INNER APPROXIMATION OF THE RANGE OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. No method for the computation of a reliable subset of the range of vector-valued functions is available today. A method for computing such inner approximations is proposed in the specific case where both domain and co-domain have the same dimension. A general sufficient condition for the inclusion of a box inside the image of a box by a continuously differentiable vector-valued is first provided. This sufficient condition is specialized to a more efficient one, which is used in a specific bisection algorithm that computes reliable inner and outer approximations of the image of a domain defined by constraints. Some experimentations are presented.

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1. INTRODUCTION

An *image set* is a subset of $\mathbb{R}^m$ that is defined as the image of a subset $\mathcal{D}$ of $\mathbb{R}^n$ by a nonlinear vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$. This paper deals with the *image set problem* which consists in computing a reliable description of the *image set*. More precisely, we want to find two subpavings (*i.e.* union of boxes) $\mathbb{Y}^- \subseteq \mathbb{R}^m$ and $\mathbb{Y}^+ \subseteq \mathbb{R}^m$ such that

$$\mathbb{Y}^- \subseteq \mathbb{Y} \subseteq \mathbb{Y}^+,$$

where $\mathbb{Y} = \{f(x) : x \in \mathcal{D}\}$ is the *image set* of $\mathcal{D}$ by f. The main contribution of this paper is to present the first method able to solve the *image set problem* for continuously differentiable nonlinear vector-valued functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (both domain and co-domain have the same dimension).

Interval analysis provides a large number of methods able to compute efficiently an outer approximation $\mathbb{Y}^+$ of the *image set* (see [16, 18, 14]). As a consequence, the outer approximation problem can be considered as solved and this is why this paper is mainly devoted to the inner characterization problem. Existing methods are unable to compute an inner approximation $\mathbb{Y}^-$ of an *image set*, except for some particular cases :

First, when f maps $\mathbb{R}^n$ into $\mathbb{R}$, methods based on Kaucher arithmetic (see [9] and extensive references) or on twin arithmetic (see [17]) have been shown to be able to solve the *image set problem*. Second, when f is vector-valued and linear the Kaucher arithmetic has been shown to be able to provide an accurate inner approximation of the *image set* (see [20, 4]). Finally, some advances [3, 5] have been carried out in the general case of constraints $\exists y \in \mathbf{y}, f(x,y) = 0$, but none of these methods is able to deal with the problems tackled in the present paper.

Although the method proposed in the present paper is restricted to functions where both domain and co-domain have the same dimension, this class of *image set problems* is of particular importance for many applications in robotics and control. Let us present two of them.

First, within a set theoretical framework state estimation alternates a prediction step and a correction step (see e.g. [19, 1]). The correction step amounts to solving a set inversion problem whereas the prediction step requires the characterization of the image of a subpaving. Existing algorithms based on interval analysis (see e.g. [11]) are unable to compute an inner approximation of the feasible set for the state vector at time k. An efficient solution for the image set problem could be really appreciated during the prediction step. Second, the workspace of a serial robot represents the set of all configurations that could be reached by the tool of the robot (see e.g. [2]). This set can be defined as an *image set* (as illustrated in the application section of this paper). A guaranteed inner approximation of the workspace is needed during the conception phase of the robot : we need to conceive the robot so that we are certain that it will be able to reach the whole region of interest.

The paper is composed as follows. Section 2 recalls the main notions of interval analysis needed for the proposed developments. A test that makes possible to prove that a box is included in an *image set* is presented in Section 3. This test is made more efficient using a preconditioning step as detailed in Section 4. Section 5 provides a bisection algorithm that computes an inner and an outer approximation of the *image set*. The efficiency of the algorithm is demonstrated on two testcases in Section 6. Finally, Section 7 presents some related work.

2. Interval analysis

The interval theory was born in the 60's aiming at rigorous computations using finite precision computers (see [15]). Since its birth, it has been developed and it proposes today original algorithms for solving problems independently of the finite precision of computer computations, although reliable computations using finite precision remain one important advantage of the interval based algorithms (see e.g. [10, 8]).

An interval is a set of real numbers $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ where $a, b \in \mathbb{R}$ such that $a \leq b$ are called respectively the lower and upper bounds of $[a,b]$. Intervals and related objects (e.g. interval functions and interval vectors and matrices) are denoted by boldface letters. The set of intervals is denoted by $\mathbb{IR}$. Interval vectors and interval matrices are defined like in the context of real numbers. They are identified with vectors of intervals and matrices of intervals. For example, the interval vector $[a,b]$, where

$a = (a_1, \cdots, a_n)^T$ and $b = (b_1, \cdots, b_n)^T$, is identified with the vector of intervals $([a_1, b_1], \cdots, [a_n, b_n])^T$. The membership is defined componentwise, e.g. $x \in [a, b]$ is equivalent to $x_i \in [a_i, b_i]$ for all $i \in [1..n]$. The interval vectors are also called boxes. Interval matrices are defined in the same way.

The boundary of an interval $\mathbf{x}$ is denoted by $\partial\mathbf{x}$ and its interior by $\operatorname{int}\mathbf{x}$ (these notations also stand for general subsets of $\mathbb{R}^n$). Also, the midpoint of an interval $[a, b]$ is denoted by $\operatorname{mid}[a, b] = (a + b)/2$, its radius (resp. its width) is denoted by $\operatorname{rad}[a, b] = (b - a)/2$ (resp. by $\operatorname{wid}[a, b] = b - a$). The same definitions hold for interval vectors and interval matrices (note that $\operatorname{rad}\mathbf{x} \in \mathbb{R}^n$ and $\operatorname{wid}\mathbf{x} \in \mathbb{R}^n$ while $\operatorname{rad}\mathbf{A} \in \mathbb{R}^{m\times n}$). The magnitude (resp. mignitude) of an interval $[a, b]$ is $||[a, b]|| = \max\{|a|, |b|\}$ (resp. $\langle [a, b]\rangle = \min\{|a|, |b|\}$). For an interval vector $[a, b]$, its magnitude $||[a, b]|| = \max_i ||[a_i, b_i]||$. Throughout the paper, we use the infinite norms $||x|| = \max_i\{|x_i|\}$ and $||A|| = \max_i \sum_j |A_{ij}|$. Also, we abbreviate $||\,|\mathbf{x}|\,||$ and $||\,|\mathbf{A}|\,||$ using $||\mathbf{x}||$ and $||\mathbf{A}||$ respectively. The distance between interval vectors is defined as the Hausdorff distance between the underlying sets of real vectors. It has the following simple expression:

$$d(\mathbf{x}, \mathbf{y}) = \max_i \max\{|\underline{\mathbf{x}}_i - \underline{\mathbf{y}}_i|, |\overline{\mathbf{x}}_i - \overline{\mathbf{y}}_i|\},$$

where $\mathbf{x} = ([\underline{\mathbf{x}}_1, \overline{\mathbf{x}}_1], \ldots, [\underline{\mathbf{x}}_n, \overline{\mathbf{x}}_n])$ and $\mathbf{y} = ([\underline{\mathbf{y}}_1, \overline{\mathbf{y}}_1], \ldots, [\underline{\mathbf{y}}_n, \overline{\mathbf{y}}_n])$. Given an interval vector $\mathbf{x} \in \mathbb{IR}^n$ (resp. an interval matrix $\mathbf{A} \in \mathbb{IR}^{n\times m}$), we define $\mathbb{I}\mathbf{x} := \{\mathbf{x}' \in \mathbb{IR}^n : \mathbf{x}' \subseteq \mathbf{x}\}$ (resp. $\mathbb{I}\mathbf{A} := \{\mathbf{A}' \in \mathbb{IR}^{n\times m} : \mathbf{A}' \subseteq \mathbf{A}\}$).

The core of the interval theory is the extension of real functions to intervals. These extensions allow computing outer approximations of the image of intervals by real functions. Formally, given a real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and an interval function $\mathbf{f} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^m$, the interval function $\mathbf{f}$ is an interval extension of f if and only if, for all $\mathbf{x} \in \mathbb{IR}^n$, $\operatorname{range}(f, \mathbf{x}) \subseteq \mathbf{f}(\mathbf{x})$, where $\operatorname{range}(f, \mathbf{x})$ is the image of $\mathbf{x}$ by f. The smallest box that contains $\operatorname{range}(f, \mathbf{x})$ is called the interval hull of $\operatorname{range}(f, \mathbf{x})$ and is denoted by $\square\operatorname{range}(f, \mathbf{x})$. It is the most accurate interval extension that one may hope to construct.

Now that the definition of interval extensions is stated, it remains to compute such interval extensions. Computing the interval hull of $\operatorname{range}(f, \mathbf{x})$ with arbitrary precision is a NP-hard problem (see [13]). Therefore lots of work has been done to provide interval extensions that are both precise and cheap. Based on the interval arithmetic (that computes exact ranges for elementary functions like $+$, $-$, $/$, $\times$, exp, cos, etc...) two interval extensions are widely used (see [18]):

- The natural extension simply consists in replacing the real operation by their interval counterparts in the expression of the function. For example, $\mathbf{f}(\mathbf{x}) = \mathbf{x}^2 - \mathbf{x}$ is an interval extension of $f(x) = x^2 - x$.
- The mean-value extension consists in rigorously linearizing the function before computing the natural extension of the linearization. For example, the mean-value extension of the function $f(x) = x^2 - x$ is

$\mathbf{f}(\mathbf{x}) = f(\text{mid } \mathbf{x}) + \Delta(\mathbf{x} - \text{mid } \mathbf{x})$ where Δ is an interval that contains $\{f'(x) \mid x \in \mathbf{x}\}$ (the interval Δ can be computed using the natural extension of f').

In order to study the asymptotic behavior of interval extensions, it is useful to introduce the Lipschitz continuity of interval functions:

Definition 2.1. An interval function $\mathbf{f} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^m$ is locally Lipschitz continuous if and only if for all $\mathbf{x}^{\text{ref}} \in \mathbb{IR}^n$ there exists $\lambda > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{I}\mathbf{x}^{\text{ref}}$,

$$d\big(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})\big) \leq \lambda \; d(\mathbf{x}, \mathbf{y}).$$

Under mild hypothesis, the natural interval extension has a linear order of convergence. Note that an interval extension $\mathbf{f}$ of a real function f which is locally Lipschitz continuous satisfies the following property: for all $\mathbf{x}^{\text{ref}} \in \mathbb{IR}^n$ there exists $\gamma > 0$ such that for all $\mathbf{x} \in \mathbb{I}\mathbf{x}^{\text{ref}}$, $||\,\text{rad } \mathbf{f}(\mathbf{x})|| \leq \gamma||\,\text{rad } \mathbf{x}||$. This follows from the following obvious overestimations:

$$\text{rad } \mathbf{f}(\mathbf{x}) \leq d(\mathbf{f}(\mathbf{x}), \mathbf{f}(\tilde{x})) \leq \lambda \; d(\mathbf{x}, \tilde{x}) \leq \lambda \,\text{wid } \mathbf{x} \leq 2\lambda \,\text{rad } \mathbf{x}.$$

Interval matrices naturally arise when the mean-value extension is used for vector-valued functions. In this case, the derivative of f is the matrix

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}.$$

The expression of the mean-value extension is then

$$f(\text{mid } \mathbf{x}) + \mathbf{J}(\mathbf{x} - \text{mid } \mathbf{x})$$

where $\mathbf{J}$ is an interval matrix that contains $\{f'(x) \mid x \in \mathbf{x}\}$. The following specific definitions about interval matrices will be used in the sequel: consider $\mathbf{A} \in \mathbb{IR}^{n\times n}$, then

- $\text{Diag}\,\mathbf{A}$ is the diagonal interval matrix whose diagonal entries are $(\text{Diag}\,\mathbf{A})_{ii} = \mathbf{A}_{ii}$.
- $\text{Diag}^{-1}\,\mathbf{A}$ is the diagonal interval matrix whose diagonal entries are $(\text{Diag}\,\mathbf{A}^{-1})_{ii} = 1/\mathbf{A}_{ii}$.
- $\text{OffDiag}\,\mathbf{A}$ is the interval matrix whose diagonal is null and whose off-diagonal entries are $(\text{OffDiag}\,\mathbf{A})_{ij} = \mathbf{A}_{ij}$.

