New Midpoint-based Approach for the Solution of n-th Order Interval Differential Equations*

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Abstract

This paper proposes a new technique to solve n-th order linear uncertain but bounded (interval) differential equations with interval initial conditions using the interval midpoint. First, the interval differential equation is solved in terms of the interval midpoint. This solution is then used to find the solution of the original interval differential equation. The method is discussed by considering various cases for the coefficients in the differential equation with examples. We have compared the results obtained by the proposed method with exact and homotopy perturbation method (HPM) to demonstrate the validity and applicability of the method.

Keywords: Interval midpoint, interval arithmetic, n-th order linear interval differential equations, generalized Hukuhara differentiability
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1 Introduction

Interval differential equations play an important role for uncertainty modelling of physical and engineering problems. Intervals represent a natural way to model the

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systems under uncertainty when the number of variables and parameters is small. For example, we might have only vague and incomplete information about the variables and parameters being a result of errors in measurement, observations, experiments, applying different operating conditions, or maintenance-induced errors, etc. To handle these uncertainties and vagueness, one may use interval or fuzzy parameters and variables in the governing differential equations. Some papers related to interval and fuzzy differential equations are given in [1, 6, 8, 9, 18, 20-32].

In this regard, Moore [15, 16] first introduced the concept of interval analysis and computations. This concept has been applied successfully by different authors for uncertainty analysis. Several books also have been written by different authors representing the scope and various aspects of interval analysis such as in [2, 10, 14-17]. These books also give an extensive review of interval computation and interval differential equations which may help the reader understand the basic concepts of interval analysis. In view of this, Lohner [11, 12] developed a comprehensive software package implementing an advanced interval Taylor series method. Lohner [13], Corliss and Rihm [7] and Nedialkov [19] have proposed modified versions of Lohner's algorithm. Detailed surveys of interval Taylor series methods for Ordinary Differential Equations (ODEs) are given in Corliss [8] and Rihm [22]. Abdelhay et al. [1] implemented a modified exponential interval technique for the solution of singularly perturbed initial value problems. A new interval method based on Taylor series expansion for the solution of linear n-th order ordinary differential equations was developed by Neher [20, 21]. An interval Hermite-Obreschkoff method has been applied by Nedialko et al. [18] for the solution of interval solution of an initial value problem. Hoffmann and Marciniak [9] solved a Poison equation using an interval difference method. Differential calculus is studied by Chalco-Cano et al. [6] for interval-valued functions by using generalized Hukuhara differentiability, which is the most general concept of differentiability for interval-valued functions. Also, the Hukuhara concept was applied by Stefanini and Bede [23, 24] in a more generalised way for interval-valued functions and interval differential equations. Stefanini and Bede [23] presented the local existence and uniqueness of two solutions with characterizations of the solutions of an interval differential equation. In Stefanini and Bede [24] the authors corrected the imprecision presentation of Stefanini and Bede [23]. As the interval differential equation may be considered as the special case of fuzzy differential equation, a few recent papers related to fuzzy differential equations are also cited here. Buckley and Feuring [5] applied two analytical methods for solving n-th order linear differential equations with fuzzy initial conditions. In the first method, they simply fuzzify the crisp solution to obtain a fuzzy function and then check whether it satisfies the differential equation. The second method was just the reverse of the first method.

The analytical method is developed by Bede [3] using the Hukuhara derivative to obtain the solution of fuzzy differential equations. Behera and Chakraverty [4] obtained the solution of uncertain impulse responses of imprecisely defined half order mechanical system. Tapaswini and Chakraverty [25, 27] developed a new technique based on Euler and improved Euler methods for the solution of fuzzy initial value problems. Also, a homotopy perturbation method is used by Tapaswini and Chakraverty [26, 28, 30] to obtain the numerical solution of fuzzy differential equations. A new double parametric form of fuzzy number is developed by Tapaswini and Chakraverty [29] and then applied to a homotopy perturbation method to get the numerical solution of uncertain beam equations. Ivaz et al. [32] investigated the numerical algorithms for the solution of fuzzy differential equations and hybrid fuzzy differential equations. Nieto et al. [31] investigated some interesting properties of the diameter and the midpoint of the solution and compared the solution with the crisp case.

For interval evaluation of derivatives, automatic differentiation has been used by Jaulin et al. [10]. This differentiation has two types viz. forward and backward (or reverse). Here Jaulin et al. [10] applied a midpoint method with automatic differentiation to obtain the solution of interval differential equations. This is a good procedure to obtain the derivative, but one may not be sure about the order in which the multiplication of the matrices is to be done, i.e., from left to right or from right to left. The computational complexity depends on whether one may go from left to right or right to left [10]. Keeping the above computational complexity of the method in mind, the present authors have developed a new analytical approach using interval midpoint with Hukuhara derivative to solve n-th order linear interval differential equations.

This paper is organized as follows. In Section 2, we give some basic preliminaries, and the proposed technique is discussed in Section 3. Also in Section 3, the basic idea of HPM is explained. In Section 4, numerical examples are solved. Finally, in the last section, conclusions are drawn.

2 Preliminaries

In this section, we present some notations, definitions and preliminaries which are used in this paper [2, 10].

Definition 2.1 (Interval midpoint)
The midpoint of an arbitrary interval $u = [\underline{u}, \overline{u}]$ is defined as $u^c = \frac{\underline{u}+\overline{u}}{2}$.

Definition 2.2 Interval arithmetic
For two intervals $\tilde{x} = [\underline{x}, \overline{x}]$ and $\tilde{y} = [\underline{y}, \overline{y}]$, and scalar k, interval arithmetic operations are defined as follows:

i. $\tilde{x} + \tilde{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$.

ii. $\tilde{x} \times \tilde{y} = \left[\min\left\{\underline{x} \times \underline{y}, \underline{x} \times \overline{y}, \overline{x} \times \underline{y}, \overline{x} \times \overline{y}\right\}, \max\left\{\underline{x} \times \underline{y}, \underline{x} \times \overline{y}, \overline{x} \times \underline{y}, \overline{x} \times \overline{y}\right\}\right]$.

iii. $kx = \begin{cases} [k\overline{x}, k\underline{x}], k < 0, \\ [k\underline{x}, k\overline{x}], k \geq 0. \end{cases}$

iv. $\tilde{x}/\tilde{y} = [\underline{x}, \overline{x}] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right]$, where $0 \notin \tilde{y}$.

Let I be the set of (closed bounded) intervals of the real line and let Θ be the usual Hukuhara difference.

