Maps II: Chasing Diagrams in Categorical Proof Theory

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Dedicated to
Jim Lambek
in friendship and admiration

Abstract

In categorical proof theory, propositions and proofs are presented as objects and arrows in a category. It thus embodies the strong constructivist paradigms of *propositions-as-types* and *proofs-as-constructions*, which lie in the foundation of computational logic. Moreover, in the categorical setting, a third paradigm arises, not available elsewhere: *logical-operations-as-adjunctions*. It offers an answer to the notorious question of the equality of proofs. So we chase diagrams in algebra of proofs.

On the basis of these ideas, the present paper investigates proof theory of *regular logic*: the $\{\wedge, \exists\}$-fragment of the first order logic with equality. The corresponding categorical structure is *regular fibration*. The examples include stable factorisations, sites, triposes. Regular logic is exactly what is needed to talk about *maps*, as total and single-valued relations. However, when enriched with proofs-as-arrows, this familiar concept must be supplied with an additional conversion rule, connecting the proof of the totality with the proof of the single-valuedness. We explain the logical meaning of this rule, and then determine precise conditions under which a regular fibration supports the principle of *function comprehension* (that each map corresponds to a unique function *via* arrow in the base), thus lifting a basic theorem from regular categories (e.g. [12, 2.132]), recently relativized to factorisation systems [22, 42]. The obtained results bring us a step closer to extending the $\wp$-set construction [20] from triposes to non-posetal fibrations, and thus closer to 'toposes' accommodating categorical proof theory.

1 Introduction

The basic ideas of constructive logic cristallised very slowly. The *Brouwer-Heyting-Kolmogorov interpretation* of proofs as constructions had evolved through the first half of this century [4, 16, 23]. It motivated the well-known debate about foundations and, indirectly, the creation of realizability in the fourties. Most of the time, however, this conception of *strong constructivism* was overshadowed by simpler ideas, boiling down to the rejection of Excluded Middle.

The search for a mathematical theory of constructions was propounded in the sixties, when computer science was making its first steps (cf. [30], especially [46]). The main result of that search is the discipline of logic as type theory, the *sine qua non* of virtually all logical frameworks for computation [18]. The underlying foundational principle is the so called *Curry-Howard isomorphism* of propositions and types. At first, this isomorphism was just a remark about a 'striking analogy between the theory of functionality and the theory of implication' in Curry's book with Feys [11,

sec. 9E] from 1958. Eleven years later, the remark was developed in a widely circulated manuscript by Howard (which awaited actual publication [17] for another eleven years). In the meantime, Laüchli's completeness theorem [29] and AUTOMATH, the very first logical framework [5, 6, §14], had already put the propositions-as-types at work. The two facets of this paradigm were recognized in the very successful type systems due to Martin-Löf on one side [36], and to Girard [13] and Reynolds [44] on the other. Synthesis of these systems led to the Theory of Constructions [10, 21], the most *comprehensive* [40] type system so far.

The categorical part of the story goes back to the sixties too. The first ideas of categorical logic were conceived in Lawvere's pursuit of foundations. Lambek's investigations in linguistics and algebra, on the other hand, led to a concrete analysis of the parallelism between proof theory and categories.

While developing his *syntactic calculus* [24, 25], Lambek noticed that generating free categories resembled proof derivation in the Gentzen-style calculi of sequents. In [26], he formalized the view of arrows as labelled proofs from and used the cut elimination to deal with some categorical questions. Category theory promptly accepted the proof-theoretical methods [34], but proof theory never really profited from categorical methods[1] — except perhaps indirectly, through categorical semantics of type theories, still based on Lambek's ideas [28].

Lawvere's ideas, on the other hand, seem to be leading beyond type theory. The keyword is *adjunction*. It reduces Curry's 'striking analogy between the theory of functionality and the theory of implication' to the fact that both the function-space constructor $(-)^A$ and the implication $P \Rightarrow (-)$ are right adjoint to some product functors — namely, to $A \times (-)$ and $P \wedge (-)$, respectively. In a *hyperdoctrine*, the structure that captures predicate logic, all logical operations were presented as adjunctions [31].

In the present paper, we shall study a structure closely related to hyperdoctrines. So let us recall that a hyperdoctrine was defined as a functor $\mathcal{P} : \mathcal{C}^{op} \to \mathsf{CAT}$, assigning to each 'set' $A \in \mathcal{C}$ a category of 'predicates' $\mathcal{P}(A)$ over it, usually cartesian closed; and to each 'function' $f : A \to B$ in $\mathcal{C}$ the 'substitution' functor

$$\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A) : Q(y) \mapsto Q(f(x)).\text{[2]}$$

The quantifiers are represented as the functors adjoint to the substitution along a projection $p : A \times B \to A$:

$$\exists y \dashv \mathcal{P}p \dashv \forall y : \mathcal{P}(A \times B) \to \mathcal{P}(A) : R(x,y) \mapsto \forall y.R(x,y).$$

This representation is justified by the derivation rules

$$\frac{P(x, y) \to R(x,y)}{P(x) \to \forall y.R(x,y)} \; (\forall) \qquad\qquad \frac{R(x,y) \to P(x, y)}{\exists y.R(x,y) \to P(x)} \; (\exists)$$

where $P(x, y)$ stands for $P(p(x,y))$. In ordinary sequent calculus, these rules speak for themselves. However, when proofs are not reduced to mere entailment, but preserved as arrows in a category, one must decide *which* particular proof $\exists y.R(x,y) \to P(x)$

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[1] It is hard to see a reason for this, other than subjective. The main obstacles to categorical proof theory, e.g. as analyzed by Dana Scott in [46], had been removed by 1970 [31, 32].

[2] This notation will be formalized in section 2.

should be assigned to a given proof $R(x,y) \rightarrow P(x,\mathbf{y})$ in ($\exists$); idem for ($\forall$). Our logical experience suggests no uniform answer to such questions: the distinctions between particular proofs appear to be just as noncanonical as formal systems in which we present them.

Questions of this kind often arise in proof theory [43]: transformations, reductions, conversions of proofs are successfully approached by type-theoretical means. Computer science amplifies all this by demanding a genuine *logic of proofs*, as opposed to the traditional *provability logic*. However, there are fundamental questions, for which our logical intuitions do not offer a sufficient lead. For instance, Girard's [13] and Martin-Löf's [36] type theories depict the quantification quite differently — and suggest, in particular, different term decorations of rule ($\exists$). Some computer scientists argue that Martin-Löf's way correctly describes the actual structure of programs [35], while others prefer Girard's way [37].

The categorical proposal is that the derivation rules like ($\exists$) and ($\forall$) should be enriched to adjunctions. This means that, say, correspondence ($\exists$) should be realized by composing with some proofs

$$\eta_R \; : \quad R(x,y) \quad \rightarrow \quad \exists y'.R(x,y',\mathbf{y}) \qquad \text{and}$$
$$\varepsilon_P \; : \quad \exists y.P(x,\mathbf{y}) \quad \rightarrow \quad P(x),$$

given uniformly in R and P (i.e. natural), and such that both composites

$$\exists y.R(x,y) \quad \rightarrow \quad \exists yy'.R(x,y',\mathbf{y}) \quad \rightarrow \quad \exists y'.R(x,y')$$
$$P(x,\mathbf{y}) \quad \rightarrow \quad \exists y'.P(x,\mathbf{y},\mathbf{y'}) \quad \rightarrow \quad P(x,\mathbf{y})$$

derived from them reduce to the identity.

In general, the notion of adjunction is thus offered as a uniform way to extend some logical concepts beyond the level of entailment and despite the shortcomings of intuition. The uniformity is, of course, not always welcome and this idea should not be used as a Procrustean bed. Linear logic, for instance, does not arise entirely in terms of adjunctions. But for the basic constructs, the adjunctions seem to be working remarkably well. In [32], Lawvere has described the comprehension scheme, assigning to each predicate $P(x)$ a set $\{x|P(x)\}$, as an adjoint functor. In [33], he has explained how maps, as total and single-valued relations, can be presented as self-adjoint bimodules. This idea is central in the present paper.

In the next three sections, we introduce the categorical setting for generalized regular logic, present some typical examples, and describe the formal concept of map. The main theorem, characterizing maps relative to a regular fibration, is stated and proved in section 5. It is meant to be a step towards feasible reasoning in strongly constructive logic. Although conceptually quite solid, the original definition of a map, in terms of adjunctions, turns out to be impracticable in general non-posetal situations: expanding and demonstrating the adjunction equations on proofs is usually too demanding to be justified in routine reasoning with maps. Our characterisation, however, replaces the adjunction equations by a more manageable condition: maps are just the total and single-valued relations which happen to be *subterminal*. Of course, while eliminating some proof-chasing from the notion of map, this simple result demands a certain amount of chasing itself. The proof is reduced to two special lemmas, worked out in section 6.

On a technical level, the message of our main theorem is that *the proofs involving maps alone are always unique*. In other words, on maps — all the proof diagrams commute. Only this insight makes some general tasks involving maps tractable. They are illustrated on examples in the subsequent sections.

Conceptually, on the other hand, the upshot of the story on maps is that *the extensional part of strongly constructivist logic is inherently posetal*: all reasoning about the base of 'sets and maps' boils down, *by its nature*, to logic of entailment.

Together with [41], the presented logical analyses are a part of an effort towards understanding universes with strongly constructivist logic, which relativize the notion of a topos. In this framework, regular fibrations generalize sites (cf. 3.2). An object in the base of such a fibration is a 'sheaf' if it is Cauchy complete, i.e. if every map to it extends to a unique arrow.[3] A regular fibration is thus 'subcanonical' if it supports the principle of function comprehension: all objects are Cauchy complete. In section 7, an analysis of the site representation of regular fibrations yields the necessary and sufficient conditions for the function comprehension. We get a subcanonicity test applicable, for instance, to triposes.

But the main direct goal of the presented study of maps is actually to allow lifting of tripos theory [20] to non-posetal situations. With maps as adjunctions, this is a completely unfeasible task. With the presented results, though, a general form of the $\wp$-set construction, that led to the Effective Topos [20], seems to be within reach. It is briefly discussed in the final section, eighth. In a forthcoming paper we shall investigate how much topos theory can be lifted to these general $\wp$-sets with strongly constructive logic.

Some abstract facts, needed throughout the paper, have been listed in the Appendix. The lemma proved in part C is the germ of our main result. In fact, they are both just symptoms of a remarkable stability of the concept of adjunction: all of its rich structure is often derived from modest and relatively simple requirements.[4] In principle, deriving proofs using adjunctions is 'simpler than it looks'. Like sentences in natural language, they can be lengthy, but easy to understand — although perhaps difficult to analyze and lay out. This experience of categorical proof theory at work is perhaps the most valuable result of the present paper.

For reading 'Maps II', no acquaintance with 'Maps I' [42] is necessary. The theme is still the same, and the results of that paper can be derived from this one; but the conceptual background is different and, in a sense, complementary. Ironically, this is reflected in the fact that even the order of composition had to be changed: the composite of $f : A \to B$ and $g : B \to C$ is here $(f; g) : A \to C$.

2   Regular fibrations

When presented by a functor $\mathcal{P} : C^{op} \to \mathsf{CAT}$, the formulas of predicate logic are stratified according to their contexts of free variables: each category $\mathcal{P}(A)$ contains just the predicates $P(x)$, where x is of type A. In order to 'chase proofs', however, we need often to have all the predicates displayed together. Using the *Grothendieck construction* [14, 1.5.], we can 'glue together' all $\mathcal{P}(A)$ into a *total category* of predicates $\wp$, and replace the functor $\mathcal{P} : C^{op} \to \mathsf{CAT}$ by a functor $\mathsf{P} : \wp \to C$.

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[3] This idea has been worked out by Walters in [49].

[4] Perhaps the only comparably robust mathematical concept is holomorphicity.

To collect the objects of $\wp$, take the disjoint union of all $\mathcal{P}(A)$. A typical object of $\wp$ is thus a pair $\langle A, P(x) \rangle$, with $A \in C$ and $P(x) \in \mathcal{P}(A)$. An arrow from $\langle A, P(x) \rangle$ to $\langle B, Q(y) \rangle$ in $\wp$ is then a pair $\langle f, \varphi \rangle$, where $f : A \to B$ is a 'function' in C and $\varphi : P(x) \longrightarrow Q(f(x))$ is a 'proof' in $\mathcal{P}(A)$. The functor P $: \wp \to C$ is the first projection.

The described passage from an *indexed category* $\mathcal{P} \in \mathsf{CAT}^{C^{op}}$ to a *fibration* P $\in$ CAT$/C$ can be understood as a point-free generalisation of the passage from a sheaf $F \in \mathsf{Set}^{O(X)^{op}}$ to the corresponding *espace étalé* $E \in \mathsf{Esp}/X$. Fibrations could be defined as the functors (equivalent to those) which are obtained from indexed categories $\mathcal{P} : C^{op} \to$ CAT by the Grothendieck construction. The other way around, from each fibration $\wp$ one can get back an indexed category $\mathcal{P}$. The object part is recovered in the obvious way: $\mathcal{P}(A)$ is just the *fibre* $\wp(A)$ of P — namely, the subcategory of $\wp$ consisting of those objects and arrows which P projects on A and its identity. To get the arrow part of $\mathcal{P}$, one uses the observation that the arrow $\langle f, id \rangle : Q(f(x)) \longrightarrow Q(y)$ in $\wp$ is terminal among all arrows to $Q(y)$ which the functor P projects on $f : A \to B$. This couniversal property determines the object $Q(f(x)) = \mathcal{P}f(Q(y))$. The arrow part of the functor $\mathcal{P}f$ follows from it. The arrow $\langle f, id \rangle$ is a *cartesian lifting* of f at Q. The *structure* of an indexed category $\mathcal{P}$ is thus encoded in the corresponding fibration P by this specific *property* of cartesian liftings.