3. A test for the inclusion in the range of a function

This section presents Theorem 3.1 which provides a sufficient condition for a box $\mathbf{y} \subseteq \mathbb{R}^n$ to be included inside the image of a box $\mathbf{x} \subseteq \mathbb{R}^n$ by a continuously differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Its corollary 3.1 will be used in the next section as the basis of a specific bisection algorithm that computes an inner approximation of $\text{range}\,(f, \mathbf{x})$.

Theorem 3.1. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{IR}^n$ *and* $f : \mathbf{x} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a continuous function continuously differentiable in* $\operatorname{int} \mathbf{x}$*. Suppose that the three following conditions are satisfied:*

(i) $\mathbf{y} \cap \operatorname{range}(f, \partial\mathbf{x}) = \emptyset$*;*
(ii) $\mathbf{y} \cap \operatorname{range}(f, \Sigma) = \emptyset$ *where* $\Sigma = \{x \in \operatorname{int} \mathbf{x} \mid \det f'(x) = 0\}$*;*
(iii) $f(\tilde{x}) \in \mathbf{y}$ *for some* $\tilde{x} \in \mathbf{x}$*.*

Then $\mathbf{y} \subseteq \operatorname{range}(f, \mathbf{x})$*.*

Proof. $\mathbf{x}$ is compact and we can apply Lemma A.2 of Appendix A. Therefore

$$\partial\big(\operatorname{range}(f, \mathbf{x})\big) \subseteq \operatorname{range}(f, \partial\mathbf{x}) \bigcup \operatorname{range}(f, \Sigma).$$

Using the conditions (i) and (ii), we have

$$\mathbf{y} \cap \partial\big(\operatorname{range}(f, \mathbf{x})\big) = \emptyset. \tag{1}$$

Denote $\operatorname{range}(f, \mathbf{x})$ by E (which is compact because $\mathbf{x}$ is compact and f is continuous) and $f(\tilde{x})$ by y. Notice that $y \in \operatorname{int} E$ because of (1). Consider any $z \in \mathbf{y}$ and suppose that $z \notin E$. There exists a path included in $\mathbf{y}$ between y and z. By Lemma A.1 of Appendix A this path intersects ∂E which is absurd by (1). Therefore $z \in E$ which concludes the proof. □

Theorem 3.1 can be used in many ways to construct some inner approximations of the range of a function. The bisection algorithm to be proposed in the next section is based on the following corollary of Theorem 3.1.

Corollary 3.1. *Let* $\mathbf{x} \in \mathbb{IR}^n$ *and* $f : \mathbf{x} \longrightarrow \mathbb{R}^n$ *be a continuous function continuously differentiable in* $\operatorname{int} \mathbf{x}$*. Consider* $\mathbf{y} \in \mathbb{IR}^n$ *and* $\tilde{x} \in \mathbf{x}$ *such that* $f(\tilde{x}) \in \mathbf{y}$*. Consider also an interval matrix* $\mathbf{J} \in \mathbb{IR}^{n\times n}$ *that contains all* $f'(x)$ *for* $x \in \mathbf{x}$*. Suppose that* $0 \notin \mathbf{J}_{ii}$ *for all* $i \in [1..n]$*. Then*

$$\tilde{x} + \Gamma\big(\mathbf{J}, (\mathbf{x} - \tilde{x}), \mathbf{y} - f(\tilde{x})\big) \subseteq \operatorname{int} \mathbf{x} \tag{2}$$

$$\Longrightarrow \mathbf{y} \subseteq \operatorname{range}(f, \mathbf{x}),$$

where $\Gamma(\mathbf{A}, \mathbf{u}, \mathbf{b}) := (\operatorname{Diag}^{-1} \mathbf{A})\big(\mathbf{b} - (\operatorname{OffDiag} \mathbf{A})\mathbf{u}\big)$*. Furthermore,* (2) *also implies that the matrix* $\mathbf{J}$ *is an H-matrix.*

Proof. It is sufficient to prove that the condition (2) implies the three conditions of Theorem 3.1.
(i) Consider an arbitrary $x \in \partial\mathbf{x}$. As $x \in \mathbf{x}$ the mean-value theorem shows that $f(x) \in f(\tilde{x}) + \mathbf{J}(x - \tilde{x})$ (this is the well-known argument that leads to the interval mean-value extension). It is therefore sufficient to prove that $\big(f(\tilde{x}) + \mathbf{J}(x - \tilde{x})\big) \bigcap \mathbf{y} \neq \emptyset$ contradicts (2). So suppose that there exist $J \in \mathbf{J}$ and $y \in \mathbf{y}$ such that $f(\tilde{x}) + J(x - \tilde{x}) = y$. Splitting J to $(\operatorname{Diag} J) + (\operatorname{OffDiag} J)$ leads to

$$x = \tilde{x} + (\operatorname{Diag}^{-1} J)\Big(y - f(\tilde{x}) - (\operatorname{OffDiag} J)(x - \tilde{x})\Big).$$

As $x \in \mathbf{x}$, $y \in \mathbf{y}$, $(\mathrm{Diag}^{-1} J) \in (\mathrm{Diag}^{-1} \mathbf{J})$ and $(\mathrm{OffDiag}\, J) \in (\mathrm{OffDiag}\, \mathbf{J})$, using the interval arithmetic leads to

$$x \in \tilde{x} + (\mathrm{Diag}^{-1} \mathbf{J})\Big(\mathbf{y} - f(\tilde{x}) - (\mathrm{OffDiag}\, \mathbf{J})(\mathbf{x} - \tilde{x})\Big).$$

As $x \in \partial \mathbf{x}$ this latter appurtenance contradicts (2).
(ii) It is sufficient to prove that (2) implies that any real matrix $J \in \mathbf{J}$ is regular (so $\Sigma = \emptyset$ and (ii) holds). The condition (2) implies

$$\tilde{x} + (\mathrm{Diag}^{-1} J)\Big(\mathbf{y} - f(\tilde{x}) - (\mathrm{OffDiag}\, J)(\mathbf{x} - \tilde{x})\Big) \subseteq \mathrm{int}\, \mathbf{x}$$

for every $J \in \mathbf{J}$. As $f(\tilde{x}) \in \mathbf{y}$ (so $0 \in \big(\mathbf{y} - f(\tilde{x})\big)$) the previous inclusion implies

$$\tilde{x} - (\mathrm{Diag}^{-1} J)\big((\mathrm{OffDiag}\, J)(\mathbf{x} - \tilde{x})\big) \subseteq \mathrm{int}\, \mathbf{x}.$$

This inclusion proves that $\mathrm{rad}\, \big(\tilde{x} - (\mathrm{Diag}^{-1} J)\big((\mathrm{OffDiag}\, J)(\mathbf{x} - \tilde{x})\big)\big) < \mathrm{rad}\ \mathbf{x}$. Using the radius rules proved on page 84 of [18], one concludes that

$$|(\mathrm{Diag}^{-1} J)(\mathrm{OffDiag}\, J)|(\mathrm{rad}\ \mathbf{x}) < \mathrm{rad}\ \mathbf{x}.$$

As $(\mathrm{Diag}^{-1} J)$ is diagonal, $|(\mathrm{Diag}^{-1} J)(\mathrm{OffDiag}\, J)| = |(\mathrm{Diag}^{-1} J)||(\mathrm{OffDiag}\, J)|$ and the i^{th} line of the latter inequality can be written

$$\sum_{i \neq j} |J_{ij}|\, \mathrm{rad}\ \mathbf{x}_j < |J_{ii}|\, \mathrm{rad}\ \mathbf{x}_i$$

As $|J_{ij}|\, \mathrm{rad}\ \mathbf{x}_j = |J_{ij}(\mathrm{rad}\ \mathbf{x}_j)|$ (because $\mathrm{rad}\ \mathbf{x}_j \geq 0$), the latter inequality means that J can be scaled to a strictly diagonally dominant matrix. As a consequence, J is an H-matrix and is eventually regular. This also proves that the interval matrix $\mathbf{J}$ is an H-matrix (because it contains only real H-matrices).
(iii) By hypothesis, $f(\tilde{x}) \in \mathbf{y}$. □

Remark. Corollary 3.1 is closely related to the existence test provided by the Hansen-Sengupta interval operator (see [6]). Indeed the latter is retrieved if $\mathbf{y}$ is set to $[0, 0]$ in the expression (2). As a consequence, Theorem 3.1 provides an elementary proof of the existence test related to the Hansen-Sengupta operator.

4. A preconditioning process

In order to succeed, the test provided by Corollary 3.1 needs the interval evaluation of the derivative of f to be an H-matrix. However, this is not the case in general. The following simple example illustrates this situation.

Example 4.1. Consider the linear function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $f(x) = Mx$ with

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Also consider $\mathbf{x} = ([-2, 2], [-2, 2])^T$ and $\mathbf{y} = ([-\epsilon, \epsilon], [-\epsilon, \epsilon])^T$. Obviously, the image of $\mathbf{x}$ is obtained from $\mathbf{x}$ by a $\pi/4$ rotation followed by a $\sqrt{2}$

inflation. Therefore $\mathbf{y}$ is a subset of the image of $\mathbf{x}$ by f if and only if $\epsilon \leq 2$. The function f has a constant derivative so M is an interval evaluation of its derivative. Now apply Corollary 3.1: on one hand choose $\tilde{x} = \operatorname{mid} \mathbf{x}$ so $\tilde{x} = 0$ and $f(\tilde{x}) = 0$. On the other hand $(\operatorname{Diag} \mathbf{J}) = I$ and $(\operatorname{OffDiag} \mathbf{J}) = M - I$. Therefore Corollary 3.1 leads to the following implication:

$$\begin{pmatrix} [-\epsilon,\epsilon] \\ [-\epsilon,\epsilon] \end{pmatrix} - \begin{pmatrix} [-2,2] \\ -[-2,2] \end{pmatrix} \subseteq \operatorname{int} \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \implies \mathbf{y} \subseteq \operatorname{range}(f, \mathbf{x}).$$

The sufficient condition is never satisfied whatever is $\epsilon > 0$. This is due to the fact that $\mathbf{J} = M$ is not an H-matrix.