Definition 2.3 [23] Let $f :]a, b] \to I$ and $t_0 \in]a, b]$. We say that f is strongly gH-differentiable (generalized Hukuhara differentiable) at t_0 if there exists an element $f'(t_0) \in I$, such that, for all $h > 0$ sufficiently small,

i. $\exists f(t_0 + h)\Theta f(t_0), f(t_0)\Theta f(t_0 - h)$ and

$$\lim_{h\to 0^+} \frac{f(t_0 + h)\Theta f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)\Theta f(t_0 - h)}{h} = f'(t_0),$$

or

ii. $\exists f(t_0)\Theta f(t_0 + h), f(t_0 - h)\Theta f(t_0)$ and

$$\lim_{h\to 0^+} \frac{f(t_0)\Theta f(t_0 + h)}{-h} = \lim_{h\to 0^+} \frac{f(t_0 - h)\Theta f(t_0)}{-h} = f'(t_0),$$

or

iii. $\exists f(t_0+h)\Theta f(t_0), f(t_0-h)\Theta f(t_0)$ and

$$\lim_{h\to 0^+}\frac{f(t_0+h)\Theta f(t_0)}{h}=\lim_{h\to 0^+}\frac{f(t_0-h)\Theta f(t_0)}{-h}=f'(t_0),$$

or

iv. $\exists f(t_0)\Theta f(t_0+h), f(t_0)\Theta f(t_0-h)$ and

$$\lim_{h\to 0^+}\frac{f(t_0)\Theta f(t_0+h)}{-h}=\lim_{h\to 0^+}\frac{f(t_0)\Theta f(t_0-h)}{h}.$$

Definition 2.4 [23] Let $f:]a,b]\to I$ be $gH-$ differentiable at $t_0\in\,]a,b]$. We say that f is (i) $gH-$ differentiable at t_0 if

i. $f'(t_0)=\left[\underline{f}'(t_0),\overline{f}'(t_0)\right]$,

and that f is (ii) $gH-$ differentiable at t_0 if

ii. $f'(t_0)=\left[\overline{f}'(t_0),\underline{f}'(t_0)\right]$.

3 Proposed Method

In this section, we propose a new method based on interval midpoints, known as the Interval Midpoints Method (IMM), to solve an n-th order linear interval differential equation. To compare the results of the interval midpoints method, we also have obtained the exact solution using interval computation.
Let us consider the n-th order linear interval differential equation

$$\tilde{y}^{(n)}(t)+a_{n-1}(t)\tilde{y}^{(n-1)}(t)+\cdots+a_1(t)\tilde{y}'(t)+a_0(t)\tilde{y}(t)=\tilde{g}(t), \tag{1}$$

where $a_i(t)$, $0\le i\le n-1$, continuous on some interval I, subject to interval initial conditions

$$\tilde{y}(0)=\tilde{b}_0, \tilde{y}'(0)=\tilde{b}_1,\ \ldots,\tilde{y}^{(n-1)}(0)=\tilde{b}_{n-1}.$$

For interval $\tilde{b}_i$, $0\le i\le n-1$, where $\tilde{y}(t)$ is the solution to be determined. Now three cases may arise:
Case 1 Coefficients $a_{n-1}(t), a_{n-2}(t),\cdots,a_1(t),a_0(t)$ are all positive.
Case 2 Coefficients $a_{n-1}(t), a_{n-2}(t),\cdots,a_1(t),a_0(t)$ are all negative.
Case 3 Coefficients $a_{n-m-1}(t), a_{n-m-2}(t),\cdots,a_1(t),a_0(t)$ are negative for $n\ge m$.
We now discuss the above three cases in detail.

Case 1 Coefficients $a_{n-1}(t), a_{n-2}(t),\cdots,a_1(t),a_0(t)$ are all positive.
First we will write Eq. (1) in terms of interval midpoints as

$$y^{c^{(n)}}(t)y^{c^{(n)}}(t)+a_{n-1}(t)y^{c^{(n-1)}}(t)+\cdots+a_1(t)y^{c^{'}}(t)+a_0(t)y^{c}(t)=g^{c}(t), \tag{2}$$

with initial conditions

$$y^c(0)=b_0^c, y^{c^{'}}(0)=b_1^c,\ \ldots,y^{c^{(n-1)}}(0)=b_{n-1}^c.$$

The solution to Eq. (2) may be found easily for y^c by any analytical method.
We may write interval differential equation (1) as

$$\begin{aligned}&\left[\underline{y}^{(n)}(t), \overline{y}^{(n)}(t)\right] + a_{n-1}(t)\left[\underline{y}^{(n-1)}(t), \overline{y}^{(n-1)}(t)\right] + \cdots + a_1(t)\left[\underline{y}'(t), \overline{y}'(t)\right] \\ &\quad + a_0(t)\left[\underline{y}(t), \overline{y}(t)\right] = \left[\underline{g}(t), \overline{g}(t)\right],\end{aligned} \tag{3}$$

subject to interval initial conditions

$$\left[\underline{y}(0), \overline{y}(0)\right] = \left[\underline{b}_0, \overline{b}_0\right], \left[\underline{y}'(0), \overline{y}'(0)\right] = \left[\underline{b}_1, \overline{b}_1\right], \ldots, \left[\underline{y}^{(n-1)}(0), \overline{y}^{(n-1)}(0)\right] = \left[\underline{b}_{n-1}, \overline{b}_{n-1}\right].$$

It may be noted that the repeated interval evaluation of derivatives of functions requires an interval solver. For example, Newton contractors, contractors based on parallel linearization, the evaluation of centred inclusion functions and magnitude all require the computation of derivatives of functions with interval arguments [10].

Automatic differentiation also has been used to handle the differentiation [10]. To overcome the difficulties in the automatic differentiation (as mentioned in the Introduction), we may use an alternative midpoint method using generalized Hukuhara derivative. Using generalized Hukuhara derivatives and interval arithmetic, one may write Eq. (3) as

$$\underline{y}^{(n)}(t) + a_{n-1}(t)\underline{y}^{(n-1)}(t) + \cdots + a_1(t)\underline{y}'(t) + a_0(t)\underline{y}(t) = \underline{g}(t), \tag{4}$$

$$\overline{y}^{(n)}(t) + a_{n-1}(t)\overline{y}^{(n-1)}(t) + \cdots + a_1(t)\overline{y}'(t) + a_0(t)\overline{y}(t) = \overline{g}(t). \tag{5}$$

Now solving either Eq. (4) or (5), one may get $\underline{y}(t)$ or $\overline{y}(t)$, respectively. Next, substituting the above value y^c and $\underline{y}(t)$ or $\overline{y}(t)$ into the definition of an interval midpoint, we may find the solution as $\underline{y} = 2y^c - \overline{y}$ or $\overline{y} = 2y^c - \underline{y}$.