A formal definition of a fibration and explanations about some related notions can be found in Appendix A. For an insider, let us reiterate that, for this paper, the essential feature of the fibrational setting is the total category $\wp$ of predicates, rather than the 'difficult' issues surrounding the canonical isomorphisms and the reduction of structures to properties. Indeed, all the logical structures considered here are given by universal properties, and we can — and shall — safely work modulo isomorphism.

DEFINITION 2.1
A functor P $: \wp \to C$ is a *regular fibration* if

- C has finite products,
- $\wp$ has finite fibrewise products,
- P is a bifibration, satisfying the Beck-Chevalley and the Frobenius conditions.

Logical notation. One of attractive features of categorical logic — particularly for computer science — is that it can be developed in a *variable-free* way. This is elegantly explained in [27]. We shall, however, proceed in the opposite direction and reintroduce the notation with variables, pursued so far informally. (It will sometimes have to be combined with the standard notation, summarized in the Appendix.)

Let a regular fibration P $: \wp \to C$ be fixed. $\wp(A)$ is its fibre over A. The metavariables $A, B, C \ldots f, g, h \ldots$ are reserved for the objects and the arrows of the base category C. The objects of the category $\wp$ are called *predicates* and written $P(x) \in \wp(A)$, $Q(y) \in \wp(B)$, $R(x,y) \in \wp(A \times B)$. The arrows of $\wp$ will be called *proofs*: they are denoted by Greek letters $\varphi, \gamma, \chi \ldots$ The superscript of a proof tells its image in C: we write φ^f when P$(\varphi) = f$. The vertical arrows (i.e., those from fibres) have no superscript.

While $A \times B$ denotes a product in the base C, $Q \wedge Q'$ is a product in a fibre $\wp(B)$. The projections are $p : A \times B \to A$ and $q : A \times B \to B$ (or $p' : A \times B \to B$), as opposed to $\pi : Q \wedge Q' \to Q$ and $\pi' : Q \wedge Q' \to Q'$. The diagonals are $d := \langle id, id \rangle : A \to A \times A$ and $\delta : Q \to Q \wedge Q$. The terminal object in C is 1, as opposed to the *truth* $\top(y) \in \wp(B)$.

The inverse images along an arrow $f : A \to B$ from C will be presented as substitution instances. For instance,

$$Q\Big(f(x)\Big) \quad := \quad f^*(Q) \qquad \text{or}$$
$$R\Big(x, f(x')\Big) \quad := \quad (id_A \times f)^*(R).$$

The usual conventions for the manipulation with variables are supported. To identify two variables of the same type, one takes an inverse image along the diagonal:

$$R\Big(x, f(x)\Big) = d_A^\pi\Big((id_A \times f)^*(R)\Big) = \langle id_A, f\rangle^*(R).$$

As mentioned in the Introduction, a predicate with a dummy variable $P(x, y\!\!\!/)$ corresponds to the inverse image $p^*(P)$ along a projection $p : A \times B \to A$. Since $P(x) \wedge Q(y)$ obviously denotes $P(x, y\!\!\!/) \wedge Q(x\!\!\!/, y) = p^*(P) \wedge q^*(Q)$, the dummies are often left implicit. In some cases, we write $P(x, y\!\!\!/)$ as $P(x) \wedge \mathsf{T}(y)$. Consequently, $R(x, y) \to S(y, z)$ means that there is a proof $R(x, y, z\!\!\!/) \to S(x\!\!\!/, y, z)$ in $\wp(A \times B \times C)$. But $R(x, y) \to P(x)$ can equivalently denote a proof over $p : A \times B \to A$.

The correspondence of the substitution and the inverse images is the well ploughed ground of categorical logic. It is fairly innocuous, although some details may be subtle. The situation becomes more complicated when it comes to the direct images. The Beck-Chevalley and the Frobenius conditions play the main roles.

Stability conditions. The idea is that the direct images should correspond to the quantification:

$$\exists y. R(x, y) := p_!(R).$$

As explained in [39], the formula $\exists y. R(x, y, z\!\!\!/)$ can now be assigned two different interpretations: $\tilde{p}_!\Big(\tilde{q}^*(R)\Big)$ and $q^*\Big(p_!(R)\Big)$ — over the pullback square $(\tilde{p}; q) = (\tilde{q}; p)$ spanned by the projections $p : A \times B \to A$ and $q : A \times C \to A$. In $\tilde{p}_!\Big(\tilde{q}^*(R)\Big)$, we first add the dummy z and then quantify over y, while in $q^*\Big(p_!(R)\Big)$ we first quantify and then add the dummy. The Beck-Chevalley condition now says that both ways lead to the same result. The Frobenius condition, on the other hand, ensures an isomorphism between the predicates $P(x) \wedge \exists y. R(x, y)$ and $\exists y. P(x) \wedge R(x, y)$.

The logical notation for regular fibrations is well-defined only when these two conditions are satisfied. They guarantee that quantifying over a variable does not interfere with other variables. The other way around, the commutativity of operations on variables, expressed by the stability conditions, is implicit to the notation of predicate logic. By hiding the logically irrelevant isomorphisms, this notation simplifies the presentation of logical constructions.

Equality. So far, we have found a logical expression for the direct images along the *projections*. Now we shall do the same for the direct image $f_!(P)$ along an arbitrary arrow $f : A \to B$.

First of all, recall Lawvere's [32] definition of the equality predicate:

$$(y' \overset{B}{\equiv} y'') \quad := \quad d_!(\mathsf{T}_B), \tag{2.1}$$

where $d : B \to B \times B$ is the diagonal. The predicate $y' \overset{B}{\equiv} y$ is the *internal equality* of a regular fibration. Applying the Beck-Chevalley condition, we get things like

$$
\begin{aligned}
(f(x) \overset{B}{\equiv} y) &= & (f \times B)^*(\overset{B}{\equiv}) &\cong & \langle id_A, f \rangle_!(\mathsf{T}_A) \\
(\exists x. f(x) \overset{B}{\equiv} y) &\cong & p_!(\langle id_A, f \rangle_!(\mathsf{T}_A)) &\cong & f_!(\mathsf{T}_A)
\end{aligned}
$$

The following special case of the last formula sheds some light on the idea behind (2.1).

$$
(y' \equiv y'') \cong \exists y. d(y) \equiv \langle y', y'' \rangle.
$$

Using the internal equality, one can express all the direct images in logical form.

PROPOSITION 2.2

For an arrow $f : A \to B$ in C and predicates $P(x) \in \wp(A)$ and $R(x, y) \in \wp(A \times B)$, we have

1. $f_!(P) \cong \left(\exists x. f(x) \equiv y \land P(x) \right)$

2. $(d_A \times B)_!(R) \cong \left(x \overset{A}{\equiv} x' \land R(x', y) \right)$

Case 1 has been proved in [32, p. 9]. Note that the *substitutivity of equality*

$$
P(x) \cong \left(\exists x'. x \equiv x' \land P(x') \right), \tag{2.2}
$$

is obtained as the special case $f = id$. On the other hand, for $S \in \wp(A \times C)$, isomorphism 1 extends to

$$
(f \times C)_!(S) \cong \left(\exists x. f(x) \equiv y \land S(x, z) \right). \tag{2.3}
$$

Case 2 is an easy exercise with the stability conditions.

Let us summarize. From every expression denoting an object of a regular fibration in the standard (variable-free) way, we can remove all the occurrences of f^* and $f_!$, and replace them by instances of the substitution or the quantification, using suitable internal equations with f. When f is a projection or a diagonal, it can be removed altogether, since the inverse images along these arrows correspond to certain operations on variables, while the direct images represent either a quantifier or an equality predicate.

We close this section with a useful lemma, easily derived from A.8(2) and B.4(3).

LEMMA 2.3

For every object B, the equality predicate $y' \overset{B}{\equiv} y$ is a subobject of the terminal object in $\wp(B \times B)$ — i.e., it is a *subterminal object*.

3 Examples

EXAMPLE 3.1

Every stable factorisation $(\mathcal{E}, \mathcal{M})$ in a finitely complete category C can be presented as the regular fibration Cod : $\mathcal{M}/C \to C$, where $\mathcal{M}/C$ is the full subcategory of the arrow category C/C, spanned by the $\mathcal{M}$-arrows, and Cod is the codomain functor. A

fibre of this fibration is thus the category $\mathcal{M}/B$, spanned by the $\mathcal{M}$-arrows in the slice C/B. An inverse image of $b \in \mathcal{M}/B$ along $f : A \to B$ is an arrow $f^*(b) \in \mathcal{M}/A$, obtained by pulling b back along f. A direct image $f_!(m)$ of $m \in \mathcal{M}/A$ along f is an $\mathcal{M}$-image of the arrow $(m; f)$.

This is the level of generality of the paper 'Maps I' [42]; all the examples considered there induce regular fibrations. Ordinary first-order logic is represented by the fibration Cod : Mon/Set $\to$ Set, where Mon is the class of monics. Of course, Set can be replaced by any topos, or even a regular category C. Indeed, a left exact category C is regular if and only if Cod : Mon/C $\to$ C is a regular fibration.

Stable factorisations thus yield an important source of regular fibrations. Without going into details, let us mention that a regular fibration P : $\wp \to C$ is equivalent to one obtained from a stable factorisation in C if and only if it is *comprehensive* — i.e., we have P $\dashv$ T $\dashv$ D : $\wp \to C$ — and, for every opcartesian arrow σ, the arrow T(D(σ)) is again opcartesian.

EXAMPLE 3.2

One of the basic ideas is that regular fibrations generalize sites. Of course, every site must induce a regular fibration.

For every $A \in C$, the domain functor Dom : $C/A \to C$ is a fibration. These and equivalent fibrations are called *representable*: by the Grothendieck construction, they correspond to the representable presheaves. By definition, *sieves* are subfibrations of representable fibrations. A sieve over A can be presented as a class $U \subseteq C/A$, such that

$$u \in U \implies (h; u) \in U, \text{ for every } h.$$

The domain functor makes the class of arrows U into a fibration.

Now let Ω be the category of all sieves in C. If $U \subseteq C/A$ and $V \subseteq C/B$ are sieves, the hom-set $\Omega(U, V)$ will by definition consist of all the triples $\langle f, U, V \rangle$ where $f \in C(A, B)$, and for every $u \in U$, $(u; f) \in V$. If we project all the sieves over A to A, and $\langle f, U, V \rangle$ to f, we get the regular fibration $\Omega : \Omega \to C$, provided that C is finitely complete. The inverse and the direct images are respectively

$$f^*(V) := \{u \in C/A \mid (u; f) \in V\} \text{ and} \tag{3.1}$$

$$f_!(U) := \{(u; f) \mid u \in U\} \tag{3.2}$$

A *topology* on C is a $\wedge$-closed subfibration J : J $\to$ C of $\Omega : \Omega \to C$. In other words, each fibre J_B consists of some sieves — the *covers* of B. If V is one of them, then $f^*(V)$ must be a cover of A, for every $f \in C(A, B)$. If V and V' are covers of B, then $V \wedge V' = V \cap V'$ must be a cover of B too.

A *site* is a pair (C, J), where C is a finitely complete category and J a topology on it. Associated with this topology, there is a *closure operator* — a cartesian functor j : $\Omega \to \Omega$, defined

$$j(V) := \{f \mid \exists U \in J. f_!(U) \subseteq V\}. \tag{3.3}$$

A sieve V is said to be *closed* if $j(V) = V$. Let Ω_J be the full subcategory of Ω, spanned by the closed sieves. The restriction $\Omega_J : \Omega_J \to C$ of the functor Ω is the regular fibration associated with the site (C, J).

To view all this in a more familiar setting, notice that both Ω and Ω_J are fibred locales. The latter is a regular sublocale of the former; in other words, the bicartesian

functor $j : \Omega \to \Omega_J$ is fibrewise a surjective morphism of complete Heyting algebras. If the size conditions are met, so that all the involved categories exist, Ω is the subobject classifier in the topos of presheaves over C, while Ω_J is the subobject classifier in the topos of J-sheaves. More precisely, they correspond to these classifiers by the Grothendieck construction.

EXAMPLE 3.3

Triposes provide another representation for toposes, rather different from sites — and a different kind of regular fibration. A tripos [20] is a *globally small* hyperdoctrine, which is fibrewise a Heyting algebra. Just as any other hyperdoctrine, it yields a regular fibration by the Grothendieck construction. And we say that a fibration $P : \wp \to C$ is globally small when there is a *generic object* $\xi \in \wp$, such that every object of $\wp$ is its inverse image along some arrow. Intuitively, the object $\Upsilon := P(\xi)$ represents the set of truth values (or propositions) of the associated topos — in the sense in which the tripos represents a topos; and the generic predicate ξ can be understood as a propositional variable.

