Modal Fracture of Higher Groups

"Differential Cohomology Hexagon"

$$\begin{tikzcd}
 & & \overset{\infty}{G} \arrow[rr] \arrow[dr] & & \mathfrak{g} \arrow[dr] & \\
 & \flat\overset{\infty}{G} \arrow[ur] \arrow[dr] & & G \arrow[ur] \arrow[dr] & & \int\mathfrak{g} \\
 & & \flat G \arrow[ur] \arrow[rr] & & \int G \arrow[ur] &
\end{tikzcd}$$

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Plan

Thm: For any crisp higher group G: (Unstable version of Schreiber §4.1.2)

$$\begin{tikzcd}
 \overset{\infty}{G} \arrow[rr] \arrow[dr] & & \mathfrak{g} \arrow[dr] & \\
 \flat\overset{\infty}{G} \arrow[u] \arrow[dr] \arrow[r, phantom, "> (a)"] & G \arrow[ur] \arrow[dr] \arrow[r, phantom, "> (b)"] & & \int\mathfrak{g} \overset{(e)}{=} \flat B\overset{\infty}{G} \\
 & \flat G \arrow[u] \arrow[r] & \int G \arrow[ur] &
\end{tikzcd}$$

i.e. (a) and (b) are pullbacks, (c) and (d) are fiber sequences, and (e): $\int\mathfrak{g} = \flat B\overset{\infty}{G}$.

0. Cohesive HoTT, a refresher
1. The universal ∞-cover $\overset{\infty}{G} \xrightarrow{\pi} G$ (proof of (a))
2. The infinitesimal remainder $G \xrightarrow{\theta} \mathfrak{g}$ (proof of (b) and (c))
3. The Modal Fracture Hexagon (proof of (d) and (e))

Cohesive HoTT - Crispness and $\flat$-comodality
(Shulman)

$$\Delta \mid \Gamma \vdash a : A$$

Add crisp variables to express discontinuous dependence

$$x :: A$$

Crisp terms: $\Delta \mid \cdot \vdash a : A$ have only crisp variables.

Comodality $\flat$: $\flat A$ is inductively generated by crisp $a :: A$.

$$\frac{\Delta \mid \cdot \vdash A : \mathrm{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathrm{Type}} \qquad \frac{\Delta \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^{\flat} : \flat A} \qquad \frac{\begin{array}{c}\Delta \mid \Gamma, x : \flat A \vdash C : \mathrm{Type} \\ \Delta \mid \Gamma \vdash a : \flat A \\ \Delta, x :: A \mid \Gamma \vdash c : C(x^{\flat})\end{array}}{\Delta \mid \Gamma \vdash \mathrm{let}\ x^{\flat} :\equiv a \text{ in } c : C(a)}$$

$$(\mathrm{let}\ x^{\flat} :\equiv a^{\flat} \text{ in } c \equiv c(a))$$

Counit: $(-)_{\flat} : \flat A \to A$

$$a^{\flat} \mapsto a$$

$$u \mapsto \mathrm{let}\ a^{\flat} :\equiv u \text{ in } a.$$

Cohesive HoTT - Shape and Unity of Opposites

We assume a modality "shape" $\int$ which satisfies:

"homotopy type" in Real cohesion

Axiom (Unity of Opposites): A crisp type $A :: \mathrm{Type}$ is $\int$-modal iff it is $\flat$-modal

$$A \xrightarrow{\sim} \int A \quad \text{iff} \quad \flat A \xrightarrow{\sim} A \quad \equiv: \text{"}A \text{ is crisply discrete"}$$

Theorem (Shulman): For $A, B :: \mathrm{Type}$,

$$\flat(A \to \flat B) \simeq \flat\left(\int A \to B\right)$$

(Rmk: We don't need $\sharp$, so this is really "Strongly ∞-connected type theory")

Cohesive HoTT - Examples:

