.4 Groupoids with Structure

Ehresmann [53] was the first to endow groupoids with additional structure, as he applied groupoids to his study of foliations. Rather than attempting to describe a general theory of "structured groupoids," we will simply mention some useful special cases.

1. **Topological groupoids:** For a topological groupoid, G and X are required to be topological spaces and all the structure maps must be continuous.

   Examples.

   - In the case of a group, this is the same as the concept of topological group.
   - The pair groupoid of a topological space has a natural topological structure derived from the product topology on $X \times X$.

   $\diamondsuit$

   For analyzing topological groupoids, it is useful to impose certain further axioms on G and X. For a more complete discussion, see [143]. Here is a sampling of commonly used axioms:

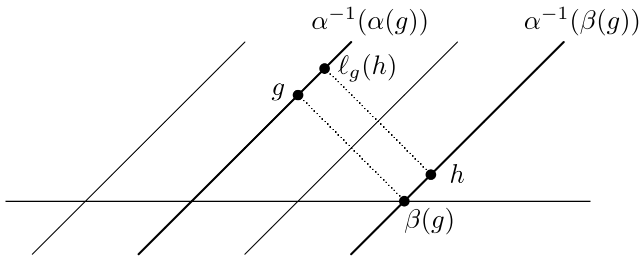
   (a) $G^{(0)} \simeq X$ is locally compact and Hausdorff.

   (b) The α- and β-fibers are locally compact and Hausdorff.

   (c) There is a countable family of compact Hausdorff subsets of G whose interiors form a basis for the topology.

   (d) G admits a **Haar system**, that is, admits a family of measures on the α-fibers which is invariant under left translations. For any $g \in G$, left translation by g is a map between α fibers

$$\begin{array}{rcl} \alpha^{-1}(\beta(g)) & \longrightarrow & \alpha^{-1}(\alpha(g)) \\ h & \overset{\ell_g}{\longmapsto} & gh \ . \end{array}$$



Example. For the pair groupoid, each fiber can be identified with the base space X. A family of measures is invariant under translation if and only if the measure is the same on each fiber. Hence, a Haar system on a pair groupoid corresponds to a measure on X. $\diamondsuit$

2. **Measurable groupoids:** These groupoids, also called **Borel groupoids**, come equipped with a σ-algebra of sets and a distinguished subalgebra (called the null sets); see [113, 120]. On each α-fiber, there is a **measure class**, which is simply a measure defined up to multiplication by an invertible measurable function.

3. **Lie groupoids** or **differentiable groupoids:** The groupoid G and the base space X are manifolds and all the structure maps are smooth. It is *not* assumed that G is Hausdorff, but only that $G^{(0)} \simeq X$ is a Hausdorff manifold and closed in G.[5] Thus we can require that the identity section be smooth. Recall that multiplication is defined as a map on $G^{(2)} \subseteq G$. To require that multiplication be smooth, first $G^{(2)}$ needs to be a smooth manifold. It is convenient to make the stronger assumption that the map α (or β) be a submersion.

   Exercise 44
   Show that the following conditions are equivalent:
   (a) α is a submersion,
   (b) β is a submersion,
   (c) the map (α, β) to the pair groupoid is transverse to the diagonal.

4. **Bundles of groups:** A groupoid for which $\alpha = \beta$ is called a bundle of groups. This is not necessarily a trivial bundle, or even a locally trivial bundle in the topological case, as the fibers need not be isomorphic as groups or as topological spaces. The orbits are the individual points of the base space, and the isotropy subgroupoids are the fiber groups of the bundle.

13.5 The Holonomy Groupoid of a Foliation

Let X be a (Hausdorff) manifold. Let $F \subseteq TX$ be an integrable subbundle, and $\mathcal{F}$ the corresponding foliation ($\mathcal{F}$ is the decomposition of X into maximal integral manifolds called **leaves**). The notion of holonomy can be described as follows. An F-path is a path in X whose tangent vectors lie within F. Suppose that $\gamma : [0,1] \to \mathcal{O}$ is an F-path along a leaf $\mathcal{O}$. Let $N_{\gamma(0)}$ and $N_{\gamma(1)}$ be cross-sections for the spaces of leaves near $\gamma(0)$ and $\gamma(1)$, respectively, *i.e.* they are two small transversal manifolds to the foliation at the end points of γ. There is an F-path near γ from each point near $\gamma(0)$ in $N_{\gamma(0)}$ to a uniquely determined point in $N_{\gamma(1)}$. This defines a local diffeomorphism between the two leaf spaces. The **holonomy** of γ is defined to be the *germ*, or direct limit, of such diffeomorphisms, between the local leaf spaces $N_{\gamma(0)}$ and $N_{\gamma(1)}$.

The notion of holonomy allows us to define an equivalence relation on the set of F-paths from x to y in X. Let $[\gamma]_H$ denote the equivalence class of γ under the relation that two paths are equivalent if they have the same holonomy.

[5] Throughout these notes, a manifold is assumed to be Hausdorff, *unless* it is a groupoid.

The **holonomy groupoid** [32], also called the **graph** of the foliation [175], is

$$H(\mathcal{F}) = \{(x, [\gamma]_H, y) \mid x, y \in X, \gamma \text{ is an } F\text{-path from } x \text{ to } y\} \ .$$

Given a foliation $\mathcal{F}$, there are two other related groupoids obtained by changing the equivalence relation on paths:

1. The **$\mathcal{F}$-pair groupoid** – This groupoid is the equivalence relation for which the equivalence classes are the leaves of $\mathcal{F}$, *i.e.* we consider any two F-paths between $x, y \in \mathcal{O}$ to be equivalent.

2. The **$\mathcal{F}$-fundamental groupoid** – For this groupoid, two F-paths between x, y are equivalent if and only if they are **F-homotopic**, that is, homotopic within the set of all F-paths. Let $[\gamma]_F$ denote the equivalence class of γ under F-homotopy. The set of this groupoid is

$$\Pi(\mathcal{F}) = \{(x, [\gamma]_F, y) \mid x, y \in X, \gamma \text{ is an } F\text{-path from } x \text{ to } y\} \ .$$

If two paths γ_1, γ_2 are F-homotopic with fixed endpoints, then they give the same holonomy, so the holonomy groupoid is intermediate between the $\mathcal{F}$-pair groupoid and the $\mathcal{F}$-fundamental groupoid:

$$[\gamma_1]_F = [\gamma_2]_F \quad \Longrightarrow \quad [\gamma_1]_H = [\gamma_2]_H \ .$$

The pair groupoid may not be a manifold. With suitably defined differentiable structures, though, we have:

Theorem 13.2 *$H(\mathcal{F})$ and $\Pi(\mathcal{F})$ are (not necessarily Hausdorff) Lie groupoids.*

For a nice proof of this theorem, and a comparison of the two groupoids, see [137]. Further information can be found in [102].

> **Exercise 45**
> Compare the $\mathcal{F}$-pair groupoid, the holonomy groupoid of $\mathcal{F}$, and the $\mathcal{F}$-fundamental groupoid for the Möbius band and the Reeb foliation, as described below.

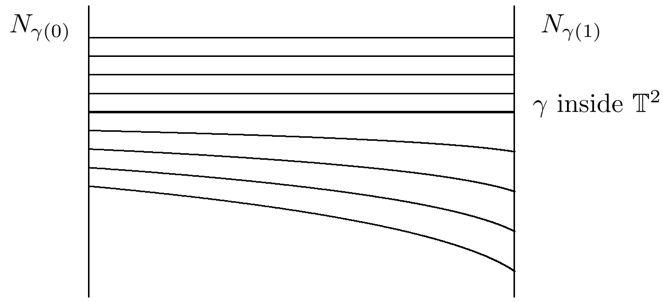
1. The **Möbius band**. Take the quotient of the unit square $[0, 1] \times [0, 1]$ by the relation $(1, x) \sim (0, 1 - x)$. Define the leaves of $\mathcal{F}$ to be images of the horizontal strips $\{(x, y) \mid y = \text{constant}\}$.

2. The **Reeb foliation** [142]. Consider the family of curves $x = c + \sec y$ on the strip $-\pi/2 < y < \pi/2$ in the xy-plane. If we revolve about the axis $y = 0$, then this defines a foliation of the solid cylinder by planes. Noting that the foliation is invariant under translation, we see that this defines a foliation of the open solid torus $D^2 \times S^1$ by planes. The foliation is smooth because its restriction to the xy-plane is defined by the 1-form $\cos^2 y \, dx + \sin y \, dy$, which is smooth even when $y = \pm\frac{\pi}{2}$. We close the solid torus by adding one exceptional leaf – the $\mathbb{T}^2$ boundary.

   Let α be a vanishing cycle on $\mathbb{T}^2$, that is, $[\alpha] \in \pi_1(\mathbb{T}^2)$ generates the kernel of the natural map $\pi_1(\mathbb{T}^2) \to \pi_1(D^2 \times S^1)$. Although α is not null-homotopic on the exceptional leaf, any perturbation of α to a nearby leaf results in a curve that is F-homotopically trivial. On the other hand, the transverse curve (the cycle given by $(c, y) \in D^2 \times S^1$ for some fixed $c \in \partial D^2$) cannot be pushed onto any of the nearby leaves.

A basic exercise in topology shows us that we can glue two solid tori together so that the resulting manifold is the 3-sphere S^3. For this gluing, the transverse cycle of one torus is the vanishing cycle of the other. (If we instead glued the two vanishing cycles and the two transverse cycles together, we would obtain $S^2 \times S^1$.)

It is interesting to compute the holonomy on each side of the gluing $\mathbb{T}^2$. Each of the two basic cycles in $\mathbb{T}^2$ has trivial holonomy on one of its sides (holonomy given by the germ of the identity diffeomorphism), and *non*-trivial holonomy on the other side (given by the germ of an expanding diffeomorphism).



This provides an example of *one-sided holonomy*, a phenomenon that cannot happen for real analytic maps. The leaf space of this foliation is not Hausdorff; in fact, any function constant on the leaves must be constant on all of S^3, since all leaves come arbitrarily close to the exceptional leaf $\mathbb{T}^2$. This foliation and its holonomy provided the inspiration for the following theorems.

Theorem 13.3 (Haefliger [79]) *S^3 has no real analytic foliation of codimension-1.*

Theorem 13.4 (Novikov [132]) *Every codimension-1 foliation of S^3 has a compact leaf that is a torus.*

14 Groupoid Algebras

Groupoid algebras include matrix algebras, algebras of functions, and group algebras. We refer the reader to [101, 135, 143] for extensive discussion of groupoid algebras as sources of noncommutative algebras in physics and mathematics.

14.1 First Examples

Let X be a locally compact space with a Borel measure μ. Let $C_c(X \times X)$ be the space of compactly supported continuous functions on $X \times X$. We define multiplication of two functions $f, g \in C_c(X \times X)$ by the following integral, representing "continuous matrix multiplication"

$$(f * g)(x, y) = \int_X f(x, z)\ g(z, y)\ d\mu(z)\ .$$

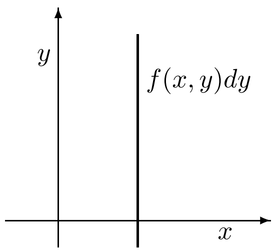
Exercise 46
Check that this multiplication is associative and that the *-operation

$$f(x, y) \longmapsto f^*(x, y) := \overline{f(y, x)}$$

is compatible with multiplication:

$$f^* * g^* = (g * f)^*\ .$$

To define our multiplication without the choice of a measure on X, we replace $C_c(X \times X)$ by the space whose elements are objects of the form $f(x, y)dy$. Such an object assigns to each point of X a measure on X.



These objects have a "matrix" multiplication as written above. Furthermore, they operate on functions on X by

$$u(\cdot) \longmapsto \int_X f(\cdot, y)\ u(y)\ dy\ .$$

However, the *-operation can no longer be described in this language.

When X is a manifold, there is a related algebra on which the *-operation can be defined intrinsically. Let $\mathcal{A}$ be the space of compactly supported $\frac{1}{2}$-densities on $X \times X$. A typical element of $\mathcal{A}$ is of the form

$$f(x, y)\sqrt{|dx|}\sqrt{|dy|}\ .$$

We multiply two elements

$$f(x, z)\sqrt{|dx|}\sqrt{|dz|}\ , \quad g(z, y)\sqrt{|dz|}\sqrt{|dy|}\ ,$$

by integrating over z:

$$\left(\int_{z\in X} f(x,z)g(z,y)|dz|\right)\sqrt{|dx|}\sqrt{|dy|}\ .$$

This algebra no longer acts on functions, but rather on $\frac{1}{2}$-densities on X. The *-operation is now defined by

$$f^*(x,y)\sqrt{|dx|}\sqrt{|dy|} = \overline{f(y,x)}\sqrt{|dy|}\sqrt{|dx|}\ .$$

Exercise 47
Give a precise definition of a generalized $\frac{1}{2}$-density which serves as an identity element for this algebra.

Implicit in these formulations is the multiplication law for the pair groupoid

$$(x,z)(z,y) = (x,y)\ .$$

From this point of view, our multiplication operation becomes convolution in the *groupoid algebra*, as we shall see in the next section.

14.2 Groupoid Algebras via Haar Systems

Let G be a locally compact groupoid over X, and let φ and ψ be compactly supported continuous functions on G. A product function $\varphi * \psi$ might be obtained in the following way: for its value at $k \in G$, we evaluate φ and ψ on all possible pairs $(g,h) \in G \times G$ satisfying $gh = k$, and then integrate the products of the values. That is, we write the integral

$$(\varphi * \psi)(k) = \int_{\{(g,h)|gh=k\}} \varphi(g)\ \psi(h)\ (...)\ ,$$

where we need a measure (...) on the set $\{(g,h) \in G \times G \mid gh = k\}$. If we rewrite $gh = k$ as $h = g^{-1}k$, we see that the domain of integration is all $g \in G$ such that $\beta(g^{-1}) = \alpha(g) = \alpha(k)$. In other words, the product above equals

$$(\varphi * \psi)(k) = \int_{g\in\alpha^{-1}(\alpha(k))} \varphi(g)\ \psi(g^{-1}k)\ (...)\ .$$

But in order to integrate, we need measures on the α-fibers. If $\{\lambda_x\}_{x\in X}$ is a family of measures on the α-fibers, then we define the **convolution** product of φ and ψ to be

$$(\varphi * \psi)(k) = \int_{g\in\alpha^{-1}(\alpha(k))} \varphi(g)\ \psi(g^{-1}k)\ d\lambda_{\alpha(k)}\ .$$

Here, we assume that the family of measures $\{\lambda_x\}$ is continuous in x. To ensure that this product is associative, we require left invariance of $\{\lambda_x\}$, *i.e.* we require that $\{\lambda_x\}$ be a **Haar system** (*cf.* Sections 13.4 and 11.4).

The vector space of bounded continuous functions φ on G for which the target map α restricted to support(φ) is a proper map, is closed under the convolution product. Its completion under a suitable norm is called the **groupoid C^*-algebra associated to the Haar system** $\{\lambda_x\}$. Since the multiplicative structure depends on the choice of $\{\lambda_x\}$, the groupoid algebra is sometimes denoted by $\mathcal{A}_\lambda$. We refer to [143] for more details about the analytic aspects of this construction.

The groupoid algebra operates on functions on the base. Let φ be a function on G, and u a function on X. Define

$$(\mathrm{Op}\ \varphi)u(x) := \int_{g\in\alpha^{-1}(x)} \varphi(g)\ u(\beta(g))\ d\lambda_x\ .$$

Intuitively, if we think of the elements of G as "arrows" on the base space X, then this integral tells us to look at all the arrows g going *into* a given point $x \in X$, evaluate the function u at the tail of each of those arrows, then move back to x and integrate over all arrows g with "weight" given by φ.

Examples.

- $G = X \times X$ – The groupoid algebra is isomorphic to the "matrix" algebra of functions on $X \times X$ (see Section 14.1). If X is finite, it really is a matrix algebra.
- $G =$ a group – The groupoid algebra is isomorphic to a subalgebra of the standard group algebra (see Chapter 11). A function on G acts on constant functions via multiplication by its integral over G.
- $G = X$, where $(f * g)(x) = f(x)g(x)$ – The groupoid algebra is the algebra of functions on X (which operates on itself by pointwise multiplication).

$\diamondsuit$

14.3 Intrinsic Groupoid Algebras

Suppose that G is a Lie groupoid over X. Denote the bundles over G of $\frac{1}{2}$-densities along the α- and β-fibers by $\Omega_\alpha^{\frac{1}{2}}$ and $\Omega_\beta^{\frac{1}{2}}$, respectively. Letting

$$\Omega = \Omega_\alpha^{\frac{1}{2}} \otimes \Omega_\beta^{\frac{1}{2}}\ ,$$

the **intrinsic groupoid algebra** of G is the completion of the space $\Gamma(\Omega)$ of compactly supported sections of Ω under a suitable norm. The term "intrinsic" refers to the fact that it does not involve the arbitrary choice of a Haar system. The multiplication on $\Gamma(\Omega)$ is defined as follows.

Suppose that $\beta(g) = \alpha(h) = x \in X \simeq G^{(0)}$. There is a natural isomorphism

$$\Omega(g) \otimes \Omega(h) \xrightarrow{\simeq} \Omega_\alpha^1(g) \otimes \Omega(gh)$$

constructed using the identifications

$$\Omega_\alpha^{\frac{1}{2}}(g) \otimes \Omega_\beta^{\frac{1}{2}}(g) \xrightarrow{1 \otimes r_h} \Omega_\alpha^{\frac{1}{2}}(g) \otimes \Omega_\beta^{\frac{1}{2}}(gh)$$

$$\Omega_\alpha^{\frac{1}{2}}(h) \otimes \Omega_\beta^{\frac{1}{2}}(h) \xrightarrow{\ell_g \otimes 1} \Omega_\alpha^{\frac{1}{2}}(gh) \otimes \Omega_\beta^{\frac{1}{2}}(h)$$

together with

$$\begin{tikzcd}
\Omega^{\frac{1}{2}}_{\alpha}(g) \arrow[r, "l_{g^{-1}}"] & \Omega^{\frac{1}{2}}_{\alpha}(x) \arrow[dr] & \\
& & \Omega^{\frac{1}{2}}(T_xG/T_xG^{(0)}) = \Omega^{\frac{1}{2}}(N_xG^{(0)}) \arrow[dl] \\
\Omega^{\frac{1}{2}}_{\beta}(h) & \Omega^{\frac{1}{2}}_{\beta}(x) \arrow[l, "\ell_h"'] &
\end{tikzcd}$$

In general, there is no natural isomorphism between $\Omega^{\frac{1}{2}}_{\alpha}$ and $\Omega^{\frac{1}{2}}_{\beta}$ at a given point in G. However, on $G^{(0)}$ an isomorphism is provided by projection along the identity section from X into $G^{(0)}$: we can identify both $\Omega^{\frac{1}{2}}_{\alpha}$ and $\Omega^{\frac{1}{2}}_{\beta}$ over $x \in G^{(0)}$ with the $\frac{1}{2}$-densities on the normal space $N_xG^{(0)}$ to $G^{(0)}$ in G at x.

We use these isomorphisms to determine the product of $\varphi, \psi \in \Gamma(\Omega)$. The product section $\varphi\psi \in \Gamma(\Omega)$ is given at a point $k \in G$ by the formula

$$(\varphi\psi)(k) = \int_{\{g|\alpha(g)=\alpha(k)\}} \varphi(g)\ \psi(g^{-1}k)$$

where we regard $\varphi(g)\psi(g^{-1}k)$ as an element of $\Omega^{1}_{\alpha}(g) \otimes \Omega(k)$, and we integrate the 1-density factor over the α-fiber through k.

Exercise 48
Check that, if we instead use the maps

$$\Omega(g) \otimes \Omega(h) \longrightarrow \Omega^{1}_{\beta}(h) \otimes \Omega(gh)\ ,$$

the resulting multiplicative structure on $\Gamma(\Omega)$ is the same.

Remark. The identifications above also provide a natural isomorphism

$$\Omega \simeq \alpha^*(\Omega^{\frac{1}{2}}E) \otimes \beta^*(\Omega^{\frac{1}{2}}E)\ ,$$

which we will use often. $\diamondsuit$

Let E be the **normal bundle** of $G^{(0)}$ in G. The smooth groupoid algebra $\Gamma(\Omega)$ acts on smooth sections of

$$\Omega^{\frac{1}{2}}E := |\wedge^{\mathrm{top}}|^{\frac{1}{2}}\ E^*\ .$$

To see this left action, take $\varphi \in \Gamma(\Omega)$ and a section γ of $\Omega^{\frac{1}{2}}E$. We can think of φ at $g \in G$ as a $\frac{1}{2}$-density on the normal space through $x = \alpha(g)$, times a $\frac{1}{2}$-density on the normal space through through $y = \beta(g)$:

$$\begin{array}{ccccc}
\Omega(g) & = & \Omega^{\frac{1}{2}}_{\alpha}(g) \otimes \Omega^{\frac{1}{2}}_{\beta}(g) & \simeq & \Omega^{\frac{1}{2}}E_x \otimes \Omega^{\frac{1}{2}}E_y \\
 & & & & \\
\varphi(g) & & \xrightarrow{\qquad\qquad\simeq\qquad\qquad} & & \varphi_{\alpha}(x) \otimes \varphi_{\beta}(y)
\end{array}$$

Since $\varphi_\beta(y)\gamma(y) \in \Omega^1 E_y \simeq \Omega^1_\alpha(y) \simeq \Omega^1_\alpha(g)$, we can consider $\varphi(g)\gamma(y)$ as an element of $\Omega^1_\alpha(g) \otimes \Omega^{\frac{1}{2}} E_x$. The new section $\varphi \cdot \gamma$ of $\Omega^{\frac{1}{2}} E$ is then given at a point $x \in X$ by

$$(\varphi \cdot \gamma)(x) = \int_{g \in \alpha^{-1}(x)} \varphi(g)\gamma(\beta(g))$$

where we integrate the Ω^1_α factor of $\varphi(g)\gamma(\beta(g)) \in \Omega^1_\alpha(g) \otimes \Omega^{\frac{1}{2}} E_x$ over the α-fiber through x.

Exercise 49
Check that this is indeed a *left* action, *i.e.* $\varphi \cdot (\psi \cdot \gamma) = (\varphi\psi) \cdot \gamma$, for any $\varphi, \psi \in \Gamma(\Omega)$, and $\gamma \in \Gamma(\Omega^{\frac{1}{2}} E)$.

We could just as well define a right action by reversing the α and β roles, namely,

$$(\gamma \cdot \varphi)(x) = \int_{h \in \beta^{-1}(x)} \varphi(h)\gamma(\alpha(h)) \ ,$$

with $\varphi(h)\gamma(\alpha(h)) \in \Omega^1 E_{\alpha(h)} \otimes \Omega^{\frac{1}{2}} E_x \simeq \Omega^1_\beta(h) \otimes \Omega^{\frac{1}{2}} E_x$.