Case 2 Coefficients $a_{n-1}(t), a_{n-2}(t), \cdots, a_1(t), a_0(t)$ are all negative.
Eq. (1) may be written in terms of interval midpoints as

$$y^{c^{(n)}}(t) - a_{n-1}(t)y^{c^{(n-1)}}(t) - \cdots - a_1(t)y^{c^{'}}(t) - a_0(t)y^c(t) = g^c(t), \tag{6}$$

with initial conditions

$$y^c(0) = b_0^c, y^{c^{'}}(0) = b_1^c, \ldots, y^{c(n-1)}(0) = b_{n-1}^c.$$

y^c may be obtained by solving Eq. (6).
As mentioned in the previous Case 1, we use Hukuhara derivatives and interval arithmetic as

$$\underline{y}^{(n)}(t) + a_{n-1}(t)\overline{y}^{(n-1)}(t) + \cdots + a_1(t)\overline{y}'(t) + a_0(t)\overline{y}(t) = \underline{g}(t), \tag{7}$$

$$\overline{y}^{(n)}(t) + a_{n-1}(t)\underline{y}^{(n-1)}(t) + \cdots + a_1(t)\underline{y}'(t) + a_0(t)\underline{y}(t) = \vec{g}(t). \tag{8}$$

Using the definition of interval midpoints, one may write Eqs. (7) and (8) as

$$\begin{aligned}&\underline{y}^{(n)}(t) + a_{n-1}(t)\left(2y^c(t) - \underline{y}(t)\right)^{(n-1)} + \cdots + a_1(t)\left(2y^c(t) - \underline{y}(t)\right)' \\ &\quad + a_0(t)\left(2y^c(t) - \underline{y}(t)\right) = \underline{g}(t),\end{aligned} \tag{9}$$

$$\overline{y}^{(n)}(t) + a_{n-1}(t)\left(2y^c(t) - \overline{y}(t)\right)^{(n-1)} + \cdots + a_1(t)\left(2y^c(t) - \overline{y}(t)\right)' \\ +a_0(t)\left(2y^c(t) - \overline{y}(t)\right) = \vec{g}(t). \tag{10}$$

It may be seen that the above differential equations are now crisp differential equations. Hence, solving one of the above crisp differential equations, one may get the solution as $\underline{y}(t)$ or $\overline{y}(t)$. Applying the definition of interval midpoints, one may get $\overline{y}(t) = \left(2y^c - \underline{y}(t)\right)$ or $\underline{y}(t) = \left(2y^c - \overline{y}(t)\right)$.

Case 3 Coefficients $a_{n-m-1}(t), a_{n-m-2}(t), \cdots, a_1(t), a_0(t)$ are negative for, $n \geq m$
In this case, we may write Eq. (1) in terms of interval midpoints as

$$y^{c^{(n)}}(t) + a_{n-1}(t)y^{c^{(n-1)}}(t) + \cdots + a_{n-m}(t)y^{c^{(n-m)}}(t) - a_{n-m-1}(t)y^{c^{(n-m-1)}}(t) \\ + \cdots - a_0(t)y^c(t) = g^c(t), \tag{11}$$

with interval initial conditions

$$y^c(0) = b_0^c, y^{c'}(0) = b_1^c, \ldots, y^{c^{(n-1)}}(0) = b_{n-1}^c.$$

Similarly we may solve for y^c.
From Eq. (1) we have,

$$\underline{y}^{(n)}(t) + a_{n-1}(t)\underline{y}^{(n-1)}(t) + \cdots + a_{n-m}(t)\underline{y}^{(n-m)}(t) \\ +a_{n-m-1}(t)\overline{y}^{(n-m-1)}(t) + \cdots + a_0(t)\overline{y}(t) = \underline{g}(t), \tag{12}$$

$$\overline{y}^{(n)}(t) + a_{n-1}(t)\overline{y}^{(n-1)}(t) + \cdots + a_{n-m}(t)\overline{y}^{(n-m)}(t) \\ +a_{n-m-1}(t)\underline{y}^{(n-m-1)}(t) + \cdots + a_0(t)\underline{y}(t) = \overline{g}(t). \tag{13}$$

Eqs. (12) and (13) are written as

$$\underline{y}^{(n)}(t) + a_{n-1}(t)\underline{y}^{(n-1)}(t) + \cdots + a_{n-m}(t)\underline{y}^{(n-m)}(t) \\ +a_{n-m-1}(t)\left(2y^c(t) - \underline{y}(t)\right)^{(n-m-1)} + \cdots + a_0(t)\left(2y^c(t) - \underline{y}(t)\right) = \underline{g}(t), \tag{14}$$

$$\overline{y}^{(n)}(t) + a_{n-1}(t)\overline{y}^{(n-1)}(t) + \cdots + a_{n-m}(t)\overline{y}^{(n-m)}(t) \\ +a_{n-m-1}(t)\left(2y^c(t) - \overline{y}(t)\right)^{(n-m-1)} + \cdots + a_0(t)\left(2y^c(t) - \overline{y}(t)\right) = \overline{g}(t). \tag{15}$$

Similar to the previous cases, here also the system of Eqs. (12) and (13) are crisp differential equations. Substituting the value of $y^c(t)$ into Eqs. (14) and (15) and solving the crisp differential equation, one may get $\underline{y}(t)$ or $\overline{y}(t)$. Finally, the interval solution is obtained by using the definition of interval midpoint.

Here the coefficients in the differential equation also play a great role. That is why we have three cases viz. Case (1) when all the coefficients are positive, Case (2) when all the coefficients are negative, and Case (3) when the coefficients are both positive and negative. Case (1) may give a direct solution, but for Cases (2) and (3), in general, we get coupled systems of equations. In that case, one may not bound all trajectories by following the extreme (i.e., the quasimonotonicity argument [35, 36]) because in these cases, we have a combination of positive and negative coefficients, and we may not uncouple the system. However, our interval midpoints method uncouples the system to provide a direct solution of interval differential equations.