Cohesion	Site	Types	"Discrete"	$\int$	$\int X$	$\flat A$
(Smooth/Cont.) Real (Shulman, Schreiber)	Euclidean Spaces (+ infinitesimals)	Smooth/Continuous ∞-Groupoids	Discrete	$Loc_{\mathbb{R}}$	Homotopy Type of X	Moduli stack of A-valued local systems
Global (Rezk) Equivariant	$\{BG$ for Finite $G\}$	Equivariant ∞ Groupoids	Fixed/Invariant	$Loc_{\{\#BG\}}$	Strict Quotient of X	Homotopy Quotient of A
Simplicial	Δ	Simplicial ∞-groupoids	Discrete	Loc_{Δ^1}	Geometric Realization	Points A_0 of A
Spectral	?	Parametrized Spectra	Space	Loc_S	Underlying Space of X	Underlying Space of A: $\int A = \flat A \equiv \natural A$

Non-Examples: Topological/Pyknotic toposes.

↳ Objects in sites are not Locally ∞-connected.

(Good fibrations trick doesn't work since $Aut(X)$ may not be discrete even when X is discrete.)

① The Universal ∞-cover

A cover $p : E \to B$ lifts uniquely against maps which are an equivalence on π_1:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

when $\int_1 f$ is an equiv.

fundamental groupoid modality

Thm (Rijke, Cherubini): For any modality !, there is an orthogonal factorization system

$\{!\text{-equiv}\} \perp \{!\text{-étale}\}$ where f is !-étale when
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & !X \arrow[d, "!f"] \\ Y \arrow[r] & !Y \end{tikzcd}$$

Def (Cherubini, M.): A map $p : E \to B$ is a cover when it is $\int_1$-étale and its fibers are sets.

See "Good Fibrations" §9

The universal cover $\pi : \tilde{B} \to B$ is a simply connected cover

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int_1 \tilde{B} = * \arrow[d] \\ B \arrow[r] & \int_1 B \end{tikzcd}$$

① The Universal ∞-cover

An ∞-cover $p : E \to B$ lifts uniquely against homotopy equivalences:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "!"] & B \end{tikzcd} \quad \text{when } \int f \text{ is an equiv.}$$

Thm (Rijke, Cherubini): For any modality $!$, there is an orthogonal factorization system

$\{!\text{-equiv}\} \perp \{!\text{-étale}\}$ where f is $!$-étale when
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & !X \arrow[d, "!f"] \\ Y \arrow[r] & !Y \end{tikzcd}$$

Def (Schreiber, Cherubini): A map $p : E \to B$ is an ∞-cover when it is $\int$-étale

The universal ∞-cover $\pi : \overset{\infty}{B} \to B$ is a $\int$-connected contractible ∞-cover

$$\begin{tikzcd} \overset{\infty}{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \overset{\infty}{B} = * \arrow[d] \\ B \arrow[r] & \int B \end{tikzcd} \qquad \overset{\infty}{B} \longrightarrow B \longrightarrow \int B$$

fiber sequence.

① The Universal ∞-cover — What is it?

The universal ∞-cover $\overset{\infty}{X}$ is a "stacky" universal cover $\tilde{X}$

Thm: For X a crisp type, $\overset{\infty}{X} \longrightarrow \tilde{X}$ is 0-connected with fiber $B\pi_{\geq 2}X$ delooping the second homotopy ∞-group $\pi_{\geq 2}X$ of X.