14.4 Groupoid Actions

A groupoid G over $X \simeq G^{(0)}$ may act on sets $M \xrightarrow{\mu} X$ that map to X. Let $G * M$ be the space

$$G * M := \{(g, m) \in G \times M \mid \beta(g) = \mu(m)\} \ .$$

A (left) **groupoid action** of G on M is defined to be a map $G * M \to M$, taking the pair (g, m) to $g \cdot m$, with the properties:

1. $\mu(g \cdot m) = \alpha(g)$,
2. $(gh) \cdot m = g \cdot (h \cdot m)$,
3. $(\varepsilon\mu(m)) \cdot m = m$.

The map $\mu : M \to X$ is sometimes called the **moment map**, by analogy with symplectic geometry.

Remark. The terms "moment map" and "momentum map" are usually used interchangeably in the literature, with different authors preferring each of these two translations of Souriau's [153] French term, "moment". By contrast, in these notes, we have used the terms in different ways. Here, a "momentum map" is a Poisson map $J : M \to \mathfrak{g}^*$ to a Lie-Poisson manifold $\mathfrak{g}^*$, generating a hamiltonian action of an underlying Lie group G on M. On the other hand, a "moment map" is a map $\mu : M \to X$ to the base X of a groupoid G which is acting on M. $\diamondsuit$

Example. A groupoid G over X acts on G by left multiplication with moment map α and on X with moment map the identity. $\diamondsuit$

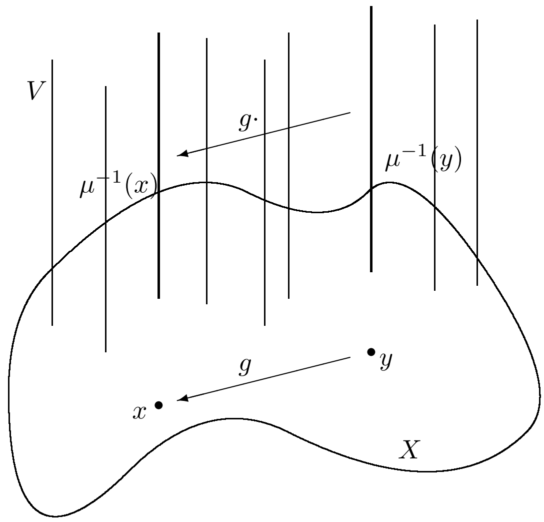
Given additional structure on G or M, we can specify special types of actions. For instance, groupoids act on vector *bundles* (rather than vector spaces). Suppose that we have a groupoid G over X and a vector bundle V also over X,

$$\begin{array}{ccc} G & \qquad & V \\ {\scriptstyle\alpha} \downarrow\downarrow {\scriptstyle\beta} & & \downarrow {\scriptstyle\mu} \\ X & & X \end{array}$$

A **representation** or **linear action** of G on V is a groupoid action of G on V whose maps

$$g\cdot : \mu^{-1}(\beta(g)) \longrightarrow \mu^{-1}(\alpha(g))$$

are linear. For more on groupoid actions, see [110].



We can think of a representation of a groupoid as a collection of representations of the isotropy subgroups together with ways of identifying these representations using different "arrows" in X.

Example. If X is a topological space, and $G = \Pi(X)$ is the fundamental groupoid, then a representation of $\Pi(X)$ on a vector bundle V would be a **flat connection** of V. By *flat* connection, we do not yet mean a differential-geometric notion, but rather a topological one, namely that parallel transport only depends on the homotopy class of the base path.

To see the flat connection, recall that $\Pi(X)$ is the collection of homotopy classes of paths in X. A representation of $\Pi(X)$ determines precisely how to parallel translate along paths to define a connection.

With this flat connection, we can look at the isotropy subgroup of loops based at a point. The fundamental group of X acts on each fiber in the usual sense, and we thus see that the representation of the fundamental groupoid on V includes the action of the fundamental group on a fiber of V.

For applications to the moduli spaces used in topological quantum field theory, see [77]. $\diamondsuit$

As with groups, the notion of groupoid representation can be formalized in terms of the following definition. The **general linear groupoid** of a vector bundle $\mu : V \to X$ is

$$\mathrm{GL}(V) = \{(x, \ell, y) \mid x, y \in X, \ell : \mu^{-1}(y) \to \mu^{-1}(x) \text{ is a linear isomorphism}\} .$$

The isotropy subgroup over any point is the general linear group of the corresponding fiber of V. A **representation** of G in V is then a groupoid homomorphism from G to $\mathrm{GL}(V)$, covering the identity map on X.

The general linear groupoid is a subset of a larger object

$$\mathfrak{gl}(V) = \{(x, \ell, y) \mid x, y \in X, \ell : \mu^{-1}(y) \to \mu^{-1}(x) \text{ is linear}\} \ ,$$

where ℓ is an arbitrary linear map between fibers. This is a generalization of the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$ of the general linear group $\mathrm{GL}(n;\mathbb{R})$.

14.5 Groupoid Algebra Actions

Example. If G is a *group*, V is a vector space, and $r : G \to \mathrm{End}(V)$ is a map, then there is an induced map $\widetilde{r} : C'(G) \to \mathrm{End}(V)$ defined by the formula

$$\varphi \longmapsto \widetilde{r}(\varphi) := \int_G r(g)\ \varphi(g)\ dg \ .$$

If r is a representation, then $\widetilde{r}$ will be a homomorphism of algebras. Hence, group representations correspond to representations of the measure group algebra. $\diamondsuit$

For a groupoid G, there is a similar correspondence. Given a representation of a groupoid G on a vector bundle V and a Haar system $\{\lambda_x\}$ on G, there is an action of the groupoid algebra $\mathcal{A}_\lambda$ on sections of V defined as follows. Let φ be any continuous compactly supported function on G, and let $u \in \Gamma(V)$. Define

$$(\varphi \cdot u)(x) = \int_{g \in \alpha^{-1}(x)} \varphi(g)\ g \cdot u(\beta(g))\ d\lambda_x(g) \ .$$

We can also describe the action of the intrinsic groupoid algebra.

Recall that, if E denotes the normal bundle to $G^{(0)}$ in G, then the intrinsic groupoid algebra is (a suitable completion of) the set of sections of

$$\Omega = \alpha^*(\Omega^{\frac{1}{2}} E) \otimes \beta^*(\Omega^{\frac{1}{2}} E) \ .$$

For a vector bundle V over $G^{(0)}$, we define

$$\mathrm{End}(V) := \alpha^*(V) \otimes \beta^*(V^*) \ ;$$

that is, $\mathrm{End}(V)$ is the bundle over G whose fiber over each point $g \in G$ is

$$V_{\alpha(g)} \otimes V^*_{\beta(g)} = \mathrm{Hom}\left(V_{\beta(g)}, V_{\alpha(g)}\right) \ .$$

Given a representation of G on V, the sections of

$$\Omega \otimes \mathrm{End}(V)$$

act naturally on sections of V. We thus build a **groupoid algebra with coefficients in a vector bundle** V,

$$\Gamma(\Omega \otimes \mathrm{End}(V)) \ .$$

Remark. In Section 14.3, we found an action of the intrinsic groupoid algebra on sections of $\Omega^{\frac{1}{2}} E$. However, this does not generally come from a representation of G on $\Omega^{\frac{1}{2}} E$ (see below).

We would have liked that the groupoid algebra acted on $\frac{1}{2}$-densities on $G^{(0)} \simeq X$ itself. However, in general, the algebra that acts on sections of $\Omega^{\frac{1}{2}}TX$ is that of sections of

$$\Omega \otimes \mathrm{End}(\Omega^{\frac{1}{2}}TX) = \alpha^*(\Omega^{\frac{1}{2}}E) \otimes \beta^*(\Omega^{\frac{1}{2}}E) \otimes \alpha^*(\Omega^{\frac{1}{2}}TX) \otimes \beta^*(\Omega^{-\frac{1}{2}}TX) \ .$$

In very special instances, there might be a natural trivialization of

$$\alpha^*(\Omega^{\frac{1}{2}}TX) \otimes \beta^*(\Omega^{-\frac{1}{2}}TX)$$

and we do obtain an action on $\frac{1}{2}$-densities on X. ♢

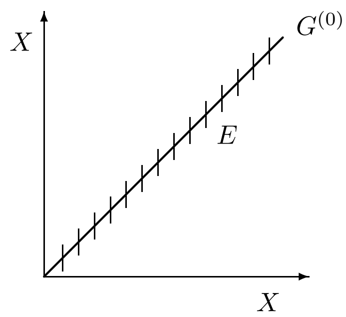
Alternatively, the intrinsic groupoid algebra itself acts on sections of

$$V \otimes \Omega^{\frac{1}{2}}E \ .$$

In order to obtain a representation of the groupoid algebra on sections of V, we hence need a representation of G on $V \otimes \Omega^{-\frac{1}{2}}E$.

Examples.

- When G is a Lie group, then $E = \mathfrak{g}$ is the Lie algebra, and there does exist a natural adjoint action of G on $\mathfrak{g}$. This gives rise to a representation of G on $\Omega^{\frac{1}{2}}E = \Omega^{\frac{1}{2}}\mathfrak{g}$ (and also on $\Omega^{-\frac{1}{2}}E = \Omega^{\frac{1}{2}}\mathfrak{g}^*$).
- At the other extreme, for the pair groupoid over a manifold X there is no natural representation of G on $\Omega^{\frac{1}{2}}E$. The normal space E along the identity section can be identified with TX, the tangent space to X.



A representation of G on E consists of an identification of T_xX with T_yX for each $(x, y) \in X \times X$. This amounts to a trivialization of the tangent bundle to X – that is, a global flat connection (with no holonomy). For an arbitrary manifold X, such a thing will not exist; even if it exists, there is no natural choice.

Similarly, to get a representation on $\Omega^{\frac{1}{2}}E$, we would need a global field of $\frac{1}{2}$-densities. This is equivalent to a global density on X, for which there is no natural choice.

♢

15 Extended Groupoid Algebras

Extended groupoid algebras encompass bisections and sections of the normal bundle to the identity section, just as distribution group algebras encompass Lie group elements and Lie algebra elements.

15.1 Generalized Sections

Recall that for a Lie group G, the algebra $C'(G)$ of measures on the group sat inside $\mathcal{D}'(G)$, the distribution group algebra (see Section 11.5). Furthermore, we saw that $\mathcal{D}'(G)$ contained G itself as the set of evaluation maps, $\mathfrak{g}$ as the dipoles at the identity, and $\mathcal{U}(\mathfrak{g})$ as the set of distributions supported at the identity element of G (see Section 12.3).

More generally, we return to the case of a Lie groupoid G over X. The intrinsic groupoid algebra is naturally identified (see Section 14.3) with the space of smooth sections of

$$\Omega =: \alpha^*(\Omega^{\frac{1}{2}}E) \otimes \beta^*(\Omega^{\frac{1}{2}}E) \ ,$$

where E is the normal bundle of $X \simeq G^{(0)}$ in G.

The **extended (intrinsic) groupoid algebra**, $\mathcal{D}'(G)$, is the dual space of the compactly supported smooth sections of

$$\Omega' =: \alpha^*(\Omega^{\frac{1}{2}}E^*) \otimes \beta^*(\Omega^{\frac{1}{2}}E^*) \otimes \Omega^1 TG \ .$$

The groupoid algebra is included in $\mathcal{D}'(G)$, as we can pair Ω and Ω' to get $\Omega^1 TG = |\wedge^{\text{top}}|\, T^*G$, and then integrate a 1-density on G (that is, a section of $|\wedge^{\text{top}}|\, T^*G$) to obtain a number. Elements of $\mathcal{D}'(G)$ are sometimes called **generalized sections** of Ω.

We may describe a typical section of Ω' along the identity section $X \simeq G^{(0)}$ of G. First, note that along X the bundle Ω' reduces to

$$\Omega'|_X = \Omega^1 E^* \otimes \Omega^1 TG|_X \ .$$

Although the tangent space of G along X can be decomposed into the tangent space of X and the normal space E, there is no *natural* choice of splitting. For densities, however, we are able to make a natural construction. Using the exact sequence

$$0 \longrightarrow TX \longrightarrow TG|_X \longrightarrow E \longrightarrow 0 \ ,$$

we see by Lemma 12.1 that

$$\begin{array}{rcl} \Omega'|_X & = & \Omega^1 E^* \otimes \Omega^1 TG|_X \\ & \simeq & \Omega^1 E^* \otimes \Omega^1 E \otimes \Omega^1 TX \\ & \simeq & \Omega^1 TX \ . \end{array}$$

Thus a section of $\Omega'|_X$ is just a 1-density on X. As a consequence, any measurable function $f : X \to \mathbb{R}$ determines a generalized section, namely

$$\varphi \in \Gamma_c(\Omega') \longmapsto \int_X f\ \varphi|_X \in \mathbb{R} \ .$$

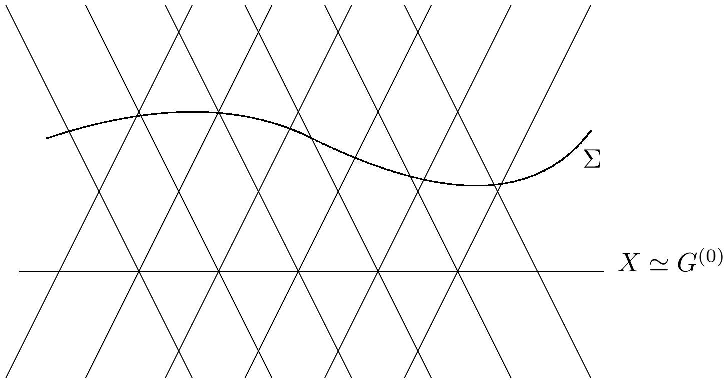
The inclusion of measurable functions on X as generalized sections is in fact a homomorphism.

We conclude that, in particular, all smooth functions on X belong to the extended intrinsic groupoid algebra:

$$C^\infty(X) \subseteq \mathcal{D}'(G) \ .$$

15.2 Bisections

The previous construction generalizes to other "sections" besides the identity section. A submanifold Σ of G such that the projections of Σ to X by α and β are isomorphisms is called a **bisection** of G or an **admissible section**.



Because we can identify the normal spaces of Σ with the tangent spaces of either the α- or the β-fibers along Σ, we see that

$$\begin{aligned}\Omega'|_\Sigma &= \Omega_\alpha^{-\frac{1}{2}}|_\Sigma \otimes \Omega_\beta^{-\frac{1}{2}}|_\Sigma \otimes \Omega^1 TG|_\Sigma \\ &= \Omega^{-1}N\Sigma \otimes \Omega^1 TG|_\Sigma \\ &\simeq \Omega^1 T\Sigma\ .\end{aligned}$$

where $N\Sigma$ is the normal bundle to Σ inside G. We can thus integrate sections of Ω' over Σ. Therefore, each bisection Σ determines an element of $\mathcal{D}'(G)$. Let $\mathfrak{B}(G)$ denote the set of smooth bisections of G. We conclude that

$$\mathfrak{B}(G) \subseteq \mathcal{D}'(G)\ .$$

Remark. Before integrating we could have multiplied by any smooth function on Σ (or X), thus obtaining other elements of $\mathcal{D}'(G)$ (see the last exercise of this section). $\diamondsuit$

Example. When G is a group, a bisection is a group element. The construction above becomes evaluation at that element. The inclusion of bisections into $\mathcal{D}'(G)$ thus extends the identification of elements of a group with elements of the distribution group algebra, as evaluation maps. The objects generalizing the Lie algebra elements will be discussed in Sections 15.4 and 15.5. $\diamondsuit$

The inclusion map from $\mathfrak{B}(G)$ to $\mathcal{D}'(G)$ is multiplicative if we define multiplication of bisections as follows.

Given two subsets A and B of a groupoid G, we form their product by multiplying all possible pairs of elements in $A \times B$,

$$AB = \{xy \in G|\ (x,y) \in A \times B \cap G^{(2)}\}\ .$$

This product defines a semigroup structure on the space 2^G of subsets of G. There are several interesting sub-semigroups of 2^G:

1. This multiplication defines a *group* structure on $\mathfrak{B}(G)$. The identity element of this group is just the identity section $X \simeq G^{(0)}$.

Exercise 50
Show that:

(a) $\mathfrak{B}(G)$ is closed under multiplication and that this multiplication satisfies the group axioms.

(b) Multiplication of bisections in $\mathfrak{B}(G)$ maps to convolution of distributions in $\mathcal{D}'(G)$ under the inclusion $\mathfrak{B}(G) \hookrightarrow \mathcal{D}'(G)$.

2. There is a larger sub-semigroup $\mathfrak{B}_{\mathrm{loc}}(G) \supseteq \mathfrak{B}(G)$ of local bisections. A **local bisection** is a subset of G for which the projection maps α, β are embeddings onto open subsets. $\mathfrak{B}_{\mathrm{loc}}(G)$ is an example of an inverse semigroup (see [135, 143]).

   Example. For the pair groupoid over X, the group $\mathfrak{B}(X \times X)$ can be identified with the group of diffeomorphisms of X, since each bisection Σ is the graph of a diffeomorphism. $\mathfrak{B}_{\mathrm{loc}}(X \times X)$ similarly corresponds to the semigroup (sometimes called a pseudogroup) of local diffeomorphisms of X.

Exercise 51
Show that the identification $\mathfrak{B}(X \times X) \to \mathrm{Diff}(X)$ is a group homomorphism (or anti-homomorphism, depending on conventions).

$\diamondsuit$

3. If we view $G \subset 2^G$ as the collection of one-element subsets, then G is not closed under the multiplication above. But if we adjoin the empty set, then $\{\emptyset\} \cup G \subseteq 2^G$ is a sub-semigroup. This is the semigroup naturally associated to a groupoid G, mentioned in Section 13.1.

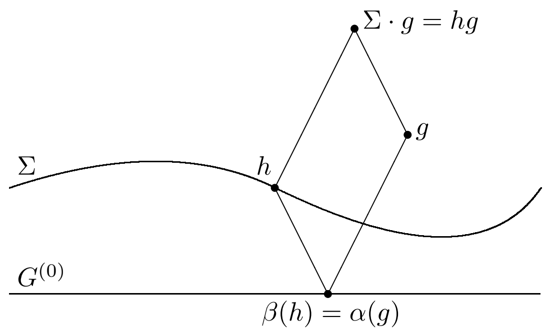
Exercise 52
The subspaces $\mathfrak{B}(G)$ and $C^\infty(X)$ of $\mathcal{D}'(G)$ generate multiplicatively the larger subspace of pairs $(\Sigma, s) \in \mathfrak{B}(G) \times C^\infty(X)$. Here we identify functions on a bisection Σ with functions on X via pull-back by α (alternatively, β). Let Σ_1, Σ_2 be bisections and $s_i \in C^\infty(\Sigma_i)$. Find an explicit formula for the product

$$(\Sigma_1, s_1) \cdot (\Sigma_2, s_2)$$

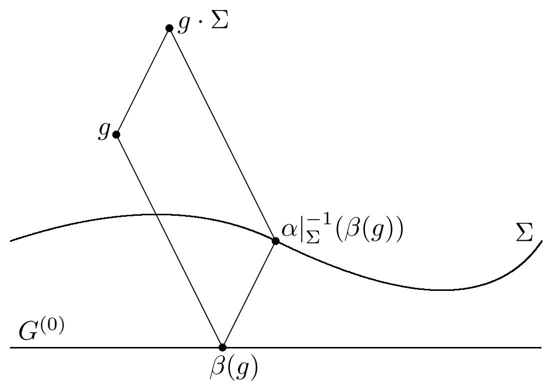
in $\mathcal{D}'(G)$.

15.3 Actions of Bisections on Groupoids

The group of bisections $\mathfrak{B}(G)$ acts on a groupoid G from the left (or from the right). To see this left action, take elements $g \in G$ and $\Sigma \in \mathfrak{B}(G)$. Because Σ is a bisection, there is a uniquely defined element $h \in \Sigma$, such that $\beta(h) = \alpha(g)$. We declare $\Sigma \cdot g := hg \in G$.



Similarly, we can define a right action of $\mathfrak{B}(G)$ on G by noting that there is also a uniquely defined element $\alpha|_\Sigma^{-1}(\beta(g)) \in \Sigma$. These actions can be thought of as "sliding" by Σ. See [2].



Exercise 53
Check that this defines a group action and that the left and right actions commute.

Remarks.

- This construction generalizes the left (or right) regular representation of a group on itself.
- We can recover the bisection Σ from its left or right action on G since

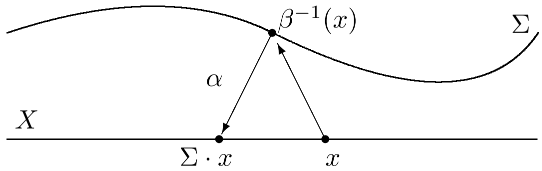
$$\Sigma = \Sigma \cdot G^{(0)} = G^{(0)} \cdot \Sigma \ .$$

$\diamondsuit$

The left action of $\mathfrak{B}(G)$ preserves the β-fibers of G, while the right action preserves the α-fibers. On the other hand, the left action of $\mathfrak{B}(G)$ maps α-fibers to α-fibers, while the right action of $\mathfrak{B}(G)$ maps β-fibers to β-fibers.

The left (respectively, right) action respects the α-fiber (respectively, β-fiber) structure even more, in the following sense. Note that $\mathfrak{B}(G)$ acts on the base space

X from the left (or from the right). For a bisection $\Sigma \in \mathfrak{B}(G)$, the (left) action on X is defined by taking $x \in X$ to $\alpha(\beta^{-1}(x))$, where $\beta^{-1}(x) \in \Sigma$ is uniquely determined.



It is easy to check that $\alpha(\Sigma \cdot g) = \Sigma \cdot \alpha(g)$, and so α is a left equivariant map from G to X with respect to the $\mathfrak{B}(G)$-actions. Similarly, β is a right equivariant map.

15.4 Sections of the Normal Bundle

As we saw in Section 15.2, the concept of bisection of a Lie groupoid generalizes the notion of Lie group element, both by its geometric definition, or when such an element is regarded as an evaluation functional at that element. From this point of view, we now explain how the objects corresponding to the Lie algebra elements are the sections of the normal bundle $E = TG|_{G^{(0)}}/TG^{(0)}$ thought of as first order perturbations of the submanifold $G^{(0)}$.