3.1 Homotopy Perturbation Method [33, 34]

To illustrate the basic idea of this method, we consider the following differential equation.

$$A(u) - f(r) = 0, r \in \Omega, \tag{16}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma, \tag{17}$$

where A is a general differential operator, B is a boundary operator,$f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω. A can be divided into two parts, L and N, where L is linear, and N is nonlinear. Eq. (16) can be written as

$$L(u) + N(u) - f(r) = 0, r \in \Omega. \tag{18}$$

By the homotopy technique, we construct a homotopy $U(r,p) : \Omega \times [0,1] \to R$, which satisfies:

$$H(U,p) = (1-p)\left[L(U) - L(v_0)\right] + p\left[A(U) - f(r)\right] = 0, p \in [0,1]\,, r \in \Omega, \tag{19}$$

or

$$H(U,p) = L(U) - L(v_0) + pL(v_0) + p\left[N(U) - f(r)\right] = 0, \tag{20}$$

where, $r \in \Omega$, and $p \in [0,1]$ is an embedding parameter, and v_0 is an initial approximation of Eq.(16). Hence, it is obvious that

$$H(U,0) = L(U) - L(v_0) = 0, \tag{21}$$

$$H(U,1) = A(U) - f(r) = 0, \tag{22}$$

and the changing process of p from 0 to 1 is just that of changing $H(U,p)$ from $L(U) - L(v_0)$ to $A(U) - f(r)$. In topology, this is called a deformation, $L(U) - L(v_0)$, and $A(U) - f(r)$ is called homotopic. Applying the perturbation technique [32, 33], due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of Eq. (19) or (20) can be expressed as a series in P

$$U = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots \tag{23}$$

when $p \to 1$, Eq. (19) or (20) corresponds to Eq.(18). Then Eq. (23) becomes the approximate solution of Eq. (18), i.e.

$$u = \lim_{p \to 1} U = u_0 + u_1 + u_2 + u_3 + \cdots \tag{24}$$

The convergence of the series (24) has been proved in [33, 34].

4 Numerical Implementation of the Interval Midpoints Method

In this section, the interval midpoints method is applied to three mathematical examples and one application problem related to an electric circuit. The results are compared with the exact and HPM solutionS to show the efficiency and applicability of the interval midpoints method.

Example 1. Let us consider the following second order interval differential equation (Case 1).

$$\tilde{y}'' + 6\tilde{y}' + 9\tilde{y} = 0 \tag{25}$$

subject to the interval initial conditions

$$\tilde{y}(0) = [1.8, 2.2],\ \tilde{y}'(0) = [-3.2,\ -2.8].$$

According to Eq. (2), the differential equation (Eq. (25)) can be written as

$$y^{c\prime\prime} + 6y^{c\prime} + 9y^{c} = 0. \tag{26}$$

Solving Eq. (26) one may obtain $y^c = (2+3t)\,e^{-3t}$.
Proceeding as Eq. (4) or (5) with the above value of y^c and solving any one of the systems gives the value of $\underline{y}(t) = \left(\frac{9}{5}\right)e^{-3t} + \left(\frac{11}{5}\right)te^{-3t}$ or $\overline{y}(t) = \left(\frac{11}{5}\right)e^{-3t} + \left(\frac{19}{5}\right)te^{-3t}$. Hence, one may have the final solution as $\tilde{y}(t) = [\underline{y}(t), \overline{y}(t)]$. Corresponding interval solution plots are given in Figs. 1 to 3 by varying t. It is interesting to note from the figures that, with increasing of time the uncertainty width of the solution gradually decreases. To show the difference of the solution bounds clearly, the plot has been made by varying the time range from 30 to 31 as given in Fig. 3. Also, the absolute errors obtained by the present and HPM methods are given in Tables 1 and 2. The solutions obtained by the present method agree well with the exact solution.

Table 1: Comparison of lower bound of exact, present and HPM solutions of Example 1

t	Exact Solution $\underline{Y}(t)$ by Bede	$\underline{y}(t)$ (HPM) $(n=3)$	$\underline{y}(t)$ (HPM) $(n=4)$	Present Solution $\underline{y}(t)$	Absolute Error by HPM $(n=3)$	Absolute Error by HPM $(n=4)$	Absolute Error by Present
0	1.8	1.8	1.8	1.8	0	0	0
0.1	1.4965	1.4959	1.4965	1.4965	0.0006	0	0
0.2	1.2293	1.2204	1.2315	1.2293	0.0089	0.0022	0
0.3	1.0002	0.95092	1.0184	1.0002	0.0493	0.0182	0
0.4	0.8072	0.63996	0.8911	0.8072	0.1672	0.0839	0
0.5	0.6470	0.21219	0.9242	0.6470	0.4349	0.2771	0
0.6	0.5157	-0.4380	1.2572	0.5157	0.9538	0.7415	0
0.7	0.409	-1.4495	2.123	0.409	1.8585	1.7140	0
0.8	0.3229	-2.9961	3.881	0.3229	3.3191	3.5580	0
0.9	0.2540	-5.2903	7.0549	0.2540	5.5443	6.8009	0
1	0.1991	-8.585	12.375	0.1991	8.7842	12.175	0

Example 2 Consider the following second order interval differential equation (case 2).