Proof:
$$\begin{tikzcd} B\pi_{\geq 2}X \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \int X\langle 1 \rangle \arrow[d] \\ \overset{\infty}{X} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & \int X \arrow[d, "|\cdot|_1"] \\ \tilde{X} \arrow[r] & X \arrow[r] & \int_1 X \end{tikzcd}$$

Cor: If X is a crisp set, then
$$\tilde{X} = \| \overset{\infty}{X} \|_0$$

Prop: For X an n-type,
$$\Omega^k \overset{\infty}{X} = \Omega^{k+1} \int X \quad \text{for } k \geq n+1$$

Proof:
$$\Omega^k \overset{\infty}{X} \to * \to \Omega^k \int X \to \cdots \to \overset{\infty}{X} \to X \to \int X$$

① The Universal ∞-cover: Proof of

$$\begin{tikzcd} & \overset{\infty}{G} \arrow[dr] & \\ b\overset{\infty}{G} \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\lrcorner \; ⓐ"] & & G \\ & bG \arrow[ur] & \end{tikzcd}$$

Add slide w/ "$\|\overset{\infty}{X}\|_0 = \tilde{X}$" or it.

Lemma (Shulman): b is left exact, and so preserves fiber sequences

Lemma: For $f :: X \to Y$, TFAE

① $$\begin{tikzcd} bX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ bY \arrow[r] & Y \end{tikzcd}$$

② $\forall y :: Y$, $\mathrm{fib}_f(y)$ is discrete.

Prop: Let X be a crisp type. Then

$$\begin{tikzcd} & \overset{\infty}{X} \arrow[dr, "\pi"] & \\ b\overset{\infty}{X} \arrow[ur, "(-)_b"] \arrow[dr, "b\pi"'] \arrow[rr, phantom, "\lrcorner"] & & X \\ & bX \arrow[ur, "(-)_b"'] & \end{tikzcd}$$

is a pullback

Proof: For $x :: X$, $\mathrm{fib}_\pi(x) = \Omega(\int X, x')$ is discrete.

$$\begin{tikzcd} \Omega\int X \arrow[r, bend right] & \overset{\infty}{X} \arrow[r] & X \arrow[r] & \int X \end{tikzcd}$$

Aside: The "good fibrations" trick

See "Good Fibrations through the Modal Prism"

Def: $\pi : E \to B$ is a $\int$-fibration if $\forall b : B$,

$$\int \mathrm{fib}_\pi(b) \longrightarrow \int E \xrightarrow{\int \pi} \int B$$

is a fiber sequence.

or: $\mathrm{fib}_\pi(b) \to \mathrm{fib}_{\int\pi}(b^{\int})$ is a $\int$-equivalence.

Thm: $\pi : E \to B$ is a $\int$-fibration iff $\int \mathrm{fib}_\pi : B \to \mathrm{Type}$ factors through $(-)^{\int} : B \to \int B$.

Prop: $\pi : E \to B$ is an ∞-cover iff it is a $\int$-fibration and its fibers are discrete.

Lemma: If F is crisply discrete, then $B\mathrm{Aut}(F)$ is. (This fails in topological examples)

Trick ("good fibrations"):

Let $\pi : E \to B$. If there is a crisp F such that $\forall b : B. \| \int \mathrm{fib}_\pi(b) = F \|$, then π is a $\int$-fibration.

② The Infinitesimal Remainder:

Lemma (Shulman): $b\|X\|_n = \|bX\|_n$

Corollary: If G is a K-commutative ∞-group, then so is bG and $bG \to G$ is a homomorphism.

Pf: Define $B^{k+1}bG \equiv bB^{k+1}G$.

Def: The *infinitesimal remainder* $\mathfrak{g}$ of G is the homotopy quotient.
(Schreiber)

$$\mathfrak{g} \equiv G/\!/bG$$

$$\mathfrak{g} \equiv b_{dR}BG$$

$$G \xrightarrow{\ominus} \mathfrak{g} \to bBG \to BG$$

Prop: $\mathfrak{g}$ is infinitesimal: $b\mathfrak{g} = *$.