By choosing a splitting of the tangent bundle over $G^{(0)}$ (for instance, with a riemannian metric)

$$TG|_{G^{(0)}} \simeq TG^{(0)} \oplus \mathcal{E} \ ,$$

we can identify the normal bundle E with a sub-bundle $\mathcal{E} \subseteq TG|_{G^{(0)}}$. Under this identification, a section $\sigma \in \Gamma(E)$ may be viewed as a vector field $v : G^{(0)} \to TG|_{G^{(0)}}$. We can find, for sufficiently small ε, a path $\psi_t : G^{(0)} \to G$ defined for $0 \leq t < \varepsilon$ and such that

$$\begin{aligned} \psi_0 &= \text{identity on } G^{(0)} \\ \left.\frac{d\psi_t}{dt}\right|_{t=0} &= \lim_{t \to 0} \frac{\psi_t - \psi_0}{t} = v \ . \end{aligned}$$

At each time t, the image of ψ_t is a bisection Σ_t (restricted to the given compact subset of G). In particular, $\Sigma_0 = G^{(0)}$ is the identity section.

The one-parameter family of bisections $\{\Sigma_t\}$ gives rise to an element, called σ, of the extended groupoid algebra $\mathcal{D}'(G)$ by the following recipe. Let φ be a compactly supported smooth section of Ω'. Each individual bisection $\Sigma_t \in \mathcal{D}'(G) = (\Gamma_c(\Omega'))'$ pairs with φ to give a number $\langle \Sigma_t, \varphi \rangle$ as described in Section 15.2. We define the new pairing by

$$\langle \sigma, \varphi \rangle := \lim_{t \to 0} \frac{\langle \Sigma_t, \varphi \rangle - \langle \Sigma_0, \varphi \rangle}{t} \ .$$

Exercise 54
Check that $\langle \sigma, \cdot \rangle$ is a well-defined linear functional on $\Gamma_c(\Omega')$, independent of the choice of $\mathcal{E}$. (Hint: notice how vector fields $v \in \Gamma(TG^{(0)})$, *i.e.* tangent to $G^{(0)}$, yield a trivial pairing.)

We conclude that
$$\Gamma(E) \subseteq \mathcal{D}'(G) \ .$$
Furthermore, these elements of the extended groupoid algebra have support in $G^{(0)}$, that is, they vanish on test sections $\varphi \in \Gamma_c(\Omega')$ with (support $\varphi) \cap G^{(0)} = \emptyset$.

If we think of $\sigma \in \Gamma(E)$ as
$$\sigma = \lim_{t \to 0} \frac{\Sigma_t - G^{(0)}}{t} \ ,$$
we can give an informal definition of a commutator bracket $[\cdot, \cdot]$ on $\Gamma(E)$. Given two sections of E
$$\sigma = \lim_{t \to 0} \frac{\Sigma_t - G^{(0)}}{t} \ , \quad \theta = \lim_{u \to 0} \frac{\Theta_u - G^{(0)}}{u} \ ,$$
we define
$$\begin{aligned} [\sigma, \theta] &= \lim_{t,u \to 0} \left\{ \frac{\Sigma_t - G^{(0)}}{t} \cdot \frac{\Theta_u - G^{(0)}}{u} - \frac{\Theta_u - G^{(0)}}{u} \cdot \frac{\Sigma_t - G^{(0)}}{t} \right\} \\ &= \lim_{t,u \to 0} \frac{\Sigma_t \Theta_u - \Theta_u \Sigma_t}{tu} \ , \end{aligned}$$
or, equivalently, the bracket evaluated on $\varphi \in \Gamma_c(\Omega')$ is
$$\langle [\sigma, \theta], \varphi \rangle = \lim_{t,u \to 0} \frac{\langle \Sigma_t \Theta_u, \varphi \rangle - \langle \Theta_u \Sigma_t, \varphi \rangle}{tu} \ .$$
Sections of E are in fact closed under the commutator bracket:
$$[\Gamma(E), \Gamma(E)] \subseteq \Gamma(E) \ ,$$
as we will see in the next section where we define the bracket properly.

The distributions on G corresponding to sections of E are sometimes known as *dipole layers*. (See the discussion of dipoles in Section 12.3.)

15.5 Left Invariant Vector Fields

Recall from Section 14.4 that there is a (left) action of the groupoid G on itself; namely, each element $g \in G$ acts on $\alpha^{-1}(\beta(g))$ by left multiplication.

The β-projection is invariant with respect to this action
$$\beta(g \cdot h) = \beta(gh) = \beta(h) \ ,$$
while α-fibers are mapped to α-fibers
$$\alpha^{-1}(\beta(g)) \xrightarrow{g \cdot} \alpha^{-1}(\alpha(g)) \ .$$

Let
$$T^\alpha G := \ker T\alpha \subseteq TG$$

be the distribution tangent to the α-fibers. The action of $g \in G$ induces a linear map

$$T^{\alpha}G|_{\alpha^{-1}(\beta(g))} \xrightarrow{Tg\cdot} T^{\alpha}G|_{\alpha^{-1}(\alpha(g))} \ .$$

Exercise 55
The left action of the group of sections $\mathfrak{B}(G)$ preserves the α-fiber structure (see Section 15.3), and hence also induces an action on $T^{\alpha}G$ by differentiation.

(a) Prove that a section of $T^{\alpha}G$ is G-left-invariant if and only if it is $\mathfrak{B}(G)$-left-invariant.

(b) If a section of TG is $\mathfrak{B}(G)$-left-invariant, then do all of its values have to lie in $T^{\alpha}G$?

A left invariant section of $T^{\alpha}G$ is called a **left invariant vector field** on the groupoid G. The set $\chi_L(G)$ of all left invariant vector fields on G has the following properties.

- $\chi_L(G)$ is closed under the bracket operation

  $$[\chi_L(G), \chi_L(G)] \subseteq \chi_L(G) \ ,$$

  and thus forms a Lie algebra.

- An element of $\chi_L(G)$ is completely determined by its values along the identity section $G^{(0)}$. Equivalently, an element is determined by its values along any other bisection.

- Every smooth section of $T^{\alpha}_{G^{(0)}}G := \ker T\alpha|_{G^{(0)}}$ can be extended to an element of $\chi_L(G)$.

Furthermore,

$$T^{\alpha}_{G^{(0)}}G \simeq E \ ,$$

where $E = T_{G^{(0)}}G/TG^{(0)}$ is the normal bundle to $G^{(0)}$ in G.

Thus we have the identifications

$$\chi_L(G) \simeq \Gamma(T^{\alpha}_{G^{(0)}}G) \simeq \Gamma(E) \ .$$

The bracket on $\chi_L(G)$ can therefore be considered as a bracket on $\Gamma(E)$; it agrees with the one defined informally in the previous section.

The left invariant vector fields on G act by differentiation on $C^{\infty}_L(G)$, the left invariant functions on G. From the identification

$$C^{\infty}_L(G) \simeq \beta^* C^{\infty}(X) \simeq C^{\infty}(X) \ ,$$

we get a map

$$\Gamma(E) \longrightarrow \chi(X) := \Gamma(TX) \ .$$

It is easy to see that this map is induced by the *bundle map*

$$\rho : E \longrightarrow TX$$

given by composition of two natural maps:

$$\begin{tikzcd} E \arrow[rr, "\simeq"] \arrow[dr, "\rho"'] & & T^{\alpha}_{G^{(0)}} G \arrow[dl, "T_{G^{(0)}}\beta"] \\ & TX & \end{tikzcd}$$

With this additional structure, E provides the typical example of a *Lie algebroid*. We study these objects in the next chapter.

Example. When G is a Lie group (X is a point), both $\chi_L(G) \simeq \mathfrak{g}$ and $E \simeq \mathfrak{g}$ are the Lie algebra, and $\rho : \mathfrak{g} \to \{0\}$ is the trivial map. $\diamondsuit$

Part VII
Algebroids

16 Lie Algebroids

Lie algebroids are the infinitesimal versions of Lie groupoids.

16.1 Definitions

A **Lie algebroid** over a manifold X is a (real) vector bundle E over X together with a bundle map $\rho : E \to TX$ and a (real) Lie algebra structure $[\cdot,\cdot]_E$ on $\Gamma(E)$ such that:

1. The induced map $\Gamma(\rho) : \Gamma(E) \to \chi(X)$ is a Lie algebra homomorphism.
2. For any $f \in C^\infty(X)$ and $v, w \in \Gamma(E)$, the following Leibniz identity holds
$$[v, fw]_E = f[v,w]_E + (\rho(v)\cdot f)w \ .$$

Remarks.

- The map ρ is called the **anchor** of the Lie algebroid. By an abuse of notation, the map $\Gamma(\rho)$ may be denoted simply by ρ and also called the anchor.
- For each $v \in \Gamma(E)$, we define E**-Lie derivative** operations on both $\Gamma(E)$ and $C^\infty(X)$ by
$$\begin{aligned} \mathcal{L}_v w &= [v,w]_E \ , \\ \mathcal{L}_v f &= \rho(v)\cdot f \ . \end{aligned}$$
We can then view the Leibniz identity as a derivation rule
$$\mathcal{L}_v(fw) = f\,(\mathcal{L}_v w) + (\mathcal{L}_v f)\,w \ .$$

$\diamondsuit$

When $(E, \rho, [\cdot,\cdot]_E)$ is a Lie algebroid over X, the kernel of ρ is called the **isotropy**. Each fiber of $\ker\rho$ is a Lie algebra, analogous to the isotropy subgroups of groupoids. To see this, let v and $w \in \Gamma(E)$ be such that $\rho(v)$ and $\rho(w)$ both vanish at a given point $x \in X$. Then, for any function $f \in C^\infty(X)$,
$$[v, fw]_E(x) = f(x)[v,w]_E(x) \ .$$
So there is a well-defined bracket operation on the vectors in any fiber of $\ker\rho$, and $\ker\rho$ is a field of Lie algebras. These form a bundle when ρ has constant rank.

On the other hand, the **image of** ρ is an integrable distribution analogous to the image of $\widetilde{\Pi}$ for Poisson manifolds. Therefore, X can be decomposed into submanifolds, called **orbits** of the Lie algebroid, whose tangent spaces are the image of ρ. There are are various proofs of this: one uses the corresponding (local) Lie groupoid, another uses a kind of splitting theorem, and a third proof involves a more general approach to integrating singular distributions. The articles of Dazord [37, 38] discuss this and related issues.

16.2 First Examples of Lie Algebroids

1. A (finite dimensional real) **Lie algebra** is a Lie algebroid over a one-point space.

2. A **bundle of Lie algebras** over a manifold X (as in Section 16.3) is a Lie algebroid over X, with $\rho \equiv 0$. Conversely, if E is any Lie algebroid with $\rho \equiv 0$, the Leibniz identity says that the bracket in $\Gamma(E)$ is a bilinear map of $C^\infty(X)$-modules and not simply of $\mathbb{R}$-modules, and hence that each fiber is a Lie algebra. (Such an E is all isotropy.)

3. We saw in Section 15.5 that the normal bundle E along the identity section of a Lie groupoid G over X carries a bracket operation and anchor $\rho : E \to TX$ satisfying the Lie algebroid conditions. This is called the **Lie algebroid of the Lie groupoid** G. The isotropy algebras of this Lie algebroid are the Lie algebras of the isotropy groups of G. The orbits are the connected components of the G-orbits.

   As for the case of Lie groups and Lie algebras, it is natural to pose the **integrability problem** (see also Sections 16.3 and 16.4):

   - When is a given Lie algebroid the Lie algebroid of a Lie groupoid?
   - If the Lie algebroid does come from a Lie groupoid, is the Lie groupoid unique?

4. The **tangent bundle** TX of a manifold X, with ρ the identity map, is a Lie algebroid over X. We can see it the Lie algebroid of the Lie groupoid $X \times X$, or of the fundamental groupoid $\Pi(X)$, or of yet other possibilities; near the identity section, $\Pi(X)$ looks like $X \times X$.

   Generally, we can say that a Lie algebroid determines and is determined by a neighborhood of the identity section in the groupoid, just as a Lie algebra determines and is determined by a neighborhood of the identity element in the corresponding Lie group.

5. Suppose that we have a right action of a Lie algebra $\mathfrak{g}$ on X, that is, a Lie algebra homomorphism $\mathfrak{g} \stackrel{\gamma}{\to} \chi(X)$. The associated **transformation Lie algebroid** $X \times \mathfrak{g}$ has anchor $X \times \mathfrak{g} \stackrel{\rho}{\to} TX$ defined by

   $$\rho(x, v) = \gamma(v)(x) \ .$$

   Combining this with the natural projections $X \times \mathfrak{g} \to X$ and $TX \to X$, we form the commutative diagram

   $$\begin{tikzcd} X \times \mathfrak{g} \arrow[r, "\rho"] \arrow[d] & TX \arrow[dl] \\ X & \end{tikzcd}$$

   A section v of $X \times \mathfrak{g}$ can be thought of as a map $v : X \to \mathfrak{g}$. We define the bracket on sections of $X \times \mathfrak{g}$ by

   $$[v, w](x) = [v(x), w(x)]_{\mathfrak{g}} + (\gamma(v(x)) \cdot w)(x) - (\gamma(w(x)) \cdot v)(x) \ .$$

When v, w are constant functions $X \to \mathfrak{g}$, we recover the Lie algebra bracket of $\mathfrak{g}$.

It is easy to see in this example that the fibers of $\ker \rho$ are the usual isotropy Lie algebras of the $\mathfrak{g}$-action. The orbits of the Lie algebroid are just the orbits of the Lie algebra action.

If γ comes from a Γ-action on X, where Γ is a Lie group with Lie algebra $\mathfrak{g}$, then $X \times \mathfrak{g}$ is the Lie algebroid of the corresponding transformation groupoid G_Γ.

6. Suppose that ρ **is injective**. This is equivalent to $E \simeq \rho(E) \subseteq TX$ being an integrable distribution, as the bracket on E is completely determined by that on TX. A universal choice of a Lie groupoid with this Lie algebroid is the holonomy groupoid of the corresponding foliation. (It might not be Hausdorff.)

   The case when ρ is surjective will be discussed in Section 17.1.

Exercise 56
Let $(v_1, \ldots, v_n)$ be a basis of sections for a Lie algebroid E such that $[v_i, v_j] = \sum_k c_{ijk} v_k$ where the c_{ijk}'s are constants. Show that E is isomorphic to a transformation Lie algebroid.

Historical Remark. Already in 1963, Rinehart [145] noted that, if a Lie algebra Γ over a field k is a module over a commutative k-algebra C, and if there is a homomorphism ρ from Γ into the derivations of C, then there is a semidirect product Lie bracket on the sum $\Gamma \oplus C$ defined by the formula

$$[(v, g), (w, h)] = ([v, w], \rho(v) \cdot h - \rho(w) \cdot g) \ .$$

Furthermore, this bracket satisfies the Leibniz identity:

$$[(v, g), f(w, h)] = f[(v, g), (w, h)] + (\rho(v) \cdot f)(w, h) \qquad \text{for } f \in C \ .$$

In the special case where $C = C^\infty(X)$, the $C^\infty(X)$-module Γ, if projective, is the space of sections of some vector bundle E over X. The homomorphism ρ and the Leibniz identity imply that ρ is induced by a bundle map $\rho : E \to TX$. The Leibniz identity for $\Gamma(E) \oplus C^\infty(X)$ also encodes the Leibniz identity for the bracket on $\Gamma(E)$ alone.

In 1967, Pradines [139] coined the term “Lie algebroid” and proved that every Lie algebroid comes from a (local) Lie groupoid. He asserted that the local condition was not needed, but this was later shown by Almeida and Molino [4] to be false. (See Section 16.4.)

Rinehart [145] proved (in a more algebraic setting) an analogue of the Poincaré-Birkhoff-Witt theorem for Lie algebroids. He showed that there is a linear isomorphism between the graded version of a universal object for the actions of $\Gamma(E) \oplus C^\infty(X)$ on vector bundles V over X, and the polynomials on the dual of the Lie algebroid E. As a result, the dual bundle of a Lie algebroid carries a Poisson structure. This Poisson structure is described abstractly in [34] as the base of the cotangent groupoid T^*G of a Lie groupoid G; it is described more explicitly in [35]. (See Section 16.5.)

The most basic instance of this phenomenon is when $E = TX$. The dual to the Lie algebroid is T^*X with its standard (symplectic) Poisson structure (see Section 6.5). The universal object is the algebra of differential operators on X, and the Rinehart isomorphism is a "symbol map". ♢

16.3 Bundles of Lie Algebras

For a first look at the integrability problem, we examine Lie algebroids for which the anchor map is zero.

A bundle of Lie groups is a bundle of groups (see Section 13.4) for which each fiber is a Lie group. Bundles of Lie algebras are vector bundles for which each fiber has a Lie algebra structure which varies continuously (or smoothly). Every bundle of Lie groups defines a bundle of Lie algebras: the Lie algebras of the individual fibers. More problematic is the question of whether we can integrate a bundle of Lie algebras to get a bundle of Lie groups.

Theorem 16.1 (Douady-Lazard [48]) *Every bundle of Lie algebras can be integrated to a (not necessarily Hausdorff) bundle of Lie groups. (Fibers and base are Hausdorff, but the bundle itself might not be.)*

Example. Given a Lie algebra $\mathfrak{g}$ with bracket $[v_i, v_j]= \sum c_{ijk} v_k$, we defined in Section 1.2 a family of Lie algebras $\mathfrak{g}_\varepsilon = (\mathfrak{g}, [\cdot,\cdot]_\varepsilon)$, $\varepsilon \in \mathbb{R}$, by the structure equations $[v_i, v_j]_\varepsilon = \varepsilon \sum c_{ijk} v_k$. This can be thought of as a bundle of Lie algebras over $\mathbb{R}$. There is a bundle of Lie groups corresponding to this bundle: the fiber over $0 \in \mathbb{R}$ is an abelian Lie group (either euclidean space, a cylinder or a torus), while the fiber over any other point $\varepsilon \in \mathbb{R}$ can be chosen to be a fixed manifold. The fiber dimensions cannot jump, but the topology may vary drastically. In the particular case of $\mathfrak{g} = \mathfrak{su}(2)$, the bundle of groups corresponding to the deformation $\mathfrak{g}_\varepsilon$ has fiber $\mathrm{SU}(2) \simeq S^3$ for $\varepsilon \neq 0$, and fiber $\mathbb{R}^3$ at $\varepsilon = 0$. Here the total space is Hausdorff, since it is homeomorphic to $\mathbb{R} \times S^3$ with a point removed from $\{0\} \times S^3$. ♢

Example. [48, p.148] Consider now the bundle of Lie algebras over $\mathbb{R}$ with fibers $\mathfrak{g}_\varepsilon = (\mathbb{R}^3, [\cdot,\cdot]_\varepsilon)$, $\varepsilon \in \mathbb{R}$, where the brackets are defined by

$$[x_\varepsilon, y_\varepsilon]_\varepsilon = z_\varepsilon \ , \qquad [x_\varepsilon, z_\varepsilon]_\varepsilon = -y_\varepsilon \ , \qquad [y_\varepsilon, z_\varepsilon]_\varepsilon = \varepsilon x_\varepsilon \ .$$

Here $x_\varepsilon, y_\varepsilon, z_\varepsilon$ denote the values at ε of a given basis of sections x, y, z for the bundle $\mathfrak{g} := \mathbb{R} \times \mathbb{R}^3$.

The corresponding simply connected Lie groups G_ε are as follows for $\varepsilon \geq 0$. G_1 is the group of unit quaternions, if we identify the basis x_1, y_1, z_1 of $\mathfrak{g}_1$ with $\frac{1}{2}i, \frac{1}{2}j, \frac{1}{2}k$, respectively. Consequently, $\exp(4\pi x_1) = e_1$ is the identity element of G_1. For $\varepsilon > 0$, $\mathfrak{g}_\varepsilon \simeq \mathfrak{g}_1$ under the isomorphism

$$x_\varepsilon \mapsto x_1 \ , \quad y_\varepsilon \mapsto \sqrt{\varepsilon}\, y_1 \ , \quad z_\varepsilon \mapsto \sqrt{\varepsilon}\, z_1 \ .$$

Taking $G_\varepsilon \simeq G_1$, we still have that $\exp(4\pi x_\varepsilon) = e_\varepsilon$ is the identity of G_ε, $\varepsilon > 0$. At $\varepsilon = 0$, G_0 is the semidirect product $\mathbb{R} \times \mathbb{R}^2$, where the first factor $\mathbb{R}$ acts on $\mathbb{R}^2$ by rotations. Here $\exp(tx_0) = (t, 0)$, thus, in particular, $\exp(4\pi x_0) \neq e_0$.

Therefore, the set of points $\varepsilon \in \mathbb{R}$ where the two continuous sections $\exp(4\pi\cdot)$ and e coincide is not closed, hence G is *not* Hausdorff.

Following this example, Douady and Lazard show that, if we replace the group G_0 by the semidirect product $S^1 \times \mathbb{R}^2$ (the group of euclidean motions of the plane), the resulting bundle of groups *is* Hausdorff. They then go on to show that a certain C^∞ bundle of semidirect product Lie algebras admits *no* Hausdorff bundle of Lie groups. They conclude by asking whether there is an analytic example. The next example answers this question. $\diamondsuit$

Example. Coppersmith [33] constructed an analytic family of 4-dimensional Lie algebras parametrized by $\mathbb{R}^2$ which cannot be integrated to a Hausdorff family of Lie groups. The fiber $\mathfrak{g}_\varepsilon$ over $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$ has basis $x_\varepsilon, y_\varepsilon, z_\varepsilon, w_\varepsilon$ and bracket given by

$$[w_\varepsilon, x_\varepsilon]_\varepsilon = [w_\varepsilon, y_\varepsilon]_\varepsilon = [w_\varepsilon, z_\varepsilon]_\varepsilon = 0 \ ,$$

$$[x_\varepsilon, z_\varepsilon]_\varepsilon = -y_\varepsilon \ , \qquad [y_\varepsilon, z_\varepsilon]_\varepsilon = x_\varepsilon \ , \qquad [x_\varepsilon, y_\varepsilon]_\varepsilon = \varepsilon_1 z_\varepsilon + \varepsilon_2 w_\varepsilon \ .$$

$\diamondsuit$

16.4 Integrability and Non-Integrability

To find Lie algebroids which are not integrable even by non-Hausdorff groupoids, we must look beyond bundles of Lie algebras.

Example. As a first attempt, take the transformation Lie algebroid $X \times \mathfrak{g}$ for an action of the Lie algebra $\mathfrak{g}$ on X (see Example 5 of Section 16.2). If the action of the Lie algebra can be integrated to an action of the group Γ, then the Γ-action on X defines a transformation groupoid G_Γ with Lie algebroid $X \times \mathfrak{g}$. Now we can make the $\mathfrak{g}$-action non-integrable by restricting to an open set $U \subseteq X$ not invariant under Γ. The Lie algebroid $X \times \mathfrak{g}$ restricts to a Lie algebroid $U \times \mathfrak{g}$. We might hope that the corresponding groupoid does not restrict. However, one property of groupoids is that they can always be restricted to open subsets of the base space.