$$\tilde{y}'' - 3\tilde{y}' - 4\tilde{y} = 0, \tag{27}$$

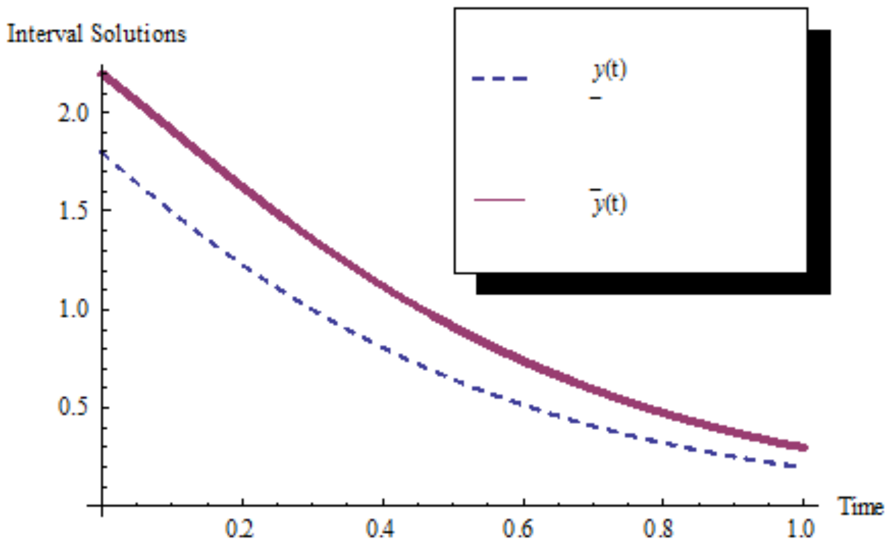

Figure 1: Interval solution of Example 1 using the interval midpoints method

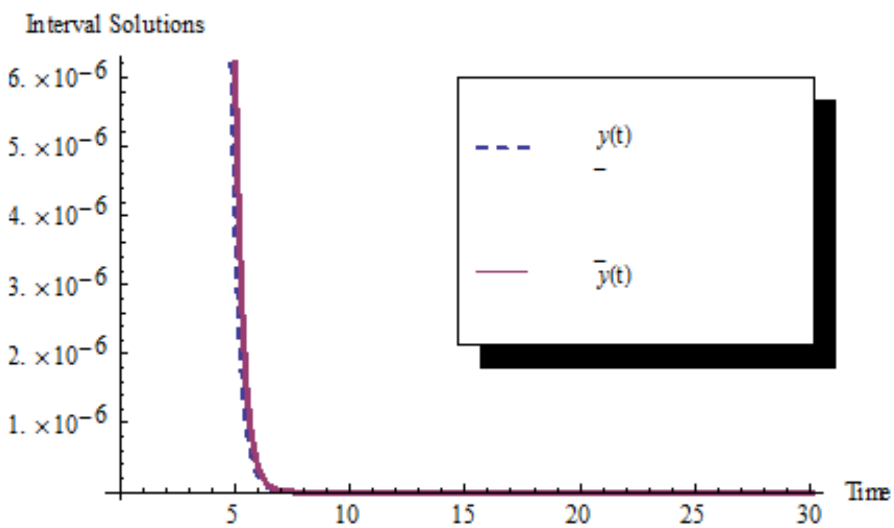

Figure 2: Interval solution of Example 1 using the interval midpoints method

Table 2: Comparison of upper bound of exact, present and HPM solutions of Example 1

t	Exact Solution $\overline{Y}(t)$ by Bede	$\overline{y}(t)$ (HPM) $(n=3)$	$\overline{y}(t)$ (HPM) $(n=4)$	Present Solution $\overline{y}(t)$	Absolute Error by HPM $(n=3)$	Absolute Error by HPM $(n=4)$	Absolute Error by Present
0	2.2	2.2	2.2	2.2	0	0	0
0.1	1.9113	1.9117	1.9113	1.9113	0.0004	0	0
0.2	1.6245	1.6287	1.6233	1.6245	0.0042	0.0012	0
0.3	1.3579	1.3729	1.3508	1.3579	0.0150	0.0071	0
0.4	1.1204	1.149	1.0983	1.1204	0.0286	0.0221	0
0.5	0.9148	0.9425	0.8700	0.9148	0.0277	0.0448	0
0.6	0.7405	0.7175	0.6842	0.7405	0.0230	0.0563	0
0.7	0.5951	0.4144	0.5891	0.5951	0.1807	0.0060	0
0.8	0.4753	-0.0525	0.6840	0.4753	0.5280	0.2087	0
0.9	0.3776	-0.7975	1.143	0.3776	1.1752	0.7653	0
1	0.2987	-1.965	2.2425	0.2987	2.2637	1.9438	0

subject to the interval initial conditions:

$$\tilde{y}(0) = [0.8, 1.2],\ \tilde{y}'(0) = [1.8, 2.2].$$

Using the interval midpoint method, we have

$$y^c = \tfrac{2}{5}e^{-t} + \tfrac{3}{5}e^{4t}.$$

Subsequently, by applying the procedure discussed previously, we get the upper and lower solution bounds respectively as

$$\underline{y}(t) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} - \frac{\sqrt{7}}{7}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right) + \left(-\frac{1}{5}\right)e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right)$$

and

$$\overline{y}(t) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t} + \frac{\sqrt{7}}{7}e^{-3t/2}\sin\left(\frac{\sqrt{7}}{2}t\right) - \left(-\frac{1}{5}\right)e^{-3t/2}\cos\left(\frac{\sqrt{7}}{2}t\right).$$

Graphs of the solution bounds by varying t of Example 2 are depicted in Fig. 4. Again, absolute errors obtained by the interval midpoint method and HPM are presented in Tables 3 and 4. Again, the solution obtained by present method exactly matches the exact solution.

Example 3 Next, let us consider the following third order interval differential equation (Case 3).

$$\tilde{y}''' - 6\tilde{y}'' + 11\tilde{y}' - 6\tilde{y} = 0, \tag{28}$$

subject to the interval initial conditions

$$\tilde{y}(0) = [0.8, 1.2],\ \tilde{y}'(0) = [0.8, 1.2],\ \tilde{y}''(0) = [1.8, 2.2].$$