In such situations, the right thing to do is to precondition the problem in such a way that the interval evaluation of the derivative becomes close to the identity matrix (and therefore possibly an H-matrix). In the present situation, preconditioning the problem consists in replacing the inclusion to be checked, i.e. $\mathbf{y} \subseteq \operatorname{range}(f, \mathbf{x})$, by the more restrictive one

$$C\mathbf{y} \subseteq \operatorname{range}(Cf, \mathbf{x}), \tag{3}$$

where $C \in \mathbb{R}^{n\times n}$ is any regular real matrix. The latter is indeed more restrictive than the former: (3) implies $Cy \in \operatorname{range}(Cf, \mathbf{x})$ for every $y \in \mathbf{y}$, that is $\big(\exists x \in \mathbf{x}\big)\big(Cy = Cf(x)\big)$. As C is regular, this finally implies $y \in \operatorname{range}(f, \mathbf{x})$ for every $y \in \mathbf{y}$, i.e. $\mathbf{y} \subseteq \operatorname{range}(f, \mathbf{x})$. As usual, $C\mathbf{J}$ is an interval evaluation of the derivative of Cf provided that $\mathbf{J}$ is an interval evaluation of the derivative of f. Therefore, once $\mathbf{J}$ is computed using the expressions of the derivative of f, one can choose $C \approx (\operatorname{mid} \mathbf{J})^{-1}$ in order to obtain a near identity interval evaluation of the derivative of Cf (so that $C\mathbf{J}$ can be an H-matrix). The preconditioned version of Corollary 3.1 leads to the following condition:

$$\begin{aligned} \tilde{x} + \Gamma\big(C\mathbf{J}, (\mathbf{x} - \tilde{x}), C\mathbf{y} - Cf(\tilde{x})\big) &\subseteq \operatorname{int} \mathbf{x} \\ \implies \mathbf{y} &\subseteq \operatorname{range}(f, \mathbf{x}). \end{aligned} \tag{4}$$

When the midpoint inverse preconditioning is used (i.e. $C \approx (\operatorname{mid} \mathbf{J})^{-1}$), it is proved in [18] that $C\mathbf{J}$ is an H-matrix if and only if $\mathbf{J}$ is strongly regular. Therefore the strong regularity of the interval evaluation of the derivative of f is a necessary condition for the success of the preconditioned version of Corollary 3.1. Example 4.1 is now revisited thanks to the presented preconditioning:

Example 4.2. Consider f, $\mathbf{x}$ and $\mathbf{y}$ as in Example 4.1. Choose $C = M^{-1}$ so that $Cf(x) = x$. Therefore the interval evaluation of the derivative of Cf is I. The preconditioned version of Corollary 3.1 leads to the following condition:

$$\begin{gathered} \begin{pmatrix} 0.5\times[-\epsilon,\epsilon] - 0.5\times[-\epsilon,\epsilon] \\ 0.5\times[-\epsilon,\epsilon] + 0.5\times[-\epsilon,\epsilon] \end{pmatrix} - \begin{pmatrix} 0\times[-2,2] \\ 0\times[-2,2] \end{pmatrix} \subseteq \operatorname{int} \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \\ \Downarrow \\ \mathbf{y} \subseteq \operatorname{range}(f, \mathbf{x}). \end{gathered}$$

Therefore $\mathbf{y} \subseteq \text{range}\,(f, \mathbf{x})$ is now proved for $\epsilon < 2$.

5. A specific bisection algorithm

This section presents a bisection algorithm that computes a paving included inside the image of a box $\mathbf{x}$ by a function f. A bisection of both the domain and co-domain aiming at a direct use of the tests provided in the previous section presents several drawbacks: on one hand, this would result in the bisection of a $2n$ dimensional space. On the other hand, the bisection would have to provide couples of boxes $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ that satisfy Corollary 3.1. This situation is not easy to reach during a bisection process. Some experimentations have shown that acting in such a way leads to instability of the bisection algorithm. The specific bisection algorithm described in Subsection 5.1 overcomes these difficulties by bisecting only the box $\mathbf{x}$. It is based on a function Inner() which is described in Subsection 5.2. The convergence of the algorithm is investigated in Subsection 5.3. Finally Subsection 5.4 provides the modifications needed for the bisection algorithm to deal with domains defined by constraints.

5.1. **The bisection algorithm.** Given the initial box $\mathbf{x} \subseteq \mathbb{R}^n$ and a function $f : \mathbf{x} \longrightarrow \mathbb{R}^n$, the bisection algorithm bisects only $\mathbf{x}$ producing a list of boxes $\mathcal{L}_{\text{Domain}} = \{\tilde{\mathbf{x}}^{(1)}, \cdots, \tilde{\mathbf{x}}^{(N)}\}$. Given $\tilde{\mathbf{x}} \in \mathcal{L}_{\text{Domain}}$, the basic idea is to compute some outer approximation $\tilde{\mathbf{y}}$ of $\text{range}\,(f, \tilde{\mathbf{x}})$ using an interval extension of f and to test if $\tilde{\mathbf{y}}$ is a subset of $\text{range}\,(f, \mathbf{x})$. This results in a list of boxes $\mathcal{L}_{\text{Inside}}$ that are proved to be subsets of $\text{range}\,(f, \mathbf{x})$. This is illustrated by Figure 1 for $n = 1$: the initial interval $\mathbf{x}$ has been bisected to twelve intervals (one $\tilde{\mathbf{x}}$ being displayed). For each of them, an outer approximation has been computed. The resulting boxes $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})^T$ are displayed in white if $\tilde{\mathbf{y}} \nsubseteq \text{range}\,(f, \mathbf{x})$ or in gray if $\tilde{\mathbf{y}} \subseteq \text{range}\,(f, \mathbf{x})$. The figure displays seven intervals that are subsets of $\text{range}\,(f, \mathbf{x})$. One can notice that these intervals actually overlap.

The core of this bisection algorithm is therefore a function Inner() that tests if a box $\tilde{\mathbf{y}}$ is a subset of the image of $\mathbf{x}$. The information that $\tilde{\mathbf{y}}$ is an outer approximation of $\text{range}\,(f, \tilde{\mathbf{x}})$ is of central importance for an efficient implementation of the function Inner() (see Subsection 5.2). Therefore, the function Inner() will have four arguments f, $\mathbf{x}$, $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. It satisfies

$$\text{Inner}(f, \mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = 1 \implies \tilde{\mathbf{y}} \subseteq \text{range}\,(f, \mathbf{x})\,.$$

The efficient implementation of this function is proposed in Subsection 5.2.

Finally, a simple but efficient bisection algorithm is then easily constructed. It is summarized in Algorithm 1. The algorithm computes both an inner and an outer approximation of $\text{range}\,(f, \mathbf{x})$: on one hand, obviously

$$\bigcup_{\mathbf{y} \in \mathcal{L}_{\text{Inside}}} \mathbf{y} \;\subseteq\; \text{range}\,(f, \mathbf{x})\,.$$

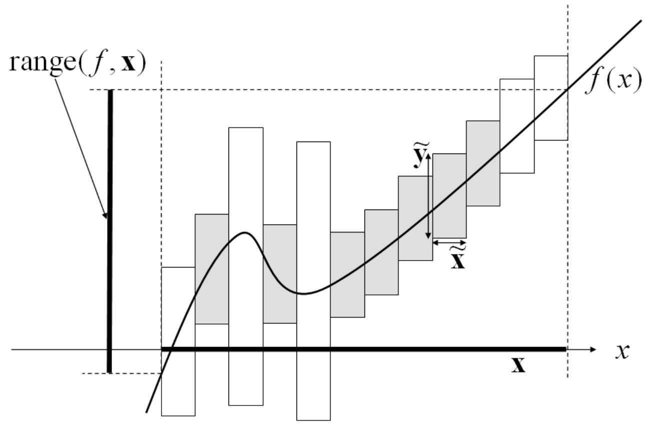

FIGURE 1. Bisection of the initial domain $\mathbf{x}$ leading to a list of boxes $\tilde{\mathbf{x}}$ and to some outer approximations $\tilde{\mathbf{y}}$ of their image.

On the other hand, the initial box $\mathbf{x}$ has been paved into a set of boxes $\tilde{\mathbf{x}}$ and for each $\tilde{\mathbf{x}}$ an outer approximation of range $(f, \tilde{\mathbf{x}})$ is stored either in the list Inside or in the list Boundary. Therefore,

$$\text{range}\,(f, \mathbf{x}) \subseteq \bigcup_{\mathbf{y} \in \left(\mathcal{L}_{\text{Inside}} \bigcup \mathcal{L}_{\text{Boundary}}\right)} \mathbf{y}.$$

The next section presents the *domain inflation* which plays a key role in the implementation of the function Inner$(f,\mathbf{x},\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. The bisection algorithm has been actually made stable thanks to the *domain inflation* (experimentations have shown that constructing some bisection algorithms without *domain inflation* leads to unstable algorithms).

5.2. **The domain inflation.** The basic idea of *domain inflation* is the following: suppose that one tries to prove that the quantified proposition

$$\big(\forall y \in \tilde{\mathbf{y}}\big)\big(\exists x \in \mathbf{x}\big)\big(\phi(x, y)\big),$$

where ϕ is a relation (in the case of the approximation of the range of a function $\phi(x, y) \equiv f(x) = y$). After the bisection of the box $\mathbf{x}$, he may happen to have to prove the quantified proposition

$$\big(\forall y \in \tilde{\mathbf{y}}\big)\big(\exists x \in \tilde{\mathbf{x}}\big)\big(\phi(x, y)\big), \tag{5}$$

where $\tilde{\mathbf{x}} \subseteq \mathbf{x}$. At this point, it is allowed and certainly useful to inflate $\tilde{\mathbf{x}}$ to a bigger box $\mathbf{x}^*$ that satisfies $\tilde{\mathbf{x}} \subseteq \mathbf{x}^* \subseteq \mathbf{x}$. Then the quantified proposition

$$\big(\forall y \in \tilde{\mathbf{y}}\big)\big(\exists x \in \mathbf{x}^*\big)\big(\phi(x, y)\big) \tag{6}$$

may be easier to validate. The trivial choice $\mathbf{x}^* = \mathbf{x}$ cannot be done in general because sufficient conditions for the validity of the quantified proposition (6) certainly work only on small enough intervals (e.g. it involves

Data: f , $\mathbf{x}$, ϵ
Result: $\mathcal{L}_{\texttt{Inside}}$ (list of boxes), $\mathcal{L}_{\texttt{Boundary}}$ (list of boxes)
$\mathcal{L}_{\texttt{Inside}}$: empty list of boxes;
$\mathcal{L}_{\texttt{Domain}}$: empty list of boxes (sorted by decreasing radius);
Store the box $\mathbf{x}$ in $\mathcal{L}_{\texttt{Domain}}$;
while $\mathcal{L}_{\texttt{Domain}}$ *not empty* **do**
  $\tilde{\mathbf{x}} \leftarrow$ Extract($\mathcal{L}_{\texttt{Domain}}$);
  $\tilde{\mathbf{y}} \leftarrow \mathbf{f}(\tilde{\mathbf{x}}) \cap \big(\mathbf{f}(\text{mid } \tilde{\mathbf{x}}) + \mathbf{f}'(\tilde{\mathbf{x}})(\tilde{\mathbf{x}} - \text{mid } \tilde{\mathbf{x}})\big)$;
  if Inner(f,$\mathbf{x}$,$\tilde{\mathbf{x}}$,$\tilde{\mathbf{y}}$) **then**
    Store the box $\tilde{\mathbf{y}}$ in $\mathcal{L}_{\texttt{Inside}}$;
  else if $||\text{rad } \tilde{\mathbf{x}}|| \geq \epsilon$ **then**
    Bisect the box $\tilde{\mathbf{x}}$ to obtain $\tilde{\mathbf{x}}'$ and $\tilde{\mathbf{x}}''$;
    Store $\tilde{\mathbf{x}}'$ and $\tilde{\mathbf{x}}''$ in $\mathcal{L}_{\texttt{Domain}}$;
  else
    Store $\tilde{\mathbf{y}}$ in $\mathcal{L}_{\texttt{Boundary}}$;
  end
end
return ($\mathcal{L}_{\texttt{Inside}}$,$\mathcal{L}_{\texttt{Boundary}}$);

Algorithm 1: The bisection algorithm

some interval extensions that are precise only for small intervals) while the initial interval $\mathbf{x}$ can be very large. Therefore, instead of trying to prove the quantified proposition (5), the task becomes to construct a box $\mathbf{x}^*$ such $\tilde{\mathbf{x}} \subseteq \mathbf{x}^* \subseteq \mathbf{x}$ and such that one can easily prove that the quantified proposition (6) is true.