② The Infinitesimal Remainder - What is it?

$$bG \to G \xrightarrow{\ominus} \mathfrak{g}$$ "Maurer-Cartan Form $g^{-1}dg$"

External Fact (Schreiber): In Formal Smooth ∞-groupoids, for G a Lie group, $\mathfrak{g} = \Lambda^1_{cl}(-;\mathfrak{g})$ classifies closed Lie algebra valued 1-forms. (I have an internal proof in a certain setting for matrix Lie groups)

Prop: Let $G \xrightarrow{\phi} H \xrightarrow{\psi} K$ be a crisp exact sequence of higher groups. Then

1. K is discrete iff $\phi_*: \mathfrak{g} \to \mathfrak{h}$ is an equivalence
2. G is discrete iff $\psi_*: \mathfrak{h} \to \mathfrak{k}$ is an equivalence.

Cor: $\overset{\infty}{G} \xrightarrow{\pi} G$ gives an equivalence $\overset{\infty}{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}$.

So: $b\overset{\infty}{G} \to \overset{\infty}{G} \to \mathfrak{g}$ is a fiber sequence

② The Infinitesimal Remainder - Proof of ⓑ

$$\begin{tikzcd}[column sep=small, row sep=small] & \mathfrak{g} \arrow[dr] & \\ G \arrow[ur] \arrow[dr] & > & \int\mathfrak{g} \\ & \int G \arrow[ur] & \end{tikzcd}$$

Lemma: If X is crisply discrete, then $B\mathrm{Aut}(X)$ is. (This fails in topological examples)

Thm: Let $\pi : E \to B$. If there is a crisply discrete F such that $\forall b : B,\ \| \mathrm{fib}_\pi(b) = F \|$, then π is an ∞-cover. (By the $\int$-fibration trick)

Cor: For G a crisp higher group,

$$\begin{tikzcd} & \mathfrak{g} \arrow[dr, "(-)^{\int}"] & \\ G \arrow[ur, "\theta"] \arrow[dr, "(-)^{\int}"'] & > & \int\mathfrak{g} \\ & \int G \arrow[ur, "\int\theta"'] & \end{tikzcd}$$

is a pullback

Proof: The fibers of θ are identifiable with $\flat G$, so it is an ∞-cover.

Cor: $\overset{\infty}{G} \to \mathfrak{g}$ is the universal ∞-cover of $\mathfrak{g}$. so

$\flat\overset{\infty}{G} \to \overset{\infty}{G} \to \mathfrak{g} \to \int\mathfrak{g}$ is a fiber sequence

Eg: $\mathbb{R} \xrightarrow{dx} \Lambda^1_{cl}$ is the universal ∞-cover of the closed 1-form classifier.

③ The Modal Fracture Hexagon

$$\begin{tikzcd} & \overset{\infty}{G} \arrow[rr] \arrow[dr] & & \mathfrak{g} \arrow[dr] & \\ \flat\overset{\infty}{G} \arrow[ur] \arrow[dr] & > & G \arrow[ur] \arrow[dr] & > & \int\mathfrak{g} \\ & \flat G \arrow[ur] \arrow[rr] & & \int G \arrow[ur] & \end{tikzcd}$$

universal ∞-cover

infinitesimal remainder

$\overset{\infty}{G} \to G \to \int G \to B\overset{\infty}{G} \to BG \to \int BG$

$\flat G \to G \to \mathfrak{g} \to \flat BG \to BG$

$\flat \downarrow$

$\int \downarrow$ Using the theory of $\int$-Fibrations

$\flat\overset{\infty}{G} \to \flat G \to \int G \to \flat B\overset{\infty}{G} \to \flat BG \to \int BG$