The original *Realizability Tripos* [20, §1.7.] can be easily transformed into a posetal regular fibration. Instead of spelling this out, let us describe a closely related non-posetal example. It is still a globally small fibration, but not locally small; and it still corresponds to a hyperdoctrine over Set. The fact that it is not locally small [3] implies that it cannot be represented in Set by a small category. Nevertheless, it is in a sense derived from a small category — namely from the category of *modest sets*, the subquotients of natural numbers. In a similar manner, the Realizability Tripos has been obtained from the lattice of subsets of natural numbers.

Following Hyland [19], let us define a modest set as a set X equipped with a surjective function $|X| \to X$, where $|X| \subseteq N$ is a set of natural numbers. Each element $x \in X$ thus comes with a disjoint *set of codes* $|x|$ — the part of $|X|$ that the function $|X| \to X$ projects to x. A function $\psi : X \to Y$ is said to be *traced* by a natural number $n \in N$ if the partial recursive function[5] $n' : N \to N$ is defined on all of $|X|$ and if it maps the codes of $x \in X$ to codes of $\psi(x) \in Y$. This will make the following diagram commute.

$$\begin{array}{ccc} |X| & \xrightarrow{n'} & |Y| \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & Y \end{array}$$

(3.4)

Let the objects of the category Mod be the set-indexed families of modest sets. If $\langle X_a \rangle = \langle X_a \rangle_{a \in A}$ and $\langle Y_b \rangle = \langle Y_b \rangle_{b \in B}$ are two such families, an arrow $\langle X_a \rangle \to \langle Y_b \rangle$ in Mod will be a pair $\langle f, \varphi \rangle$, where $f : A \to B$ is a function in Set, while $\varphi = \langle \varphi_a \rangle_{a \in A}$ is a family of functions $\varphi_a : X_a \to Y_{f(a)}$ which are traced by some number $n \in N$ — the same for all of them.

Projecting $\langle X_a \rangle$ to A and $\langle f, \varphi \rangle$ to f, we get the fibration $M : Mod \to Set$. The inverse images are simply

$$f^* \langle Y_b \rangle := \langle Y_{f(a)} \rangle.$$

---

[5]More familiar notation would be Kleene's $\{n\}$, or φ_n; but Gödel's n' is more convenient for a later development.

Using the basic properties of the encoding of partial recursive functions by natural numbers, one easily shows that Mod is fibrewise cartesian closed. We shall just sketch how the direct images $f_!\langle X_a\rangle$ are constructed.

For $f : A \to B$ and an A-indexed family $\langle X_a\rangle$, the image $f_!\langle X_a\rangle$ will be the B-indexed family $\langle X_b'\rangle$, where each set X_b' is obtained in the following pushout (in Set).

$$
\begin{array}{ccc}
\coprod_{f(a)=b} |X_a| & \longrightarrow\!\!\!\!\!\longrightarrow & \coprod_{f(a)=b} X_a \\
\downarrow & & \downarrow \\
\bigcup_{f(a)=b} |X_a| & \longrightarrow\!\!\!\!\!\longrightarrow & X_b'
\end{array}
$$

$$(3.5)$$

In words, the set X_b' is the quotient of the disjoint union of all X_a, such that $f(a) = b$, by the transitive closure of the relation

$$\langle a, x\rangle \sim \langle \tilde{a}, \tilde{x}\rangle :\Longleftrightarrow \exists n \in |x| \cap |\tilde{x}|.$$

To make X_b' into a modest set, take

$$|X_b'| := \bigcup_{f(a)=b} |X_a|,\qquad\qquad (3.6)$$

and use the surjection at the bottom of diagram (3.5) as the encoding $|X_b'| \to X_b'$.

The opcartesian arrow $\sigma^f : \langle X_a\rangle \to f_!\langle X_a\rangle$ will be the pair $\langle f, \eta\rangle$, where the components $\eta_a : X_a \to X_{f(a)}'$ of η are obtained by composing the obvious inclusion $X_a \hookrightarrow \coprod_{f(c)=f(a)} X_c$ with the quotient map $\coprod_{f(c)=f(a)} X_c \to X_{f(a)}'$. All η_a are traced by a code of the identity function on N — which restricts to the inclusion $|X_a| \hookrightarrow |X_{f(a)}'| = \bigcup_{f(c)=f(a)} |X_a|$. Checking that σ^f is indeed opcartesian is left to the reader.

A choice of all the direct and all the inverse images will, of course, determine the adjoint functors $f_! \dashv f^* : \mathrm{Mod}_B \to \mathrm{Mod}_A$. Note that f^* has also a right adjoint $f_* : \mathrm{Mod}_A \to \mathrm{Mod}_B$. The *right* direct image[6] $f_*\langle X_a\rangle$ will be the family $\langle X_b''\rangle$, where each X_b'' is obtained in the surjective-injective factorisation on the next diagram.

$$
\begin{array}{ccc}
\bigcap_{f(a)=b} |X_a| & \longrightarrow\!\!\!\!\!\longrightarrow & X_b'' \\
\downarrow & & \downarrow \\
\prod_{f(a)=b} |X_a| & \longrightarrow\!\!\!\!\!\longrightarrow & \prod_{f(a)=b} X_a
\end{array}
$$

$$(3.7)$$

---

[6]A description of the universal property which characterises it can be found in [38, II.3.2].

EXAMPLE 3.4

Every fibration $F : \mathcal{D} \to C$ with the finite limits in C and the finite fibrewise products in $\mathcal{D}$ induces a *free regular fibration*

$$\mathsf{Cod} : \mathsf{F}/C \to C : \langle R, r : \mathsf{F}(R) \to A \rangle \mapsto A.$$

The observation that the comma category F/C freely adds the stable direct images to the fibred category $\mathcal{D}$ is due to Bénabou [2]. These direct images are obtained simply by composing:

$$f_!\langle R, r \rangle := \langle R, (r; f) \rangle$$

Since the finite fibrewise products can be freely added to any fibration, this construction actually yields a left adjoint to the inclusion of regular fibrations in the general ones, over a finitely complete base.

4 Maps relative to a regular fibration

In the rest of the paper, we shall try to see some regular logic at work. The question will be: Given a regular fibration $P : \wp \to C$, *what does the category of 'predicates' $\wp$ 'know' about the category of 'sets' C?* Can it capture the notion of 'function', which comes with the arrows of C? Of course, every arrow $f : A \to B$ in C induces a predicate $f(x){\equiv}y$ in $\wp(A \times B)$. Is there an *intrinsic* characterisation of the predicates which come from C in this way? — Well, in set theory, a predicate represents a function if and only if it is total and single-valued. These concepts can easily be expressed in the setting of regular fibrations.

DEFINITION 4.1

A predicate $R(x,y) \in \wp(A \times B)$ is said to be

- *total* (in x) if there is a proof

$$\bar{\eta} : \mathsf{T}(x) \longrightarrow \exists y. R(x, y), \tag{4.1}$$

- *single-valued* (in y) if there is a proof

$$\bar{\varepsilon} : R(x, y') \wedge R(x, y) \longrightarrow y'{\equiv}y. \tag{4.2}$$

Predicates in the form $f(x){\equiv}y$ are always total and single-valued. In 'ordinary' logic of subsets, i.e. with respect to the fibration $\mathsf{Cod} : \mathsf{Mon}/\mathsf{Set} \to \mathsf{Set}$, the converse holds too: every total and single-valued predicate — a *map* — must be in the form $f(x){\equiv}y$ for a unique function f. However, for the regular fibration $\mathsf{Cod} : \mathsf{Set}/\mathsf{Set} \to \mathsf{Set}$, e.g., this will not be the case. In this fibration, a predicate $\langle a, b \rangle : R \to A \times B$ is total if and only if a is a split epi; and it is single-valued if and only if $a(x) = a(y)$ implies $b(x) = b(y)$. (Cf. [42, lemma 5.2].) Not all the predicates which satisfy these conditions are in the form $f(x){\equiv}y$, which is in fact $\langle id, f \rangle : A \to A \times B$. The situation is not hopeless, since each total, single-valued predicate $\langle a, b \rangle$ does induce a unique $f : A \to B$, by composing b with the splitting of a; but each f is induced by a proper class of such predicates.

The idea how to narrow down the total, single-valued predicates, so as to better approximate the 'functions' from C, follows the observation that the totality and the single-valuedness of a predicate are, in a sense, *dual* properties. This becomes

apparent when the proofs $\bar{\eta}$ and $\bar{\varepsilon}$ from 4.1 are slightly modified

$$\eta \quad : \quad x \equiv x' \longrightarrow \exists y. R(x,y) \wedge R(x',y) \tag{4.3}$$

$$\varepsilon \quad : \quad \exists x. R(x,y') \wedge R(x,y) \longrightarrow y' \equiv y. \tag{4.4}$$

Note that nothing is lost: $\bar{\eta}$ can be recovered from η, and $\bar{\varepsilon}$ from ε. The idea is now that η and ε should express an adjunction, i.e. that $R(x,y)$ is self-adjoint. Instead of being separate proofs of independent facts, η and ε would be tied together by the adjunction equations. In the present situation, they are dual to each other, and boil down to forcing the composite

$$R(x,y) \longrightarrow \exists x'y'. R(x,y') \wedge R(x',y') \wedge R(x',y) \longrightarrow R(x,y)$$

to reduce to the identity.

This strengthened notion of a *map* turns out to work much better. For instance, a predicate $\langle a, b \rangle : R \to A \times B$ in the regular fibration $\mathrm{Cod} : \mathrm{Set}/\mathrm{Set} \to \mathrm{Set}$ is self-adjoint if and only if it is isomorphic to one in the form $\left(f(x) \equiv y \right) = \langle id, f \rangle$. This remains true when Set is replaced by an arbitrary finitely complete category C and, more generally, when instead of the whole arrow category C/C, any subcategory $\mathcal{M}/C$, spanned by the abstract monics $\mathcal{M}$ of a stable factorisation system is considered [42]. And yet, it remains unclear *why* adding the adjunction equations is a step in the right direction: *what is their logical significance?* Moreover, they tend to be quite complicated to work with: *is there a simpler form?*

To approach these questions, let us describe in more detail the setting in which the maps are defined.

Bicategory of predicates. A regular fibration P induces a bicategory of predicates, just like a regular category induces an allegory [12]. For a formal introduction to bicategories, the reader is referred to [2].

By definition, the objects of the bicategory of predicates $\mathcal{R} = \mathcal{R}_\mathsf{P}$, associated with the regular fibration $\mathsf{P} : \wp \to C$, are the objects of C. The hom-category from A to B consists of the predicates on $A \times B$:

$$\mathcal{R}(A,B) := \wp(A \times B).$$

A predicate $R(x,y) \in \wp(A \times B)$ thus appears as the 1-cell $R : A \nrightarrow B$ in $\mathcal{R}$; a proof $\alpha : R(x,y) \to R'(x,y)$ as a 2-cell. (Note the slash on the arrow denoting a 1-cell.) The objects of C are the 0-cells.

The composite $RS : A \nrightarrow C$ of the 1-cells $R : A \nrightarrow B$ and $S : B \nrightarrow C$ will be the relation

$$RS(x,z) := \exists y. R(x,y) \wedge S(y,z) \quad \left(= p_! \left(r^*(R) \wedge s^*(S) \right) \right). \tag{4.5}$$

Exactly the same formula defines the 'horizontal' composite $\alpha\beta : RS(x,z) \to R'S'(x,z)$ of 2-cells $\alpha : R(x,y) \to R'(x,y)$ and $\beta : S(y,z) \to S'(y,z)$. The 'vertical' composite of $\alpha : R(x,y) \to R'(x,y)$ and $\alpha' : R'(x,y) \to R''(x,y)$ is, of course, the proof $(\alpha; \alpha') : R(x,y) \to R''(x,y)$ in $\wp(A \times B)$.

Composition (4.5) is associative modulo the isomorphisms that arise from the stability conditions. Furthermore, proposition 2.21, implies that the equality predicate $x \overset{\Delta}{\equiv} x'$ can play the role of the identity relation $\imath : A \nrightarrow A$.

The fibrewise products from P yield the finite products in the hom-categories of $\mathcal{R}$. The products from C induce in $\mathcal{R}$ a tensor, with (lax natural) diagonals, codiagonals etc. — the structure known from [9]. There is also a duality

$$(-)^o : \mathcal{R}^{op} \to \mathcal{R} \tag{4.6}$$

which leaves the 0-cells unchanged, and maps the hom-category $\mathcal{R}(A, B)$ to $\mathcal{R}(B, A)$ by the functor

$$c^* : \wp(A \times B) \longrightarrow \wp(B \times A),$$

where $c : B \times A \to A \times B$ is the twist — so that we actually have

$$R^o(y, x) := c^*(R).$$

In writing, we often neglect the difference between $R^o(y, x)$ and $R(x, y)$.