Exercise 57
Let G be any groupoid over X and U an open subset of X. Then $H = \alpha^{-1}(U) \cap \beta^{-1}(U)$ is a subgroupoid of G with base space U.

We conclude that this restriction H of the transformation groupoid $X \times \Gamma$ has Lie algebroid $U \times \mathfrak{g}$. $\diamondsuit$

Example. We could instead look for incomplete vector fields that cannot even be completed by inserting into a bigger manifold. One example begins with $X = \mathbb{R}^2$ and $U = \mathbb{R}^2 \setminus \{0\}$, where the vector field $\frac{\partial}{\partial x}$ is incomplete. If $\mathbb{R}^2$ is identified with $\mathbb{C}$, then the subspace U has a double cover defined by the map $z \mapsto z^2$. If we pull $\frac{\partial}{\partial x}$ back to the double cover, there is no way to smoothly "fill in the hole" to a complete vector field. Could this then give an example of a non-integrable Lie algebroid? Unfortunately, this type of construction is also doomed to fail, if non-Hausdorff groupoids are allowed. $\diamondsuit$

Theorem 16.2 (Dazord [40]) *Every transformation Lie algebroid is integrable.*

Exercise 58
Find a groupoid which integrates the Lie algebroid in the previous example.

Historical Remark. Important work on integrability of Lie algebra actions was done by Palais [134] in 1957. In that manuscript, he proved results close to Dazord's theorem, but without the language of groupoids.

The following example of a non-integrable Lie algebroid is due to Almeida and Molino [3, 4]. It is modeled on an example of a non-integrable Banach Lie algebra due to Douady-Lazard [48]. Mackenzie [110] had already developed an obstruction theory to integrating Lie algebroids, but never wrote a non-zero example. ♢

Example. We will construct a Lie algebroid E which has the following form as a bundle

$$0 \longrightarrow \mathbb{L} \longrightarrow \underbrace{TX \times \mathbb{R}}_{E} \longrightarrow TX \longrightarrow 0 ,$$

where $\mathbb{L}$ is the trivial real line bundle over X. We define a bracket on sections of E, $\Gamma(E) = \chi(X) \times C^\infty(X)$, by

$$[(v, f), (w, g)]_{E,\Omega} = ([v, w]_{TX}, v \cdot g - w \cdot f + \Omega(v, w)) ,$$

where Ω is a given 2-form on X. The bracket $[\cdot, \cdot]_{E,\Omega}$ satisfies the Jacobi identity if and only if Ω is closed.

Each integral 2-cycle $\gamma \in H_2(X; \mathbb{Z})$ gives rise to a *period*

$$\int_\gamma \Omega \in \Lambda .$$

If the set of periods of Ω is not cyclic in $\mathbb{R}$, and if X is simply connected, then one can show that E does not come from a groupoid [3].[6] In this way we obtain a non-integrable Lie algebroid. ♢

Remark. There is still a sort of Lie groupoid corresponding to this Lie algebroid. As a bundle over $X \times X$, it has structure group $\mathbb{R}/\Lambda$, where Λ is generated by two numbers which are linearly independent over $\mathbb{Q}$. There are no nonconstant differentiable functions on $\mathbb{R}/\Lambda$, but there is a notion of smooth curves, if one uses Souriau's notion of *diffeological space* [154]. In general, a map $M \to \mathbb{R}/\Lambda$ is said to be smooth if it (locally) lifts to a smooth map $M \to \mathbb{R}$.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d] \\
M \arrow[ur] \arrow[r] & \mathbb{R}/\Lambda
\end{tikzcd}$$

Examples of Lie algebroids which are "even more" non-integrable can also be constructed [39]. ♢

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[6]For instance, on $X = S^2 \times S^2$ with projections

$$\begin{array}{c} S^2 \times S^2 \\ {\scriptstyle \pi_1} \downarrow\downarrow {\scriptstyle \pi_2} \\ S^2 \end{array}$$

define the 2-form $\Omega = c_1 \pi_1^* \omega + c_2 \pi_2^* \omega$, where ω is the standard volume on S^2 and c_1, c_2 are rationally independent constants, Then the periods of Ω do not lie in a cyclic subgroup of $\mathbb{R}$.

16.5 The Dual of a Lie Algebroid

Let $x_1, \cdots, x_n$ be local coordinates on a manifold X, and let $e_1, \cdots, e_r$ be a local basis of sections of a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$ over it. With respect to these coordinates and basis, the Lie bracket and anchor map are described by **structure functions** $c_{ijk}, b_{ij} \in C^\infty(X)$ as

$$\begin{aligned} [e_i, e_j]_E &= \sum_k c_{ijk} e_k \\ \rho(e_i) &= \sum_j b_{ij} \frac{\partial}{\partial x_j} \ . \end{aligned}$$

Exercise 59
The Leibniz identity and Jacobi identity translate into differential equations for the c_{ijk} and b_{ij}. Write out these differential equations.

Let $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$ be the associated coordinates on the dual bundle E^*, where $\xi_1, \cdots, \xi_n$ are the linear functions on E^* defined by evaluation at $e_1, \cdots, e_r$.

We define a bracket $\{\cdot,\cdot\}_E$ on $C^\infty(E^*)$ by setting

$$\begin{aligned} \{x_i, x_j\}_E &= 0 \\ \{\xi_i, \xi_j\}_E &= \sum_k c_{ijk} \xi_k \\ \{\xi_i, x_j\}_E &= -b_{ij} \end{aligned}$$

Proposition 16.3 *The bracket $\{\cdot,\cdot\}_E$ defines a Poisson structure on E^*.*

Exercise 60
Show that the Jacobi identity for $\{\cdot,\cdot\}_E$ follows from the Lie algebroid axioms for E.

Remark. Although the Poisson bracket $\{\cdot,\cdot\}_E$ is defined in terms of coordinates and a basis, it is independent of these choices. Hence, the passage between the Lie algebroid structure on E and the Poisson structure on E^* is intrinsic. $\diamondsuit$

Examples.

1. When X is a point, and $E = \mathfrak{g}$ is a Lie algebra, then the Poisson bracket on $E^* = \mathfrak{g}^*$ regarded as the dual of a Lie algebroid, coincides with the Lie-Poisson bracket defined in Section 3.1.

   In general, the Poisson bracket $\{\cdot,\cdot\}_E$ on the dual of a Lie algebroid is sometimes also called a **Lie-Poisson bracket**.

2. When $E = TX$, we can choose $e_i = \frac{\partial}{\partial x_i}$ to give the standard basis of vector fields induced by the choice of coordinates on X, so that $c_{ijk} = 0$ and $b_{ij} = \delta_{ij}$.

   The Poisson structure on the dual bundle $E^* = T^*X$ as a dual of a Lie algebroid is the one induced by the canonical symplectic structure $\sum dx_i \wedge d\xi_i$ because
   $$\{x_i, \xi_j\}_E = -\{\xi_j, x_i\}_E = \delta_{ij} \ .$$

3. When $E = T^*X$ is the Lie algebroid of a Poisson manifold X (see Section 17.3), we obtain on the tangent bundle $E^* = TX$ the **tangent Poisson structure**; see [5].

Exercise 61

(a) Let E_1 and E_2 be Lie algebroids over X. Show that a bundle map $\varphi : E_1 \to E_2$ is a **Lie algebroid morphism** (*i.e.* compatible with brackets and anchors) if and only if $\varphi^* : E_2^* \to E_1^*$ is a Poisson map.

(b) Show that the dual of the anchor map of a Lie algebroid is a Poisson map from T^*X to E^*.

(c) Use the result of part (a) to suggest a definition of morphism between Lie algebroids over *different* base manifolds. See Proposition 6.1 in [111].

Exercise 62
Let $x_1, \cdots, x_n$ be coordinates on X, $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ and $dx_1, \cdots, dx_n$ the induced local bases of TX and T^*X, $x_1, \cdots, x_n, v_1, \cdots, v_n$ associated coordinates on TX, and $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$ associated coordinates on T^*X.
Express the Poisson bracket for the tangent Poisson structure on TX in terms of the Poisson bracket on X given by $\pi_{ij}(x) = \{x_i, x_j\}$.
Check that, although $TX \to X$ is *not* a Poisson map, the map $\tilde{\Pi} : T^*X \to TX$ *is* a Poisson map.

$\diamondsuit$

16.6 Complex Lie Algebroids

It can be interesting to work over $\mathbb{C}$ even if X is a real manifold. To define a **complex Lie algebroid** geometrically, take a complex vector bundle E over X and a complex bundle map $\rho : E \to T_{\mathbb{C}}X$ to the complexified tangent bundle.[7] The immediate generalization of our definition in Section 16.1 amounts to imposing that the space of sections of E be a *complex* Lie algebra satisfying the (complex versions of the) two axioms.

Example. Let X be a (real) manifold with an almost complex structure $J : TX \to TX$, *i.e.* J is a bundle map such that $J^2 = -\mathrm{id}$. The graph of $-iJ$ in $T_{\mathbb{C}}X = TX \oplus iTX$ is the sub-bundle

$$E = \{v - iJ(v) |\ v \in TX\} \subseteq T_{\mathbb{C}}X \ .$$

The bracket operation on TX extends by linearity to a bracket on $T_{\mathbb{C}}X$. To endow E with a Lie algebroid structure, we need the sections of E to be closed under that bracket:

$$[\Gamma(E), \Gamma(E)] \subseteq \Gamma(E) \ .$$

This holds if and only if J is an integrable structure. That is, by the Newlander-Nirenberg theorem [131], we have a complex Lie algebroid structure on E if and only if J comes from a complex structure on X. A **complex structure** on a

[7] Algebraically, we have changed our "ground ring" from $C^\infty(X)$ to $C^\infty(X; \mathbb{C})$.

manifold is, in this way, a typical example of a complex Lie algebroid. The natural questions arise: When does such a Lie algebroid come from a complex Lie groupoid? What *is* a complex Lie groupoid? $\diamondsuit$

Example. Let X be a manifold of dimension $2n-1$. Suppose that $F \subseteq TX$ is a codimension-1 sub-bundle with an almost complex structure $J : F \to F$ (J is linear and $J^2 = -\mathrm{id}$). As before, define a sub-bundle E of the complexified F to be the graph of $-iJ$

$$E = \{v - iJ(v) |\ v \in F\} \subset F_{\mathbb{C}} = F \oplus iF \ .$$

If $\Gamma(E)$ is closed under the bracket operation, *i.e.* if E is a Lie algebroid over X, then (F, J) is called a **Cauchy-Riemann structure** or **CR-structure** on X.

To explain the motivation behind this construction, we consider the special situation when $X^{2n-1} \subset Y^{2n}$ is a real submanifold of a complex n-manifold Y. At a point $x \in X \subset Y$, the tangent space T_xY is a vector space over $\mathbb{C}$. We denote by J_Y the complex multiplication by i in this space. Because X has odd real dimension, T_xX cannot be equal to $J_Y(T_xX)$ as subspaces of T_xY, and thus the intersection $F_x := T_xX \cap J_Y(T_xX)$ must have codimension 1 in T_xX. Then F is the maximal complex sub-bundle of TX.

Functions on a CR-manifold annihilated by the sections of E are called **CR-functions**. In the case where X is a hypersurface in a complex manifold Y, they include (and sometimes coincide with) the restrictions to X of holomorphic functions on one side of X in Y.

This construction opens several questions, including:

- What is the Lie groupoid in this case? However, at this point it is not clear what it means to integrate a complex Lie algebroid.
- What does the analytic theory of complex Lie algebroids look like? It seems to be at least as complicated as that of CR-structures, which is already very delicate [87].
- The *cohomology theory of Lie algebroids* can be applied to complex Lie algebroids. In the examples above, we recover the usual $\overline{\partial}$ cohomology and boundary $\overline{\partial}$ cohomology on complex and CR-manifolds, respectively. What does complex Lie algebroid cohomology look like in more general cases?

$\diamondsuit$

Remark. When X is a complex manifold, it is tempting to impose the Lie algebroid axioms on the space of *holomorphic* sections of a holomorphic vector bundle E. This idea fails in general, for the following reason. In the real case, sections of E always exist. On the other hand, the only holomorphic functions on a compact complex manifold are the constant functions. Similarly, it is possible that there are *no* non-zero holomorphic sections for a complex vector bundle. It is therefore more appropriate to look instead at the *sheaf of local sections.* Atiyah's (see [9] and Section 17.1) study of the obstructions to the existence of holomorphic connections on principal $\mathrm{GL}(n;\mathbb{C})$-bundles over complex manifolds used this approach to the "Atiyah algebroid". $\diamondsuit$

17 Examples of Lie Algebroids

Lie algebroids with surjective anchor map are called **transitive Lie algebroids**, **Atiyah algebras**, or **Atiyah sequences** because of Atiyah's work mentioned below. When a corresponding groupoid exists, it will be (locally) transitive, in the sense that its orbits are open.

17.1 Atiyah Algebras

In 1957, Atiyah [9] constructed in the setting of vector bundles the Lie algebroid of the following key example of a locally transitive groupoid. Suppose that we have a principal bundle P over a manifold X

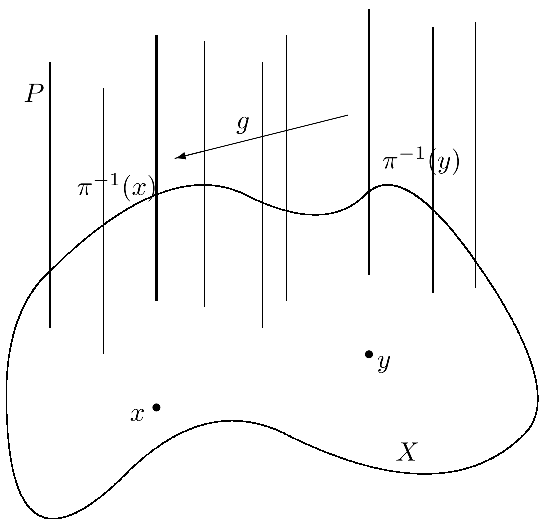
$$\begin{array}{ccc} P & \hookleftarrow & H \\ \pi \downarrow & & \\ X & & \end{array}$$

with structure group H acting on the right. The quotient $G = (P \times P)/H$ of the product groupoid by the diagonal action of H is a groupoid over X. An element $g = [p, q]$ of this groupoid is an equivalence class of pairs of points $p \in \pi^{-1}(x), q \in \pi^{-1}(y)$ in P; it is the graph of an equivariant map from the fiber $\pi^{-1}(y)$ to $\pi^{-1}(x)$.

A bisection of this groupoid corresponds to a gauge transformation, that is, an automorphism (*i.e.* an H-equivariant diffeomorphism) of the principal bundle. For this reason, we call G the *gauge groupoid* of P. The group of bisections $\mathfrak{B}(G)$ and the gauge group $\mathcal{G}$ are thus isomorphic. The infinitesimal generators of $\mathcal{G}$ are the H-invariant vector fields. Since H acts on the fibers of π freely and transitively, H-invariant vector fields are determined by their values on one point of each fiber, so they can be identified with sections of

$$E = TP/H$$

considered as a bundle over X. The bracket on E is that induced from $\chi(P)$; this is well-defined because the bracket of two H-invariant vector fields is H-invariant.



The projection π commutes with the H-action, and so there is a bundle map $T\pi$

$$\begin{tikzcd} E \arrow[rr, "T\pi"] \arrow[dr, "\pi"'] & & TX \arrow[dl] \\ & X & \end{tikzcd}$$

which is surjective. The induced map on sections is a Lie algebra homomorphism.

The kernel $\ker T\pi$ consists of the vertical part of TP/H. The sections of $\ker T\pi$ are the H-invariant vector fields on the fibers. Although each fiber of TP/H is isomorphic to the Lie algebra $\mathfrak{h}$ of H, there is no *natural* way to identify these two Lie algebras. In fact, $\ker T\pi$ is the bundle associated to the principal bundle P by the adjoint representation of H on $\mathfrak{h}$.

Exercise 63
Show that, when P is the bundle of frames for a vector bundle $V \to X$, then the gauge groupoid $(P \times P)/H$ of P is naturally isomorphic to the general linear groupoid $\mathrm{GL}(V)$ (see Section 14.4). Also show that the Lie algebroid TP/H is naturally contained in $\mathfrak{gl}(V)$.

17.2 Connections on Transitive Lie Algebroids

We can use the Atiyah algebroid above to extend the notion of connection from bundles to transitive Lie algebroids (see [110]).

A connection on the principal bundle P is a field of H-invariant direct complement subspaces to the fiber tangent spaces. Equivalently, a connection is simply a splitting φ of the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \ker T\pi \arrow[r] & E \arrow[r, shift left, "T\pi"] & TX \arrow[l, shift left, "\varphi"] \arrow[r] & 0 . \end{tikzcd}$$

For any transitive Lie algebroid

$$\begin{tikzcd} 0 \arrow[r] & \ker\rho \arrow[r] & E \arrow[r, "\rho"] & TX \arrow[r] & 0 , \end{tikzcd}$$

we define a **connection** on E to be a linear splitting

$$\begin{tikzcd} E & TX \arrow[l, "\sigma"] \end{tikzcd}$$

of the sequence above, that is, a cross-section of ρ. The corresponding projection

$$\begin{tikzcd} \ker\rho & E \arrow[l, two heads] \end{tikzcd}$$

is called the **connection form**.

The **curvature** of a connection σ is its deviation from being a Lie algebra homomorphism. Specifically, for $v, w \in \Gamma(TX)$, define the **curvature form** to be

$$\Omega(v,w) = [\sigma(v), \sigma(w)]_E - \sigma[v,w]_{TX} \in \Gamma(\ker\rho) .$$

An application of the Leibniz identity shows that Ω is "tensorial," *i.e.*

$$\Omega(v, fw) = f\Omega(v, w) \ .$$

One can verify that Ω is a skew-symmetric bundle map $TX \times TX \to \ker \rho$, *i.e.* Ω is indeed a 2-form on X with values in $\ker \rho$.

Exercise 64
Show that every (real-valued) 2-form on X is the curvature of a transitive Lie algebroid

$$0 \longrightarrow X \times \mathbb{R} \longrightarrow E \longrightarrow TX \longrightarrow 0 \ .$$

(Hint: See Section 16.4.)

17.3 The Lie Algebroid of a Poisson Manifold

The symplectic structure on a symplectic manifold (X, ω) induces an *isomorphism*

$$T^*X \xrightarrow{\widetilde{\Pi} = \widetilde{\omega}^{-1}} TX \ ,$$

where $\widetilde{\omega}(v) = \omega(v, \cdot)$. Pulling back the standard bracket on $\chi(X)$ by $\widetilde{\Pi}$, we define a bracket operation $\{\cdot, \cdot\}$ on differential 1-forms $\Omega^1(X) = \Gamma(T^*X)$. This makes T^*X into a Lie algebroid with anchor $\rho = -\widetilde{\Pi}$, called the **Lie algebroid of the symplectic manifold**.

Furthermore, the bracket on 1-forms relates well to the Poisson bracket on functions. Recall that the bracket of hamiltonian vector fields $X_f = \widetilde{\Pi}(df)$ and $X_g = \widetilde{\Pi}(dg)$ satisfies (see Section 3.5)

$$[X_f, X_g] = -X_{\{f,g\}} \ .$$

We may pull the bracket back to $\Gamma(T^*X)$ by $-\widetilde{\Pi}$, and will denote by $[\cdot, \cdot]$ the bracket on 1-forms. From the following computation

$$\begin{aligned} -\widetilde{\Pi}[df, dg] &= [-\widetilde{\Pi}(df), -\widetilde{\Pi}(dg)] \\ &= [X_f, X_g] \\ &= -X_{\{f,g\}} \\ &= -\widetilde{\Pi}(d\{f, g\}) \ . \end{aligned}$$

we conclude that for exact 1-forms

$$[df, dg] = d\{f, g\} \ .$$

Now let (X, Π) be a Poisson manifold. The Poisson bivector field Π still induces a map (see Section 4.2)

$$T^*X \xrightarrow{\widetilde{\Pi}} TX \ ,$$

though not necessarily an isomorphism. Nonetheless, there is a generalization of the symplectic construction. This is the content of the following proposition, which has been discovered many times, apparently first by Fuchssteiner [62].

Proposition 17.1 *There is a natural Lie bracket* $[\cdot,\cdot]$ *on* $\Omega^1(X)$ *arising from a Poisson structure on* X*, which satisfies*

- $[df, dg] = d\{f, g\}$,
- $\widetilde{\Pi} : \Omega^1(X) \to \chi(X)$ *is a Lie algebra anti-homomorphism.*

Proof. For general elements $\alpha, \beta \in \Omega^1(X)$, this bracket is defined by

$$[\alpha, \beta] := -\mathcal{L}_{\widetilde{\Pi}(\alpha)}\beta + \mathcal{L}_{\widetilde{\Pi}(\beta)}\alpha - d\Pi(\alpha, \beta) \ .$$

To check this definition, we first note that the map $\widetilde{\Pi}$ was defined by

$$\widetilde{\Pi}(\alpha) \lrcorner \beta = \beta(\widetilde{\Pi}(\alpha)) = \Pi(\beta, \alpha) \ .$$

If we then apply Cartan's magic formula

$$\mathcal{L}_X \eta = X \lrcorner d\eta + d(X \lrcorner \eta) \ ,$$

we can rewrite the bracket operation as

$$[\alpha, \beta] = -\widetilde{\Pi}(\alpha) \lrcorner d\beta + \widetilde{\Pi}(\beta) \lrcorner d\alpha + d\Pi(\alpha, \beta) \ .$$

When $\alpha = df$ and $\beta = dg$, it is easy to see that

$$[df, dg] = d\Pi(df, dg) = d\{f, g\} \ .$$

Exercise 65
Show that this bracket on $\Gamma(T^*X)$ satisfies the Leibniz identity

$$[\alpha, f\beta] = f[\alpha, \beta] + (-\widetilde{\Pi}(\alpha) \cdot f)\beta \ .$$

It is also easy to show that this bracket satisfies the Jacobi identity if we first check it for df, dg, dh using $[df, dg] = d\{f, g\}$. Since any $\alpha \in \Gamma(T^*X)$ can be written in a coordinate basis as

$$\alpha = \sum u_i df_i \ ,$$

we may use the Leibniz identity to extend the Jacobi identity to arbitrary 1-forms.

Exercise 66
Check that $\widetilde{\Pi}$ defines a Lie algebra anti-homomorphism from $\Gamma(T^*X)$ to $\Gamma(TX)$. Using the Leibniz identity, it suffices to check that $\widetilde{\Pi}$ is an anti-homomorphism on exact 1-forms.