Following the interval midpoints method, the solution is

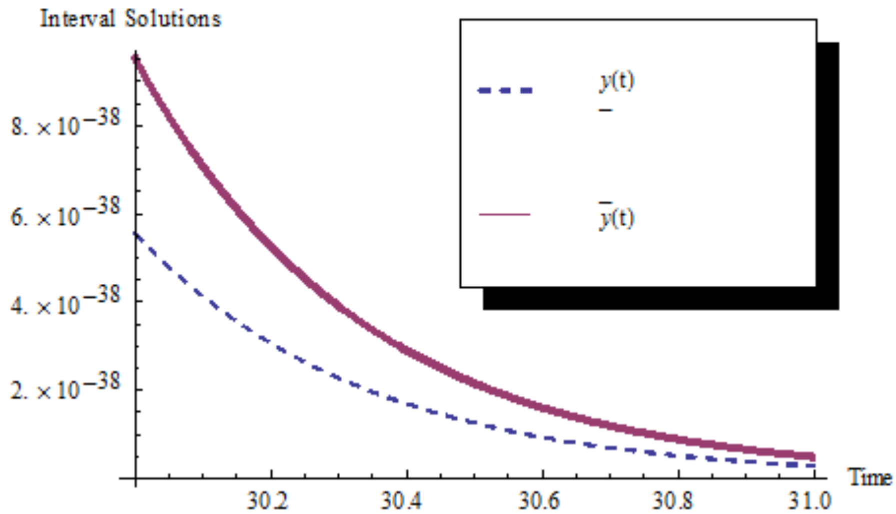

Figure 3: Interval solution of Example 1 using the interval midpoints method

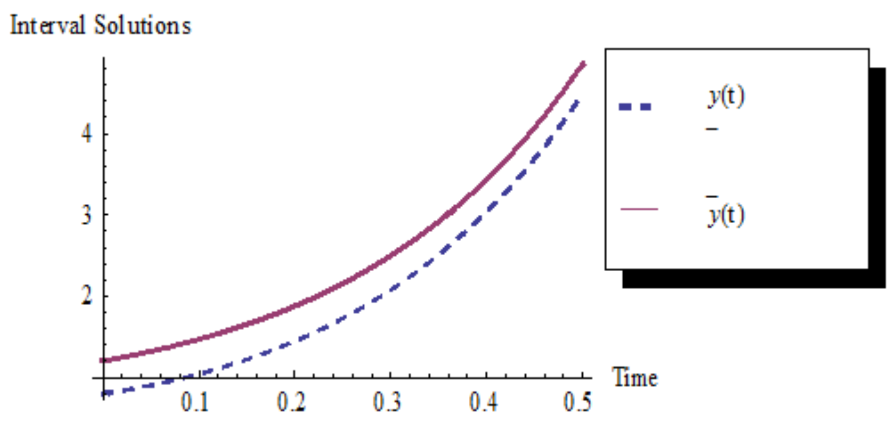

Figure 4: Interval solution of Example 2 using the interval midpoints method

Table 3: Comparison of lower bound of exact, present and HPM solutions of Example 2

t	Exact Solution $\underline{Y}(t)$ by Bede	$\underline{y}(t)$ (HPM) $(n = 3)$	$\underline{y}(t)$ (HPM) $(n = 4)$	Present Solution $\underline{y}(t)$	Absolute Error by HPM $(n = 3)$	Absolute Error by HPM $(n = 4)$	Absolute Error by Present
0	0.8	0.8	0.8	0.8	0	0	0
0.1	1.0435	1.043	1.0435	1.0435	0.0005	0	0
0.2	1.4466	1.4379	1.4458	1.4466	0.0087	0	0
0.3	2.0776	2.0282	2.0704	2.0776	0.0494	0.0008	0
0.4	3.0405	2.8633	3.0051	3.0405	0.1772	0.0354	0
0.5	4.4918	3.9992	4.367	4.4918	0.4926	0.1248	0
0.6	6.6669	5.4987	6.3076	6.6669	1.1682	0.3593	0
0.7	9.9176	7.4315	9.0187	9.9176	2.4861	0.8989	0
0.8	14.77	9.8749	12.738	14.77	4.8951	2.0320	0
0.9	22.011	12.914	17.758	22.011	9.0970	4.2530	0
1	32.813	16.64	24.43	32.813	16.1730	8.3830	0

Table 4: Comparison of upper bound of exact, present and HPM solutions of Example 2

t	Exact Solution $\overline{Y}(t)$ by Bede	$\overline{y}(t)$ (HPM) $(n = 3)$	$\overline{y}(t)$ (HPM) $(n = 4)$	Present Solution $\overline{y}(t)$	Absolute Error by HPM $(n = 3)$	Absolute Error by HPM $(n = 4)$	Absolute Error by Present
0	1.2	1.2	1.2	1.2	0	0	0
0.1	1.4706	1.4702	1.4705	1.4706	0.0004	0.0001	0
0.2	1.879	1.8721	1.878	1.879	0.0069	0.0010	0
0.3	2.4992	2.4583	2.4902	2.4992	0.0409	0.0090	0
0.4	3.4394	3.2894	3.3969	3.4394	0.1500	0.0425	0
0.5	4.8603	4.4342	4.7133	4.8603	0.4261	0.1470	0
0.6	7	5.97	6.5845	7	1.0300	0.4155	0
0.7	10.213	7.9833	9.1901	10.213	2.2297	1.0229	0
0.8	15.028	10.57	12.749	15.028	4.4580	2.2790	0
0.9	22.232	13.835	17.525	22.232	8.3970	4.7070	0
1	32.999	17.893	23.832	32.999	15.1060	9.1670	0

$$\underline{y}(t) = \tfrac{3}{2}e^{t} + \tfrac{1}{2}e^{3t} - e^{2t} + \left(-\tfrac{3}{5}\right)e^{-3t} + \left(\tfrac{8}{5}\right)e^{-2t} + \left(-\tfrac{6}{5}\right)e^{-t},$$

$$\overline{y}(t) = \tfrac{3}{2}e^{t} + \tfrac{1}{2}e^{3t} - e^{2t} - \left(-\tfrac{3}{5}\right)e^{-3t} - \left(\tfrac{8}{5}\right)e^{-2t} - \left(-\tfrac{6}{5}\right)e^{-t}.$$

Corresponding solution bounds by varying t are now depicted in Fig. 5. The results are shown in Tables 5 and 6 using the exact, HPM, and interval midpoints method, respectively. Again, the results obtained by the interval midpoints method agree well with the exact solution.

Table 5: Comparison of lower bound of exact, present and HPM solutions of Example 3

t	Exact Solution $\underline{Y}(t)$ by Bede	$\underline{y}(t)$ (HPM) $(n=3)$	$\underline{y}(t)$ (HPM) $(n=4)$	Present Solution $\underline{y}(t)$	Absolute Error by HPM $(n=3)$	Absolute Error by HPM $(n=4)$	Absolute Error by Present
0	0.8	0.8	0.8	0.8	0	0	0
0.1	0.89096	0.89092	0.89096	0.89096	0.00004	0	0
0.2	1.0121	1.011	1.012	1.0121	0.0011	0.0001	0
0.3	1.1776	1.1693	1.1772	1.1776	0.0083	0.0004	0
0.4	1.4061	1.3711	1.404	1.4061	0.0350	0.0021	0
0.5	1.7225	1.6156	1.7154	1.7225	0.1069	0.0071	0
0.6	2.162	1.8944	2.1428	2.162	0.2676	0.0192	0
0.7	2.7737	2.1903	2.7301	2.7737	0.5834	0.0436	0
0.8	3.6263	2.475	3.539	3.6263	1.1513	0.0873	0
0.9	4.8159	2.7085	4.6565	4.8159	2.1074	0.1594	0
1	6.4763	2.8372	6.205	6.4763	3.6391	0.2713	0