In Algorithm 1, the *domain inflation* is obligatory as the quantified proposition (5) is generally false: indeed the box $\tilde{\mathbf{y}}$ is an outer approximation of range$(f, \tilde{\mathbf{x}})$ so the quantified proposition (5) cannot be true. Therefore, the box $\tilde{\mathbf{x}}$ has to be inflated before applying Corollary 3.1. This inflation process is illustrated by Figure 2: a box $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})^T$ is displayed in the left hand side graphic. Of course $\tilde{\mathbf{y}} \not\subseteq \text{range}\,(f, \tilde{\mathbf{x}})$ because $\tilde{\mathbf{y}}$ is an outer approximation of range$(f, \tilde{\mathbf{x}})$ by construction. So $\tilde{\mathbf{x}}$ is enlarged to $\mathbf{x}^*$ in the right hand side graphic in such a way that $\tilde{\mathbf{y}} \subseteq \text{range}\,(f, \mathbf{x}^*)$ holds.

Corollary 3.1 offers an easy and efficient way to enlarge $\tilde{\mathbf{x}}$ to $\mathbf{x}^*$: we compute the fixed point iteration

$$\begin{array}{rcl} \mathbf{x}^{(0)} & = & \tilde{\mathbf{x}}, \\ \mathbf{u}^{(k+1)} & = & \Gamma\big(C\mathbf{f}'(\mathbf{x}^{(k)}), \mathbf{x}^{(k)} - \tilde{x}, C\tilde{\mathbf{y}} - Cf(\tilde{x})\big), \\ \mathbf{x}^{(k+1)} & = & \tilde{x} + \tau \mathbf{u}^{(k+1)}, \end{array} \tag{7}$$

where $\tau > 1$ (experimentations have shown that $\tau = 1.01$ is an efficient choice), $\tilde{x} := \text{mid } \tilde{\mathbf{x}}$ and $C = f'(\tilde{x})^{-1}$ (note that $\tilde{x}$ and C are fixed during the iteration). We expect this iteration to be contracting, therefore it is stopped as soon as the distance between consecutive iterated does not decrease of at

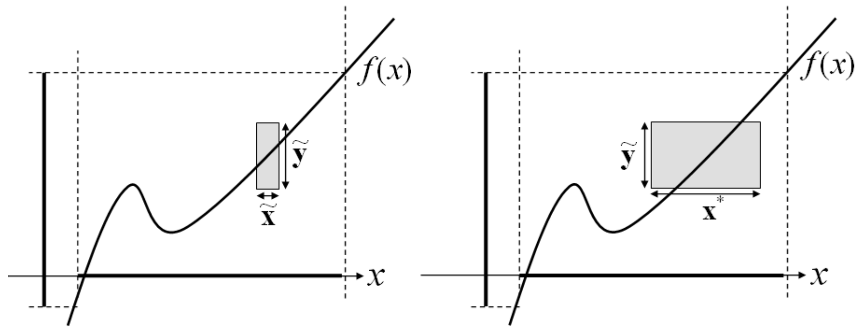

FIGURE 2. Enlargement of the box $\tilde{\mathbf{x}}$ to $\mathbf{x}^*$ aiming at a successful application of Corollary 3.1 applied to $\mathbf{x}^*$ and $\tilde{\mathbf{y}}$.

least a given ratio μ ($\mu = 0.9$ has been chosen based on experimentations). In this case, the process fails proving $\tilde{\mathbf{y}} \subseteq \operatorname{range}(f, \mathbf{x})$. At each step of the iteration, we check if Corollary 3.1 can prove $\tilde{\mathbf{y}} \subseteq \operatorname{range}\big(f, \mathbf{x}^{(k)}\big)$.

Remark. The inflation factor τ is introduced for the following reason: the limit $\mathbf{x}^{(\infty)}$ of the iteration satisfies

$$\Gamma\big(C\mathbf{f}'(\mathbf{x}^{(\infty)}), \mathbf{x}^{(\infty)} - \tilde{x}, C\tilde{\mathbf{y}} - Cf(\tilde{x})\big) \;=\; \tau^{-1}(\mathbf{x}^{(\infty)} - \tilde{x}).$$

As the inclusion

$$\Gamma\big(C\mathbf{f}'(\mathbf{x}^{(\infty)}), \mathbf{x}^{(\infty)} - \tilde{x}, C\tilde{\mathbf{y}} - Cf(\tilde{x})\big) \subseteq \operatorname{int}(\mathbf{x}^{(\infty)} - \tilde{x})$$

is required to apply Corollary 3.1, we need $\tau > 1$ so that the limit of the iteration satisfies Corollary 3.1.

This implementation of the function Inner(f,$\mathbf{x}$,$\tilde{\mathbf{x}}$,$\tilde{\mathbf{y}}$) is described in Algorithm 2. It is correct as it returns true only if $\tilde{\mathbf{x}} \subseteq \mathbf{x}$ (Line 6) and if $\tilde{x} + \Gamma(C\mathbf{f}'(\tilde{\mathbf{x}}), \tilde{\mathbf{x}} - \tilde{x}, C\tilde{\mathbf{y}} - Cf(\tilde{x})) \subseteq \tilde{\mathbf{x}}$ (Line 8). The latter inclusion allows applying Corollary 3.1 to prove that $\tilde{\mathbf{y}} \subseteq \operatorname{range}(f, \tilde{\mathbf{x}})$ holds. Finally, termination of Algorithm 2 is obvious: as computations are performed with floating point numbers, the fixed point iteration cannot be μ-contracting forever, and hence the condition $d_k \leq \mu d_{k-1}$ will be false after a finite number of steps.

5.3. **Convergence of the algorithm.** The convergence of the algorithm is now investigated. The test based on the preconditioned version of Corollary 3.1 cannot succeed if the interval evaluation of the derivative of f is not regular. Therefore, if $\det f'(x) = 0$ for some $x \in \mathbf{x}$ then it will not be possible to prove that $f(x)$ is inside $\operatorname{range}(f, \mathbf{x})$ using this x. However, there may exist another $x' \in \mathbf{x}$ such that both $\det f'(x') \neq 0$ and $f(x') = f(x)$ and $f(x)$ may eventually be proved to belong to $\operatorname{range}(f, \mathbf{x})$. To formalize this idea, the following subset of $\operatorname{range}(f, \mathbf{x})$ is defined:

$$\operatorname{range}^*(f, \mathbf{x}) \;\overset{\text{def}}{=}\; \big\{ y \in \mathbb{R}^n \mid \big(\exists x \in \operatorname{int}\mathbf{x}\big)\big(y = f(x) \wedge \det f'(x) \neq 0\big) \big\}.$$

Data: f, $\mathbf{x}$, $\tilde{\mathbf{x}}$,$\tilde{\mathbf{y}}$
Result: Boolean
$\tau \leftarrow 1.01$; $\mu \leftarrow 0.9$;
$\tilde{x} \leftarrow \text{mid } \tilde{\mathbf{x}}$;
$C \leftarrow f'(\tilde{x})^{-1}$;
$\mathbf{b} \leftarrow C\tilde{\mathbf{y}} - Cf(\tilde{x})$;
$d_k \leftarrow +\infty$; $d_{k-1} \leftarrow +\infty$; `/* The while loop is run at least twice */`
while $d_k \leq \mu d_{k-1} \ \wedge \ \tilde{\mathbf{x}} \subseteq \mathbf{x}$ **do**
  $\mathbf{u} \leftarrow \Gamma(C\mathbf{f}'(\tilde{\mathbf{x}}), \tilde{\mathbf{x}} - \tilde{x}, \mathbf{b})$;
  if $\tilde{x} + \mathbf{u} \subseteq \tilde{\mathbf{x}}$ **then return** True;
  $d_{k-1} \leftarrow d_k$;
  $d_k \leftarrow d(\tilde{\mathbf{x}}, \tilde{x} + \tau\mathbf{u})$;
  $\tilde{\mathbf{x}} \leftarrow \tilde{x} + \tau\mathbf{u}$;
end
return False;

Algorithm 2: Function Inner()

Then, the following theorem shows that Algorithm 1 converges *asymptotically* to $\text{range}^*(f, \mathbf{x})$. That is, ignoring the finite precision of the computations, any $y \in \text{range}^*(f, \mathbf{x})$ is finally proved to be inside $\text{range}^*(f, \mathbf{x})$ provided that a small enough ϵ is chosen.

Theorem 5.1. *Suppose that the interval extensions* $\mathbf{f}$ *and* $\mathbf{f}'$ *used in algorithms 1 and 2 locally Lipschitz continuous*[1]*. Then for all* $y \in \text{range}^*(f, \mathbf{x})$ *there exists* $\epsilon > 0$ *such that Algorithm 1 succeeds proving that* $y \in \text{range}(f, \mathbf{x})$.

Proof. Let us consider an arbitrary $y^* \in \text{range}^*(f, \mathbf{x})$. So, there exists $x^* \in \text{int}\,\mathbf{x}$ such that $y^* = f(x^*)$ and $\det f'(x^*) \neq 0$. The theorem is proved by contradiction: we suppose that, no matter how small ϵ is chosen, y^* is not proved to be inside $\text{range}(f, \mathbf{x})$ by the algorithm. Therefore, we can pick up a sequence of boxes $\mathbf{x}^{(k)}$ such that $x^* \in \mathbf{x}^{(k)}$, $||\text{wid } \mathbf{x}^{(k)}|| \leq 1/k$ and $\text{Inner}(f, \mathbf{x}, \mathbf{x}^{(k)}, \mathbf{y}^{(k)})$ fails for all k, where $\mathbf{y}^{(k)}$ is defined as in Line 6 of Algorithm 1, i.e.

$$\mathbf{y}^{(k)} := \mathbf{f}(\mathbf{x}^{(k)}) \cap \big(\mathbf{f}(\tilde{x}^{(k)}) + \mathbf{f}'(\mathbf{x}^{(k)})(\mathbf{x}^{(k)} - \tilde{x}^{(k)})\big),$$

with $\tilde{x}^{(k)} := \text{mid } \mathbf{x}^{(k)}$. To build a contradiction, we just have to find a $l \geq 0$ for which $\text{Inner}(f, \mathbf{x}, \mathbf{x}^{(l)}, \mathbf{y}^{(l)})$ succeeds.

For $\delta \geq 0$ let us define $\mathbf{x}^\delta := x^* \pm (\delta e)$, where $e_i = 1$ for all $i \in \{1, \ldots, n\}$. By Proposition A.1 (cases (2) and (3)), there exists $\delta^*, \kappa^*, c_1 > 0$ such that, for all $\tilde{x} \in \mathbf{x}^{\delta^*}$, $f'(\tilde{x})$ is nonsingular and $||f'(\tilde{x})^{-1}|| \leq \kappa^*$, and for all

[1]The natural interval extensions can be used under mild hypothesis like $\sqrt{\text{abs}(x)}$ is not evaluated for intervals which contain 0.