Bicategories of predicates could be axiomatized as a non-posetal version of Carboni and Walters' [9] abstract 'bicategory of relations', with tensor, diagonals, and so on. Duality (4.6) is definable in terms of adjunctions. To every such bicategory $\mathcal{R}$ one could associate a regular fibration P : $\wp \to C$. In a suitable sense, this construction is right adjoint to the building a bicategory of predicates from a regular fibration, as we did above. But the direction from bicategories to fibrations is more complicated, and it goes beyond the needs of this paper. We shall nevertheless outline it, since it uncovers important correlations.

The base of the regular fibration P associated to a bicategory of predicates $\mathcal{R}$ would be the category C of *maps* in $\mathcal{R}$. The fibres $\wp(A)$ would be the hom-categories $\mathcal{R}(I, A)$, where I is the unit of the tensor from $\mathcal{R}$. This tensor becomes in C the cartesian product, and provides the fibrewise products in $\wp$. The direct image of a 1-cell $P \in \wp(A)$ along a map $R \in \mathcal{R}(A, B)$ will be the 1-cell $PR \in \wp(B)$. The inverse image of $Q \in \wp(B)$ will be the 1-cell $QR^* \in \wp(A)$, where R^* is the right adjoint of R, which makes it into a map.

Maps. In [33], Lawvere has defined a *map* as a 1-cell $R : A \leftrightarrow B$, such that $R^o : B \leftrightarrow A$ is its right adjoint. This just means that there are proofs (4.3) and (4.4), satisfying

$$(\eta R \; ; \; R\varepsilon) \; = \; id_R. \tag{4.7}$$

The other adjunction equation $(R^o\eta; \varepsilon R^o) = id_{R^o}$ follows by dualizing.

Actually, one could start from a more general definition of a map — as a 1-cell with *some* right adjoint. By the same reasoning as in [42, section 3], it could then be proved that, in an arbitrary bicategory of predicates, a right adjoint $R^* : B \leftrightarrow A$ of a 1-cell $R : A \leftrightarrow B$ must be isomorphic to $R^o : B \leftrightarrow A$. This is a consequence of the fact that, in arbitrary bicategory, the 2-cells $\alpha : R \to S$ and $\alpha^* : R^* \to S^*$ must be isomorphisms if $R \dashv R^*$ and $S \dashv S^*$, and if α and α^* commute with the units and the counits of these adjunctions [42, lemma 3.2]. In the bicategory of predicates, each adjunction $R \dashv R^*$ induces suitable 2-cells $\alpha : R \to (R^*)^o$ and $\alpha^* : R^* \to R^o$. Constructing them runs parallel to the proof of [42, proposition 3.3]. The logical

content of the construction is reflected in the following derivation.

$$\frac{\dfrac{\eta:\ x\equiv x' \longrightarrow \exists y.R(x,y)\wedge R^*(y,x')}{\mathsf{T}(x)\longrightarrow \exists y.R(x,y)}}{\dfrac{x\equiv x'\longrightarrow \exists y.R(x,y)\wedge R^\circ(y,x')}{R^*(y',x')\to \exists x\, y.R^*(y',x)\wedge R(x,y)\wedge R^\circ(y,x')}}$$

$$\frac{\epsilon:\ \exists x.R^*(y',x)\wedge R(x,y)\longrightarrow y'\equiv y}{\exists y\, x.R^*(y',x)\wedge R(x,y)\wedge R^\circ(y,x')\to R^\circ(y',x')}$$

$$\alpha^*:\ R^*(y',x')\longrightarrow R^\circ(y',x')$$

We shall not chase these proofs here. We could proceed exactly as in [42] again, and show that

PROPOSITION 4.2

For maps $R(x,y)$ and $S(x,y)$ in the bicategory of predicates, every proof $\alpha : R \to S$ must be an isomorphism.

The upshot of this is that, in the bicategory of predicates $\mathcal{R}$, the maps modulo isomorphism form an ordinary category $\widehat{C}$. Its objects will be the same as the objects of C and $\mathcal{R}$; and the hom-set $\widehat{C}(A,B)$ will be the skeleton of the subcategory of $\mathcal{R}(A,B)$, spanned by the maps. The proof of proposition 4.2 depends only on the fact that for any 2-cell $\alpha : R \to S$ and proofs of single-valuedness $\varepsilon_R : R^\circ R \to \imath$ and $\varepsilon_S : S^\circ S \to \imath$ the equation

$$\Big((\alpha^\circ\alpha)\ ;\ \varepsilon_S\Big) = \varepsilon_R$$

must hold. And this is an immediate consequence of lemma 2.3.

For general bicategories, proposition 4.2 may not be true. For instance, the multiplicative monoid $\mathbf{N}$ of natural numbers can be regarded as a bicategory with a single 0-cell, with the numbers as 1-cells and the binary relations on $m \times n$ as the 2-cells from m to n. It is easy to see that every 1-cell m is a map, with $\eta : 1 \to m \times m$ and $\varepsilon : m \times m \to 1$ relating the only element of 1 with all pairs $\langle x,x \rangle \in m \times m$. On the other hand, there are obviously non-trivial 2-cells. This is a typical *compact closed category*.

On the other hand, in bicategories of predicates, more is actually true than stated in 4.2. In the next section, we shall see that any two proofs $\alpha, \alpha' : R \to S$ must be equal as soon as S is a map. This will follow from 5.3(1)$\Rightarrow$2. The identification of maps along isomorphisms is thus *coherent*, so that the subcategory of $\mathcal{R}(A,B)$, spanned by the maps, is equivalent with its skeleton $\widehat{C}(A,B)$. This means that the category of maps $\widehat{C}$ is biequivalent with a subbicategory of $\mathcal{R}$, fully on the 2-cells.

5 Characterizing maps

In this section, we describe some conditions, intuitively or technically simpler than (4.7), that can be equivalently imposed on a total, single-valued relation — to yield a map. Let us begin with what seems to be the weakest sensible commutativity condition that can be put on the proof of totality. It will turn out to be equivalent to (4.7).

DEFINITION 5.1

A predicate $R(x,y)$ is *strongly total* if there is a proof

$$\tilde{\eta} : \mathsf{T}(x) \longrightarrow \exists y.R(x,y)$$

such that the opcartesian arrow $\sigma^p : R(x,y) \longrightarrow \exists y.R(x,y)$ factorizes through it as $\sigma^p = (\tau^p; \bar{\eta})$.

LEMMA 5.2

If $R(x,y)$ is strongly total, then for every ϕ there is a proof η which makes the next diagram commute.
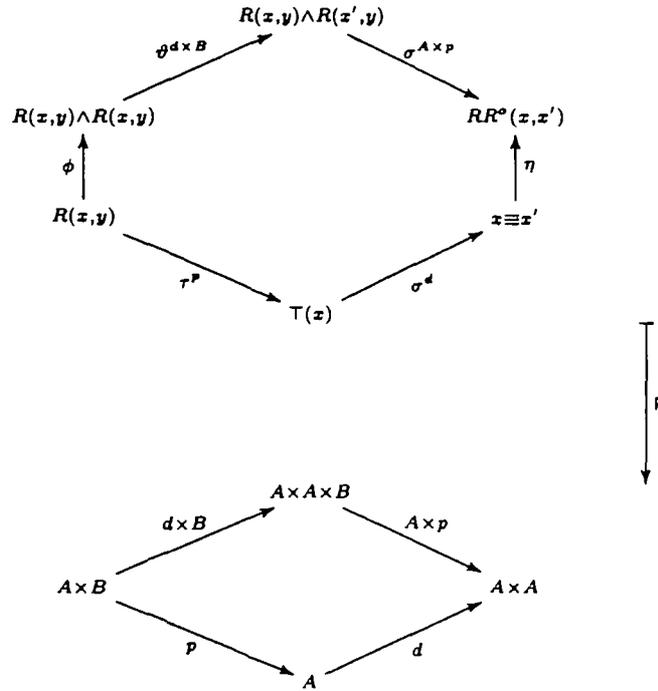
$$(5.1)$$

PROOF. The desired proof η is the unique vertical factorisation of the arrow

$$(\bar{\eta}; \exists y.\phi; \vartheta^d) : \top(x) \to \exists y.R(x,y) \to \exists y.R(x,y) \wedge R(x,y) \to RR^o(x,x')$$

through the opcartesian arrow σ^d.   ∎

THEOREM 5.3

The following four conditions are equivalent for every predicate $R(x,y) \in \wp(A \times B)$,

1. R is a map.
2. $R(x,y)$ is total, single-valued and subterminal.
3. $R(x,y)$ is total and single-valued, and $\exists y.R(x,y)$ is terminal.
4. $R(x,y)$ is strongly total and single-valued.

PROOF. 1$\Rightarrow$2 From the explanations in section 4, it is clear that every map is total and single-valued. It is subterminal, because $R(x,y) \wedge R(x,y)$ *is a map as soon as* $R(x,y)$ *is a map.* Let us postpone the proof of this fact for the next section. Assuming

it, we can use proposition 4.2 to conclude, say, that the projection $\pi : R \wedge R \to R$ is an isomorphism. Lemma B.4 then tells that R is subterminal.

2⇒3 If the projection $\pi_R : R \wedge R \to R$ (either one of them) is an iso, then the proof

$$\exists y.\pi_R : \exists y.R(x,y) \wedge R(x,y) \longrightarrow \exists y.R(x,y),$$

is obviously an iso too. On the other hand, *since $R(x,y)$ is single-valued, the functor $\exists y$ preserves the product $R(x,y) \wedge R(x,y)$ and takes the projection π_R to the projection*

$$\pi_{\exists y.R} : \exists y.R(x,y) \wedge \exists y.R(x,y) \longrightarrow \exists y.R(x,y).$$

We defer the proof of this to the next section again. This preservation granted, we conclude that the projection $\pi_{\exists y.R} = \exists y.\pi_R$ is an isomorphism. By B.4 again, the predicate $\exists y.R(x,y)$ is thus subterminal.

On the other hand, $R(x,y)$ is total, i.e. there is a section $\bar{\eta} : \mathsf{T}(x) \longrightarrow \exists y.R(x,y)$. The predicate $\exists y.R(x,y)$ is thus weakly terminal too. — But then it must be a terminal object.

3⇒4 $\exists y.R(x,y) \cong \mathsf{T}(x)$ means, of course, that the arrow $\tau^p : R(x,y) \to \mathsf{T}(x)$ must be opcartesian. (Cf. A.2.) The proof $\bar{\eta} : \mathsf{T}(x) \longrightarrow \exists y.R(x,y)$, required by condition 4 is thus the unique vertical factorisation of σ^p through τ^p.

4⇒1 Here we must show that a strongly total, single-valued predicate $R(x,y)$ must be self-adjoint. Lemma C.1 in the Appendix reduces this task to showing that the 2-cell $(\eta R \ ; R\varepsilon)$, obtained from the proofs of the totality and the single-valuedness — is a *split epi*. Dually, the 2-cell $(R^o\eta \ ; \varepsilon R^o) = (\eta R \ ; R\varepsilon)^o$ will then be a split epi too. Lemma C.1 now says that these two split epis suffice for an adjunction $R \dashv R^o$.

A left splitting of $(\eta R; R\varepsilon)$ comes about in the following diagram.

$$R(x,y')\wedge R(x',y')\wedge R(x',y) \xrightarrow{\quad R\wedge\tilde{\varepsilon}\quad} R(x,y')\wedge y'\equiv y$$

The diagram (5.2) with faces (‡), (I), (II):

- Top-left node: $R(x,y')\wedge R(x',y')\wedge R(x',y)$
- arrow $R\wedge\tilde{\varepsilon}$ to $R(x,y')\wedge y'\equiv y$
- $\sigma^{p\times B}$ down to $R(x,y)$
- $\varphi^{d\times d'}$, (‡), σ^q
- $\exists y'.R(x,y')\wedge R(x',y')\wedge R(x',y) \xrightarrow{\tilde{\sigma}^{A\times p'}} RR^{\circ}R(x,y)$
- (I)
- $R\varepsilon$
- $\eta\wedge R$, (II), ηR
- $R(x,y) \xrightarrow{\gamma^{d\times B}} x\equiv x'\wedge R(x',y) \xrightarrow{\sigma^{A\times p'}} R(x,y)$

Lower diagram:

- $A\times A\times B\times B \xrightarrow{q'} A\times B\times B$
- $d\times d'$, q, $p\times B$
- $A\times B \xrightarrow{d\times B} A\times A\times B \xrightarrow{A\times p'} A\times B$ (5.2)
- P
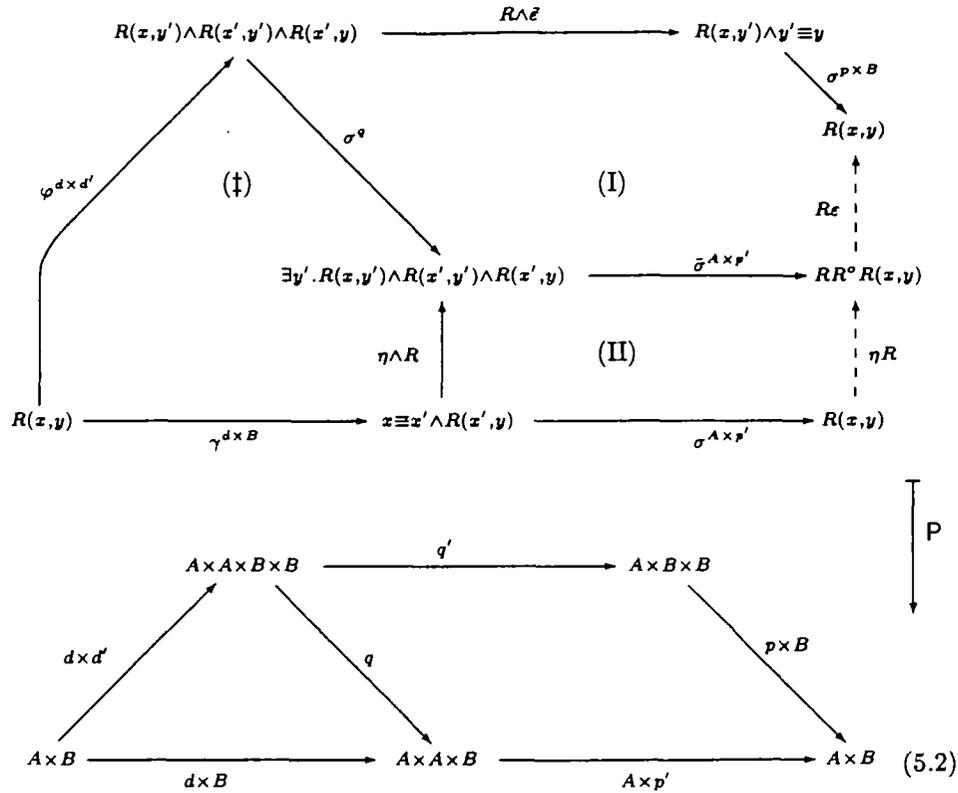
For the strongly total, single-valued predicate $R(x, y)$, we shall construct:

(i) a proof $\varphi^{d\times d'}$ — which is left inverse of $\left((R\wedge\tilde{\varepsilon});\sigma^{p'\times B}\right)$, and

(ii) proofs η and $\gamma^{d\times B}$ — such that face (‡) commutes.

Condition (ii) will make diagram (5.2) commutative, since faces (I) and (II) commute by definition (4.5) of composites ηR and $R\varepsilon$. Condition (i) will then imply that the arrow $(\gamma^{d\times B};\sigma^{A\times p})$ must be a left inverse of $(\eta R \;;\; R\varepsilon)$.

Construction (i).   Consider the diagram:

$$(5.3)$$

By proposition A.8(2), face (I) is a pullback. Face (II) is a pullback because of lemma A.4.

Since $y' \equiv y$ is subterminal (lemma 2.3), any two arrows from $R(x, y)$ to $y' \equiv y$ must be equal. The fact that (II) is a pullback implies that there is a unique arrow $\psi^{A \times d'}$, such that

$$\left(\psi^{A \times d'} \; ; \; (R \wedge \varepsilon) \right) \;=\; \sigma^{A \times d'} \tag{5.4}$$

$$\left(\psi^{A \times d'} \; ; \; \pi^{p' \times B} \right) \;=\; \left(\delta^{A \times d'} \; ; \; \sigma^{p' \times B} \right) \tag{5.5}$$

Since (I) is a pullback, equation (5.5) induces a unique arrow $\varphi^d \times d'$ with

$$\left(\varphi^{d \times d'} \; ; \; \sigma^{r'} \right) \;=\; \psi^{A \times d'} \tag{5.6}$$

$$\left(\varphi^{d \times d'} \; ; \; \pi^{q'} \right) \;=\; \delta^{A \times d'} \tag{5.7}$$

From (5.4) and (5.6) follows

$$\left(\varphi^{d \times d'} \; ; \; \sigma^{r'} \; ; \; (R \wedge \varepsilon) \right) = \sigma^{A \times d'}. \tag{5.8}$$

Since $\Big((A \times d'); (p \times B) \Big) = id_{A \times B}$, there is a unique arrow $\sigma^{p \times B}$ such that

$$\Big(\sigma^{A \times d'} \ ; \ \sigma^{p \times B} \Big) = id_R.$$

This arrow must be opcartesian, because both id_R and $\sigma^{A \times d'}$ are. Postcomposing on both sides of (5.8), we get

$$(\varphi^{d \times d'} \ ; \ \sigma^{r'} \ ; \ R \wedge \varepsilon \ ; \ \sigma^{p \times B}) = id_R. \qquad (5.9)$$

Since by proposition A.8(2) holds

$$\bar{\varepsilon} := (\sigma^{p' \times B}; \varepsilon) \quad \Longrightarrow \quad (R \wedge \bar{\varepsilon}) = \Big(\sigma^{r'}; (R \wedge \varepsilon) \Big),$$

equation (5.9) tells that $\varphi^{d \times d'}$ satisfies requirement (i).

Construction (ii). To construct $\gamma^{d \times B}$, we need a diagram similar to (5.3).
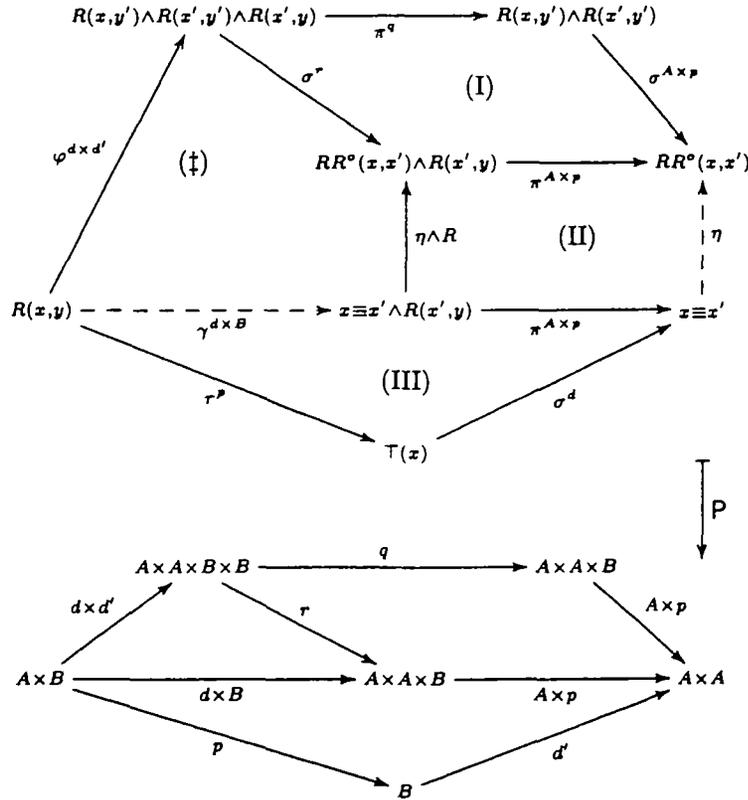
$$(5.10)$$

The task is to construct a proof η and then $\gamma^{d \times B}$, such that face (‡) commutes.

Let us denote by ϕ the unique vertical arrow $R(x,y) \to R(x,y) \wedge R(x,y)$ such that

$$(\phi \; ; \; \vartheta^{A \times d'}) = (\varphi^{d \times d'} \; ; \; \pi^q).$$

Since $R(x,y)$ is strongly total, lemma 5.2 yields a proof η such that the two external paths around the upper part of (5.10) are equal.

Square (II) has thus been formed. Lemma A.4 implies that it is a pullback. Since the two external paths in (5.10) are equal, and (I) commutes (by A.8(2), it is even a pullback) — a unique arrow $\gamma^{d \times B}$ is induced, making both (III) and ($\ddagger$) commutative.

■

6 Lemmas

In this section, we fill the gaps left in the first two parts of theorem 5.3. The setting is still the bicategory of predicates $\mathcal{R}$.

LEMMA 6.1 $(2{\Rightarrow}3)$
If the predicates $R(x,y), R'(x,y) \in \wp(A \times B)$ are *jointly single-valued* — in the sense that there is a proof $\bar{\varepsilon} : R(x,y) \wedge R'(x,y') \longrightarrow y{\equiv}y'$ — then the functor $\exists y$ preserves the product $R(x,y) \wedge R'(x,y)$.

PROOF. Given jointly single-valued predicates R and R', we must construct an isomorphism

$$\exists y.R(x,y) \wedge R'(x,y) \cong \exists y.R(x,y) \wedge \exists y.R'(x,y), \tag{6.1}$$

The Frobenius condition, on the other hand, implies that for *any* pair of predicates $R, R' \in \wp(A \times B)$, there is an isomorphism

$$\exists yy'.R(x,y) \wedge R'(x,y') \cong \exists y.R(x,y) \wedge \exists y.R'(x,y). \tag{6.2}$$

For (6.1), it is thus sufficient to show that the proof α, defined on diagram (6.3) by the requirement that face (I) commutes — is an isomorphism.

$$R(x,y) \wedge R'(x,y)$$

$$\vartheta^{A \times d'} \qquad \sigma^p$$

$$\text{(I)}$$

$$\exists y.R(x,y) \wedge R'(x,y)$$

$$\alpha$$

$$R(x,y) \wedge R'(x,y') \xrightarrow{\quad \sigma^q \quad} \exists yy'.R(x,y) \wedge R'(x,y')$$

$$\langle id,\bar\varepsilon \rangle \qquad \text{(II)} \qquad \exists yy'.\langle id,\bar\varepsilon \rangle$$

$$R(x,y) \wedge R'(x,y') \wedge y \equiv y' \xrightarrow{\quad \sigma^q \quad} \exists yy'.R(x,y) \wedge R'(x,y') \wedge y \equiv y'$$

$$p$$

$$A \times B$$

$$A \times d' \qquad p$$

$$A \times B \times B \xrightarrow{\quad q \quad} A$$
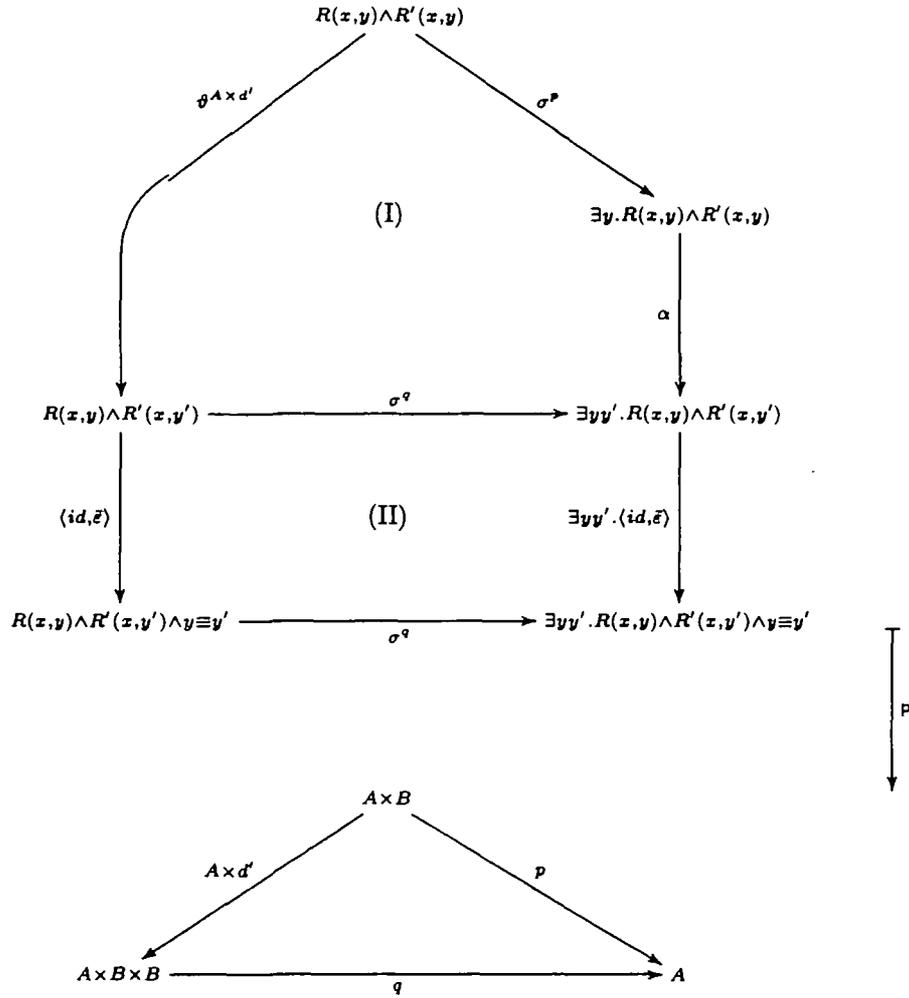
(6.3)

Square (II), of course, commutes by the definition of $\exists yy'.\langle id,\bar\varepsilon \rangle$.