$\square$

It was observed in [167] that the bracket on 1-forms makes T^*X into a Lie algebroid whose anchor is $-\widetilde{\Pi}$. This is called the **Lie algebroid of the Poisson manifold** (X, Π). The orbits of this Lie algebroid are just the symplectic leaves of X. The isotropy at a point x – those cotangent vectors contained in $\ker \widetilde{\Pi}$ – is the conormal space to the symplectic leaf $\mathcal{O}_x$. The Lie algebra structure which is inherited from the Lie algebroid T^*X is exactly the transverse Lie algebra structure from Section 5.2. Thus the Lie algebroid contains much of the information associated with the Poisson structure! More on the Lie algebroid of a Poisson manifold can be found in [162].

Exercise 67
How canonical is this construction?
Specifically, if $\varphi : X \to Y$ is a Poisson map, is the induced map $\varphi^* : \Omega^1(Y) \to \Omega^1(X)$ a Lie algebra homomorphism?

The Lie algebroid T^*X is not always integrable to a Lie groupoid. However, when it is integrable, at least one of its associated Lie groupoids carries a natural symplectic structure compatible with the groupoid structure. Such an object is called a *symplectic groupoid* (see [167]).

17.4 Vector Fields Tangent to a Hypersurface

Let Y be a hypersurface in a manifold X. Denote by $\chi_Y(X)$ the space of vector fields on X which are tangent to Y; $\chi_Y(X)$ is closed under the bracket $[\cdot,\cdot]$ of vector fields, it is a module over $C^\infty(X)$, and it acts on $C^\infty(X)$ by derivation. The following theorem asserts that $\chi_Y(X)$ is the space of sections of some vector bundle. This result was probably noticed earlier than the cited reference.

Theorem 17.2 (Melrose [117][8]) *There is a vector bundle whose space of sections is isomorphic to $\chi_Y(X)$ as a $C^\infty(X)$-module.*

This is a consequence of $\chi_Y(X)$ being a locally free module over $C^\infty(X)$. The corresponding vector bundle A can be constructed from its space of sections and it is called the **Y-tangent bundle** of X.

The Y-tangent bundle A comes equipped with a Lie algebroid structure over X. To see the anchor map at the level of sections, introduce local coordinates $x, y_2, \ldots, y_n$ in a neighborhood $U \subseteq X$ of a point in Y, and adapted to the submanifold Y in the sense that $U \cap Y$ is defined by $x = 0$.

A vector field

$$v = a\frac{\partial}{\partial x} + \sum_{i=2}^{n} b_i \frac{\partial}{\partial y_i} \ , \qquad a, b_i \in C^\infty(U) \ ,$$

over U is the restriction of a vector field in $\chi_Y(X)$ if and only if the coefficient a vanishes when $x = 0$, that is, if and only if the smooth function a is divisible by x. Hence, with respect to these coordinates, the vector fields

$$x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}$$

form a local basis for $\chi_Y(X)$ as a module over $C^\infty(X)$. Call these local basis vectors $e_1, e_2, \ldots, e_n$. They satisfy $[e_i, e_j] = 0$, just like a local basis for the tangent bundle.

The difference between A and the tangent bundle lies in the anchor map $\rho : \chi_Y(X) \to \chi(X)$, which is the inclusion

$$\rho(e_1) = x\frac{\partial}{\partial x} \ , \quad \rho(e_j) = \frac{\partial}{\partial y_j} \ , \ j \geq 2 \ .$$

This induces an anchor map ρ at the level of vector bundles. Together, these data form a Lie algebroid

$$(A, \rho, [\cdot,\cdot]) \ .$$

[8] In [117] Melrose handles the case $Y = \partial X$, the boundary of X, but the idea works for any hypersurface.

The orbits of A (orbits were defined in Section 16.1) are the connected components of Y and of $X \setminus Y$.

The isotropy of A, *i.e.* the kernel of $\rho : A \to TX$, is trivial over $X \setminus Y$. Over Y, the isotropy $\ker \rho|_Y$ is the real line bundle spanned by e_1. This is clearly the trivial line bundle $Y \times \mathbb{R}$ when Y is cooriented (meaning that the normal bundle NY is trivial, or equivalently that Y is a two-sided hypersurface). But even if Y is not cooriented, $x\frac{\partial}{\partial x}$ still provides a trivialization of $\ker \rho|_Y$, as this section is invariant under change of orientation of NY over U.

Restricting the vector bundle A to Y, we obtain the exact sequence

$$0 \longrightarrow \ker \rho|_Y \longrightarrow A|_Y \xrightarrow{\ \rho\ } TY \longrightarrow 0 \ .$$

Therefore, a typical section of $\ker \rho|_Y = Y \times \mathbb{R}$ has the form

$$v = a(y) \cdot x \cdot \frac{\partial}{\partial x}$$

for some bundle morphism $a : NY \to NY$, expressing the rate at which v grows as we move across Y. We conclude that sections of $\ker \rho|_Y$ coincide with endomorphisms of the normal bundle of Y. Note that $A|_Y$ is the gauge algebroid (or Atiyah algebroid) of NY; see also the first remark at the end of Section 17.5.

17.5 Vector Fields Tangent to the Boundary

The construction of the previous section extends to the case where X is a manifold with boundary $Y = \partial X$.

Recall that the tangent space to X at a point in the boundary is just the usual tangent space as if the manifold was enlarged by a collar extension so that the point became interior.

Let $\varphi : X \to [0,1]$ be a defining function for the boundary Y; *i.e.*

$\varphi^{-1}(0) = Y$,
$d\varphi \neq 0$ on Y, and
$\varphi \equiv 1$ off a tubular neighborhood of Y .

With respect to the coordinates $x, y_2, \ldots, y_n$ above, we define a map $m_\varphi \oplus 1$ on vector fields by

$$a\frac{\partial}{\partial x} + \sum b_i \frac{\partial}{\partial y_i} \ \longmapsto\ \varphi \cdot a\frac{\partial}{\partial x} + \sum b_i \frac{\partial}{\partial y_i} \ .$$

We extend $m_\varphi \oplus 1$ as the identity map outside the tubular neighborhood of Y. Then

$$m_\varphi \oplus 1 : \chi(X) \to \chi_{_Y}(X)$$

is an isomorphism of $C^\infty(X)$-modules.

This isomorphism of the $C^\infty(X)$-modules $\chi(X) = \Gamma(TX)$ and $\chi_{_Y}(X) = \Gamma(A)$ induces an isomorphism between the underlying vector bundles

$$TX \simeq A$$

which we interpret over a tubular neighborhood of Y as $TX \simeq \nu \oplus \tau$, where ν and τ are the pull-back to the tubular neighborhood of the normal bundle $NY \simeq Y \times \mathbb{R}$ and of the tangent bundle TY, respectively.

Remarks.

1. When $Y = \partial X$, A is the Lie algebroid of a groupoid over X, namely the groupoid built from the pair groupoid of $X \setminus Y$ together with the gauge groupoid of the normal bundle of Y in X.

> **Exercise 68**
> What if Y is not the boundary of X?

2. In general, if the hypersurface Y is not the boundary of X, then the Y-tangent bundle A might be *not* isomorphic to the tangent bundle TX.

   For example, let X be a circle and let Y be one point. Then the Y-tangent bundle is a Möbius band rather than the trivial bundle. A similar construction works when X is a 2-torus and Y is a single homologically nontrivial closed curve.

   Notice that, if Y is two points on a circle X, then the Y-tangent bundle is again the trivial bundle.

   It would be interesting to understand how much of the structure of the Y-tangent bundle is determined by the cohomology class dual to Y (and the original tangent bundle).

$\diamondsuit$

18 Differential Geometry for Lie Algebroids

A useful way to view a Lie algebroid E over X is as an "alternative tangent bundle" for X, endowing X with a "peculiar differentiable structure". The Lie algebroid axioms allow us to carry out virtually all of the usual differential-geometric constructions, replacing TX by E. The reader may wish to keep the example $E = TX$ in mind during a first reading of this chapter.

18.1 The Exterior Differential Algebra of a Lie Algebroid

Let $(E, \rho, [\cdot,\cdot]_E)$ be a Lie algebroid over X, and let $\wedge^\bullet E^*$ be the exterior algebra of its dual E^*. Sections of $\wedge^\bullet E^*$ are called **E-differential forms** on X, or simply **E-forms** on X.

If $\theta \in \Gamma(\wedge^k E^*)$, we say that θ is **homogeneous**, and furthermore that its **degree** is $|\theta| = k$. In this case θ is called an **E-k-form**.

We define a differential operator taking an E-k-form θ to an E-$(k+1)$-form $d_E\theta$, which at E-vector fields $v_1, \ldots, v_{k+1} \in \Gamma(E)$ is

$$\begin{aligned} d_E\theta(v_1,\ldots,v_{k+1}) &= \sum_i (-1)^{i+1}\rho(v_i)\cdot\theta(v_1,\ldots,\hat{v}_i,\ldots,v_{k+1}) \\ &\quad + \sum_{i<j}(-1)^{i+j}\theta([v_i,v_j]_E, v_1,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_{k+1}) \ . \end{aligned}$$

The Lie algebroid axioms for E imply the following properties for d_E:

1. d_E is $C^\infty(X)$-multilinear,
2. $d_E^2 = 0$, and
3. d_E is a superderivation of degree 1, *i.e.*

$$d_E(\theta\wedge\nu) = d_E\theta\wedge\nu + (-1)^{|\theta|}\theta\wedge d_E\nu \ .$$

The triple $(\Gamma(\wedge^\bullet E^*), \wedge, d_E)$ forms a differential graded algebra, like the usual algebra of differential forms. We can recover the Lie algebroid structure on E from $(\Gamma(\wedge^\bullet E^*), \wedge, d_E)$:

- the anchor map ρ is obtained from d_E on functions by the formula:

$$\rho(v)\cdot f = (d_E f)(v) \ , \quad \text{for } v \in \Gamma(E) \text{ and } f \in C^\infty(X) \ ;$$

- the Lie bracket $[\cdot,\cdot]_E$ is determined by

$$\begin{aligned} [v,w]_E \lrcorner \theta &= \rho(v)\cdot\theta(w) - \rho(w)\cdot\theta(v) - d_E\theta(v,w) \\ &= v\lrcorner d_E(w\lrcorner\theta) - w\lrcorner d_E(v\lrcorner\theta) - (v\wedge w)\lrcorner d_E\theta \end{aligned}$$

  for $v, w \in \Gamma(E)$ and $\theta \in \Gamma(E^*)$.

We conclude that there is a one-to-one correspondence between Lie algebroid structures on E and differential operators on $\Gamma(\wedge^\bullet E^*)$ satisfying properties 1-3.

Remark. The space of sections of $\wedge^\bullet E^*$ can be regarded as the space of functions on a *supermanifold*.

In this language, d_E is an *odd* (since the degree is 1) *vector field* (since it is a derivation), which is *integrable* because its superbracket with itself vanishes:

$$[d_E, d_E] = d_E d_E - (-1)^1 d_E d_E = 2d_E^2 = 0 \ .$$

Hence, we may say that a Lie algebroid is a *supermanifold with an odd integrable supervector field*. This idea permits one to apply to Lie algebroids some of the intuition attached to ordinary vector fields. See [161]. $\diamondsuit$

The exterior differential algebra $(\Gamma(\wedge^\bullet E^*), \wedge, d_E)$ associated to a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$ determines de Rham cohomology groups, called the **Lie algebroid cohomology** of E or E**-cohomology**.

Examples.

1. When $E = \mathfrak{g}$ is a Lie algebra (*i.e.* a Lie algebroid over a one-point space), the cohomology of the differential complex

$$(\wedge^\bullet \mathfrak{g}^*, \wedge, d_\mathfrak{g}) : \mathbb{R} \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \wedge \mathfrak{g}^* \longrightarrow \dots$$

   is the standard **Lie algebra cohomology** with trivial coefficients, also known as **Chevalley cohomology**.

   Notice that the first arrow is the zero map and the second arrow is the usual **cobracket** with the opposite sign:

   for $\mu \in \mathfrak{g}^*$, $d_\mathfrak{g}\mu$ is the element of $\mathfrak{g}^* \wedge \mathfrak{g}^*$
   which at $v, w \in \mathfrak{g}$ gives $d_\mathfrak{g}\mu(v,w) = -\mu([v,w])$.

   The higher differentials are determined by the first two and the derivation property.

2. When $E = TX$ is a tangent bundle of a manifold X, the cohomology computed by $(\Gamma(\wedge^\bullet E^*), \wedge, d_E) = (\Omega^\bullet(X), \wedge, d_{\mathrm{deRham}})$ is the usual de Rham cohomology.

$\diamondsuit$

Exercise 69
Compute the Lie algebroid cohomology for the Y-tangent bundle of a manifold X where $Y = \partial X$ is the boundary (see Sections 17.4 and 17.5 and [117], proposition 2.49).

Remark. There have been several theories of characteristic classes associated to Lie algebroids. We refer to [85] for a recent study of these with ample references to earlier literature. $\diamondsuit$

18.2 The Gerstenhaber Algebra of a Lie Algebroid

Sections of the exterior algebra $\wedge^\bullet E$ of a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$ are called **Lie algebroid multivector fields** or E**-multivector fields**. If $v \in \Gamma(\wedge^k E)$, then v is called **homogeneous** with **degree** $|v| = k$.

We extend the bracket $[\cdot,\cdot]_E$ to arbitrary E-multivector fields by setting it, on homogeneous E-multivector fields v, w, to be

$$\begin{array}{rcl} [v,w]_E \lrcorner \theta & = & (-1)^{(|v|-1)(|w|-1)} v \lrcorner d_E (w \lrcorner \theta) - w \lrcorner d_E (v \lrcorner \theta) \\ & & -(-1)^{|v|-1} (v \wedge w) \lrcorner d_E \theta \end{array}$$

where $\theta \in \Gamma(\wedge^\bullet E^*)$. If $\theta \in \Gamma(\wedge^k E^*)$, then $[v,w]_E \lrcorner \theta$ is homogeneous of degree $k - (|v| + |w| - 1)$. For $[v,w]_E \lrcorner \theta$ to be a function, the degree of θ should be $k = |v| + |w| - 1$. Therefore, $[v,w]_E$ has degree $|v| + |w| - 1$, and $[\cdot,\cdot]_E$ is a bracket of degree -1.

Remark. In order to obtain a bracket of degree 0, we can redefine the grading on $\Gamma(\wedge^\bullet E)$, and let the new degree be the old degree minus 1:

$$(v) := |v| - 1 = k - 1 \ , \qquad \text{for } v \in \Gamma(\wedge^k E) \ .$$

For the $(\cdot)$ grading, we have

$$\Big([v,w]_E\Big) = \Big|[v,w]_E\Big| - 1 = |v| + |w| - 2 = (v) + (w) \ .$$

$\diamondsuit$

The bracket $[\cdot,\cdot]_E$ on E-mutivector fields has the following properties:

1. $[\cdot,\cdot]_E$ allows us to extend to arbitrary elements of $v, w \in \Gamma(\wedge^\bullet E)$ the E-Lie derivative operation defined for E-vector fields in Section 16.1:
$$\mathcal{L}_v w := [v,w]_E \ .$$
2. $[\cdot,\cdot]_E$ is a super-Lie algebra (or "graded" Lie algebra) structure for the $(\cdot)$ grading:
$$[v,w]_E = -(-1)^{(v)(w)}[w,v]_E = -(-1)^{(|v|-1)(|w|-1)}[w,v]_E \ .$$
In words, $[v,w]_E$ is symmetric in v and w when both $|v|$ and $|w|$ are even and is antisymmetric otherwise.
3. $[\cdot,\cdot]_E$ satisfies a super-Jacobi identity:
$$\begin{array}{rcl} [v,[w,y]_E]_E & + & (-1)^{(|y|-1)(|v|+|w|)}[y,[v,w]_E]_E \\ & + & (-1)^{(|v|-1)(|w|+|y|)}[w,[y,v]_E]_E \quad = \quad 0 \ . \end{array}$$
4. $[v,\cdot]_E$ satisfies a super-Leibniz identity (notice that both gradings appear here):
$$[v, w \wedge y]_E = [v,w]_E \wedge y + (-1)^{(v)|w|} w \wedge [v,y]_E \ .$$

The triple $(\Gamma(\wedge^\bullet E), \wedge, [\cdot,\cdot]_E)$ is called the **Gerstenhaber algebra of the Lie algebroid** $(E, \rho, [\cdot,\cdot]_E)$, or just the **$E$-Gerstenhaber algebra**. We will refer to the bracket $[\cdot,\cdot]_E$ on $\Gamma(\wedge^\bullet E)$ as the **E-Gerstenhaber bracket**.

In general, a **Gerstenhaber algebra** $(\mathfrak{a}, \wedge, [\cdot,\cdot])$ is the following structure:

1. a graded vector space
$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \ldots$$
together with

2. a supercommutative associative multiplication of degree 0

$$\mathfrak{a}_i \wedge \mathfrak{a}_j \subseteq \mathfrak{a}_{i+j}$$

and

3. a super-Lie algebra structure of degree -1

$$[\mathfrak{a}_i, \mathfrak{a}_j] \subseteq \mathfrak{a}_{i+j-1}$$

satisfying the super-Leibniz identity

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c] \ .$$

Historical Remark. Gerstenhaber found such a structure in 1963 [66] in the Hochschild cohomology of an associative algebra (see Sections 19.1 and 19.2). ♢

Remark. For a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$, the pull-back by ρ

$$\Gamma(\wedge^\bullet E^*) \xleftarrow{\ \rho^*\ } \Gamma(\wedge^\bullet T^*X)$$

satisfies

$$\rho^* \circ d = d_E \circ \rho^* \ ,$$

hence induces a map in cohomology. On the other hand, the wedge powers of ρ

$$\Gamma(\wedge^\bullet E) \xrightarrow{\ \wedge^\bullet \rho\ } \Gamma(\wedge^\bullet TX)$$

form a morphism of Gerstenhaber algebras. ♢

To summarize, from a Lie algebroid structure on E

$$(E, \rho, [\cdot,\cdot]_E) \ ,$$

we obtain a differential algebra structure on $\Gamma(\wedge^\bullet E^*)$

$$(\Gamma(\wedge^\bullet E^*), \wedge, d_E) \ ,$$

and from that we get a Gerstenhaber algebra structure on $\Gamma(\wedge^\bullet E)$

$$(\Gamma(\wedge^\bullet E), \wedge, [\cdot,\cdot]_E) \ .$$

This process can be reversed, so these structures are equivalent.

For more on this material, see [84, 98, 162, 179].

18.3 Poisson Structures on Lie Algebroids

Example. For the tangent bundle Lie algebroid

$$(E, \rho, [\cdot,\cdot]_E) = (TX, \mathrm{id}, [\cdot,\cdot]) \ ,$$

d_E is the de Rham differential and the E-Gerstenhaber bracket is usually called the **Schouten-Nijenhuis bracket** on multivector fields (*cf.* Sections 3.2 and 3.3).[9]

A bivector field $\Pi \in \Gamma(\wedge^2 TX)$ is called a **Poisson bivector field** if and only if $[\Pi, \Pi] = 0$ (*cf.* Section 3.3). This condition is equivalent to the condition $d_\Pi^2 = 0$ for the differential operator $d_\Pi := [\Pi, \cdot]$.

If Π is a Poisson bivector field on X, then T^*X is a Lie algebroid with anchor $-\widetilde{\Pi}$ (as seen in Section 17.3), and d_Π is the induced differential on multivector fields. $\diamondsuit$

The notion of Poisson structure naturally generalizes to arbitrary Lie algebroids as follows. Let $(E, \rho, [\cdot,\cdot]_E)$ be a Lie algebroid over X. An element $\Pi \in \Gamma(\wedge^2 E)$ is called an E**-Poisson bivector field** when $[\Pi, \Pi]_E = 0$, where $[\cdot,\cdot]_E$ is the E-Gerstenhaber bracket.

Example. When $E = \mathfrak{g}$ is a Lie algebra, a $\mathfrak{g}$-Poisson bivector field $\Pi \in \mathfrak{g} \wedge \mathfrak{g}$ corresponds to a left-invariant Poisson structure on the underlying Lie group G. The equation $[\Pi, \Pi]_\mathfrak{g} = 0$ is called the **classical Yang-Baxter equation**. $\diamondsuit$

Remarks.

1. The push-forward $\rho_*\Pi$ of an E-Poisson bivector field Π by the anchor $\rho : \Gamma(\wedge^2 E) \to \Gamma(\wedge^2 TX)$ defines an ordinary Poisson structure on the manifold X.

2. By the Jacobi identity, an arbitrary (not necessarily Poisson) element $\Theta \in \Gamma(\wedge^2 E)$ satisfies
$$d_\Theta^2 + [\tfrac{1}{2}[\Theta, \Theta]_E, \cdot]_E = 0 \ .$$
Notice the resemblance to the equation for a flat connection.

$\diamondsuit$

An E-Poisson bivector field $\Pi \in \Gamma(\wedge^2 E)$ is called an E**-symplectic structure** when the induced bundle morphism
$$\widetilde{\Pi} : E^* \to E$$
is an isomorphism. As in Section 17.3, $\widetilde{\Pi}$ satisfies
$$\alpha(\widetilde{\Pi}_x(\beta)) = \Pi_x(\alpha, \beta)$$
for $\alpha, \beta \in E_x^*$ and $x \in X$.

An E-symplectic structure defines an element $\omega_\Pi \in \Gamma(\wedge^2 E^*)$ by
$$\omega_\Pi(v, w) = \Pi(\widetilde{\Pi}^{-1}v, \widetilde{\Pi}^{-1}w)$$
for $v, w \in \Gamma(E)$. This E-2-form on X is non-degenerate and E-closed:
$$d_E \omega_\Pi = 0 \ .$$
Hence, ω_Π is called an E**-symplectic form**.

[9] According to the definitions of Section 18.2, the signs here differ from the conventions of Vaisman [162].

18.4 Poisson Cohomology on Lie Algebroids

In this section, we study Poisson cohomology on general Lie algebroids, but the most interesting case is of course that where $E = TM$. This "ordinary" Poisson cohomology, introduced by Lichnerowicz [105], was studied from a general homological viewpoint by Huebschmann [83].

An E-Poisson structure Π on a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$ over X induces an operator

$$d_\Pi = [\Pi, \cdot]_E$$

on $\Gamma(\wedge^\bullet E)$ (see Section 18.3). The super-Jacobi identity for $[\cdot,\cdot]_E$, together with the property $[\Pi,\Pi]_E = 0$, imply that

$$d_\Pi^2 = 0 \ ,$$

so $(\Gamma(\wedge^\bullet E), d_\Pi)$ forms a differential complex. The cohomology of this complex is called the **Lie algebroid Poisson cohomology** or **E-Π-cohomology**. We will next interpret the corresponding cohomology groups $H_\Pi^\bullet$.