$=$

$\flat G \to \int G \to \int\mathfrak{g} \to \flat BG \to \int BG$

So, $\int\mathfrak{g} = \flat B\overset{\infty}{G}$

③ The Modal Fracture Hexagon – $BU(1)$

We can continue the modal fracture hexagon as long as G can be delooped:

$$
\begin{tikzcd}
 & \overset{\infty}{G} \arrow[r] \arrow[dr] & & \mathfrak{g} \arrow[dr] \arrow[ddrr, bend left] & & B\overset{\infty}{G} \arrow[r] \arrow[dr] & & B\mathfrak{g} \arrow[dr] \arrow[ddrr, bend left] & & B^2\overset{\infty}{G} \arrow[dr] \arrow[r, dashed] & & {} \\
\flat\overset{\infty}{G} \arrow[ur] \arrow[dr] & & G \arrow[ur] \arrow[dr] & & \int\mathfrak{g} = \flat B\overset{\infty}{G} \arrow[ur] \arrow[dr] & & BG \arrow[ur] \arrow[dr] & & \int B\mathfrak{g} = \flat B^2\overset{\infty}{G} \arrow[ur] \arrow[dr] & & B^2G \arrow[ur, dashed] \arrow[dr, dashed] & \\
 & \flat G \arrow[ur] \arrow[r] & & \int G \arrow[ur] \arrow[uurr, bend right] & & \flat BG \arrow[ur] \arrow[r] & & \int BG \arrow[ur] \arrow[uurr, bend right] & & \flat B^2G \arrow[ur] \arrow[r, dashed] & & {}
\end{tikzcd}
$$

Eg: $G \equiv U(1)$

$BU(1) \equiv \left\{ \begin{array}{l} \text{1-dim } \mathbb{C}\text{-vector spaces} \\ \text{with Hermitian } \langle , \rangle \end{array} \right\}$

$B\mathbb{R} \equiv \{\text{1-dim } \mathbb{R} \text{ affine spaces}\}$

$\Lambda^1_{cl} \to \Lambda^1 \xrightarrow{d} \Lambda^2_{cl}$

$\leadsto B\Lambda^1_{cl} \to B\Lambda^1$ So: $B\Lambda^1_{cl} = \Lambda^2_{cl} /\!/ \Lambda^1$

$$
\begin{tikzcd}
 & B\mathbb{R} \arrow[rr, "Bd"] \arrow[dr, "B\exp"] & & B\Lambda^1_{cl} \arrow[dr, "\substack{\text{Flat } \nabla \\ \text{on trivial} \\ \text{affine gerbe}}"] & \\
\flat B\mathbb{R} \arrow[ur, "\substack{\text{forget} \\ \text{Flat } \nabla}"] \arrow[dr, "\flat B\exp"'] & & BU(1) \arrow[ur, "\text{curv}"] \arrow[dr, "c_1"] & & \flat B^2\mathbb{R} \\
 & \flat BU(1) \arrow[ur, "\substack{\text{forget} \\ \text{flat } \nabla}"] \arrow[rr, "\text{Bockstein}"'] & & B^2\mathbb{Z} \arrow[ur, "\flat B^2\text{inclusion}"'] &
\end{tikzcd}
$$

Via de Rham, interpret as closed 2-form mod exact

③ The Modal Fracture Hexagon – Other Examples

Cor (spectral cohesion): Any ∞-group G of parametrized spectra is the product $G = \natural G \times G_1$ of its underlying index group and the spectrum indexed at the identity

Q: What does it mean in the other cohesions?

Eg:

$$
\begin{tikzcd}
 & \begin{array}{c}\text{Stabilizer} \\ \text{Group of } 1\end{array} \arrow[rr] \arrow[dr] & & \begin{array}{c}\text{Quotient of } G \text{ by} \\ \text{The action of} \\ \text{its homotopy Quotient}\end{array} \arrow[dr] & \\
\begin{array}{c}\text{Homotopy} \\ \text{Quotient} \\ \text{of Stabilizer}\end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c}\text{Equivariant} \\ \infty\text{-Group } G\end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c}\text{Delooping of} \\ \text{Homotopy Quotient} \\ \text{of Stabilizer}\end{array} \\
 & \begin{array}{c}\text{Fixed} \\ \text{Homotopy} \\ \text{Quotient} \\ \flat G\end{array} \arrow[ur] \arrow[rr] & & \begin{array}{c}\text{Strict Quotient} \\ \int G\end{array} \arrow[ur] &
\end{tikzcd}
$$