$\delta \leq \delta^*$ and all $\tilde{x} \in \mathbf{x}^\delta$, $f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x}^\delta) \subseteq I \pm (c_1\,\delta\,E)$, where $E_{ij} = 1$ for all $i, j \in \{1, \ldots, n\}$. Define $\mathbf{x}^* := \mathbf{x}^{\delta^*}$ and

$$\mathbf{b}^* := \square\{f'(\tilde{x})^{-1}\mathbf{f}(\tilde{x}) - f'(\tilde{x})^{-1}f(\tilde{x}) + f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x}^*)(\mathbf{x}^* - \tilde{x}) : \tilde{x} \in \mathbf{x}^*\}$$

(which is well defined because $||f'(\tilde{x})^{-1}|| \leq \kappa^*$). Note that for all $\tilde{x} \in \mathbf{x}^*$ we have $\mathbf{b}^* \supseteq f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x}^*)(\mathbf{x}^* - \tilde{x})$.

For $\tilde{x} \in \mathbf{x}^*$, consider the function $\mathbf{A}_{\tilde{x}} : \mathbb{I}\mathbf{x}^* \longrightarrow \mathbb{IR}^{n\times n}$ defined by $\mathbf{A}_{\tilde{x}}(\mathbf{x}) := f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x})$. As $\mathbf{f}'$ is locally Lipschitz continuous, it is Lipschitz continuous inside $\mathbb{I}\mathbf{x}^*$, for some Lipschitz constant λ^*. As $||f'(\tilde{x})^{-1}|| \leq \kappa^*$, $\mathbf{A}_{\tilde{x}}$ is $(\kappa^*\lambda^*)$-Lipschitz continuous inside $\mathbb{I}\mathbf{x}^*$.

For some arbitrary $\delta \leq \min\{1/2, \delta^*\}$, $\tilde{x} \in \mathbf{x}^\delta$, $\mathbf{b} \subseteq \mathbf{b}^*$, define

$$\begin{array}{rccl} \mathbf{g}_{\delta,\mathbf{b},\tilde{x}} : & \mathbb{I}\mathbf{x}^\delta & \longrightarrow & \mathbb{IR}^n \\ & \mathbf{x} & \longmapsto & \Gamma(\mathbf{A}_{\tilde{x}}(\mathbf{x}), \mathbf{x} - \tilde{x}, \mathbf{b}). \end{array}$$

Proposition A.2 shows that there exists $c_2 > 0$ such that the functions $\mathbf{g}_{\delta,\mathbf{b},\tilde{x}}$ are $c_1\,c_2\,\delta$-Lipschitz. Furthermore, using Proposition A.1 (case (1)) and Lemma A.4, we show that there exists $c_3 > 0$ such that

$$\forall \mathbf{x} \in \mathbb{I}\mathbf{x}^*\ ,\ \forall \tilde{x}, \tilde{y} \in \mathbf{x}^*\ ,\ d\big(\mathbf{g}_{\delta,\mathbf{b},\tilde{x}}(\mathbf{x}), \mathbf{g}_{\delta,\mathbf{b},\tilde{y}}(\mathbf{x})\big) \leq c_3\,||\tilde{x} - \tilde{y}||. \tag{8}$$

Then, define

$$\delta^+ = \min\{1/2, \delta^*, (c_1\tau(n+1))^{-1}, \mu(c_1c_2\tau)^{-1}\}$$

so that both $\tau\mathbf{A}_{\tilde{x}}(\mathbf{x}^{\delta^+})$ is strictly diagonally dominant and $\delta \leq \delta^+$ implies that the functions $\tau\mathbf{g}_{\delta,\mathbf{b},\tilde{x}}$ are μ-Lipschitz.

Define $\mathbf{x}^+ := \mathbf{x}^{\delta^+} \subseteq \mathbf{x}^*$. Lemma A.5 shows that there exist $\mathbf{b}^+ \in \mathbb{IR}^n$ such that $\operatorname{rad}\mathbf{b}^+ > 0$, $0 \in \operatorname{int}\mathbf{b}^+$, $\mathbf{b}^+ \subseteq \mathbf{b}^*$ and

$$\tau\mathbf{g}_{\delta^+,\mathbf{b}^+,x^*}(\mathbf{x}^+) = \mathbf{x}^+ - x^*. \tag{9}$$

Now define $\mathbf{b}^{(k)} := f'(\tilde{x}^{(k)})^{-1}\mathbf{y}^{(k)} - f'(\tilde{x}^{(k)})^{-1}f(\tilde{x}^{(k)})$ which satisfies $\mathbf{b}^{(k)} \subseteq \mathbf{b}^*$ (see the definition of $\mathbf{b}^*$ and note that $\mathbf{y}^{(k)} \subseteq \mathbf{f}(\tilde{x}^{(k)}) + \mathbf{f}'(\mathbf{x}^{(k)})(\mathbf{x}^{(k)} - \tilde{x}^{(k)})$). As $\mathbf{f}$ is locally Lipschitz continuous, there exists $c > 0$ such that $||\operatorname{wid}\mathbf{y}^{(k)}|| \leq c||\operatorname{wid}\mathbf{x}^{(k)}|| = c/k$, which implies $||\operatorname{wid}\mathbf{b}^{(k)}|| \leq c\kappa^*/k$. Finally, define

$$l' := \left\lceil \frac{c\kappa^* + c_3 + 1}{m^+} \right\rceil \quad \text{and} \quad l = \max\{l', \lceil 1/\delta^+ \rceil\} \tag{10}$$

where $m^+ := \min_i |\mathbf{b}^+_i|$ (which satisfies $m^+ > 0$ as $0 \in \operatorname{int}\mathbf{b}^+$ and $\pm(m^+e) \subseteq \mathbf{b}^+$), so that both $\mathbf{x}^{(l)} \subseteq \mathbf{x}^+$ (because $\operatorname{wid}\mathbf{x}^{(l)} \leq 1/l \leq \delta^+ = \operatorname{rad}\mathbf{x}^+$ while $x^* \in \mathbf{x}^{(l)}$ and $\mathbf{x}^+ = x^* \pm (\delta^+e)$) and $l \geq l'$.

We are now in position to prove that $\operatorname{Inner}(f, \mathbf{x}, \mathbf{x}^{(l)}, \mathbf{y}^{(l)})$ succeeds. Fix $\tilde{x} = \tilde{x}^{(l)}$ and define $\mathbf{h} : \mathbb{I}\mathbf{x}^+ \longrightarrow \mathbb{IR}^n$ by

$$\mathbf{h}(\mathbf{x}) \ := \ \tilde{x} + \tau\mathbf{g}_{\delta^+,\mathbf{b}^{(l)},\tilde{x}}(\mathbf{x}) \ = \ \tilde{x} + \tau\Gamma(f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x}), \mathbf{x} - \tilde{x}, \mathbf{b}^{(l)}).$$

Note that $\tilde{\mathbf{x}}$ computed at Line 11 of Algorithm 2 is actually $\tilde{\mathbf{x}} \leftarrow \mathbf{h}(\tilde{\mathbf{x}})$. We have proved that $\mathbf{h}$ is μ-Lipschitz continuous inside $\mathbb{I}\mathbf{x}^+$ and we now prove that for all $\mathbf{x} \in \mathbb{I}\mathbf{x}^+$, $\mathbf{h}(\mathbf{x}) \subseteq \mathbf{x}^+$. First, note that by (10) we have

$l \geq c\kappa^*/m^+$, which implies that $\alpha := m^+ - c\kappa^*/l \geq 0$. For all $i \in \{1,\dots,n\}$ we have

$$\alpha + \operatorname{wid}\, \mathbf{b}_i^{(l)} \;\leq\; \alpha + c\kappa^*/l \;=\; m^+.$$

As $0 \in \mathbf{b}^{(l)}$, we conclude that $\mathbf{b}_i^{(l)} \pm \alpha \subseteq \pm m^+ \subseteq \mathbf{b}_i^+$, and hence $\mathbf{b}^{(l)} \pm (\alpha e) \subseteq \mathbf{b}^+$. Now,

$$\begin{array}{rcl}
\mathbf{h}(\mathbf{x}) & = & \tilde{x} + \tau \mathbf{g}_{\delta+,\mathbf{b}^{(l)},\tilde{x}}(\mathbf{x}) \\
\text{(using (8))} & \subseteq & \tilde{x} + \tau \mathbf{g}_{\delta+,\mathbf{b}^{(l)},x^*}(\mathbf{x}) \pm (c_3/l\, e) \\
 & \subseteq & x^* + \tau \mathbf{g}_{\delta+,\mathbf{b}^{(l)},x^*}(\mathbf{x}) \pm ((1+c_3)/l\, e) \\
\text{(using (10))}^2 & \subseteq & x^* + \tau \mathbf{g}_{\delta+,\mathbf{b}^{(l)},x^*}(\mathbf{x}) \pm (\alpha e) \\
(1 \in (f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x}))_{ii}) & \subseteq & x^* + \tau \mathbf{g}_{\delta+,\mathbf{b}^{(l)},x^*}(\mathbf{x}) + (\pm(\alpha e))/(f'(\tilde{x})^{-1}\mathbf{f}'(\mathbf{x}))_{ii} \\
\text{(basic properties of IA)} & = & x^* + \tau \mathbf{g}_{\delta+,\mathbf{b}^{(l)}\pm(\alpha e),x^*}(\mathbf{x}) \\
\text{(inclusion isotonicity)} & \subseteq & x^* + \tau \mathbf{g}_{\delta+,\mathbf{b}^+,x^*}(\mathbf{x}) \\
\text{(using (9))} & = & \mathbf{x}^+.
\end{array}$$

Therefore, we can apply the Banach fixed point theorem which proves that the fixed point iteration $\tilde{\mathbf{x}} \leftarrow \mathbf{h}(\tilde{\mathbf{x}})$ converges to the unique solution of $\tilde{\mathbf{x}} = \mathbf{h}(\tilde{\mathbf{x}})$. From the proof of the Banach fixed point theorem, we know that the distance of consecutive iterated decreases of at least a factor μ. Therefore, the condition at Line 6 always succeeds. It just remains to prove that the condition at Line 8 eventually succeeds. This condition can be written $\tilde{x} + \mathbf{g}(\tilde{\mathbf{x}}) \subseteq \tilde{\mathbf{x}}$ with $\mathbf{g}(\tilde{\mathbf{x}}) := \mathbf{g}_{\delta+,\mathbf{b}^{(l)},\tilde{x}}(\tilde{\mathbf{x}})$.