For $f \in C^\infty(X)$ and $\theta \in \Gamma(E^*)$,

$$[\Pi, f]_E \lrcorner \theta = -\Pi \lrcorner (d_E f \wedge \theta) = -\Pi(d_E f, \theta) = \widetilde{\Pi}(d_E f) \lrcorner \theta = X_f \lrcorner \theta \ ,$$

where the vector field

$$X_f := \widetilde{\Pi}(d_E f)$$

is called the **hamiltonian vector field** of f with respect to Π (similar to Section 4.5).

The computation above shows that

$$X_f = [\Pi, f]_E = d_\Pi f \ ,$$

so the *image* of $d_\Pi : C^\infty(X) \to \Gamma(E)$ is precisely the space of hamiltonian vector fields.

Exercise 70
Check that ρ maps the hamiltonian vector field of f with respect to Π to the ordinary hamiltonian vector field of f with respect to $\rho_*\Pi$.

The **Poisson bracket** of functions $f, g \in C^\infty(X)$ with respect to an E-Poisson structure Π

$$\{f, g\} = \Pi(d_E f, d_E g) \ ,$$

coincides with the ordinary Poisson bracket with respect to $\rho_*\Pi$

$$\{f, g\} = (\rho_*\Pi)(df, dg) \ .$$

Exercise 71
Check this assertion.

Hence the *kernel* of $d_\Pi : C^\infty(X) \to \Gamma(E)$ is the set of usual **Casimir functions**.

For an E-vector field v, we have

$$[\Pi, v]_E = -[v, \Pi]_E = -\mathcal{L}_v \Pi$$

where $\mathcal{L}_v$ is the E-Lie derivative (defined in Sections 16.1 and 18.2). We naturally call **Poisson vector fields** those $v \in \Gamma(E)$ satisfying $\mathcal{L}_v \Pi = 0$; these form the *kernel* of $d_{\Pi} : \Gamma(E) \to \Gamma(\wedge^2 E)$.

A synopsis of these observations is

$$H^0_{\Pi} = \text{Casimir functions}$$

$$H^1_{\Pi} = \frac{\text{Poisson vector fields}}{\text{hamiltonian vector fields}}$$

The next two sections demonstrate how H^2_{Π} and H^3_{Π} are related to deformations of the Poisson structure Π.

Exercise 72
Compute the Π-cohomology for the following Poisson manifolds:

(a) $\mathfrak{g}^*$ with its Lie-Poisson structure,

(b) the 3-torus $\mathbb{T}^3$ with a translation-invariant regular Poisson structure (see [81]),

(c) $\mathbb{R}^2$ with $\{x, y\} = x^2 + y^2$ (see [70, 123]).

Remark. Let Π be a Poisson structure on a Lie algebroid E. The operator d_{Π} induces a Lie algebroid structure on E^*, hence a bracket on $\Gamma(\wedge^\bullet E^*)$. The E^*-de Rham complex $(\Gamma(\wedge^\bullet E), d_{E^*})$ coincides with the Π-complex for E, $(\Gamma(\wedge^\bullet E), d_{\Pi})$. Therefore, the E-Π-cohomology equals the E^*-cohomology.

The canonical cohomology class $[\Pi] \in H^2_{\Pi}$ is zero if and only if there exists $X \in \Gamma(E)$ such that $\mathcal{L}_X \Pi = \Pi$. An element $\Pi \in \Gamma(E \wedge E)$ satisfying $\mathcal{L}_X \Pi = \Pi$ for some $X \in \Gamma(E)$ is called **exact**; X is called a **Liouville vector field** for Π (as in the symplectic case). $\diamondsuit$

Exercise 73
Find an example of an exact Poisson structure on a compact manifold (see [81]).

18.5 Infinitesimal Deformations of Poisson Structures

Let $\Pi(\varepsilon)$ be a smooth family of sections of $\wedge^2 E$ for a Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$. Write

$$\Pi(\varepsilon) = \Pi_0 + \varepsilon \Pi_1 + \varepsilon^2 \Pi_2 + \dots$$

as a formal power series expansion.

The equation for each $\Pi(\varepsilon)$ to be a Poisson structure is

$$\begin{aligned} 0 &= [\Pi(\varepsilon), \Pi(\varepsilon)]_E \\ &= [\Pi_0, \Pi_0]_E + 2\varepsilon [\Pi_0, \Pi_1]_E + \varepsilon^2 (2[\Pi_0, \Pi_2]_E + [\Pi_1, \Pi_1]_E) + \dots \end{aligned} \qquad (\star)$$

Assume that $\Pi(0) = \Pi_0$ is a Poisson structure, so that $[\Pi_0, \Pi_0]_E$ vanishes.

The coefficient Π_1 is called an **infinitesimal deformation** of Π_0 when

$$d_{\Pi_0} \Pi_1 = [\Pi_0, \Pi_1]_E = 0 \ .$$

This is a cocycle condition in the complex $(\Gamma(\wedge^\bullet E), d_{\Pi_0})$.

Suppose that

$$\Pi_1 = d_{\Pi_0} v = [\Pi_0, v]_E = -\mathcal{L}_v \Pi_0$$

for some $v \in \Gamma(E)$. Then Π_1 is considered a **trivial infinitesimal deformation** of Π_0.

Remark. The term "trivial" is suggested by the tangent bundle $E = TX$ case with the (local) flow φ_t of $-v$. For each $t = \varepsilon$, the pull-back $\varphi_\varepsilon^* \Pi_0$ is again a Poisson structure. Furthermore,

$$\frac{d}{d\varepsilon} \varphi_\varepsilon^* \Pi_0 \bigg|_{\varepsilon=0} = \mathcal{L}_{-v} \Pi_0 = \Pi_1 \ .$$

The infinitesimal deformation Π_1 is trivial in the sense that all Poisson structures $\Pi(\varepsilon) = \varphi_\varepsilon^* \Pi_0$ are essentially the same expressed in different coordinates. The interpretation of this infinitesimal triviality for general Lie algebroids (with or without using an associated groupoid) is not so clear. ♢

We conclude that

$$H_\Pi^2 = \frac{\text{infinitesimal deformations of } \Pi}{\text{trivial infinitesimal deformations of } \Pi}$$

The group H_Π^2 is a candidate for the tangent space at Π of the moduli space of Poisson structures on E modulo isomorphism.

18.6 Obstructions to Formal Deformations

Returning to the equation ($\star$) of the previous section, suppose that $[\Pi_0, \Pi_0]_E = [\Pi_0, \Pi_1]_E = 0$. To eliminate the ε^2 term, we need the vanishing of

$$[\Pi_0, \Pi_2]_E + \tfrac{1}{2}[\Pi_1, \Pi_1]_E \ ,$$

i.e. having found Π_1, we need to solve for Π_2 in the non-homogeneous differential equation

$$d_{\Pi_0} \Pi_2 = -\tfrac{1}{2}[\Pi_1, \Pi_1]_E \ .$$

By the super-Jacobi identity,

$$d_{\Pi_0}([\Pi_1, \Pi_1]_E) = 0 \ ,$$

so $[\Pi_1, \Pi_1]_E$ determines an element of $H_{\Pi_0}^3$. This element is zero if and only if the solution Π_2 of $d_{\Pi_0} \Pi_2 = -\frac{1}{2}[\Pi_1, \Pi_1]_E$ exists. Therefore, $H_{\Pi_0}^3$ is the home of obstructions to continuing infinitesimal deformations.

In general, the recursive solution of equation ($\star$) involves at each step working out an equation of type

$$d_{\Pi_0} \Pi_n = \text{ quadratic expression in the } \Pi_i\text{'s with } i < n \ .$$

Exercise 74
Let Π be a Poisson structure on E.
Show that Π induces, via $\widetilde{\Pi} : E^* \to E$, a chain map

$$(\Gamma(\wedge^\bullet E^*), d_E) \longrightarrow (\Gamma(\wedge^\bullet E), d_\Pi) \ .$$

Hence, Π induces a map from E-Π-cohomology to E-cohomology.
Show that, if Π is symplectic, then all the maps above are isomorphisms, so E-Π-cohomology and E-cohomology are the same.

In view of the exercise, we conclude that, in the symplectic case, the obstructions to formal deformations of a Poisson structure lie in H^2_{deRham} and H^3_{deRham} (see below).

The bracket $[\cdot,\cdot]_E$ on $\Gamma(\wedge^\bullet E)$ passes to E-Π-cohomology. In particular, it gives rise to a **squaring map**

$$\tfrac{1}{2}[\cdot,\cdot]_E : H^2_\Pi \longrightarrow H^3_\Pi \ .$$

This is a quadratic map whose zeros are the infinitesimal deformations which can be extended to second order in ε.

Exercise 75
Show that the squaring map is zero when Π is symplectic.

The exercise implies that, in the symplectic case, any infinitesimal deformation can be extended to second order. In fact, since symplectic structures are open in the vector space of closed 2-forms, there are no obstructions to extending an infinitesimal deformation: one may invert the Poisson structure, extend the resulting deformation of symplectic structure, and invert back.

Remark. If a formal power series $\Pi(\varepsilon)$ satisfies all the stepwise equations for $[\Pi(\varepsilon),\Pi(\varepsilon)]_E = 0$, there remains the question of whether there exists a *smooth* deformation corresponding to that power series. It is not known how or if this problem can be answered in terms of the E-Π-cohomology groups. ♢

Part VIII

Deformations of Algebras of Functions

19 Algebraic Deformation Theory

Let V be a vector space (or just a module over a ring). We will study product-type structures associated to V.

19.1 The Gerstenhaber Bracket

For $k = 0, 1, 2, \ldots$, consider the set of all k-multilinear maps on V:

$$M^k(V) = \{m : \underbrace{V \times \ldots \times V}_{k} \mid m \text{ is linear in each argument } \} \ .$$

Let $A^k(V) \subseteq M^k(V)$ be the subset of *alternating* k-multilinear maps on V.

Candidates for an associative product structure on V lie in $M^2(V)$.
Candidates for a Lie bracket structure on V lie in $A^2(V)$.

For $a \in M^k(V)$ and $b \in M^\ell(V)$, let

$$(a \circ_i b)(x_1, x_2, \ldots, x_{k+\ell-1}) := a(x_1, \ldots, x_{i-1}, b(x_i, \ldots, x_{i+\ell-1}), x_{i+\ell}, \ldots, x_{k+\ell-1})$$

where $x_1, x_2, \ldots, x_{k+\ell-1} \in V$. Then let

$$a \diamond b := N \cdot \sum_i (-1)^{(i-1)(\ell-1)} a \circ_i b$$

where N is a combinatorial factor not relevant to our study. The **Gerstenhaber bracket** $[\cdot, \cdot]_{\mathcal{G}}$ (see [66]) is defined to be

$$[a, b]_{\mathcal{G}} := a \diamond b - (-1)^{(k-1)(\ell-1)} b \diamond a \ .$$

Theorem 19.1 (Gerstenhaber [66]) *The bracket $[\cdot, \cdot]_{\mathcal{G}}$ satisfies the super-Jacobi identity if we declare elements of $M^k(V)$ to have degree $k - 1$.*

When $a, b \in M^2(V)$ are bilinear maps,

$$\begin{aligned}
(a \diamond b)(x, y, z) &= a(b(x, y), z) - a(x, b(y, z)) \\
[a, b]_{\mathcal{G}}(x, y, z) &= a(b(x, y), z) - a(x, b(y, z)) \\
&\quad + b(a(x, y), z) - b(x, a(y, z) \\
\tfrac{1}{2}[a, a]_{\mathcal{G}}(x, y, z) &= a(a(x, y), z) - a(x, a(y, z))
\end{aligned}$$

Writing $x \cdot y$ for $a(x, y)$, we obtain

$$\tfrac{1}{2}[a, a]_{\mathcal{G}}(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \ .$$

Therefore, associative algebra structures on V are the solutions of the quadratic equation

$$[a,a]_{\mathcal{G}} = 0\ , \quad a \in M^2(V)\ .$$

In terms of the squaring map (similar to the one mentioned in Section 18.6)

$$\begin{array}{rcccl} \mathrm{sq} & : & M^2(V) & \longrightarrow & M^3(V) \\ & & a & \longmapsto & \frac{1}{2}[a,a]_{\mathcal{G}} \end{array}$$

the associative algebra structures on V are the elements of $\ker(\mathrm{sq})$.

Given an associative multiplication $m \in M^2(V)$, $[m,m]_{\mathcal{G}} = 0$, we denote the multiplication by

$$x \cdot y := m(x,y)\ .$$

We may then define a **cup product** on $M^\bullet(V)$ by the formula

$$(a \cup b)(x_1, x_2, \ldots, x_{k+\ell}) = a(x_1, \ldots, x_k) \cdot b(x_{k+1}, \ldots, x_{k+\ell})$$

where $a \in M^k(V)$, $b \in M^\ell(V)$ and $x_1, \ldots, x_{k+\ell} \in V$.

The associativity of the cup product follows from the associativity of m. Notice that, while the Gerstenhaber bracket is defined on any vector space V, the cup product structure depends on the choice of a multiplication on V.

Remark. $A^\bullet(V)$ is not closed under $[\cdot,\cdot]_{\mathcal{G}}$. However, using anti-symmetrization, we find a similar bracket on $A^\bullet(V)$ for which the equation $[a,a]_{\mathcal{G}} = 0$ amounts to the Jacobi identity for $a \in A^2(V)$. In the case of *symmetric* multilinear maps on V, $S^\bullet(V) \subseteq M^\bullet(V)$, we may use symmetrization to obtain a bracket. ♢

19.2 Hochschild Cohomology

Suppose that m is an associative multiplication on V, *i.e.* $m \in M^2(V)$ and $[m,m]_{\mathcal{G}} = 0$. Define the map

$$\delta_m := [m,\cdot]_{\mathcal{G}} : M^\bullet(V) \to M^{\bullet+1}(V)\ .$$

By the super-Jacobi identity, we have

$$\delta_m^2 = 0\ .$$

We hence obtain a complex $(M^\bullet(V), \delta_m)$, called the **Hochschild complex** of (V,m).

The cohomology of $(M^\bullet(V), \delta_m)$ is known as **Hochschild cohomology**. The cohomology groups are denoted by $HH_m^\bullet$.

Remark. For the alternating version of the bracket $[\cdot,\cdot]_{\mathcal{G}}$, consider $\delta_a := [a,\cdot]_{\mathcal{G}} : A^\bullet(V) \to A^{\bullet+1}(V)$ where $a \in A^2(V)$, $[a,a]_{\mathcal{G}} = 0$. The corresponding complex $(A^\bullet(V), \delta_a)$ is the **Chevalley complex** of (V,a) and its cohomology is known as **Chevalley cohomology**, or **Lie algebra cohomology** or **Chevalley-Eilenberg cohomology** [69]. For the case of symmetric multilinear maps $S^\bullet(V)$, we obtain **Harrison cohomology** [69]. ♢

Repeating the computations and definitions of Sections 18.4 and 18.5, we find that:

$$HH^0_m = \text{center of the algebra } (V, m)$$

$$HH^1_m = \frac{\text{derivations of the algebra } (V,m)}{\text{inner derivations of the algebra } (V,m)}$$

$$HH^2_m = \frac{\text{infinitesimal deformations of } m}{\text{trivial infinitesimal deformations of } m}$$

Exercise 76
Check the assertions above.

The groups $HH^\bullet_m$ have the following algebraic structures:

1. The **Gerstenhaber bracket** $[\cdot,\cdot]_{\mathcal{G}}$ passes to $HH^\bullet_m$, since it commutes with δ_m. Notice that $[\cdot,\cdot]_{\mathcal{G}}$ is independent of the algebra structure on V, while $HH^\bullet_m$ is defined for a particular choice of $m \in M^2(V)$ with $[m,m]_{\mathcal{G}} = 0$.

2. In particular, the Gerstenhaber bracket on Hochschild cohomology induces a **squaring map**
$$\tfrac{1}{2}[\cdot,\cdot]_{\mathcal{G}} : HH^2_m \to HH^3_m \ .$$
This map describes the obstructions to extending infinitesimal deformations of m as we will see in Section 19.4.

3. The **cup product** operation on $M^\bullet(V)$, for a fixed associative multiplication m, satisfies a derivation law with respect to $[\cdot,\cdot]_{\mathcal{G}}$ which passes to $HH^\bullet_m$:
$$[\mathbf{a}, \mathbf{b} \cup \mathbf{c}]_{\mathcal{G}} = [\mathbf{a},\mathbf{b}]_{\mathcal{G}} \cup \mathbf{c} + (-1)^{(|\mathbf{a}|-1)|\mathbf{b}|}\mathbf{b} \cup [\mathbf{a},\mathbf{c}]_{\mathcal{G}}$$
where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are Hochschild cohomology classes.
Since, for $a, b, c \in M^\bullet(V)$, we have
$$a \diamond \delta_m b - \delta_m(a \diamond b) + (-1)^{|b|-1}\delta_m a \diamond b = (-1)^{|b|-1}(b \cup a - (-1)^{|a||b|} a \cup b) \ ,$$
on cohomology we have **supercommutativity**
$$\mathbf{a} \cup \mathbf{b} = (-1)^{|\mathbf{a}||\mathbf{b}|}\mathbf{b} \cup \mathbf{a} \ .$$

   Remark. Notice that the cup product is supercommutative only in cohomology, whereas the Gerstenhaber bracket $[\cdot,\cdot]_{\mathcal{G}}$ was supercommutative already before passing to cohomology. ♢

4. The action of the **permutation (or symmetric) groups** on the spaces $M^k(V)$ gives rise to a finer structure in Hochschild cohomology, analogous to the Hodge decomposition [69].

Remark. There is a groupoid related to HH^1 and HH^2. It is the transformation groupoid of the category whose objects are the associative multiplications on V, and whose morphisms are the triples (m_1, φ, m_2), where m_1, m_2 are objects and φ is a linear isomorphism with $m_1 = \varphi^* m_2$. ♢

19.3 Case of Functions on a Manifold

In the case where $V = C^\infty(M)$ for some manifold M, HH^0 is the center $C^\infty(M)$, while $HH^1 = \chi^1(M)$, since every derivation comes from a vector field, and the only inner derivation is 0. More generally, we have the following result, after an algebraic version by Hochschild, Kostant and Rosenberg [82].

Theorem 19.2 (Cahen-Gutt-De Wilde [21]) *The subcomplex of $M^\bullet(C^\infty(M))$ consisting of those multilinear maps which are differential operators in each argument, has cohomology*

$$HH^k_{\text{diff}}(C^\infty(M)) \simeq \chi^k(M) = \Gamma(\wedge^k TM) \ ,$$

and the Gerstenhaber bracket becomes the Schouten-Nijenhuis bracket.

The theorem is saying that:

1. Every k-cocycle is cohomologous to a skew-symmetric cocycle.
2. Every skew-symmetric cocycle is given by a k-vector field.
3. A k-vector field is a coboundary only if it is zero.

The inclusion

$$(\chi^\bullet(M), 0) \longrightarrow (M^\bullet_{\text{loc}}(C^\infty(M)), \delta)$$

is a linear isomorphism on the level of cohomology, but it is not a morphism for the Gerstenhaber bracket. Kontsevich has recently [97] proven his **formality conjecture**, which states that the inclusion can be deformed to a morphism of differential graded Lie algebras which still induces an isomorphism on cohomology. As a consequence of this theorem, Kontsevich establishes an equivalence between the classification of formal deformations of the standard associative multiplication on $C^\infty(M)$ and formal deformations of the zero Poisson structure on M. We discuss these issues from a "pre-Kontsevich" viewpoint in the remainder of these notes.

19.4 Deformations of Associative Products

The equation for a formal series in $M^2(V)$

$$m(\varepsilon) = m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots$$

to be associative, identically in ε, is

$$\begin{aligned} 0 &= [m(\varepsilon), m(\varepsilon)]_{\mathcal{G}} \\ &= [m_0, m_0]_{\mathcal{G}} + 2\varepsilon[m_0, m_1]_{\mathcal{G}} + \varepsilon^2(2[m_0, m_2]_{\mathcal{G}} + [m_1, m_1]_{\mathcal{G}}) + \dots \qquad (\star) \end{aligned}$$

cf. Section 18.5. We will try to solve this equation stepwise:

We first need the term m_0 to be associative, *i.e.* $[m_0, m_0]_{\mathcal{G}} = 0$. Next, for the coefficient of ε in $(\star)$ to vanish, we need

$$0 = [m_0, m_1]_{\mathcal{G}} = \delta_{m_0} m_1 \ .$$

Writing

$$x \cdot y := m_0(x, y) \ ,$$

$\delta_{m_0} m_1$ is:

$$\delta_{m_0} m_1(x,y,z) = x \cdot m_1(y,z) - m_1(x \cdot y, z) + m_1(x, y \cdot z) - m_1(x,y) \cdot z \ .$$

If m_1 were a biderivation (*i.e.* a derivation in each argument), this would become

$$\begin{array}{rcl} \delta_{m_0} m_1(x,y,z) & = & x \cdot m_1(y,z) - x \cdot m_1(y,z) - m_1(x,z) \cdot y \\ & & +y \cdot m_1(x,z) + m_1(x,y) \cdot z - m_1(x,y) \cdot z \\ & = & -m_1(x,z) \cdot y + y \cdot m_1(x,z) \ . \end{array}$$

If m_0 is symmetric (*i.e.* commutative), then every biderivation m_1 is a cocycle with respect to δ_{m_0}.

Suppose that m_1 is antisymmetric.[10] We then have

$$\begin{array}{rcl} \delta_{m_0} m_1(x,y,z) & = & x \cdot m_1(y,z) - m_1(x \cdot y, z) + m_1(x, y \cdot z) - m_1(x,y) \cdot z \\ \delta_{m_0} m_1(x,z,y) & = & x \cdot m_1(z,y) - m_1(x \cdot z, y) + m_1(x, z \cdot y) - m_1(x,z) \cdot y \\ \delta_{m_0} m_1(z,x,y) & = & z \cdot m_1(x,y) - m_1(z \cdot x, y) + m_1(z, x \cdot y) - m_1(z,x) \cdot y \end{array}$$

Writing

$$\{x,y\} := m_1(x,y) \ ,$$

and assuming that m_0 is symmetric, we obtain

$$\begin{array}{l} \frac{1}{2}[\delta_{m_0} m_1(x,y,z) - \delta_{m_0} m_1(x,z,y) + \delta_{m_0} m_1(z,x,y)] \\ \qquad = x \cdot \{y,z\} + \{x,z\} \cdot y - \{x \cdot y, z\} \ . \end{array}$$

The vanishing of this expression is the **Leibniz identity** for m_1 with respect to m_0.