We first claim that the arrow $(\vartheta^{A \times d'}; \langle id,\bar\varepsilon \rangle)$ is opcartesian. By proposition 2.22, there is indeed an opcartesian arrow $\sigma^{A \times d'}$ from $R(x,y) \wedge R'(x,y)$ to $R(x,y) \wedge R'(x,y') \wedge y \equiv y'$. Proposition A.81, on the other hand, implies that every opcartesian lifting of a monic must also be cartesian. Hence the following diagram.

$$R(x,y) \wedge R'(x,y) \xrightarrow{\ \vartheta^{A \times d'}\ } R(x,y) \wedge R'(x,y')$$

$$(A \times d')^* \langle id, \bar{\varepsilon} \rangle \Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow \langle id, \bar{\varepsilon} \rangle$$

$$R(x,y) \wedge R'(x,y) \xrightarrow[\ \sigma^{A \times d'}\ ]{\ \vartheta^{A \times d'} =\ } R(x,y) \wedge R'(x,y') \wedge y \equiv y'$$

$$(6.4)$$

Since the inverse images preserve the fibrewise products, hence the pairing the arrow $(A \times d')^* \langle id, \bar{\varepsilon} \rangle$ is in fact the pair

$$\langle id, (A \times d')^* \bar{\varepsilon} \rangle : R \wedge S \longrightarrow R \wedge S \wedge \mathsf{T},$$

clearly an iso. The arrow $(\vartheta^{A \times d'}; \langle id, \bar{\varepsilon} \rangle)$ is thus isomorphic to an opcartesian arrow — thus opcartesian itself.

The arrow $(\vartheta^{A \times d'}; \langle id, \bar{\varepsilon} \rangle; \sigma^q)$ is thus an opcartesian lifting of $p = \left((A \times d'); q \right)$ — just like σ^p is. As an arrow between two opcartesian liftings of the same arrow, $\left(\alpha \ ; \ \exists yy'.\langle id, \bar{\varepsilon} \rangle \right)$ must be an iso — and $\exists yy'.\langle id, \bar{\varepsilon} \rangle$ is a split epi. But $\exists yy'.\langle id, \bar{\varepsilon} \rangle$ is surely a split monic, because $\langle id, \bar{\varepsilon} \rangle$ is. So $\exists yy'.\langle id, \bar{\varepsilon} \rangle$ is an iso. Since its composition with α is an iso too, α must be an iso.

A straightforward application of proposition A.8(2) now shows that $\exists y$ takes projections to projections. ∎

LEMMA 6.2 (1⇒2)
If $R(x,y)$ is a map, then $R^2(x,y) := R(x,y) \wedge R(x,y)$ must be a map too.

PROOF. Given the proofs η and ε, which make $R(x,y)$ a map, we must define

$$\eta_2 \ : \ x \equiv x' \longrightarrow \exists y. R^2(x,y) \wedge R^2(x',y) \text{ and}$$
$$\varepsilon_2 \ : \ \exists x. R^2(x,y') \wedge R^2(x,y) \longrightarrow y' \equiv y,$$

making $R^2(x,y)$ into a map. So let the unit η_2 be the composite

$$\eta_2 := \left(\langle \eta, \eta \rangle; \kappa \right) : x \overset{\scriptscriptstyle A}{\equiv} x' \ \longrightarrow \ \left(\exists y. R(x,y) \wedge R(x',y) \right)^2$$
$$\longrightarrow \ \exists y. R^2(x,y) \wedge R^2(x',y), \qquad\qquad (6.5)$$

where the isomorphism $\kappa \ : \ \left(\exists y. R(x,y) \wedge R(x',y) \right)^2 \longrightarrow \exists y. R^2(x,y) \wedge R^2(x',y)$ is derived from lemma 6.1. The counit ε_2, on the other hand, can be obtained in the opcartesian-vertical factorisation of the 2-cell

$$\bar{\varepsilon}_2 := \left(\rho; (\bar{\varepsilon} \wedge \bar{\varepsilon}); \pi \right) : R^2(x,y') \wedge R^2(x,y) \ \longrightarrow \ (R(x,y') \wedge R(x,y))^2$$
$$\longrightarrow \ (y' \equiv y)^2$$
$$\longrightarrow \ y' \equiv y. \qquad\qquad (6.6)$$

The arrow $\rho := \langle \pi_1, \pi_3, \pi_2, \pi_4 \rangle$ is the 'middle two exchange', and π is just a projection. Both these arrows are isomorphisms, the latter by lemmas 2.3 and B.4.

The composite $(\eta_2 R^2; R^2 \varepsilon_2)$ can be obtained using the universal property of opcartesian arrows on the following diagram.

$$(6.7)$$

The composite $(\eta R; R\varepsilon)$ appears, of course, in a diagram which can be obtained from (6.7) by dropping the '2' everywhere. We can thus put this diagram for R^2 on top of the diagram for R, and connect each pair of the corresponding objects by the arrows derived from, say, the first projection $\pi : R^2(x, y) \to R(x, y)$. The lateral squares of the prism obtained in this way will all be commutative: the ones built from the opcartesian arrows and the projections will commute as special cases of (A.1); those built by projecting $\eta_2 \wedge R^2$ to $\eta \wedge R$ and $R^2 \wedge \bar{\varepsilon}_2$ to $R \wedge \bar{\varepsilon}$ will commute by the definitions of η_2 and $\bar{\varepsilon}_2$, respectively. (The latter, in fact, already by lemma 2.3.) Chasing this prism and using the universal property of opcartesian arrows, we conclude that the square

$$R^2(x,y) \xrightarrow{\;\eta_2 R^2\;} R^2(R^\circ)^2 R^2(x,y) \xrightarrow{\;R^2 \varepsilon_2\;} R^2(x,y)$$

$$\downarrow{\scriptstyle \pi} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \pi}$$

$$R(x,y) \xrightarrow[\;\eta R\;]{} RR^\circ R(x,y) \xrightarrow[\;R\varepsilon\;]{} R(x,y)$$

$$(6.8)$$

must commute. The assumption that $R \dashv R^\circ$ means that the composite at the bottom reduces to the identity. Hence

$$(\eta_2 R^2 \; ; \; R^2 \varepsilon_2 \; ; \pi) \;=\; \pi.$$

Another equation like this is obtained by projecting (6.7) on the corresponding diagram for R once again, but this time using everywhere the second projection π' : $R^2 \to R$ instead of π. Putting these two equations together, we get

$$
\begin{aligned}
(\eta_2 R^2 \; ; \; R^2 \varepsilon_2) \;&=\; \langle (\eta_2 R^2; R^2 \varepsilon_2; \pi), (\eta_2 R^2; R^2 \varepsilon_2; \pi') \rangle \\
&=\; \langle \pi, \pi' \rangle \;=\; id_{R^2} \qquad\qquad\qquad (6.9)
\end{aligned}
$$

∎

7   Applications

Using theorem 5.3, we can now analyze the correlation between the maps relative to a regular fibration $P : \wp \to C$ and its functions, *viz* the arrows of C. The terms of reference are similar as in [42]. Following Lawvere [33], we say that an object $B \in C$ is *Cauchy complete* (a P-sheaf) if for every map $R(x,y) \in \wp(A \times B)$ there is a unique arrow $f : A \to B$ such that $(f(x){\equiv}y) \cong R(x,y)$. The fibration P itself is *function comprehensive* (subcanonical) if every object of C is Cauchy complete.

Examples. Recall from 3.4 that $\mathrm{Cod} : F/C \to C$ is the free regular fibration generated by a fibration $F : D \to C$, where C has the finite limits, while D has the finite fibrewise products. Direct calculations show that a binary predicate $\langle R, r \rangle \in F/C$ — where r is thus a pair $\langle a, b \rangle : F(R) \to A \times B$ — will be

- total — if and only if $a : F(R) \to A$ has a left inverse $j : A \to F(R)$, such that there is a vertical arrow $\top_A \to j^*(R)$ in D;

- single-valued — if and only if $(h;a) = (k;a)$ implies $(h;b) = (k;b)$, for all h, k, and

- subterminal — if and only if R is subterminal in the fibre D_{FR}, while r is a monic in C.

According to 5.3, these three conditions characterize maps. Recombining them, one concludes that $\langle R, r \rangle \in F/C$ will be a map if and only if a is iso and R is terminal in D_{FR}. In other words, the maps in F/C are (isomorphic to the objects) in the form $\langle \top_A, \langle id, f \rangle : A \to A \times B \rangle$. The free regular fibrations are thus function comprehensive.

The fibration M : Mod → Set, described in 3.3, is function comprehensive too. Namely, the subterminal objects in a fibre $\text{Mod}_{A \times B}$ are just the subsets of $A \times B$, and the maps boil down to actual functions in Set. On the other hand, the regular fibration corresponding to the Realizability Tripos is posetal, so that *all* the predicates in it are subterminal. Although the fibration M may seem richer, the Realizability Tripos defines more maps. It is thus very far from being function comprehensive: its category of maps is significantly larger then the base category Set.

Given a site (C, J), an object $B \in C$ is a J-sheaf if and only if it is Cauchy complete with respect to the regular fibration $\Omega_J : \Omega_J \to C$, defined in 3.2. This is immediate from the internal definition of a sheaf. The fibration Ω_J is thus function comprehensive if and only if the site (C, J) is subcanonical. And this is known [1, A.19] to be equivalent with the requirement that J-covers are jointly regular epi.

Given a stable factorisation $(\mathcal{E}, \mathcal{M})$ in C, one can fabricate a topology on C by saying that every sieve containing an $\mathcal{E}$-arrow is a cover. In this way, the results of [42], proved in the bicategory of predicates (i.e. in the calculus of relations) induced by the fibration $\text{Cod} : \mathcal{M}/C \to C$ from 3.1 — could also be derived using sites and the posetal fibration from 3.2.

Site representation. In fact, *every* regular fibration $P : \wp \to C$ over a left exact category C induces a topology $J : J \to C$. The J-covers of $A \in C$ are the sieves collected in

$$\mathbf{J}_A := \left\{ U \subseteq C/A \mid \text{there is } e \in U \text{ with } (\exists z.e(z) \equiv x) \cong \top(x) \right\}. \tag{7.1}$$

As outlined in 3.2, the fibred category Ω_J, associated with the site (C, J), will consist of the j-closed sieves, where the closure of a sieve $V \subseteq C/B$ is

$$j(V) = \left\{ f \in C/B \mid \text{there is } e \text{ with } (\exists z.e(z) \equiv x) \cong \top(x) \text{ and } (e; f) \in V \right\}. \tag{7.2}$$

Now we can define the site representation of the regular fibration $P : \wp \to C$ in $\Omega_J : \Omega_J \to C$. It is realized by the functor $\overline{(-)} : \wp \to \Omega_J$, which assigns to each predicate $Q(y) \in \wp(B)$ the sieve

$$\overline{Q(y)} := j \{ f \in C/B \mid \top(x) \to Q(f(x)) \}. \tag{7.3}$$

This is a product-preserving cartesian functor from P to Ω_J. In general, though, it is not opcartesian. So it may not preserve maps. When it does, the Cauchy complete objects with respect to P are J-sheaves.

PROPOSITION 7.1
The regular fibration $P : \wp \to C$ is function comprehensive if and only if

- its site representation preserves maps, and
- for every $e \in C$, whenever $(\exists z.e(z) \equiv x) \cong \top(x)$, then e must be a regular epi.

PROOF. ($\Rightarrow$) First suppose that P is function comprehensive. To show that the site representation preserves the single-valuedness, it suffices to prove that it preserves the equality, i.e. that

$$\overline{y' \stackrel{B}{\equiv} y} = d_! \left(\overline{\top}_B \right) \quad \left(:= j\{\langle f, f \rangle \mid f \in C/B\} \right).$$

Since $\overline{\mathsf{T}}_B \subseteq d^*\left(\overline{y'\equiv y}\right)$, the inclusion $d_!\left(\overline{\mathsf{T}}_B\right) \subseteq \overline{y'\equiv y}$ holds by the definition of $d_!$. The other way around, suppose $\langle h', h\rangle \in \overline{y'\equiv y}$. By (7.3) and (7.2), this means that there is a covering arrow e, such that $\mathsf{T}(x) \longrightarrow h'(e(z))\equiv h(e(z))$. From this proof, we derive $h'(e(z))\equiv y \longrightarrow h(e(z))\equiv y$. Since both predicates here are maps, the proof connecting them must be an isomorphism, by proposition 4.2. Since P is function comprehensive, it follows that $(e; h') = (e; h)$ must hold. Hence $\langle h', h\rangle \in \mathsf{j}\{\langle f, f\rangle \mid f \in C/B\}$.

The site representation thus preserves equality. Since it always preserves the fibre-wise <u>products</u>, it will take a proof $\bar{\varepsilon}$ of the single-valuedness of $R(x,y)$ into a proof that $\overline{R(x,y)}$ is single-valued.

If $R(x,y) \in \wp(A \times B)$ is, furthermore, a map, it must be (isomorphic to one) in the form $f(x)\equiv y$. For arbitrary $a \in C/A$, there is thus $b := (a; f) \in C/B$ such that $R(a(z), b(z)) \cong \mathsf{T}(z)$. This implies that the sieve

$$p_!\left(\overline{R}\right) = \mathsf{j}\left\{a \in C/A \mid \langle a, b\rangle \in \overline{R} \text{ for some } b \in C/B\right\}$$

is all of C/A — so that $\overline{R}$ satisfies condition 3 of theorem 5.3.