Let us explicitly enumerate the iterations computed in Algorithm 2 by $\tilde{\mathbf{x}}^{(i)}$, and denote by $\tilde{\mathbf{x}}^*$ the unique interval vector which satisfies $\tilde{\mathbf{x}}^* = \mathbf{h}(\tilde{\mathbf{x}}^*)$ (which hence satisfies $\tau\mathbf{g}(\tilde{\mathbf{x}}^*) = \tilde{\mathbf{x}}^* - \tilde{x}$). By the Banach fixed point theorem, we know that $d(\tilde{\mathbf{x}}^{(i)}, \tilde{\mathbf{x}}^*) \leq \mu^i/(1-\mu) d(\tilde{\mathbf{x}}^{(0)}, \tilde{\mathbf{x}}^{(1)}) =: \beta^{(i)}$ and thus $d(\mathbf{g}(\tilde{\mathbf{x}}^{(i)}), \mathbf{g}(\tilde{\mathbf{x}}^*)) \leq \mu\tau^{-1}\beta^{(i)}$ (because $\tau\mathbf{g}$ is μ-Lipschitz continuous). Then,

$$\begin{array}{rcl}
\tilde{x} + \mathbf{g}(\tilde{\mathbf{x}}^{(i)}) & \subseteq & \tilde{x} + \mathbf{g}(\tilde{\mathbf{x}}^*) \pm (\mu\tau^{-1}\beta^{(i)}e) \\
 & = & \tilde{x} + \tau^{-1}\tau\mathbf{g}(\tilde{\mathbf{x}}^*) \pm (\mu\tau^{-1}\beta^{(i)}e) \\
 & = & \tilde{x} + \tau^{-1}(\tilde{\mathbf{x}}^* - \tilde{x}) \pm (\mu\tau^{-1}\beta^{(i)}e) \\
 & \subseteq & \tilde{x} + \tau^{-1}(\tilde{\mathbf{x}}^{(i)} \pm (\beta^{(i)}e) - \tilde{x}) \pm (\mu\tau^{-1}\beta^{(i)}e) \\
 & = & \tilde{x} + \tau^{-1}(\tilde{\mathbf{x}}^{(i)} - \tilde{x}) \pm (\gamma^{(i)}e)
\end{array}$$

with $\gamma^{(i)} := \tau^{-1}(1+\mu)\beta^{(i)}$. Now note that $[-m^+, m^+] \subseteq \mathbf{b}_j^{(l)}$ for all $j \in \{1,\dots,n\}$. Using direct computations, we show that $\tilde{x} \in \tilde{\mathbf{x}}^{(i)}$ implies both $\tilde{x} \in \tilde{\mathbf{x}}^{(i+1)}$ and $\mathbf{b}^{(l)} \subseteq \tilde{\mathbf{x}}^{(i+1)}$. By induction, we thus prove that for all $i \geq 1$ both $\tilde{x} \in \tilde{\mathbf{x}}^{(i)}$ and $\mathbf{b}^{(l)} \subseteq \tilde{\mathbf{x}}^{(i)}$ hold. Therefore, we can write $\tilde{\mathbf{x}}^{(i)} = \tilde{x} + [a^{(i)}, b^{(i)}]$ with $a_j^{(i)} \leq -m^+$ and $m^+ \leq b_j^{(i)}$ for all $j \in \{1,\dots,n\}$. We

[2]Note that (10) implies $1 + c_3 + c\kappa^* \leq lm^+$ which implies $(1+c_3)/l \leq \alpha$.

obtain

$$\tilde{x} + \mathbf{g}(\tilde{\mathbf{x}}^{(i)}) \quad \subseteq \quad \tilde{x} + [a^{(i)}/\tau - \gamma^{(i)} \ , \ b^{(i)}/\tau + \gamma^{(i)}].$$

Now because $\tau \geq 1$, multiplying $m^+ \leq b_j^{(i)}$ by $(1-1/\tau)$ gives rise to $b^{(i)}/\tau \leq b^{(i)} - (1-1/\tau)m^+$, while multiplying $m^+ \geq a^{(i)}$ by $(1-1/\tau)$ gives rise to $a^{(i)}/\tau \geq a^{(i)} + (1-1/\tau)m^+$. Therefore, we have proved that $\tilde{x} + \mathbf{g}(\tilde{\mathbf{x}}^{(i)})$ is a subset of

$$\tilde{x} + [a^{(i)} + (1-1/\tau)m^+ - \gamma^{(i)} \ , \ b^{(i)} - (1-1/\tau)m^+ + \gamma^{(i)}].$$

Finally, as $\gamma^{(i)}$ can be made arbitrary small and $(1-1/\tau)m^+ > 0$, we can choose i such that $\gamma^{(i)} \leq (1-1/\tau)m^+$. For this i, we have $\tilde{x} + \mathbf{g}(\tilde{\mathbf{x}}^{(i)}) \subseteq \tilde{\mathbf{x}}^{(i)}$, and the condition in Line 8 succeeds. $\square$

The experimentations presented in Section 6 illustrate this convergence.

5.4. **Domains described by constraints.** The presented bisection algorithm can be easily extended in order to deal with domains that are defined by constraints. Constrained domains that are considered are of the following form:

$$\mathcal{D} = \mathbf{x} \bigcap E,$$

where $\mathbf{x} \in \mathbb{IR}^n$ and $E \subseteq \mathbb{R}^n$. The constraint will be actually described by the following three-valued interval constraint:

$$c : \mathbb{IR}^n \longrightarrow \{0, 1, \{0,1\}\}.$$

This function has the following semantic:

- $c(\tilde{\mathbf{x}}) = 1 \Longrightarrow \tilde{\mathbf{x}} \subseteq \mathcal{D}$;
- $c(\tilde{\mathbf{x}}) = 0 \Longrightarrow \tilde{\mathbf{x}} \bigcap \mathcal{D} = \emptyset$;
- $c(\tilde{\mathbf{x}}) = \{0,1\}$ is interpreted as "don't know" so $\tilde{\mathbf{x}}$ will have to be bisected.

Example 5.1. Suppose $E = \{x \in \mathbb{R}^2 | g(x) \leq 0\}$. Then an interval extension $\mathbf{g}$ of g can be used in order to obtain the corresponding function c:

- $c(\tilde{\mathbf{x}}) = 1$ if $\mathbf{g}(\tilde{\mathbf{x}}) \leq 0$ and $\tilde{\mathbf{x}} \subseteq \mathbf{x}$;
- $c(\tilde{\mathbf{x}}) = 0$ if $\mathbf{g}(\tilde{\mathbf{x}}) > 0$ or $\tilde{\mathbf{x}} \cap \mathbf{x} = \emptyset$;
- $c(\tilde{\mathbf{x}}) = \{0,1\}$ otherwise.

Finally, the bisection algorithm is easily updated in order to deal with the constrained domain:

(1) The constrained domain has to be taken into account in the function $\text{Inner}(f, \mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. This is done changing the condition $\tilde{\mathbf{x}} \subseteq \mathbf{x}$ in Line 6 of Algorithm 2 to

$$\tilde{\mathbf{x}} \subseteq \mathbf{x} \ \wedge \ c(\tilde{\mathbf{x}}) = 1.$$

As a consequence, the function $\text{Inner}(f, \mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ returns true only if $\tilde{\mathbf{y}}$ is an inner approximation of range $(f, \tilde{\mathbf{x}})$ and $\tilde{\mathbf{x}}$ is inside $\mathcal{D}$.

(2) Line 12 of Algorithm 1 is modified so that only boxes satisfying $c(\tilde{\mathbf{x}}') \neq 0$ ($c(\tilde{\mathbf{x}}'') \neq 0$ respectively) are stored in the list. As a consequence, boxes $\tilde{\mathbf{x}}'$ ($\tilde{\mathbf{x}}''$ respectively) such that $c(\tilde{\mathbf{x}}) = 0$ ($c(\tilde{\mathbf{x}}'') = 0$ respectively) are not used anymore. These boxes do not belong to the domain $\mathcal{D}$. Therefore the lists $\mathcal{L}_{\text{Inside}}$ and $\mathcal{L}_{\text{Outside}}$ keep their respective semantics.

Note that the asymptotic convergence proved by Theorem 5.1 also holds in the case of constrained domains provided that the interval extensions used to define c are convergent.

6. Experimentations

This section presents some applications of Algorithm 1. The algorithm has been implemented in C/C++ using the PROFIL/BIAS [12] interval library and executed on a PentiumM 1.4Ghz processor. In order to provide clearer results, a regularisation of the overlapping pavings computed by Algorithm 1 has been performed leading to regular pavings. Indicated timings concern only the execution of Algorithm 1. The quality of the couple inner/outer approximations is defined by

$$\text{qual}(Inner, Outer) := \sqrt[n]{\frac{\text{vol}(Inner)}{\text{vol}(Outer)}}, \tag{11}$$

if $\text{vol}(Outer) \neq 0$. The closer to one the quality is, the closer inner and outer approximations are.

Example 6.1 shows the approximation of the range of a simple function with a constrained domain. It was proposed in [14] where a non-reliable approximation was obtained.

Example 6.1. Consider the function

$$f(x) = \begin{pmatrix} xy \\ x+y \end{pmatrix}$$

and the domain $\mathcal{D} = \{(x,y)^T \in \mathbb{R}^2 \mid x^2 + y^2 \in [1,2]\}$, which is represented in the left hand side graph of Figure 3. The following table displays the time needed to compute the approximations and their quality measure for some different values of epsilon.

ϵ	0.1	0.05	0.025	0.0125	0.00625
qual	0.37	0.63	0.80	0.89	0.94
time	0.09	0.28	0.69	1.9	5.1

In these experimentations, the quality of the approximation obviously improves linearly as ϵ decreases, leading to 1 when ϵ goes to zero. This is an experimental evidence of the truthfulness of Theorem 5.1. On the other hand, the computation time seems to increase quadratically with ϵ: $t(\epsilon) \approx 0.000097\epsilon^{-2} + 0.017\epsilon^{-1} - 0.10$. One may be surprised as the boundary of the set to be approximated has dimension 1. However, two pieces of this

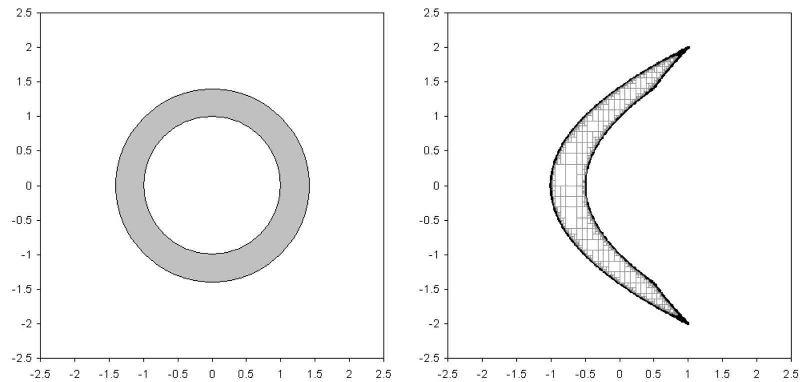

FIGURE 3. Domain $\mathcal{D}$ (left hand side graph) and the reliable approximation of its image (right hand side graph).

boundary are built folding the domain and are therefore made of singularities. On these singular boundaries, the algorithm accumulates on a thin surface around the boundary, hence the quadratic dependence w.r.t. ϵ^{-1}.

The right hand side graphic of Figure 3 shows the approximation of range $(f, \mathcal{D})$ obtained with Algorithm 1 for $\epsilon = 0.0125$.

A more realistic situation is now investigated in Example 6.2.