Hence, assuming that m_0 is symmetric and m_1 is antisymmetric, if m_1 is a δ_{m_0}-cocycle, then m_1 is a biderivation.

Similarly, we find

$$\tfrac{1}{2}[m_1, m_1]_{\mathcal{G}}(x,y,z) = \{\{x,y\},z\} - \{x,\{y,z\}\} \ .$$

The equation for eliminating the ε^2 coefficient in $(\star)$ is

$$\delta_{m_0} m_2 + \tfrac{1}{2}[m_1, m_1]_{\mathcal{G}} = 0 \ , \ \textit{i.e.}$$

$$\{\{x,y\},z\} - \{x,\{y,z\}\} + x \cdot m_2(y,z) - m_2(x \cdot y, z) + m_2(x, y \cdot z) - m_2(x,y) \cdot z = 0 \ .$$

Assume that m_0 is symmetric, m_1 is antisymmetric and m_2 is symmetric:

$$\begin{array}{rcl} x \cdot y & = & y \cdot x \\ \{x,y\} & = & -\{y,x\} \\ m_2(x,y) & = & m_2(y,x) \end{array}$$

The equation for the vanishing of the coefficient of ε^2 of in $(\star)$ added to itself under cyclic permutations (x,y,z) yields:

$$\{\{x,y\},z\} + \{\{y,z\},x\} + \{\{z,x\},y\} = 0 \ ,$$

that is, the Jacobi identity for $\{\cdot,\cdot\}$.

We conclude that the extendibility of the deformation to second order, with the (anti)symmetry conditions imposed above, is equivalent to

$$\begin{array}{rl} & [m_1, m_1]_{\mathcal{G}} \text{ is a coboundary} \\ \Longleftrightarrow & \text{jacobiator for } m_1 \text{ is zero} \\ \Longleftrightarrow & \text{Jacobi identity for } m_1 \ . \end{array}$$

[10] For local cochains on $C^\infty(M)$, this can always be arranged by subtracting a coboundary from m_1.

19.5 Deformations of the Product of Functions

We now apply the observations of the previous section to the case where $V = C^\infty(M)$ is the space of smooth functions on a Poisson manifold (M, Π) (see also Section 19.3).

Let m_0 be pointwise multiplication of functions, and let m_1 be the Poisson bracket $\{\cdot,\cdot\}$.

Take a formal deformation of m_0 with linear term m_1. The formal variable ε is traditionally replaced by $\frac{i\hbar}{2}$, where the symbol $\hbar$ plays the role of Planck's constant from physics. We redefine $m_1 = \frac{i}{2}\{\cdot,\cdot\}$, and take $\varepsilon = \hbar$ instead. The formal deformation is then

$$m(\hbar) = m_0 + \hbar m_1 + \hbar^2 m_2 + \ldots$$

The equation for $m(\hbar)$ to be an associative product for each "value" of $\hbar$ is

$$[m(\hbar), m(\hbar)]_{\mathcal{G}} = 0 \ ,$$

cf. Sections 18.5 and 19.4.

For these particular m_0 and m_1, we have

$$\begin{array}{rcl} [m_0, m_0]_{\mathcal{G}} = 0 & \Longleftrightarrow & m_0 \text{ is associative} \\ [m_0, m_1]_{\mathcal{G}} = 0 & \Longleftrightarrow & m_1 \text{ satisfies the Leibniz identity} \\ \exists m_2 : \ 2[m_0, m_2]_{\mathcal{G}} + [m_1, m_1]_{\mathcal{G}} = 0 & \Longleftrightarrow & m_1 \text{ satisfies the Jacobi identity .} \end{array}$$

Hence, the coefficients of $\hbar^0$, $\hbar^1$ and $\hbar^2$ in $[m(\hbar), m(\hbar)]_{\mathcal{G}}$ vanish. To eliminate the coefficient of $\hbar^3$, we need

$$[m_0, m_3]_{\mathcal{G}} + [m_1, m_2]_{\mathcal{G}} = 0 \ .$$

This is equivalent to requiring the δ_{m_0}-cocycle $[m_1, m_2]_{\mathcal{G}}$ to be a δ_{m_0}-coboundary:

$$\delta_{m_0} m_3 = -[m_1, m_2]_{\mathcal{G}} \ .$$

The obstruction to solving the equation lies hence in $HH^3_{m_0}(C^\infty(M))$.

Exercise 77
Check that $\delta_{m_0}[m_1, m_2]_{\mathcal{G}} = 0$.

Historical Remarks. The program of quantizing a symplectic manifold M with a $*$-product, that is an associative multiplication on formal power series $C^\infty(M)[[\hbar]]$, was first set out by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in the 70's [12].

In 1983 [43], De Wilde and Lecomte showed that every symplectic manifold admits a formal deformation quantization. Their proof involved rather complicated calculations which made the result look quite technical.

Some later versions of the existence proof relied on patching together local Weyl algebras with nonlinear coordinate changes. In [92], Karasev and Maslov gave further details of a proof, whose first outline was sketched in [91], which reduces the patching to standard sheaf-theoretic ideas.

Another proof of the existence of deformation quantization which uses patching ideas was given by Omori, Maeda and Yoshioka [133]. Although their proof still involved substantial computations, it used a fundamental idea which is also basic in the proof of Fedosov (who discovered it independently). Each tangent space of a Poisson manifold M can be viewed as an affine space with a constant Poisson structure, so it carries a natural Moyal-Weyl quantization (see Section 20.1). In this way, the tangent bundle TM becomes a Poisson manifold with the fibrewise Poisson bracket, and with a fibrewise quantization. To quantize M itself, we may try to identify a subalgebra of the quantized algebra $C^\infty(TM)[[\hbar]]$ with the vector space $C^\infty(M)[[\hbar]]$ in such a way that the induced multiplication on $C^\infty(M)[[\hbar]]$ gives a deformation quantization of M. Such an identification is called a *Weyl structure* in [133].

In Chapter 21 we will discuss Fedosov's proof of existence of deformation quantization on symplectic manifolds.

For the history of these developments, see [14, 60]. $\diamondsuit$

20 Weyl Algebras

Let (E, Π) be a Poisson vector space. We will regard the Poisson structure $\Pi \in E \wedge E$ as a bivector field on E with constant coefficients.

20.1 The Moyal-Weyl Product

For local canonical coordinates $(q_1, \ldots, q_k, p_1, \ldots, p_k, c_1, \ldots, c_l)$ (defined in Sections 3.4 and 4.2), we use the symbols

$$\frac{\overleftarrow{\partial}}{\partial q_j} \quad \text{and} \quad \frac{\overleftarrow{\partial}}{\partial p_j}$$

for differential operators acting on functions to their left, and

$$\frac{\overrightarrow{\partial}}{\partial q_j} \quad \text{and} \quad \frac{\overrightarrow{\partial}}{\partial p_j}$$

for differential operators acting on functions to their right, so that

$$\{f, g\} = f \underbrace{\sum_j \left(\frac{\overleftarrow{\partial}}{\partial q_j} \frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j} \frac{\overrightarrow{\partial}}{\partial q_j} \right)}_{P} g \ .$$

Let m_1 be the following bidifferential operator on $C^\infty(E)$:

$$m_1 = \frac{i}{2} P = \frac{i}{2} \sum_j \left(\frac{\overleftarrow{\partial}}{\partial q_j} \frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j} \frac{\overrightarrow{\partial}}{\partial q_j} \right) \ .$$

The operator P is closely related to an operator on functions on the product space

$$\widehat{P} : C^\infty(E \times E) \longrightarrow C^\infty(E \times E)$$

defined in coordinates $(q', p', c', q'', p'', c'')$ on $E \times E$ as

$$\widehat{P} = \sum_j \left(\frac{\partial}{\partial q'_j} \frac{\partial}{\partial p''_j} - \frac{\partial}{\partial p'_j} \frac{\partial}{\partial q''_j} \right) \ .$$

Consider the maps

$$C^\infty(E) \otimes C^\infty(E) \longrightarrow C^\infty(E \times E)$$

$$f(q, p, c) \otimes g(q, p, c) \longrightarrow f(q', p', c')\, g(q'', p'', c'')$$

and

$$C^\infty(E \times E) \xrightarrow{\ \Delta\ } C^\infty(E)$$

$$f(q', p', c', q'', p'', c'') \longrightarrow f(q, p, c, q, p, c)$$

The bidifferential operator P is the composition

$$C^\infty(E) \otimes C^\infty(E) \longrightarrow C^\infty(E \times E) \xrightarrow{\widehat{P}} C^\infty(E \times E) \xrightarrow{\Delta} C^\infty(E) \ .$$

Powers P^k are defined by taking $\widehat{P}^k$ in this composition. Adding all the powers (with the usual factorial coefficients), we define the formal power series of operators

$$\begin{array}{ccc} C^\infty(E) \otimes C^\infty(E) & \longrightarrow & C^\infty(E) \\ & & \\ f \otimes g & \longmapsto & f \star_\hbar g \end{array}$$

by the formula

$$\begin{aligned} f \star_\hbar g \quad &:= \quad \sum_{j=0}^{\infty} \tfrac{1}{j!} (\tfrac{i\hbar}{2})^j f P^j g \\ &=: \quad f \cdot e^{\frac{i\hbar}{2} P} \cdot g \ . \end{aligned}$$

This is called the **Moyal-Weyl product** [122, 174] or simply the **Weyl product**.

Remark. This exponential series is analogous to the Taylor expansion

$$f(x_0 + \varepsilon) = (e^{\varepsilon \frac{d}{dx}} f)(x_0) \ ,$$

which converges for small ε only when f is real analytic. $\diamondsuit$

Similarly, the Moyal-Weyl product will not converge in general, so we must regard it as a formal power series in $\hbar$. The **formal Weyl algebra** is the algebra of formal series in $q, p, c, \hbar$ equipped with the Moyal-Weyl product defined as above. Note that, in the formal Weyl algebra:

- the polynomials in $q, p, c, \hbar$ form a subalgebra,
- the variables c and $\hbar$ commute with everything, and
- $q_j p_j - p_j q_j = \frac{i\hbar}{2}$,

whence the following relations:

$$\begin{array}{rclcrcl} [q_j, p_j] & = & \frac{i\hbar}{2} & \qquad\qquad & [c_i, \cdot] & = & 0 \\ [q_i, q_j] & = & [p_i, p_j] = 0 & & [\hbar, \cdot] & = & 0 \end{array}$$

where $[\cdot, \cdot]$ is the usual commutator bracket.

The affine functions on E

$$\mathfrak{h} := E^* \oplus \mathbb{R}^* = (E \oplus \mathbb{R})^*$$

form a Lie algebra. When Π is non-degenerate, $\mathfrak{h}$ is the **Heisenberg algebra**, with central element $\hbar$. The universal enveloping algebra $\mathcal{U}(\mathfrak{h})$ may be identified by symmetrization with the polynomial algebra $\mathrm{Pol}(E \oplus \mathbb{R})$.

20.2 The Moyal-Weyl Product as an Operator Product

Let (E, Π) be a *symplectic* Poisson vector space with canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. The Moyal-Weyl product on (E, Π) (defined in the previous section) can be interpreted as an operator product for operators on $\mathbb{R}^{2n}$. (This is in fact how it originated [174].)

The following map $\mathrm{Op}(\cdot)$ from the coordinate functions $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $\mathbb{R}^{2n}$ to operators on $\mathbb{R}^n$ equipped with coordinates $(x_1, \ldots, x_n)$:

$$\begin{array}{ccll} q_j & \longmapsto & \mathrm{Op}(q_j) = \widehat{q_j} & := \text{multiplication by } x_j \\ p_j & \longmapsto & \mathrm{Op}(p_j) = \widehat{p_j} & := \frac{\hbar}{i}\frac{\partial}{\partial x_j} \\ 1 & \longmapsto & \mathrm{Op}(1) & := \text{multiplication by } 1 \end{array}$$

satisfies

$$[\mathrm{Op}(q_j), \mathrm{Op}(p_j)] = i\hbar \mathrm{Op}(1) = i\hbar \mathrm{Op}(\{q_j, p_j\}) \ .$$

Remark. In the language of Dirac and Schrödinger, we are mapping the *classical observables* q and p to the corresponding *quantum operators* $\widehat{q}$ and $\widehat{p}$. The Poisson bracket of classical observables maps to the commutator of operators. ♢

In order to avoid ordering ambiguity, products of observables $q_j p_j = p_j q_j$ may be mapped to $\frac{1}{2}(\widehat{q_j}\widehat{p_j} + \widehat{p_j}\widehat{q_j})$. For arbitrary functions $f(q,p)$, a device of Weyl extends this symmetric ordering. Write $f(q,p)$ in terms of its Fourier transform as

$$f(q,p) = \int_{(\mathbb{R}^{2n})^*} e^{i\sum(q_j Q_j + p_j P_j)} \, (\mathcal{F}f)\,(Q,P)\, dQ\, dP$$

where Q and P are variables on $(\mathbb{R}^{2n})^*$ dual to q and p on $\mathbb{R}^{2n}$. Restricting to Schwartz functions on $\mathbb{R}^{2n}$, we may set

$$\mathrm{Op}(f) := \int_{(\mathbb{R}^{2n})^*} e^{i\sum(Q_j \mathrm{Op}(q_j) + P_j \mathrm{Op}(p_j))} \, (\mathcal{F}f)\,(Q,P)\, dQ\, dP$$

since the exponential factor is a unitary operator. The function f is called the **Weyl symbol** [174] of the operator $\mathrm{Op}(f)$.

We then define

$$f \star_\hbar g := \mathrm{Op}^{-1}(\mathrm{Op}(f), \mathrm{Op}(g)) \ .$$

Here f and g are Schwartz functions, and Op^{-1} is the map taking an operator to its Weyl symbol. For this new (noncommutative) product of functions, the map $f \mapsto \mathrm{Op}(f)$ is an algebra homomorphism.

Remark. An integral formula for $\star_\hbar$ in the symplectic case was found by von Neumann [128] (well before Moyal):

$$(f \star_\hbar g)(x) = \left(\frac{1}{\pi\hbar}\right)^{2n} \int f(y)\, g(z)\, e^{\frac{i}{\hbar}S(x,y,z)} dy\, dz \ ,$$

where $S(x,y,z)$ is 4 times the symplectic area of the triangle with vertices x, y and z.

The von Neumann integral formula gives a well-defined product on various spaces of functions, including Schwartz functions, smooth functions whose partial derivatives of all orders are bounded, and Λ-periodic smooth functions on E

where Λ is a lattice, *i.e.* smooth functions on a torus E/Λ. This product does not extend to continuous functions on E/Λ, but it is possible to complete $C^\infty(E/\Lambda)$ to a noncommutative C^* algebra called "the continuous functions on a quantum torus" [144]. $\diamondsuit$

20.3 Affine Invariance of the Weyl Product

The Weyl product on a Poisson vector space (E, Π) is invariant under affine Poisson maps, *i.e.* if $A : E \to E$ is an affine symplectic map, then the induced pull-back map

$$A^* : C^\infty(E)[[\hbar]] \longrightarrow C^\infty(E)[[\hbar]]$$

is an algebra automorphism for the Weyl product.

By affine invariance, the Weyl product (on the Weyl algebra or any of the other related spaces of functions mentioned at the end of the previous section) passes to any Poisson manifold locally modeled on E, as long as we only allow affine coordinate changes. This condition on E amounts to the existence of a flat connection without torsion, for which parallel transport preserves the Poisson structure.

The infinitesimal counterpart of affine invariance is that, for every polynomial function k on E of degree less than or equal to 2,

$$\{f \star_\hbar g, k\} = f \star_\hbar \{g, k\} + \{f, k\} \star_\hbar g \ .$$

In words, $\{\cdot, k\}$ is a derivation not just of the pointwise product (Leibniz identity) and of the Poisson bracket (Jacobi identity), but of the whole $\star$-product.

Remark. Dirac's [44] quantum Poisson bracket

$$[f, g]_\star := \frac{f \star_\hbar g - g \star_\hbar f}{i\hbar}$$

satisfies the derivation law

$$[f \star_\hbar g, k]_\star = f \star_\hbar [g, k]_\star + [f, k]_\star \star_\hbar g$$

just as a consequence of associativity. The similar property for $\{\cdot, k\}$ is explained by the fact that, for a polynomial k on E of degree ≤ 2, we have $[\cdot, k]_\star = \{\cdot, k\}$.

In particular, for k_1 and k_2 polynomials of degree ≤ 2, we have $[k_1, k_2]_\star = \{k_1, k_2\}$, which shows that polynomials of degree ≤ 2 form a Lie algebra. $\diamondsuit$

20.4 Derivations of Formal Weyl Algebras

Let $F(E)[[\hbar]]$ be the space of formal power series on the vector space E, thought as an algebra over $\mathbb{C}[[\hbar]]$.

A theorem of E. Borel states that every formal power series is the Taylor expansion of some function. This implies that the space $F(E)$ of formal power series on the vector space E is isomorphic to $C^\infty(E)$ modulo the functions which vanish to infinite order at 0.

Theorem 20.1 *Suppose that* Π *is a non-degenerate Poisson structure on* E*. Then every derivation* D *of* $F(E)[[\hbar]]$ *such that* $D\hbar = 0$ *is of the form* $[\cdot, f]_\star$ *for some* $f \in F(E)[[\hbar]]$.

Exercise 78
Prove this theorem. Hints:
A derivation D is determined by its effect on generators of the algebra $q_1, \ldots, q_n, p_1, \ldots, p_n$. Notice that q_i, p_i have degree ≤ 2. Suppose that $D = [\cdot, f]_\star$ were a inner derivation. Then

$$\begin{array}{rcccccc} Dq_i & = & [q_i, f]_\star & = & \{q_i, f\} & = & \frac{\partial f}{\partial p_i} \\ Dp_i & = & [p_i, f]_\star & = & \{p_i, f\} & = & -\frac{\partial f}{\partial q_i} \end{array}$$

To find the element f, we must solve

$$df = (Dq_i)dp_i - (Dp_i)dq_i$$

for f. If the right-hand side is closed, then the left-hand side will be determined up to an element in the center $\mathbb{C}[[\hbar]]$ of $C^\infty(E)[[\hbar]]$. Let us check that the right-hand side is closed:

$$\begin{array}{rcl} \frac{\partial}{\partial q_j}(Dq_i) + \frac{\partial}{\partial p_i}(Dp_j) & = & -\{p_j, Dq_i\} + \{q_i, Dp_j\} \\ & = & [Dq_i, p_j]_\star + [q_i, Dp_j]_\star \\ & = & D[q_i, p_j]_\star = D(\delta_{i,j}) = 0 \ . \end{array}$$

To finish the proof that $D = [\cdot, f]_\star$, consider the filtration of $F(E)$ by ideals A^k generated by the homogeneous polynomials of degree k. Show that, if D is a derivation, then $DA^k \subseteq A^{k-1}[[\hbar]]$.

Let (E, Π) be a Poisson vector space, and let φ be an automorphism of the Weyl algebra $C^\infty(E)[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-algebra..

The term in φ of 0-th order in $\hbar$ shows that φ induces an automorphism of $C^\infty(E)$, hence a diffeomorphism of E.

The term in φ of first order in $\hbar$ shows that this diffeomorphism is a Poisson automorphism of (E, Π).

We hence obtain an exact sequence

$$1 \longrightarrow \mathcal{I} \longrightarrow \mathrm{Aut}(C^\infty(E)[[\hbar]]) \longrightarrow \mathcal{P}(E, \Pi) \longrightarrow 1$$

where $\mathcal{P}(E, \Pi)$ is the set of Poisson automorphisms of (E, Π). The kernel $\mathcal{I}$ of the third arrow is the group of inner automorphisms of $C^\infty(E)[[\hbar]]$ corresponding to invertible elements of $C^\infty(E)[[\hbar]]$ [59].

20.5 Weyl Algebra Bundles

Let $(E, \rho, [\cdot, \cdot]_E)$ be a Lie algebroid over a manifold M, with symplectic structure $\omega \in \Gamma(\wedge^2 E^*)$. The symplectic E-2-form ω is non-degenerate and $d_E\omega = 0$; it determines an E-Poisson structure Π (see Section 18.3) by $\widetilde{\Pi} = \widetilde{\omega}^{-1}$, and an ordinary Poisson structure $\rho(\Pi)$ on TM.

Let WE be the **Weyl algebra bundle** over M whose fiber at $x \in M$ is the formal Weyl algebra of the symplectic (hence Poisson) vector space E_x. The smooth sections of WE are those for which the coefficient of each term is a smooth function on M; they form an algebra under fiberwise multiplication. We think of $\Gamma(WE)$ as "functions on the quantized E". Locally, we write a typical section as $f(x, y, \hbar)$, where $x \in M$, y is a formal variable in E_x, and $\hbar$ is another formal parameter. (The constant $\hbar$ is taken the same on each fiber, just as Planck's constant is a universal constant.)

From now on, to simplify, we will analyze the case where $E = TM$ is the tangent bundle of M. Everything works for the general Lie algebroid case [126].

Interpret $\Gamma(WTM)$ as the space of smooth functions on the "quantized tangent bundle"

$$\Gamma(WTM) = C^\infty({}^Q TM) \ .$$

The zero section is the map

$$C^\infty({}^Q TM) \longrightarrow C^\infty(M)[[\hbar]]$$

given by evaluation at $y = 0$. We may think of ${}^Q TM$ as an infinitesimal neighborhood of the zero section.

In the next chapter, we will describe the quantization method of Fedosov, in which $C^\infty(M)[[\hbar]]$ is identified with a subalgebra of $\Gamma(WTM)$. The Weyl product is then carried back to $C^\infty(M)[[\hbar]]$ to give a deformation quantization.

Geometrically, a subalgebra of $\Gamma(WTM)$ annihilated by a Lie algebra of derivations corresponds to a "foliation" of ${}^Q TM$. The foliation is transverse to the fibers when the derivations are of the form ∇_X as X ranges over the vector fields on M, defining a flat connection on the bundle WTM itself.

When the foliation is transverse to the zero section as well, parallel sections of WTM are in one-to-one correspondence with elements of $C^\infty(M)[[\hbar]]$. Notice that a flat linear connection on TM would not work: parallel sections of a flat connection on $C^\infty(TM)$ correspond to functions on a tangent fiber, not $C^\infty(M)[[\hbar]]$ as we need.