To see that $\exists z.e(z)\equiv x$ implies that $e : D \to A$ is a regular epi, take an arbitrary arrow $g : D \to B$, such that $(h; e) = (k; e)$ implies $(h; g) = (k; g)$, for all h, k. We claim that

$$R(x,y) := \exists z.e(z)\equiv x \wedge g(z)\equiv y$$

is a map. The assumption about g implies that R is single-valued. (Exercise. Form a pullback C of $B \times \langle e, g\rangle$ and $\langle g, e\rangle \times B$; show that the obtained arrow $c : C \to B \times A \times B$ must be in the form $c = \langle b, a, b\rangle$. But $R^\circ R$ is the direct image of T_C along $\langle b, b\rangle$.) On the other hand, the assumption $\exists z.e(z)\equiv x$ implies that $R(x,y)$ is total in the strong sense:

$$
\begin{aligned}
\exists y.R(x,y) &= \exists yz.e(z)\equiv x \wedge g(z)\equiv y \\
&\cong \exists z.e(z)\equiv x \wedge \exists y.g(z)\equiv y \\
&\cong \exists z.e(z)\equiv x \wedge \mathsf{T}(z) \\
&\cong \mathsf{T}(x).
\end{aligned}
$$

The assumption that P is function comprehensive now yields a unique $f : A \to B$ such that $R(x,y) \cong (f(x)\equiv y)$. Applying the substitutivity and the uniqueness part of the function comprehension, we can derive from the definition of $R(x,y)$ that $(e; f) = g$. ($\Leftarrow$) If the site representation preserves maps, an object $B \in C$ is P-Cauchy complete whenever it is Ω_J-Cauchy complete — i.e., whenever it is a J-sheaf. And we know that every object of C is a J-sheaf if and only if every J-cover is jointly regular epi. The second assumption implies that this must be the case here, since every J-cover by definition contains an arrow e such that $(\exists z.e(z)\equiv x) \cong \mathsf{T}(x)$.　■

The two conditions <u>from</u> the preceding proposition are independent. For instance, the site representation $\overline{(-)}$ of any factorisation system $(\mathcal{E}, \mathcal{M})$ is an opcartesian functor, and always preserves maps. But according to [42], the regular fibration induced by $(\mathcal{E}, \mathcal{M})$ is function comprehensive if and only if all $\mathcal{E}$-arrows are regular epi. And they need not always be. On the other hand, a set-theoretical function e is surjective whenever the Realizability Tripos says so — i.e., whenever $(\exists z.e(z)\equiv x) \cong \mathsf{T}(x)$. But this tripos is far from being function comprehensive. Hence, its site representation cannot preserve maps.

In fact, the second condition echoes (and subsumes) the standard characterisation of subcanonicity for sites [1, prop. I.4.3], as well as some more recent results about factorisations [22, 42]. For sites and factorisations, the first condition is satisfied automatically. The Effective Topos shows how interesting can be those 'slightly distorted' situations when it is not satisfied.

At any rate, the upshot of the function comprehension of $P : \wp \to C$ is that it provides for sound reasoning about C in $\wp$. It ensures that an arrow $u \in C$ is a regular epi as soon as it is P-surjective — in the sense that $\exists x.u(x){\equiv}y$ is provable in $\wp$. The converse does not hold: the function comprehension does not guarantee that every regular epi is P-surjective, as witnessed by any factorisation where the class $\mathcal{E}$ is *strictly* contained in the regular epis. On the other hand, whenever u is monic, it must be P-injective: a proof $u(x){\equiv}y \wedge u(x'){\equiv}y \longrightarrow x{\equiv}x'$ exists even without the function comprehension. The uniqueness part of the function comprehension (i.e., each map induces at most one arrow) suffices for proving that a P-injective arrow must be monic.

In bicategorical terms, a map $R \in \wp(A \times B)$ is surjective in the above sense if and only if the counit $\varepsilon : R^\circ R \to \imath_B$ of the adjunction $R \dashv R^\circ$ is iso; and R is injective if and only if the unit $\eta : \imath_A \to RR^\circ$ is iso. The former equivalence can be demonstrated using a diagram dual to (5.1). For the latter, notice that both $R(x,y) \wedge R(x',y)$ and $x{\equiv}x'$ are subterminal; and then use lemma 6.2 to show that RR° must be subterminal too.

8 Future work

Instead of speculating about great promises of categorical proof theory, let us try to touch upon a vanishing point of the presented material. In [7, 45], the Effective Topos is obtained by certain universal constructions. Now it turns out that the original construction of $\wp$-sets [20] has a universal property too. The presented analysis of maps suggests that a non-posetal version of $\wp$-sets might be feasible and meaningful.

First of all, every regular fibration $P : \wp \to C$ has a *function comprehensive completion* $P^\flat : \wp^\flat \to C^\flat$, defined as follows. The base $C^\flat$ is the category of maps relative to P. While $C^\flat$ has the same objects as C, the total category $\wp^\flat$ of the completion will have the same objects as $\wp$. The arrows from $P(x) \in \wp(A)$ to $Q(y) \in \wp(B)$ will be pairs $\langle R(x,y), \varrho \rangle$, where $R(x,y) \in \wp(A \times B)$ is a map, and $\varrho : P(x) \longrightarrow \exists y.R(x,y) \wedge Q(y)$ is a proof in $\wp(A)$. On the objects, the functor $P^\flat$ is the same as P; and it projects each arrow $\langle R, \varrho \rangle$ to the map R. The reader may wish to check that the inverse and the direct images are respectively provided by:

$$R^*(Q) \quad := \quad \exists y.R(x,y) \wedge Q(y) \text{ and} \tag{8.1}$$

$$R_!(P) \quad := \quad \exists x.P(x) \wedge R(x,y). \tag{8.2}$$

Clearly, $P^\flat$ is function comprehensive. Using theorem 5.3, it is not hard to show that a regular fibration is function comprehensive if and only if it is equivalent to its own completion. The construction $(-)^\flat$ is left (bi)adjoint to the inclusion of function comprehensive fibrations in the regular ones.

The Effective Topos is, of course, much bigger than the function comprehensive completion of the Realizability Tripos. To get a step closer to what it actually is, let us mention another free construction, which can be applied to any fibration $P : \wp \to C$

with finite products in C and in all fibres. Let us call such fibrations *left exact*.

An *equivalence relation* with respect to a left exact fibration P is a predicate $S(x, x') \in \wp(A \times A)$, which is reflexive, symmetric and transitive in the sense of $\wp$. We say that this equivalence relation is *effective* if there are an object B and an arrow $f : A \to B$ in C such that $S(x, x') \cong (f(x) \equiv f(x'))$. The fibration $P : \wp \to C$ is said to be effective itself if all its equivalence relations are. This terminology is consistent with the standard categorical usage: an internal equivalence relation in a left exact category C is just a reflexive, symmetric and transitive predicate with respect to the canonical fibration Cod : Mon/$C \to C$; the notions of the effectiveness coincide. Recalling (from 3.1) that the notions of regularity coincide too, and observing that C is an exact category [1] if and only if this fibration is regular and effective, we define *exact fibrations* as those which are regular and effective.

The definition of the *effective completion* $\mathsf{P}^\sharp : \wp^\sharp \to C^\sharp$ of a left exact fibration $P : \wp \to C$ can be lifted from [8]. We only describe the base $C^\sharp$. Its objects are thus the equivalence relations with respect to P. To describe morphisms from $S(x, x') \in \wp(A \times A)$ to $T(y, y') \in \wp(B \times B)$, consider the arrows $f : A \to B$ in C, for which there is a proof $S(x, x') \longrightarrow T(f(x), f(x'))$. A morphism $S \to T$ in $C^\sharp$ is the equivalence class of such arrows $A \to B$ modulo the relation which identifies f and g if $T(f(x), g(x))$ is provable in $\wp(A)$. Applying this construction to the basic fibration Cod : $C/C \to C$ yields the exact completion of C, from [8]. For general left exact fibrations, however, it does not give the regularity for free. But it does produce exact fibrations from regular fibrations.

Note, finally, that the function comprehensive completion leaves the objects unchanged, and only adds more arrows (the maps); while the exact completion, in a sense, adds more objects (the quotients) and changes the arrows as little as possible. As a synthesis of the two, the $\wp$-set construction, adds more of everything. The objects of the category $C^\natural$ of $\wp$-sets derived from a regular fibration $P : \wp \to C$ will be the *partial* equivalence relations, i.e. the $\wp$-predicates which are symmetric and transitive. The arrows will be suitable maps [20, 2.4].

An analysis based on 5.3 strongly suggests that the fibration $\mathsf{P}^\natural : \wp^\natural \to C^\natural$, obtained by proceeding in this way, will be the *comprehensive completion* of the regular fibration $P : \wp \to C$. A fibration $P : \wp \to C$ is said to be *comprehensive* [32, 38, III.3] if the terminal objects functor $T : C \to \wp$, right adjoint to P, has a further right adjoint $\{x|-\} : \wp \to C$, assigning to each predicate $P(x)$ its extent $\{x|P(x)\}$. A fibrewise cartesian closed fibration is comprehensive if and only if it is locally small, in the sense of [3]. If we begin with a globally small such fibration (i.e., all the objects of its total category are inverse images of a generic one, as in a tripos), the $\wp$-set construction will lead to a small fibred category [*op. cit.*]. This is where the object of truth values in the Effective Topos comes from. On the other hand, since we are in the·realm of regular fibrations, the obtained small category will always have the small (fibrational) coproducts. Thus, the $\wp$-set construction will transform a globally small (non-posetal) regular fibration into a small (nondegenerate) category with small coproducts. By [41], it will also small products too. Such (co)completeness phenomena in small categories are extremely rare and exciting.

In dealing with the non-posetal $\wp$-set construction, our technical results will probably be very helpful: maps as adjunctions become completely unmanageable here. It is, for instance, quite complicated to prove directly that the comprehensive comple-

tion of the fibration M : Mod → Set from 3.3 just displays modest sets in a familiar setting: as a small complete and cocomplete category fibred over encoded sets [41] (i.e. assemblies [12, 2.153]). A typical result of the general $\wp$-set construction, this setting can be thought of as a universe with strongly constructivist logic: a 'topos' in which the object of truth values is a genuine category, rather than poset.

Appendix

A Fibrations

The main references about fibrations are still [15] and [14]. Here we survey the terminology and some basic facts, following [38].

Let P : $\wp$ → C be a functor. We say that an object or arrow $X \in \wp$ is *over* P(X) ∈ C. A *fibre* of P over $A \in C$ is the subcategory $\wp(A) \subseteq \wp$, consisting of all the objects over A and all the arrows over id_A. The arrows over identities are called *vertical*.

A $\wp$-arrow $\vartheta : Q' \to Q$ is said to be *cartesian* with respect to P if for every $\varphi : P \to Q$ such that P(φ) = P(ϑ) there is a unique vertical arrow $\varphi' : P \to Q'$ with $\varphi = (\varphi'; \vartheta)$.

The functor P is a fibration if the vertical and the cartesian arrows that it induces form a factorisation system. In other words,

DEFINITION A.1
A functor P : $\wp$ → C is a *fibration* if for every arrow $f : A \to B$ in C, and every $Q \in \wp(B)$ there is a cartesian arrow $\vartheta^f : f^*(Q) \to Q$ over f, and if the cartesian arrows are moreover closed under composition.

The object $f^*(Q)$ — determined by this definition up to an isomorphism — is the *inverse image* of Q along f. For instance, let C be the category Set of sets and functions, and let the objects of $\wp$ be pairs $\langle A, P \rangle$, where A is a set and $P \subseteq A$. Let the arrows from $\langle A, P \rangle$ to $\langle B, Q \rangle$ be the functions $f : A \to B$ which map P to Q. P is the projection. Clearly $\wp(A)$ is now the lattice of subsets of A, and $f^* : \wp(B) \to \wp(A)$ takes each $Q \subseteq B$ to its inverse image in the usual sense. These data determine a functor $\mathcal{P}$: Set^{op} → CAT, which is the corresponding *indexed category*. But take another example: let $\wp$ be the arrow category C/C of a finitely complete C and let P : $C/C \to C$ be the codomain functor. The inverse images are now obtained as pullbacks. To extract the functors $f^* : \wp(B) \to \wp(A)$ for the arrow part of the corresponding indexed category $\mathcal{P} : C^{op} \to$ CAT, one in principle needs the axiom of choice. The same holds for any fibration P : $\wp \to C$: the arrow part of the corresponding indexed category $\mathcal{P} : C^{op} \to$ CAT requires choice. Moreover, the couniversal property of the inverse images ensures a natural isomorphism $g^* f^* \cong (f; g)^*$ — but not the strict equality. An indexed category is generally not a functor, but only a *pseudo*functor: it preserves the composition only up to a canonical isomorphism.

Finitary structures are lifted from ordinary category theory to fibrations in *fibrewise* way: each fibre must possess the structure *and* this structure must be stable under the inverse images. For instance, we say that the fibred category $\wp$ has binary *fibrewise products* if each fibre $\wp(B)$ has products $Q \wedge Q'$, and moreover

$$f^*(Q) \wedge f^*(Q') \cong f^*(Q \wedge Q')$$

holds for every $f : A \to B$ and for any choice of the inverse images. (Of course, the projections and diagonals must be preserved as well.) We say that $\wp$ has *fibrewise terminal objects* if each $\wp(B)$ has terminal object T_B, and $f^*(\mathsf{T}_B)$ is terminal in $\wp(A)$. Of course, if $\wp$ has fibrewise terminal objects and binary products, then it has all finite fibrewise products.