Equivariant Modal Fracture

References

David Jaz Myers:
- Modal Fracture of Higher Groups (In prep)
- Good Fibrations through the Modal Prism (arXiv:1908.08034)

Urs Schreiber:
- Differtial Cohomology in a Cohesive ∞-topos (arXiv:1310.7390)
- Differential Cohesion and Idelic Structure (nLab)

Mike Shulman:
- Brouwer's Fixed Point Theorem in Real-Cohesive HoTT (arXiv:1509.07584)

Egbert Rijke:
- Classifying Types (arXiv:1906.09435)

Felix Cherubini:
- Cohesive Covering Theory
- Modal Descent (arXiv:2003.09713)

Rezk: Global Homotopy Theory and Cohesion

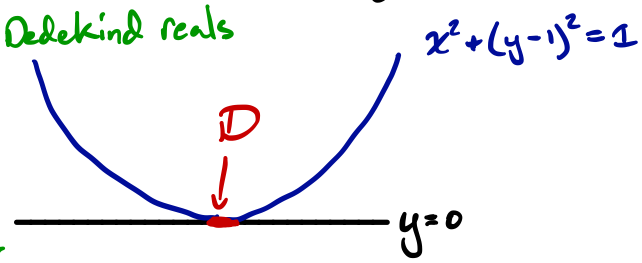
④ Differential Cohomology

Idea: Cohesive HoTT + Synthetic Diff. geometry + Tiny Infinitesimals + Axiom of Constancy ⇒ (Ordinary) Differential Cohomology.

Synthetic Differential Geometry:
- $\mathbb{R}$ is a local, ordered field ⋯ not the Dedekind reals
- $\mathbb{D} :\equiv \{r : \mathbb{R} \mid r^2 = 0\}$ satisfies

$$\mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}^{\mathbb{D}}$$
$$(a,b) \longmapsto \lambda\varepsilon.\, a + b\varepsilon$$

"every function of a first-order infinitesimal is linear"



Tiny Infinitesimals:
- $\Pi_{\mathbb{D}} : \mathrm{Type}^{\mathbb{D}} \to \mathrm{Type}$ has an external right adjoint

↳ implies $\#(X^{\mathbb{D}} \to Y) = \#(X \to Y^{1/\mathbb{D}})$

↳ Then can define $\Lambda^1 \overset{eq}{\hookrightarrow} \mathbb{R}^{1/\mathbb{D}} \underset{\text{on } \mathbb{D}}{\overset{\text{act on } \mathbb{R}}{\rightrightarrows}} (\mathbb{R}^{1/\mathbb{D}})^{\mathbb{R}}$ (Kock) "$\omega(rv) = r\omega(v)$" implies linearity!

so that $\#(X \to \Lambda^1) = \#\{\text{1-forms on } X\}$.

④ Differential Cohesion

Def: $d : \mathbb{R} \to \Lambda^1$ is the transpose of
$$\begin{aligned} \mathbb{R}^{\mathbb{D}} &\longrightarrow \mathbb{R} \\ v &\longmapsto \dot{v}(0) \end{aligned}$$

Axiom of Constancy: Let $f : \mathbb{R} \to \mathbb{R}$ if $df = 0$, then f is constant.

$$df :\equiv \mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{d} \Lambda^1.$$

Thm: Given the axiom of constancy, we have that
$$\operatorname{Ker} d = \flat\mathbb{R}$$

proof: The axiom says that $\mathrm{const} : \operatorname{Ker} d \to (\mathbb{R} \to \operatorname{Ker} d)$ is an equiv.
So $\operatorname{Ker} d$ is a crisp, discrete subgroup of $\mathbb{R}$, so $\operatorname{Ker} d \subseteq \flat\mathbb{R}$.
But by transposing, we see that $\flat\mathbb{R} \subseteq \operatorname{Ker} d$.