Example 6.2. Consider the serial robot presented in the left hand side graphic of Figure 4. Both abscissa a and the angle θ can be controlled in their respective domains $a \in [0, 4]$ and $\theta \in [-2, 2]$. An obstacle (the dashed disk of radius r centered at $C = (C_x, C_y)$) is placed in the robot environment so

$$\mathcal{D} = \left\{(a, \theta) \in \mathbb{R}^2 \mid a \in [0, 4], \theta \in [-2, 2], c(a, \theta)\right\}.$$

The constraint $c(a, \theta)$ is true if and only if the whole robot arm does not meet the obstacle for the controls a and θ. To obtain an expression of this constraint, we can note that $d^2(M, A, B)$, the square of the distance between a point M to a segment $[A, B]$, is given by

$$\begin{array}{rl} \langle M-A|M-A\rangle & \text{if } \langle B-A|M-A\rangle < 0 \\ \langle M-B|M-B\rangle & \text{if } \langle A-B|M-B\rangle < 0 \\ \langle M-A|M-A\rangle - \frac{\langle B-A|M-A\rangle^2}{\langle B-A|B-A\rangle} & \text{otherwise,} \end{array}$$

where $\langle\,.\,|\,.\,\rangle$ is the scalar product (writing explicitly these constraints w.r.t. the variables of the problem allows some useful formal simplification). An interval expression of this constraint is easily obtained. The function that computes the working point coordinates from the commands is

$$f(a, \theta) = \begin{pmatrix} a + L\cos(\theta) \\ a + L\sin(\theta) \end{pmatrix}.$$

The image of $\mathcal{D}$ by f is the workspace of the robot. For our numerical experimentation, we choose $C = (3, 1)$, $r = 1$ and $L = 2$. The following

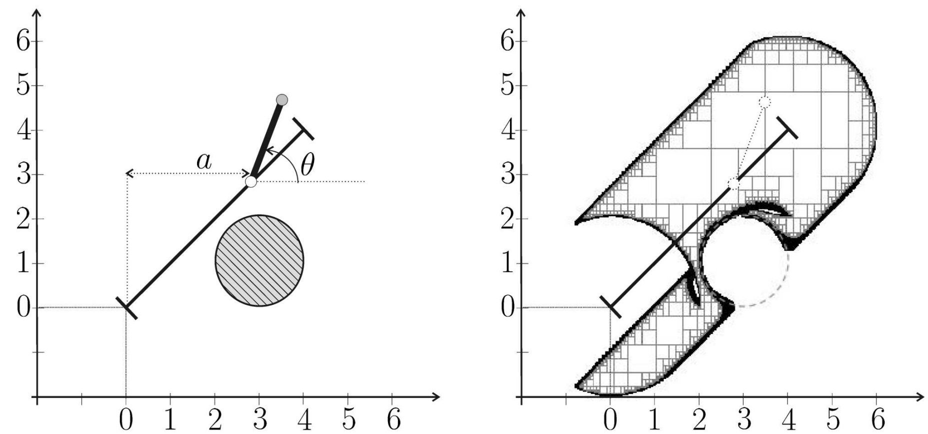

FIGURE 4. A serial robot with an obstacle (left hand side graphic) and its workspace reliable approximation (right hand side graph). The dotted circle in the workspace approximation has been added to represent the obstacle.

table displays the time needed to compute the approximations and their quality measure for some different values of epsilon.

ϵ	0.1	0.05	0.025	0.0125	0.00625
qual	0.80	0.9	0.95	0.975	0.987
time	0.95	2.3	5.9	15.3	40.2

As previously, the quality of the approximation obviously improves linearly as ϵ decreases, leading to 1 when ϵ goes to zero. This is again an experimental evidence of the truthfulness of Theorem 5.1. The dependence of the computational time w.r.t. ϵ^{-1} is once more quadratic ($t(\epsilon) \approx 0.00072\epsilon^{-2} + 0.14\epsilon^{-1} - 0.69$). Here again a piece of the boundary is made of singularities and the algorithm accumulates on a thin surface in this area.

The results are plotted on the right hand side of Figure 4 and have been computed using $\epsilon = 0.025$.

7. RELATED WORK

In the context of the study of serial robots, some work has been done for providing some description of the workspace of the robot. These works naturally concern the *image set problem* as workspaces of serial robots consists in the computation of the *image set* of a vector valued function.

One of the main technique proposed in this framework is to compute the boundary of the workspace (i.e. the boundary of the *image set*) using some continuation method (see [7, 2]). The method proposed in the present paper has several advantages: on one hand, the continuation method does not provide a reliable approximation while Algorithm 1 does. On the other hand, Algorithm 1 can deal with constrained domains while the continuation method proposed in [7] cannot.

8. Conclusion and perspectives

This paper has presented a method able to compute an inner approximation of an image set in the case where the involved function is vector-valued (with domain and co-domain of the same dimension) nonlinear and differentiable. Although this problem appears in many applications, to our knowledge, it has never been solved before. A general reliable sufficient condition for a box to be a subset of the image of a given box has been provided (Theorem 3.1). It has been specialized to an efficient test (Corollary 3.1) that turned out to be closely related to the existence test associated to the Hansen-Sengupta interval operator. Our algorithm has been made efficient and stable thanks to a preconditioning process and an original *domain-inflation* phase based on a fixed-point iteration. The good behavior of our algorithm has been demonstrated on two academic examples.

The main forthcoming work will be to deal with domain and co-domain that do not have the same dimension. When the domain has a dimension smaller than the one of the co-domain, the *image set* has a zero volume and an inner approximation is not needed. When the domain has a greater dimension than the co-domain, the image set has an interior and the presented method is not able to compute an inner approximation of the *image set* without serious adaptations. Now, there exist some applications where the domain has a greater dimension than co-domain: workspaces of over-actuated robots (i.e., its number of motors in greater than its number of degrees of freedom) are *image sets* of vector-valued functions with greater domain dimension than co-domain dimension.

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Appendix A. Technical results

A.1. Technical results for Section 3.

Lemma A.1. *Let E be compact in $\mathbb{R}^n$ and $y \in \operatorname{int} E$ and $z \notin E$. Any continuous path connecting y to z intersects ∂E.*

Proof. As E is compact we have $\overline{E} \subseteq E$. By definition $\partial E = \mathbb{R}^n \backslash (\operatorname{int} E \cup \mathcal{C}\overline{E})$ which is equal to $\mathbb{R}^n \backslash (\operatorname{int} E \cup \mathcal{C}E)$. We will prove that if the continuous path $w : [0,1] \longrightarrow \mathbb{R}^n$ such that $w(0) = y$ and $w(1) = z$ does not intersect ∂E then it is disconnected. This fact being absurd because the image of the connected set $[0,1]$ by the continuous function w must be connected. On one hand, $\operatorname{int} E$ is open in $\mathbb{R}^n$ so $w([0,1]) \cap \operatorname{int} E$ is open inside $w([0,1])$. Also, $y \in w([0,1]) \cap \operatorname{int} E$ and therefore this set is nonempty. On the other hand, $\mathcal{C}E$ is open in $\mathbb{R}^n$ (because E is compact) so $w([0,1]) \cap \mathcal{C}E$ is open inside $w([0,1])$. Also, $z \in w([0,1]) \cap \mathcal{C}E$ and therefore this set is nonempty. Therefore, $w([0,1]) \subseteq (\operatorname{int} E \cup \mathcal{C}E)$ implies $w([0,1])$ is disconnected, which is absurd. Therefore, $w([0,1]) \cap \partial E \neq \emptyset$. $\square$

Lemma A.2. *Let $E \subseteq \mathbb{R}^n$ be compact and $f : E \longrightarrow \mathbb{R}^n$ be a continuous function continuously differentiable in* $\operatorname{int} E$. *Denote* $\{x \in \operatorname{int} E \mid \det f'(x) = 0\}$ *by Σ. Then we have $\partial\big(f(E)\big) \subseteq f(\partial E) \cup f(\Sigma)$.*

Proof. Consider any $y \in \partial\big(f(E)\big)$. As E is compact and f is continuous, $f(E)$ is also compact. Therefore $\partial\big(f(E)\big) \subseteq f(E)$ and there exists $x \in E$ such that $y = f(x)$. Now suppose that $x \in \operatorname{int} E$ and $x \notin \Sigma$. We now prove that it is absurd. As $x \in \operatorname{int} E$ we have $\big(\exists \alpha > 0\big)\big(B(x,\alpha) \subseteq E\big)$. As $x \notin \Sigma$ we can apply the inverse function theorem and we conclude that there exists β such that f is a diffeomorphism in $B(x,\beta)$. Define $\gamma := \min\{\alpha,\beta\}$ and $F := f\big(B(x,\gamma)\big)$. Notice that $y \in F$. On one hand $F \subseteq f\big(B(x,\alpha)\big) \subseteq f(E)$. On the other hand, F is the preimage of $B(x,\gamma)$ by $(f|_{B(x,\beta)})^{-1}$. This latter being continuous, F is open in $\mathbb{R}^n$. As a conclusion $y \in \operatorname{int}\big(f(E)\big)$ which contradicts $y \in \partial\big(f(E)\big)$. As a conclusion we have $x \in \partial E$ or $x \in \Sigma$. And eventually $\partial\big(f(E)\big) \subseteq f(\partial E) \cup f(\Sigma)$. $\square$

A.2. **Technical results for Section 5.** In the rest of the subsection, we define $\mathbf{x}^\delta := x^* \pm (\delta e)$ (as previously, $e \in \mathbb{R}^n$ and $E \in \mathbb{R}^{n\times n}$ satisfy respectively $e_i = 1$ for all $i \in \{1,\ldots,n\}$ and $E_{ij} = 1$ for all $i,j \in \{1,\ldots,n\}$). Basic properties of interval arithmetic can be found in [18] and won't be explicitly mentioned each time they are used.

Proposition A.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuously differentiable, and $x^* \in \mathbb{R}^n$ such that* $\det f'(x^*) \neq 0$. *Consider a locally Lipschitz continuous interval extension $\mathbf{f}'$ of f', e.g. its natural extension. Then, there exist $\delta^* > 0$, $\kappa^* > 0$ and $c > 0$ such that:*

(1) *$f'(x)^{-1}$ is Lipschitz continuous inside $\mathbf{x}^{\delta^*}$,*
(2) *for all $x \in \mathbf{x}^{\delta^*}, ||f'(x)^{-1}|| \leq \kappa^*$,*
(3) *for all $\delta \leq \delta^*, \forall x \in \mathbf{x}^\delta, f'(x)^{-1}\mathbf{f}'(\mathbf{x}^\delta) \;\subseteq\; I \pm (c\delta E)$.*

Proof. (1) It is well known that the function $M \longmapsto M^{-1}$ is continuously differentiable where it is defined. Therefore, $f'(x)^{-1}$, which is the composition of two continuously differentiable functions, is continuously differentiable where it is defined. As $f'(x^*)$ is nonsingular, there exists a neighborhood $\mathbf{x}^{\delta^*}$ of x^* inside which $f'(x)^{-1}$ is continuously differentiable. As $\mathbf{x}^{\delta^*}$ is compact, $f'(x)^{-1}$ is Lipschitz continuous inside $\mathbf{x}^*$.

(2) $||f'(x)^{-1}||$ is the composition of two continuous functions, and is therefore continuous. As a consequence, it is bounded above by some $\kappa^* > 0$ inside the compact set $\mathbf{x}^*$.