Table 6: Comparison of upper bound of exact, present and HPM solutions of Example 3

t	Exact Solution $\overline{Y}(t)$ by Bede	$\overline{y}(t)$ (HPM) $(n=3)$	$\overline{y}(t)$ (HPM) $(n=4)$	Present Solution $\overline{y}(t)$	Absolute Error by HPM $(n=3)$	Absolute Error by HPM $(n=4)$	Absolute Error by Present
0	1.2	1.2	1.2	1.2	0	0	0
0.1	1.3316	1.3316	1.3316	1.3316	0	0	0
0.2	1.4906	1.4903	1.4904	1.4906	0.0003	0	0
0.3	1.6873	1.6853	1.6848	1.6873	0.0020	0.0025	0
0.4	1.9384	1.9294	1.9248	1.9384	0.0090	0.0136	0
0.5	2.2688	2.2395	2.2169	2.2688	0.0293	0.0519	0
0.6	2.7137	2.6368	2.5595	2.7137	0.0769	0.1542	0
0.7	3.3233	3.1468	2.935	3.3233	0.1765	0.3883	0
0.8	4.1675	3.8006	3.3016	4.1675	0.3669	0.8659	0
0.9	5.3434	4.6359	3.583	5.3434	0.7075	1.7604	0
1	6.9859	5.6981	3.6563	6.9859	1.2878	3.6563	0

Example 4 Finally, let us consider the electrical circuit shown in Fig. 6 [5], where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 20\cos t$. If Q is the charge on the capacitor at time $t > 0$, then

$$\tilde{Q}''(t) + 2\tilde{Q}'(t) + 4\tilde{Q}(t) = 50\cos t, \tag{29}$$

subject to the interval initial conditions

$$\tilde{Q}(0) = [4, 6],\ \tilde{Q}'(0) = [0, 2].$$

By following the interval midpoints method, we get the solution for Eq. (29) as

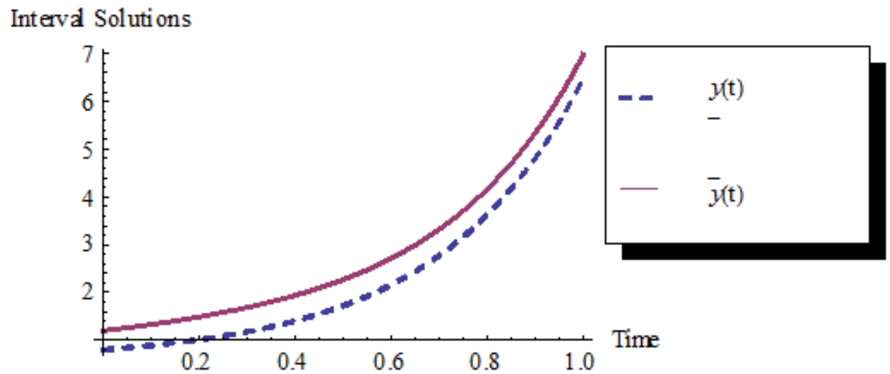

Figure 5: Interval solution of Example 3 using the interval midpoints method

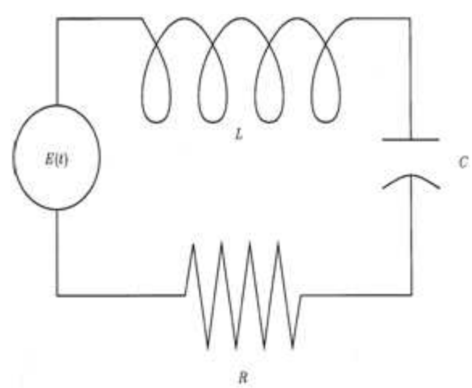

Figure 6: Electrical circuit [5] in Example 4

$$\underline{Q}(t) = \tfrac{2}{39}e^{-t}\sin(\sqrt{3}t)(-99)\sqrt{3} + e^{-t}\cos(\sqrt{3}t)\left(-\tfrac{98}{13}\right) + \left(\tfrac{150}{13}\right)\cos(t) + \left(\tfrac{100}{13}\right)\sin(t),$$

$$\overline{Q}(t) = -\tfrac{2}{39}e^{-t}\sin(\sqrt{3}t)(73)\sqrt{3} + e^{-t}\cos(\sqrt{3}t)\left(-\tfrac{72}{13}\right) + \left(\tfrac{150}{13}\right)\cos(t) + \left(\tfrac{100}{13}\right)\sin(t).$$

Figs.7 to 9 represent the corresponding interval plots for this example. Similarly, the width of the solution gradually deceases with increasing time t. Fig. 9 represents the solution bounds for the time range from 4 to 4.1. The results for this problem are shown in Tables 7 and 8, using the method of Bede [3], HPM, and the interval midpoints method, respectively. Again there is good agreement between the results obtained by the present and by the exact methods.

Table 7: Comparison of lower bound of exact, present and HPM solutions of Example 4

t	Exact Solution $\underline{Q}(t)$ by Bede	$\underline{Q}(t)$ (HPM) ($n=3$)	$\underline{Q}(t)$ (HPM) ($n=4$)	Present Solution $\underline{Q}(t)$	Absolute Error by HPM ($n=3$)	Absolute Error by HPM ($n=4$)	Absolute Error by Present
0	4	4	4	4	0	0	0
0.1	4.1585	4.1579	4.1585	4.1585	0	0	0
0.2	4.5869	4.5772	4.5878	4.5869	0.0097	0.0009	0
0.3	5.2141	5.1635	5.2207	5.2141	0.0506	0.0066	0
0.4	5.9701	5.8062	5.9994	5.9701	0.1639	0.0293	0
0.5	6.7888	6.3802	6.8823	6.7888	0.4086	0.0935	0
0.6	7.6094	6.7479	7.8525	7.6094	0.8615	0.2431	0
0.7	8.3783	6.7608	8.9257	8.3783	1.6175	0.5474	0
0.8	9.0498	6.2619	10.158	9.0498	2.7879	1.1082	0
0.9	9.5865	5.0878	11.655	9.5865	4.4987	2.0685	0
1	9.9594	3.0715	13.578	9.9594	6.8879	3.6186	0

For all the above examples, the results obtained by the interval midpoints method are exactly same as the computed exact interval solutions using the method of Bede [3] for the special case $r = 0$ in their method. Here, the main aim is to develop a new analytical method to handle n-th order interval differential equations giving all the possible cases. The known differential equations are solved as test problems to have confidence in the interval midpoints method. The solution by the interval midpoints method in all the test problems exactly matches the exact solution. The interval midpoints method gives us a straightforward, alternate and computationally efficient way to handle n-th order linear interval differential equations.