Example. Let (M, ω) be a symplectic vector space with coordinates x. Define the connection by

$$\nabla_{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial x_i} - \frac{\partial}{\partial y_i} \ ,$$

where y are the tangent coordinates induced by x. Lift functions $u(x)$ and $v(x)$ on M to $u(x+y)$ and $v(x+y)$ on TM. To evaluate $(u \star_\hbar v)(x_0)$, freeze the x variable at x_0, take the Weyl product with respect to y, and then set $y = 0$ to obtain a function on M. This recipe reproduces the usual Weyl product. $\diamondsuit$

21 Deformation Quantization

On a general Poisson manifold, if the rank of the Poisson tensor Π is constant, then by a theorem of Lie the Poisson manifold is locally isomorphic to a vector space with constant Poisson structure (see Section 3.4). Such Poisson manifolds, which are called regular, are always *locally* deformation quantizable using the Moyal-Weyl product in canonical coordinates; the problem is to patch together the local deformations to produce a global $\star$-product.

21.1 Fedosov's Connection

There is one case in which the patching together of local quantizations is easy. Since the Moyal-Weyl product on a vector space V with constant Poisson structure is invariant under all the affine automorphisms of V, we can construct a global quantization of any Poisson manifold (M, Π) covered by canonical coordinate systems in the general case for which the transition maps are affine. Such a covering exists when M admits a flat torsionless linear connection for which the covariant derivative of Π is zero.

Fedosov overcomes the difficulty of patching together local Weyl structures by making the canonical coordinate neighborhoods "infinitely small". To understand his idea, we should first think of elements of the deformed algebra $C^\infty(TM)[[\hbar]]$ as sections of the bundle WTM over M whose fiber at $x \in M$ is WT_xM.

Of course we are most interested in dealing with the case where (M, Π) does not admit a flat Poisson connection, and this is where the most interesting part of Fedosov's proof comes in. He shows (in other terms) that the tangent bundle of every symplectic (or regular Poisson) manifold *does* admit a flat Poisson connection, if one gives the appropriate extended meaning to that concept, namely admitting "nonlinear quantum maps" as the structure group.

Fedosov's connection is constructed on the bundle WTM of Weyl algebras. The "structure Lie algebra" of this connection, in which the connection forms take values, is $W\mathbb{R}^{2n}$ acting on itself by the adjoint representation of its Lie algebra structure. Since the full Weyl algebra is used, and not just the quadratic functions which generate linear symplectic transformations, the structure group allows non-linear transformations of the (quantized) tangent spaces. Since linear generating functions are included, the structure group even allows translations.

In fact (this idea was also used in [133]), it is not the full Weyl algebra of $\mathbb{R}^{2n}$ which serves as the typical fiber, but only the formal Weyl algebra $F(W\mathbb{R}^{2n})$, consisting of formal Taylor expansions at the origin. Geometrically, one can think of this step as the replacement of the (quantized) tangent bundle by a formal neighborhood of the zero section ${}^{Q}TM$.

Remark. Since ${}^{Q}TM$ is an infinitesimal neighborhood of the zero section, parallel transport does not go anywhere. This step may hence appear to be inconsistent with the inclusion of translations in the structure group, since these do not leave the origin fixed. In fact, the effect is to force us to forget the group and to work only with the structure Lie algebra. A beneficial, and somewhat surprising, result of this effect is that a parallel section with respect to a flat connection is not determined by its value at a single point. This situation is very close to that in formal differential geometry, where the bundle of infinite jets of functions on a manifold M has a flat connection whose sections are the lifts of functions on M. (See [160, Section 1] for a nice exposition with references.) $\diamondsuit$

Fedosov uses an iterative method for "flattening" a connection which is similar to that used in many differential geometric problems. (See [119] for an example, and [147] for a recent survey.) Over the domain of a local trivialization of a principal G-bundle, a connection is given by a 1-form ϕ with values in the Lie algebra $\mathfrak{g}$; the curvature of the connection is the Lie algebra valued 2-form

$$\Omega_\phi = d\phi + \frac{1}{2}[\phi, \phi] \ .$$

If the curvature is not zero, we may try to "improve" the connection by adding another Lie algebra valued 1-form α. The curvature zero condition for $\phi + \alpha$ is the quadratic equation

$$d\alpha + [\phi, \alpha] = -\Omega_\phi - \frac{1}{2}[\alpha, \alpha] \ .$$

Rather than trying to solve this equation exactly, we linearize it by dropping the term $-\frac{1}{2}[\alpha, \alpha]$. The operator $d + [\phi, \cdot]$ is the covariant exterior derivative D_ϕ, so our linearized equation has the form

$$D_\phi \alpha = -\Omega_\phi \ .$$

From the Bianchi identity, $D_\phi \Omega_\phi = 0$, it appears that the obstruction to solving the equation above for α lies in a cohomology space. This is not quite correct, since $D_\phi^2 = [\Omega_\phi, \cdot]$, which is not zero because the connection ϕ is not yet flat.

Up to now, we have essentially been following Newton's method for solving nonlinear equations. At this point, we add an idea similar to one often attributed to Nash and Moser. (See [155, Section III.6] for an exposition of this method with original references.) Since the linear differential equation we are trying to solve is only an approximation to the nonlinear one which we really want to solve, we do not have to solve it precisely. It is enough to solve it approximately and to compensate for the error in the later iterations which will in any case be necessary to take care of the neglected quadratic term $-\frac{1}{2}[\alpha, \alpha]$. Such approximate solutions are constructed by some version of the Hodge decomposition. In the differential geometric applications mentioned above, the full story involves elliptic differential operators, Sobolev spaces, and so on, but in the case at hand, it turns out that the "Hodge theory" is purely algebraic and quite trivial.

21.2 Preparing the Connection

We now start the construction of a flat connection on the bundle of Weyl algebras by an iteration procedure. All the constructions are intrinsic, but for simplicity we will describe them in local canonical coordinates.

Step 1 We begin with an arbitrary (linear) Poisson connection on the tangent bundle of the symplectic manifold M.

The connection induces a covariant differentiation operator on the dual bundle, *i.e.* on the linear functions on fibers. In coordinates $(x_1, \ldots, x_m)$ on M:

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ijk} \omega_{k\ell} \frac{\partial}{\partial x_\ell} \ .$$

We introduce the coefficients $(\omega_{k\ell})$ of the symplectic form to lower the last index. For convenience, we assume that $\omega_{k\ell}$ is constant (*i.e.* the x_i's are Darboux coordinates).

If the connection has torsion, we can make it torsion-free by symmetrization [59]

$$\Gamma_{ijk} \rightsquigarrow \frac{\Gamma_{ijk} + \Gamma_{jik}}{2} \ .$$

Because this is a symplectic connection, symmetry in the last two indices comes for free: $\Gamma_{ijk} = \Gamma_{ikj}$.

Step 2 The connection form is a 1-form with values in the Lie algebra of the symplectic group $\mathfrak{sp}(m)$. The elements of $\mathfrak{sp}(m)$ may be identified with linear hamiltonian vector fields on the manifold and hence with quadratic functions. Thus the connection form can be written as

$$\phi_{-1} = \frac{1}{2} \sum \Gamma_{ijk} y_i y_j \otimes dx_k \ ,$$

where $(y_1, \ldots, y_m)$ is a basis of linear functions on the fibers corresponding to the coordinates $(x_1, \ldots, x_m)$ on M.

Step 3 The symplectic connection ∇ lifts to the Weyl algebra bundle. A covariant differentiation D on the Weyl algebra bundle is described with respect to a local trivialization by

$$Du = du + \psi u$$

for a local section u, where ψ is a 1-form with values in $\mathrm{Der}(WTM)$. We can rewrite this local expression in the form

$$Du = du + [\phi, u]_\star \ ,$$

where now ϕ is a 1-form with values in WTM itself, and $[\cdot,\cdot]_\star$ is $(1/i\hbar)$ times the commutator bracket; the bracket $[\cdot,\cdot]_\star$ is the quantum Poisson bracket of Dirac [44]defined in Section 20.3. The generator ϕ of this "inner derivation" is determined up to a 1-form on M with values in the center $\mathbb{C}[[\hbar]]$ of the Weyl algebra.

Step 4 If we consider the form ϕ_{-1} (with the y_i's now interpreted as formal variables) as taking values in the bundle $FW(TM)$,

$$\phi_{-1} \in \Gamma(T^*M \otimes FW(TM))$$

becomes the connection form for the associated connection on that bundle. Even if this connection were flat, it would not be the correct one to use for quantization, since its parallel sections would not be identifiable in any reasonable way with functions on M. Instead we must use for our first approximation

$$\phi_0 = (\sum \omega_{kj} y_j + \tfrac{1}{2} \sum \Gamma_{ijk} y_i y_j) \otimes dx_k \ .$$

Step 5 To start the recursion, one calculates, using the fact that the connection is symplectic and torsion free (see [57]), that its curvature is

$$\begin{aligned}\Omega_0 &= d\phi_0 + \tfrac{1}{2}[\phi_0, \phi_0] \\ &= -\tfrac{1}{2}\sum \omega_{ir} \otimes dx_i \wedge dx_r + \tfrac{1}{4}\sum R_{ijkl} y_i y_j \otimes dx_k \wedge dx_l \\ &= -1 \otimes \omega + R\ ,\end{aligned}$$

where R is the curvature of the original linear symplectic connection, considered as a 2-form with values in the Lie algebra of quadratic functions. The term linear in y vanishes because the torsion is zero. The term $-1\otimes\omega$ appears even when the linear connection is flat, but it causes no trouble because it is a central element of the Weyl Lie algebra and therefore acts trivially in the adjoint representation.

21.3 A Derivation and Filtration of the Weyl Algebra

The coefficients of the connection forms which we are using are sections of the bundle $FW(TM)$. Rather than measuring the size of these forms by the usual Sobolev norms involving derivatives, we shall use a pointwise algebraic measurement.

In the formal Weyl algebra $FW(V)$ of a Poisson vector space V, we assign degree 2 to the variable $\hbar$ and degree 1 to each linear function on V. We denote by $FW_r(V)$ the ideal generated by the monomials of degree r. Because the kth term in the expansion of the $\star$-product involves $2k$ derivatives and multiplication by $\hbar^k$, we obtain a filtration of the algebra $FW(V)$. We will also occasionally use the classical grading, compatible with the commutative multiplication but not with the $\star$-product, which assigns degree 0 to $\hbar$ and 1 to each linear function on V.

The Lie algebra structure which we use for the formal Weyl algebra is the quantum Poisson bracket of Dirac [44] defined in Section 20.3. The factor $(1/i\hbar)$ makes the quantum bracket reduce to the classical one (rather than to zero) when $\hbar \to 0$. In addition, the quantum and classical brackets are equal when one of the entries contains only terms linear or quadratic in the variable on V, and they share the property

$$[FW_r(V), FW_s(V)] \subseteq FW_{r+s-2}(V)\ ,$$

so that the adjoint action of any element of $FW_2(V)$ preserves the filtration.

We introduce the algebra

$$\mathcal{W}(V) = FW(V) \otimes \wedge^*(V)$$

whose elements may be regarded as differential forms on the "quantum space whose algebra of functions is $FW(V)$". The algebra $\mathcal{W}(V)$ inherits a filtration by subspaces $\mathcal{W}_r(V)$ from the formal Weyl algebra, and a grading from the exterior algebra. We can also consider $\mathcal{W}(V)$ as the algebra of infinite jets at the origin of differential forms on the classical space V, in which case we generally use the classical grading. In this way, $\mathcal{W}(V)$ inherits the exterior derivative operator, which we denote by δ. Remarkably, δ is also a derivation for the quantized algebra structure on $\mathcal{W}(V)$.

We may describe the operator δ in terms of linear coordinates $(x_1, \ldots, x_m)$ on V. With an eye toward the case where V is a tangent space, we denote the corresponding formal generators of $FW(V)$ by $(y_1, ..., y_m, \hbar)$ and the generators of

$\wedge^*(V)$ by $(dx_1, \ldots, dx_m)$. Then $\mathcal{W}(V)$ is formally generated by the elements $y_i \otimes 1$, $\hbar \otimes 1$, and $1 \otimes dx_i$, and we have

$$\delta(y_i \otimes 1) = 1 \otimes dx_i \ , \ \delta(\hbar \otimes 1) = 0 \ , \ \text{ and } \ \delta(1 \otimes dx_i) = 0 \ .$$

Notice that δ decreases the Weyl algebra filtration degree by 1 while it increases the exterior algebra grading by 1.

Since δ is essentially the de Rham operator on a contractible space, we expect the cohomology of the complex which it defines to be trivial. Fedosov makes this explicit by introducing the dual operator δ^* of contraction with the Euler vector field $\sum_i y_i \otimes \frac{\partial}{\partial x_i}$. More precisely, δ^* maps the monomial $y_{i_1} \cdots y_{i_p} \otimes dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ to

$$\sum_k (-1)^{k-1} y_{i_1} \cdots y_{i_p} y_{j_k} \otimes dx_{j_1} \wedge \cdots \wedge \widehat{dx_{j_k}} \wedge \cdots \wedge dx_{j_q} \ .$$

(This operator is *not* a derivation for the quantized algebra structure.) A simple computation (or the Cartan formula for the Lie derivative by the Euler vector field) shows that, on the monomial above, we have

$$\delta\delta^* + \delta^*\delta = (p+q)\mathrm{id} \ ,$$

so that if we define the operator δ^{-1} to be $\frac{1}{p+q}\delta^*$ on the monomial above, and 0 on $1 \otimes 1$, we find that each element u of $\mathcal{W}(V)$ has the decomposition

$$u = \delta\delta^{-1}u + \delta^{-1}\delta u + \mathcal{H}u \ ,$$

where the "harmonic" part $\mathcal{H}u$ of u is the part involving only powers of $\hbar$ and no y_i's or dx_i's, *i.e.* the pullback of u by the constant map from V to the origin. In other words, we have reproduced the usual proof of the Poincaré lemma via a homotopy operator δ^{-1} from $\mathcal{H}$ to the identity.

When the Poisson vector space V is symplectic, the operator δ has another description. For any $a \in FW(V)$, $[y_i, a] = \{y_i, a\} = \sum_j \pi_{ij}(\partial a/\partial y_j)$. If (ω_{ij}) is the matrix of the symplectic structure, inverse to (π_{ij}), we get $\partial a/\partial y_i = [\sum_j \omega_{ij} y_j, a]$, and hence

$$\delta(a \otimes 1) = \sum_i (\partial a/\partial y_i) \otimes dx_i = [\sum_{ij} \omega_{ij} y_j \otimes dx_i, a \otimes 1] \ .$$

It follows from the derivation property that a similar equation holds for any element of $\mathcal{W}(V)$; *i.e.* the operator δ is equal to the adjoint action of the element $\sum_{ij} \omega_{ij} y_j \otimes dx_i$ (which is just the symplectic structure itself).

Of course, all the considerations above apply when V is replaced by a symplectic vector bundle E and $\mathcal{W}(V)$ by the space of sections of the associated bundle

$$\mathcal{W}(E) = FW(E) \otimes \wedge^*(E) \ .$$

In particular, when E is the tangent bundle of a symplectic manifold M, the operator δ and its relatives act on the algebra of differential forms on M with values in $FW(TM)$. These operators are purely algebraic with respect to the variable in M, with δ being just the adjoint action of the symplectic structure viewed as an $FW(TM)$-valued 1-form.

21.4 Flattening the Connection

Following Section 21.2, we next try to construct a convergent (with respect to the filtration) sequence ϕ_n of connections whose curvatures Ω_n tend to the central element $-1 \otimes \omega$. Fedosov calls this central element the **Weyl curvature** of the limit connection; to simplify notation, we will write $\widehat{\Omega} = \Omega + 1 \otimes \omega$ for the form which should be zero, and we call this the **effective curvature**.

Step 6 As suggested in Section 21.1, we let

$$\phi_{n+1} = \phi_n + \alpha_{n+1} \ ,$$

where α_{n+1} is a section of $\mathcal{W}(TM)$.

The corresponding curvature is

$$\begin{aligned} \Omega_{n+1} &= d\phi_{n+1} + \tfrac{1}{2}[\phi_{n+1}, \phi_{n+1}] \\ &= \Omega_n + \underbrace{d\alpha_{n+1} + [\phi_n, \alpha_{n+1}]}_{D_n \alpha_{n+1}} + \tfrac{1}{2}[\alpha_{n+1}, \alpha_{n+1}] \end{aligned}$$

where $D_n = D_{\phi_n} = d + [\phi_n, \cdot]$. Instead of solving

$$D_n \alpha_{n+1} = -\Omega_n - \tfrac{1}{2}[\alpha_{n+1}, \alpha_{n+1}] - 1 \otimes \omega \ ,$$

we drop the quadratic term and look at the simpler equation

$$D_n \alpha_{n+1} = -\Omega_n - 1 \otimes \omega \ .$$

This would solve approximately the linearized equation for zero effective curvature

$$D_n \alpha_{n+1} + \widehat{\Omega}_n = 0 \ .$$

Step 7 The operator $D_n = D_{\phi_n}$ will have the form $d + \delta + [c_n, \cdot]$, where c_n is an $FW(TM)$-valued 1-form. We will try to arrange for c_n to lie in $F\mathcal{W}_2(TM)$ so that the operator $[c_n, \cdot]$, like d, is filtration preserving. Since δ *lowers* the filtration degree by 1, the principal part of the differential operator D_n will actually be the algebraic operator δ (and not d as it would be if we measured forms by the size of their derivatives.)

We cannot even solve

$$\delta \alpha_{n+1} + \widehat{\Omega}_n = 0$$

exactly, because the Bianchi identity gives $D_n \widehat{\Omega}_n = 0$ instead of $\delta \widehat{\Omega}_n = 0$. (The term $1 \otimes \omega$ is killed by both operators.) Nevertheless, we define

$$\alpha_{n+1} = -\delta^{-1}(\widehat{\Omega}_n) \ ,$$

and take care of the errors later.

Step 8 From the recursion relation

$$\Omega_{n+1} = \Omega_n + D_n \alpha_{n+1} + \tfrac{1}{2}[\alpha_{n+1}, \alpha_{n+1}] \ ,$$

we find after a straightforward calculation using the decompositions

$$D_n = d + \delta + [c_n, \cdot] \quad \text{and} \quad u = \delta\delta^{-1}u + \delta^{-1}\delta u + \mathcal{H}u$$

that

$$\begin{aligned}\widehat{\Omega}_{n+1} &= \delta^{-1}\delta\widehat{\Omega}_n \\ &\quad +\mathcal{H}\widehat{\Omega}_n + d\alpha_{n+1} + [c_n, \alpha_{n+1}] + \tfrac{1}{2}[\alpha_{n+1}, \alpha_{n+1}] \;.\end{aligned}$$

Using $D_n = d + \delta + [c_n, \cdot]$ again, we can rewrite this as

$$\begin{aligned}\widehat{\Omega}_{n+1} &= \delta^{-1}D_n\widehat{\Omega}_n - \delta^{-1}d\widehat{\Omega}_n - \delta^{-1}[c_n, \widehat{\Omega}_n] \\ &\quad +\mathcal{H}\widehat{\Omega}_n + d\alpha_{n+1} + [c_n, \alpha_{n+1}] + \tfrac{1}{2}[\alpha_{n+1}, \alpha_{n+1}] \;.\end{aligned}$$

By the Bianchi identity $D_n\Omega_n = 0$, we get

$$\widehat{\Omega}_{n+1} = \mathcal{H}\widehat{\Omega}_n - \delta^{-1}d\widehat{\Omega}_n - \delta^{-1}[c_n, \widehat{\Omega}_n] + d\alpha_{n+1} + [c_n, \alpha_{n+1}] + \tfrac{1}{2}[\alpha_{n+1}, \alpha_{n+1}] \;.$$

Suppose now that $\widehat{\Omega}_n \in \mathcal{W}_r(TM)$ with $r \geq 1$. Then $\mathcal{H}\widehat{\Omega}_n = 0$ and $\alpha_{n+1} \in \mathcal{W}_{r+1}(TM)$, so that $c_n \in \mathcal{W}_2(TM)$ and hence all the terms on the right hand side of the equation above belong to $\mathcal{W}_{r+1}(TM)$.

Step 9 Since $\widehat{\Omega}_0 = R$ has filtration-degree 2, we conclude that $\widehat{\Omega}_n$ has degree at least $n+2$, and α_{n+1} has degree at least $n+3$, so the sequence ϕ_n converges to a connection form

$$\phi = \phi_0 + \alpha_1 + \alpha_2 + \ldots$$

for which the curvature is $\Omega = -1 \otimes \omega$. This curvature is a central section, so the connection on $FW(TM)$ associated to ϕ by the adjoint representation $FW(TM)$ is flat. Since the adjoint action is by derivations of the multiplicative structure, the space of parallel sections is a subalgebra of the space of all sections.

Step 10 The last step in Fedosov's construction is to show by a recursive construction, similar to the one above, that each element of $C^\infty(M)[[\hbar]]$ is the harmonic part of a unique parallel section of $FW(TM)$, so that $C^\infty(M)[[\hbar]]$ is identified with the space of parallel sections and thus inherits from it an algebra structure, which is easily shown to be a deformation quantization associated with the symplectic structure ω.

21.5 Classification of Deformation Quantizations

Fedosov [59] showed that his iterative construction of a connection on $FW(TM)$ can be modified so that the curvature becomes $\sum \hbar^j \otimes \omega_j$, for any sequence of closed 2-forms ω_j such that ω_0 is the original symplectic structure ω. He also showed that the isomorphism class of the resulting $\star$-product depends precisely on the sequence of de Rham cohomology classes $[\omega_j] \in H^2(M, \mathbb{R})$ and in particular is independent of the initial choice of connection ϕ_0.

In summary, the relevant data for an equivalence class of deformation quantizations on a manifold M is

$$\omega \;,\; [\omega_1] \;,\; [\omega_2] \;,\; \ldots$$

A representative of such an equivalence class is called a **Fedosov quantization** of M.

This left open the question of whether every $\star$-product is isomorphic to one obtained by Fedosov's construction. A positive answer to this question has been given by Nest and Tsygan. Using a noncommutative version of Gel'fand-Fuks cohomology, they construct in [124] for each deformation quantization a characteristic class in $H^2(M,\mathbb{R})[[\hbar]]$ with constant term ω. In [125], they show that this class determines the $\star$-product up to isomorphism and that it agrees with Fedosov's curvature for the $\star$-products constructed by his method. By Moser's classification [121] of nearby symplectic structures by their cohomology classes, the isomorphism classes of $\star$-products on a symplectic manifold are thus in one-to-one correspondence with isomorphism classes of formal deformations of the symplectic structure. Other references concerning this classification are Bertelson-Cahen-Gutt [15] Kontsevich [97], and Weinstein-Xu [173].

One consequence of this classification is that there is (up to isomorphism) a unique deformation quantization whose characteristic class is independent of $\hbar$. Although one might think that this special quantization is somehow the natural one, there is considerable evidence that the others are important as well. For instance, [54] suggests that $\star$-products with nonconstant characteristic classes may be related to geometric phases and deformations of symplectic forms which arise in the analysis of coupled wave equations [107].

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Index