Now we list some easy lemmas about the fibrewise structure, used throughout the paper. The proofs can be found in [38, II.2.2].

LEMMA A.2
Suppose that $\wp$ has fibrewise terminal objects. For every $P \in \wp(A)$ and each $f : A \to B$ in C, there is a unique arrow

$$\tau^f := (\tau_P; \vartheta^f) : P \to \mathsf{T}_A \to \mathsf{T}_B$$

where τ_P is the unique vertical arrow from P to $\top_A$.

LEMMA A.3

Suppose $\wp$ has fibrewise binary products. For each pair of arrows $\psi^f : P \to Q$ and $\bar{\psi}^f : P \to \bar{Q}$ there is a unique arrow

$$\langle \psi, \bar{\psi} \rangle^f := \left(\langle \psi, \bar{\psi} \rangle; \vartheta^f \right) : P \longrightarrow f^*(Q) \wedge f^*(\bar{Q}) \longrightarrow Q \wedge \bar{Q}$$

such that the equations $\left(\langle \psi, \bar{\psi} \rangle; \pi \right) = \psi$ and $\left(\langle \psi, \bar{\psi} \rangle; \bar{\pi} \right) = \bar{\psi}$ hold, where π and $\bar{\pi}$ are the vertical projections.

LEMMA A.4

Let a commutative square S in a fibred category $\wp$ have two opposite sides cartesian. Then S is a pullback in $\wp$ if and only if the square P(S) is a pullback C. In particular, a commutative square consisting of two opposite vertical and two opposite cartesian arrows is always a pullback.

In an indexed category $\mathcal{P} : C^{op} \to \mathbf{CAT}$, the existential quantification and the C-small coproducts are represented by the left adjoints of the functors $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$. For a fibration $P : \wp \to C$, the inverse image functors $f^* : \wp(B) \to \wp(A)$ will turn out to have left adjoints if and only if the opposite functor $P^{op} : \wp^{op} \to C^{op}$ is a fibration too.

DEFINITION A.5

A functor $P : \wp \to C$ is a *bifibration* if both P and P^{op} are fibrations.

The cartesian arrows with respect to the fibration $P^{op} : \wp^{op} \to C^{op}$ can be characterized as arrows of $\wp$ by a property dual to cartesianness. They are called *opcartesian* arrows with respect to the *opfibration* $P : \wp \to C$. An opcartesian lifting of $f \in C(A, B)$ is generically denoted by $\sigma^f : P \to f_!(P)$. The induced *direct image* functor $f_! : \wp(A) \to \wp(B)$ is left adjoint to f^*.

DEFINITION A.6 (Lawvere)

A bifibration $P : \wp \to C$ satisfies the *Beck-Chevalley condition* if for every pullback square $(\bar{s}; t) = (\bar{t}; s)$ in C and for every object R over the domain of s, the following canonical arrow (induced by the universal property of direct images) is an iso:

$$\bar{s}_! \left(\bar{t}^*(R) \right) \longrightarrow t^* \left(s_!(R) \right).$$

DEFINITION A.7 (Lawvere)

A bifibration $P : \wp \to C$ with binary fibrewise products satisfies the *Frobenius condition* if for every arrow $f : A \to B$ in C and objects $P \in \wp(A)$ and $Q \in \wp(B)$, the following canonical arrow is an iso:

$$f_!(P \wedge f^*(Q)) \to f_!(P) \wedge Q.$$

Since they are defined using the universal properties, the canonical arrows in both preceding definitions form natural families. In the latter case, for instance, this means that for every arrow $\alpha : Q \to Q'$ in $\wp(B)$ and any $P \in \wp(A)$, the following square will commute.

$$
\begin{array}{ccc}
P \wedge f^*(Q) & \xrightarrow{\;\sigma^f\;} & f_!(P) \wedge Q \\
\downarrow{\scriptstyle P \wedge f^*(\alpha)} & & \downarrow{\scriptstyle f_!(P) \wedge \alpha} \\
P \wedge f^*(Q') & \xrightarrow[\;\sigma^f\;]{} & f_!(P) \wedge Q'
\end{array}
$$

$$(\mathrm{A.1})$$

PROPOSITION A.8

1. In any bifibration, the Beck-Chevalley condition is satisfied if and only if the opcartesian arrows are stable under pullbacks along the cartesian arrows.

$$
\begin{array}{ccc}
i^*(R) & \xrightarrow{\;\sigma^i\;} & \mathfrak{s}_!(i^*(R)) \\
& & \cong \\
& & t^*(\mathfrak{s}_!(R)) \\
\vartheta^i \downarrow & & \downarrow \vartheta^t \\
R & \xrightarrow{\;\sigma^s\;} & \mathfrak{s}_!(R)
\end{array}
$$

$$(A.2)$$

2. In any bifibration with fibrewise products, the Frobenius condition is satisfied if and only if the opcartesian arrows are stable under pullbacks along the vertical projections.

$$
\begin{array}{ccc}
P \wedge f^*(Q) & \xrightarrow{\;\partial^f\;} & f_!(P \wedge f^*(Q)) \\
& & \cong \\
& & f_!(P) \wedge Q \\
\check{\pi} \downarrow & & \downarrow \pi \\
P & \xrightarrow{\;\sigma^f\;} & f_!(P)
\end{array}
$$

$$(A.3)$$

A morphism between fibrations $P' : \wp' \to C$ and $P : \wp \to C$ is a *cartesian* functor $F : \wp' \to \wp$, preserving the cartesian arrows, and with $P' = FP$. An *opcartesian* functor preserves, of course, the opcartesian arrows.

B Subterminal objects

DEFINITION B.1

An object R is *subterminal* if there is at most one arrow $X \to R$ from any object X.

REMARK B.2

An object R is said to be *weakly terminal* if for every X there is *at least* one arrow $X \to R$. Thus an object is terminal if and only if it is subterminal and weakly terminal.

LEMMA B.3

In a category with a terminal object $\top$, an object R is subterminal if and only if it is a subobject of the terminal object.

LEMMA B.4

In a category with binary products $P \wedge Q$ the following conditions are equivalent.
1. R is subterminal.
2. The projections $p, p' : R \wedge R \to R$ are equal: $p = p'$.
3. The first (equivalently, the second) projection $p : R \wedge R \to R$ is an isomorphism.
4. The diagonal $d : R \to R \wedge R$ is an isomorphism.

COROLLARY B.5

In a regular fibration, every inverse image of a subterminal object is subterminal.

C   A lemma on adjunctions

Here we prove a general lemma on adjunctions, used in the proof of theorem 5.3. It is formulated in an arbitrary bicategory, but it can be understood in the framework of ordinary category theory, where 1-cells are functors and 2-cells are natural transformations. This is possible because the 'horizontal' composition is indeed functorial in both arguments. A composite $\psi\psi'$ of 2-cells

$$
\begin{array}{ccc}
\xrightarrow{H} & & \xrightarrow{H'} \\[-4pt]
\xrightarrow[K]{\downarrow\psi} & & \xrightarrow[K']{\downarrow\psi'}
\end{array}
$$

(C.1)

must therefore be equal both to $(\psi H');(K\psi')$ and to $(H\psi');(\psi K')$. The equality of these two is sometimes called the *Godement law*. The diagram

$$
\begin{array}{ccc}
HH' & \xrightarrow{\psi H'} & KH' \\
{\scriptstyle H\psi'}\downarrow & & \downarrow{\scriptstyle K\psi'} \\
HK' & \xrightarrow[\psi K']{} & KK'
\end{array}
$$

(C.2)

thus commutes, and the 2-cells ψ and ψ' appear to be *natural.*

LEMMA C.1
Let $\mathcal{R}$ be a bicategory. Suppose we are given 0-cells A and B, 1-cells $F : A \mapsto B$ and $G : B \mapsto A$, and 2-cells $\eta : id_A \to FG$ and $\varepsilon : GF \to id_B$. Then F is left adjoint to G if and only if the 2-cells

$$
\begin{array}{rclcl}
\alpha & := & (\eta F; F\varepsilon) & : & F \to FGF \to F \\
\beta & := & (G\eta; \varepsilon G) & : & G \to GFG \to G
\end{array}
$$

(C.3)

are both split epi (or both split monic).

PROOF. The 'only if'-direction is trivial: $\alpha : F \to F$ and $\beta : G \to G$ are then identities.

Suppose, towards a proof of the 'if'-direction, that α and β are split epis. Let α' and β' be their respective left inverses. We claim that the 2-cells

$$
\begin{array}{rclcl}
\bar{\eta} & := & \eta & : & id_A \longrightarrow FG \\
\bar{\varepsilon} & := & (\beta' F; \varepsilon) & : & GF \longrightarrow id_B
\end{array}
$$

(C.4)

satisfy the adjunction equations, making $F \dashv G$. The first one is obvious:

$$
\begin{array}{ccccc}
G & \xrightarrow{G\bar{\eta}} & GFG & \xrightarrow{\bar{\varepsilon}G} & G \\
{\scriptstyle\beta'}\downarrow & & {\scriptstyle\beta'FG}\downarrow & \nearrow{\scriptstyle\varepsilon G} & \\
G & \xrightarrow[G\eta]{} & GFG & &
\end{array}
$$

(C.5)

The triangle commutes by definition, the square by the naturality. The lower path is the identity by the definitions of β and β'. Hence $G\bar{\eta}; \bar{\varepsilon}G = id$.

The second adjunction equation follows from a similar diagram.

$$\begin{array}{ccccc}
F & \xrightarrow{\ \bar{\eta}F\ } & FGF & \xrightarrow{\ F\bar{\varepsilon}\ } & F \\
\alpha' \downarrow & & FG\alpha' \Vert\; \Vert F\beta'F & \nearrow F\varepsilon & \\
F & \xrightarrow[\ \eta F\ ]{} & FGF & &
\end{array}$$
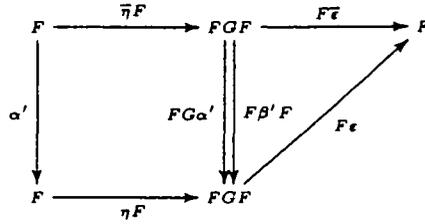
$$\text{(C.6)}$$

The triangle and the square commute for the same reasons as above; the lower path is clearly identity again. We claim, furthermore, that the equation

$$(FG\alpha' \; ; \; F\varepsilon) = (F\beta'F \; ; \; F\varepsilon) \tag{C.7}$$

holds. It implies that (C.6) is a commutative diagram. Hence $(\bar{\eta}F; F\bar{\varepsilon}) = id$.

To prove (C.7), first consider the following diagram:

$$\begin{array}{ccc}
FGF & \xrightarrow{\ F\beta F\ } & FGF \\
FG\alpha \downarrow \quad \searrow FG\eta F & & \downarrow F\varepsilon \\
 & FGFGF \xrightarrow{\ F\varepsilon GF\ } FGF & \\
 & FGF\varepsilon \downarrow \qquad\quad \downarrow F\varepsilon & \\
 & FGF \xrightarrow[\ F\varepsilon\ ]{} F &
\end{array}$$
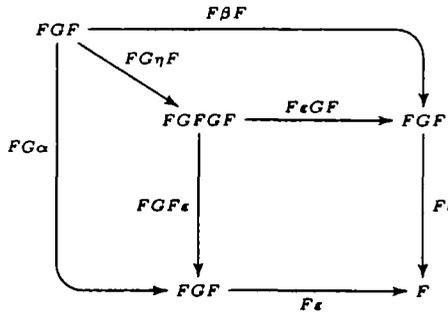
$$\text{(C.8)}$$

The smaller square commutes by the naturality again. The rest — by definitions of α and β. Now we know that the lower square in (C.9) commutes.

$$
\begin{array}{c}
FGF \xrightarrow{\ \ \iota d\ \ } \\
\quad F\beta'F \\
FGF \xrightarrow{\ F\beta F\ } FGF \\
F\beta'F \quad FG\alpha' \quad FG\alpha' \\
FGF \xrightarrow{\ F\beta F\ } FGF \\
FG\alpha \quad F\epsilon \\
FGF \xrightarrow{\ F\epsilon\ } F
\end{array}
$$
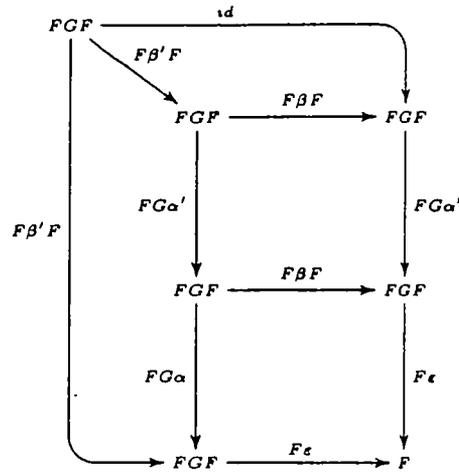
(C.9)

The upper square is commutative by the naturality. The two external paths yield (C.7). ∎

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