(3) As $\mathbf{f}'$ is locally Lipschitz continuous, there exists c' such that for all $\delta \leq \delta^*$ we have $||\operatorname{rad} \mathbf{f}'(\mathbf{x}^\delta)|| \leq c'\delta$. Consider some arbitrary $\delta \leq \delta^*$ and

$x \in \mathbf{x}^\delta$. Then we have

$$|| \,\mathrm{rad}\; f'(x)^{-1}\mathbf{f}'(\mathbf{x}^\delta)|| \le ||f'(x)^{-1}||\,||\,\mathrm{rad}\;\mathbf{f}'(\mathbf{x}^\delta)|| \le c'\kappa^*\delta.$$

Because $f'(x) \in \mathbf{f}'(\mathbf{x}^\delta)$, we have $I \in f'(x)^{-1}\mathbf{f}'(\mathbf{x}^\delta)$. From the last two results, we obtain $f'(x)^{-1}\mathbf{f}'(\mathbf{x}^\delta) \subseteq I \pm (2c'\kappa^*\delta E)$. As the latter holds for all $x \in \mathbf{x}^\delta$, this concludes the proof with $c = 2c'\kappa^*$. □

Lemma A.3. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}'' \in \mathbb{IR}$ *be intervals such that* $\mathbf{a}' \subseteq \mathbf{a}$, $\mathbf{a}'' \subseteq \mathbf{a}$, $\mathbf{b}' \subseteq \mathbf{b}$ *and* $\mathbf{b}'' \subseteq \mathbf{b}$. *Then*

$$d(\mathbf{a}'\mathbf{b}', \mathbf{a}''\mathbf{b}'') \le |\mathbf{a}|d(\mathbf{b}', \mathbf{b}'') + |\mathbf{b}|d(\mathbf{a}', \mathbf{a}'').$$

Proof. We have $d(\mathbf{a}'\mathbf{b}', \mathbf{a}''\mathbf{b}'') \le d(\mathbf{a}'\mathbf{b}', \mathbf{a}'\mathbf{b}'') + d(\mathbf{a}'\mathbf{b}'', \mathbf{a}''\mathbf{b}'')$ by the triangular inequality. Then $d(\mathbf{a}'\mathbf{b}', \mathbf{a}'\mathbf{b}'') \le |\mathbf{a}'|d(\mathbf{b}', \mathbf{b}'') \le |\mathbf{a}|d(\mathbf{b}', \mathbf{b}'')$, and $d(\mathbf{a}'\mathbf{b}'', \mathbf{a}''\mathbf{b}'') \le |\mathbf{b}''|d(\mathbf{a}', \mathbf{a}'') \le |\mathbf{b}|d(\mathbf{a}', \mathbf{a}'')$. □

Lemma A.4. *Let* $\mathbf{A}^* = I \pm (\delta E)$ *with* $\delta \le 1/2$, *and* $\mathbf{x}^*, \mathbf{b}^* \in \mathbb{IR}^n$. *Then, there exist* $\lambda', \lambda'' > 0$ *such that for all* $\delta \le 1/2$, $\Gamma(\,.\,,\,.\,,\mathbf{b}) : \mathbb{I}\mathbf{A}^* \times \mathbb{I}\mathbf{x}^* \longrightarrow \mathbb{IR}^n$ *is* $\lambda'\delta$*-Lipschitz w.r.t.* $\mathbf{A}$ *and* $\lambda''\delta$*-Lipschitz w.r.t.* $\mathbf{x}$, *where* λ' *and* λ'' *depend only on* $\mathbf{x}^*$ *and* $\mathbf{b}^*$.

Proof. Let us compute the Lipschitz constant w.r.t. $\mathbf{A}$, the Lipschitz constant w.r.t. $\mathbf{x}$ being similar and simpler to compute. Let $\mathbf{A}, \tilde{\mathbf{A}} \subseteq \mathbf{A}^*$ and $\mathbf{x} \subseteq \mathbf{x}^*$. Then $d(\Gamma(\mathbf{A}, \mathbf{x}, \mathbf{b}^*), \Gamma(\tilde{\mathbf{A}}, \mathbf{x}, \mathbf{b}^*))$ is equal to

$$\max_i d\big(\ \mathbf{A}_{ii}^{-1}(\mathbf{b}_i^* - \sum_{j\ne i}\mathbf{A}_{ij}\mathbf{x}_j)\ ,\ \tilde{\mathbf{A}}_{ii}^{-1}(\mathbf{b}_i^* - \sum_{j\ne i}\tilde{\mathbf{A}}_{ij}\mathbf{x}_j)\ \big).$$

Now, by Lemma A.3 the latter is less than

$$\max_i\ |\mathbf{A}_{ii}^{*-1}|\, d\big(\mathbf{b}_i^* - \sum_{j\ne i}\mathbf{A}_{ij}\mathbf{x}_j, \mathbf{b}_i^* - \sum_{j\ne i}\tilde{\mathbf{A}}_{ij}\mathbf{x}_j\big) + |\mathbf{b}_i^* - \sum_{j\ne i}\tilde{\mathbf{A}}_{ij}^*\mathbf{x}_j|\, d(\mathbf{A}_{ii}^{-1}, \tilde{\mathbf{A}}_{ii}^{-1}).$$

Note that $|\mathbf{A}_{ii}^{*-1}| = \langle \mathbf{A}_{ii}^* \rangle^{-1} = (1-\delta)^{-1}$ while $|\mathbf{b}_i^* - \sum_{j\ne i}\tilde{\mathbf{A}}_{ij}^*\mathbf{x}_j| \le ||\mathbf{b}^*|| + (n-1)\delta||\mathbf{x}^*||$. Now

$$\begin{aligned} d\big(\mathbf{b}_i^* - \sum_{j\ne i}\mathbf{A}_{ij}\mathbf{x}_j, \mathbf{b}_i^* - \sum_{j\ne i}\tilde{\mathbf{A}}_{ij}\mathbf{x}_j\big) &\le \sum_{j\ne i} d(\mathbf{A}_{ij}\mathbf{x}_j, \tilde{\mathbf{A}}_{ij}\mathbf{x}_j) \\ &\le \sum_{j\ne i} |\mathbf{x}_j^*|\, d(\mathbf{A}_{ij}, \tilde{\mathbf{A}}_{ij}) \\ &\le \sum_{j\ne i} ||\mathbf{x}^*||\, 2\delta \ = \ 2(n-1)||\mathbf{x}^*||\delta, \end{aligned}$$

while

$$d(\mathbf{A}_{ii}^{-1}, \tilde{\mathbf{A}}_{ii}^{-1}) \le |\mathbf{A}_{ii}^{*\,-2}| d(\mathbf{A}_{ii}, \tilde{\mathbf{A}}_{ii}) \le (1-\delta)^{-2}\, 2\delta.$$

So far, we have proved that $\Gamma(\,.\,,\,.\,,\mathbf{b})$ is Lipschitz continuous of constant

$$2(n-1)||\mathbf{x}^*||(1-\delta)^{-1}\delta \ + \ 2(||\mathbf{b}^*|| + (n-1)\delta||\mathbf{x}^*||)(1-\delta)^{-2}\delta$$

w.r.t. $\mathbf{A}$. The proof is concluded noting that $\delta \le 1/2$ which implies $(1-\delta)^{-1} \le 2$. □

Proposition A.2. *Let* $\mathbf{x}^* \in \mathbb{IR}^*$ *and* $\mathbf{A} : \mathbb{I}\mathbf{x}^* \longrightarrow \mathbb{I}\big(I \pm (\delta E)\big)$, *with* $\delta \leq 1/2$, *be* λ'''*-Lipschitz. Let* $\tilde{x} \in \mathbb{R}^n$ *and* $\mathbf{b}^* \in \mathbb{IR}^n$ *and an arbitrary* $\mathbf{b} \subseteq \mathbf{b}^*$*. Define* $\mathbf{T} : \mathbb{I}\mathbf{x}^* \longrightarrow \mathbb{IR}^n$ *by*

$$\mathbf{T}(\mathbf{x}) := \Gamma\big(\mathbf{A}(\mathbf{x}), \mathbf{x} - \tilde{x}, \mathbf{b}\big).$$

Then, there exists $\lambda > 0$ *such that for all* $\delta \leq 1/2$, $\mathbf{T}$ *is* $\lambda\delta$*-Lipschitz continuous, where* λ *depends only on* $\mathbf{x}^*$ *and* $\mathbf{b}^*$*.*

Proof. The proposition is proved using Lemma A.4. Fix an arbitrary $\delta \leq 1/2$. Then,

$$\begin{array}{cl} & d\big(\ \Gamma(\mathbf{A}(\mathbf{x}), \mathbf{x} - \tilde{x}, \mathbf{b})\ ,\ \Gamma(\mathbf{A}(\mathbf{y}), \mathbf{y} - \tilde{x}, \mathbf{b}\big) \\ \leq & \begin{array}{cl} & d\big(\ \Gamma(\mathbf{A}(\mathbf{x}), \mathbf{x} - \tilde{x}, \mathbf{b})\ ,\ \Gamma(\mathbf{A}(\mathbf{y}), \mathbf{x} - \tilde{x}, \mathbf{b}\big) \\ + & d\big(\ \Gamma(\mathbf{A}(\mathbf{y}), \mathbf{x} - \tilde{x}, \mathbf{b})\ ,\ \Gamma(\mathbf{A}(\mathbf{y}), \mathbf{y} - \tilde{x}, \mathbf{b}\big) \end{array} \\ \leq & \lambda'\delta\, d(\mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{y}))\ + \lambda''\delta\, d(\mathbf{x} - \tilde{x}, \mathbf{y} - \tilde{x}) \\ \leq & \lambda'\lambda'''\delta\, d(\mathbf{x}, \mathbf{y})\ +\ \lambda''\delta\, d(\mathbf{x}, \mathbf{y}). \end{array}$$

Thus, the proposition is proved with $\lambda = \lambda'\lambda''' + \lambda''$. □

The proof of the following technical lemma involves generalized intervals and the Kaucher arithmetic (see [20] and references therein). Details on this generalization of the interval arithmetic are not given here as it is used only in the following short proof.

Lemma A.5. *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *be a strictly diagonally dominant interval matrix and* $0 < \delta \in \mathbb{R}$*. Define* $\mathbf{x} \in \mathbb{IR}^n$ *by* $\mathbf{x}_i = \pm\delta$*. Then there exists an unique* $\mathbf{b} \in \mathbb{IR}^n$ *such that* $\operatorname{rad} \mathbf{b}_i > 0$ *and* $\Gamma(\mathbf{A}, \mathbf{x}, \mathbf{b}) = \mathbf{x}$*. Furthermore,* $\operatorname{mid} \mathbf{b} = 0$ *and* $\mathbf{b} \subseteq \mathbf{A}\mathbf{x}$*.*

Proof. Using the Kaucher arithmetic group properties, $\Gamma(\mathbf{A}, \mathbf{x}, \mathbf{b}) = \mathbf{x}$ is equivalent to, for all $i \in \{1, \ldots, n\}$

$$\begin{aligned} \mathbf{b}_i &= (\text{dual } \mathbf{A}_{ii})\mathbf{x}_i + \sum_{j\neq i}(\text{dual } \mathbf{A}_{ij})(\text{dual } \mathbf{x}_j) \\ &= \pm\langle \mathbf{A}_{ii}\rangle\delta + \sum_{j\neq i} \mp|\mathbf{A}_{ij}|\,\delta, \end{aligned}$$

where $\mp\epsilon := \text{dual } \pm\epsilon$. Now, $\operatorname{rad} \mathbf{b}_i = \delta(\mathbf{A}_{ii} - \sum_{j\neq i}|\mathbf{A}_{ij}|)$, which is strictly positive as $\mathbf{A}$ is strictly diagonally dominant, so $\mathbf{b}$ is proper. Furthermore $\operatorname{mid} \mathbf{b}_i = 0$ as it is the sum of zero-centered intervals. Finally, by inclusion monotonicity we have $\mathbf{b}_i \subseteq \mathbf{A}_{ii}\mathbf{x}_i + \sum_{j\neq i}\mathbf{A}_{ij}\mathbf{x}_j = \mathbf{A}\mathbf{x}$. □

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