Cor: Every function $f : \mathbb{R} \to \mathbb{R}$ admits a unique primitive $\int_0^x f$ with $\int_0^0 f = 0$.

$$\begin{tikzcd} * \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{R} \arrow[d, "d"] \\ \mathbb{R} \arrow[r, "f\,dx"'] \arrow[ur, dashed, "\exists!"] & \Lambda^1_{cl} \end{tikzcd}$$

4 Differential Cohomology

Assume we have the following exact sequences of additive abelian groups:

$$0 \to b\mathbb{R} \to \mathbb{R} \xrightarrow{d} \Lambda^1_{cl} \to 0$$

$$0 \to \Lambda^1_{cl} \to \Lambda^1 \xrightarrow{d} \Lambda^2_{cl} \to 0$$

(Can be constructed using tiny infinitesimals + "$f:\mathbb{R}\to\mathbb{R}$ const iff $df=0$")

And that Λ^1 is an $\mathbb{R}$-vector space

Def (Schreiber): Moduli Stack of $U(1)$-bundles with connection:

$$\begin{tikzcd}
B_\nabla U(1) \arrow[r, "curv"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^2_{cl} \arrow[d] \\
BU(1) \arrow[r] & B\Lambda^1_{cl}
\end{tikzcd}$$

so that we have fiber sequences

$\Lambda^1 \to B_\nabla U(1) \to BU(1)$ "connection 1-form" and "$B_\nabla U(1) \simeq \Lambda^1 /\!/ U(1)$"

$bBU(1) \to B_\nabla U(1) \to \Lambda^2_{cl}$ "flat connection iff vanishing curvature"

Lem: $\int \Lambda^2_{cl} = bB^2\mathbb{R}$ and $\int B_\nabla U(1) = B^2\mathbb{Z}$

Proof: Since Λ^1 is a vector space, $\int \Lambda^1 = *$, so:

$$\int \Lambda^2_{cl} \xrightarrow{\sim} \int B\Lambda^1_{cl} \to \int B\Lambda^1 \quad \text{and} \quad \int \Lambda^1 \to \int B_\nabla U(1) \xrightarrow{\sim} \int BU(1)$$

In General: $\int \Lambda^n_{cl} = bB^n\mathbb{R}$

4 Differential Cohomology

Define

$$\begin{tikzcd}
B^2_\nabla U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B\Lambda^2_{cl} \arrow[d] \\
B^2 U(1) \arrow[r] & B^2 U(1)
\end{tikzcd}.$$

Then

$$\begin{tikzcd}
b_{dR}B^2_\nabla U(1) \arrow[rr, "\sim"] \arrow[ddd] \arrow[dr] & & \Lambda^2_{cl} \arrow[ddd] \arrow[dr] & & \\
& bB^2_\nabla U(1) \arrow[rr] \arrow[ddd, "2"'] \arrow[dr] & & * \arrow[ddd, equal] \arrow[dr] & \\
& & B^2_\nabla U(1) \arrow[rr] \arrow[ddd] & & B\Lambda^2_{cl} \arrow[ddd] \\
BU(1) \arrow[rr, equal] \arrow[dr] & & BU(1) \arrow[dr] & & \\
& bB^2U(1) \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\
& & B^2U(1) \arrow[rr] & & B^2U(1)
\end{tikzcd}$$

$$\begin{tikzcd}
& B_\nabla\mathbb{R} \arrow[rr] \arrow[dr] & & \Lambda^2_{cl} \arrow[dr] & \\
bB\mathbb{R} \arrow[ur] \arrow[dr] & & B_\nabla U(1) \arrow[ur] \arrow[dr] & & bB^2\mathbb{R} \\
& bBU(1) \arrow[ur] \arrow[rr] & & B^2\mathbb{Z} \arrow[ur] &
\end{tikzcd}$$