5 Conclusions

This paper investigates the solution of n-th order linear interval differential equations by proposing a new method. First, the interval differential equation is solved in terms of midpoints, then this solution is used to get the required solution of the original n-th order interval differential equation. The interval midpoints method makes the procedure straightforward and efficient. The obtained results are compared with the HPM and exact solutions. The results obtained from the interval midpoints method are found to be in good agreement with the exact solutions.

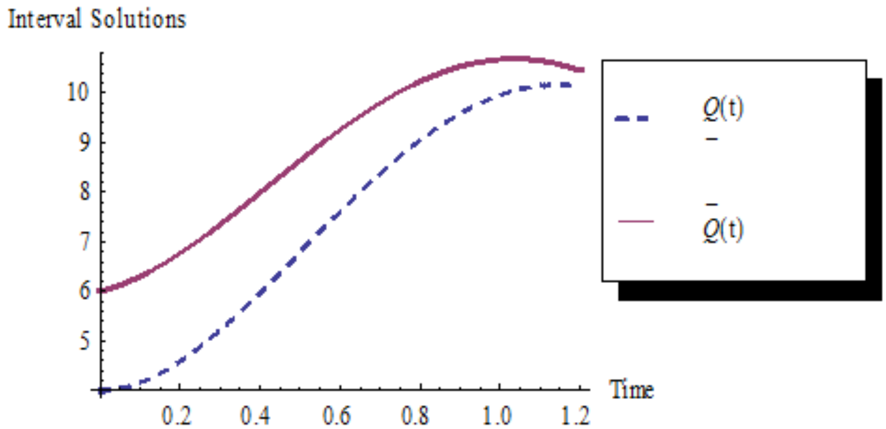

Figure 7: Interval solution of Example 4 using the interval midpoints method

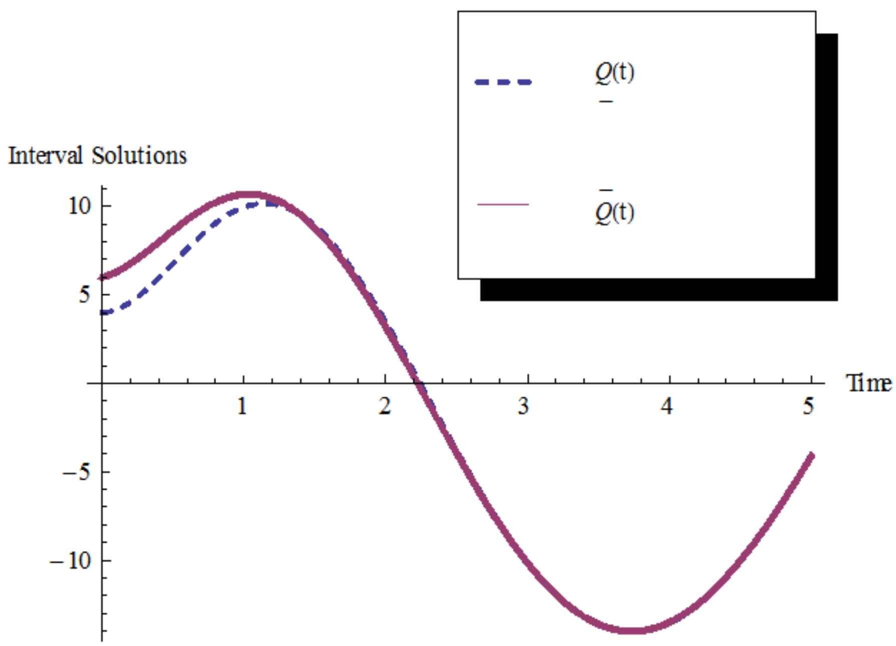

Figure 8: Interval solution of Example 4 using the interval midpoints method

Table 8: Comparison of upper bound of exact, present and HPM solutions of Example 4

t	Exact Solution $\overline{Q}(t)$ by Bede	$\overline{Q}(t)$ (HPM) $(n = 3)$	$\overline{Q}(t)$ (HPM) $(n = 4)$	Present Solution $\overline{Q}(t)$	Absolute Error by HPM $(n = 3)$	Absolute Error by HPM $(n = 4)$	Absolute Error by Present
0	6	6	6	6	0	0	0
0.1	6.3012	6.3008	6.3012	6.3012	0.0004	0	0
0.2	6.7691	6.7629	6.7696	6.7691	0.0062	0.0005	0
0.3	7.3497	7.3177	7.3539	7.3497	0.0320	0.0042	0
0.4	7.9904	7.8878	8.0088	7.9904	0.1026	0.0184	0
0.5	8.6417	8.3886	8.7001	8.6417	0.2531	0.0584	0
0.6	9.2583	8.7303	9.409	9.2583	0.5280	0.1507	0
0.7	9.8006	8.8206	10.137	9.8006	0.9800	0.3364	0
0.8	10.235	8.5668	10.91	10.235	1.6682	0.6750	0
0.9	10.535	7.8789	11.783	10.535	2.6561	1.2480	0
1	10.68	6.6715	12.842	10.68	4.0085	2.1620	0

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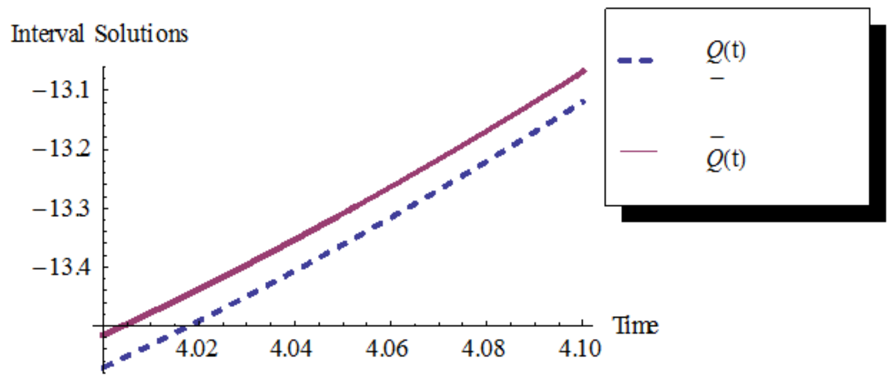

Figure 9: Interval solution of Example 4 using the interval midpoints method

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