Trustworthy, Useful Languages for Probabilistic Modeling and Inference

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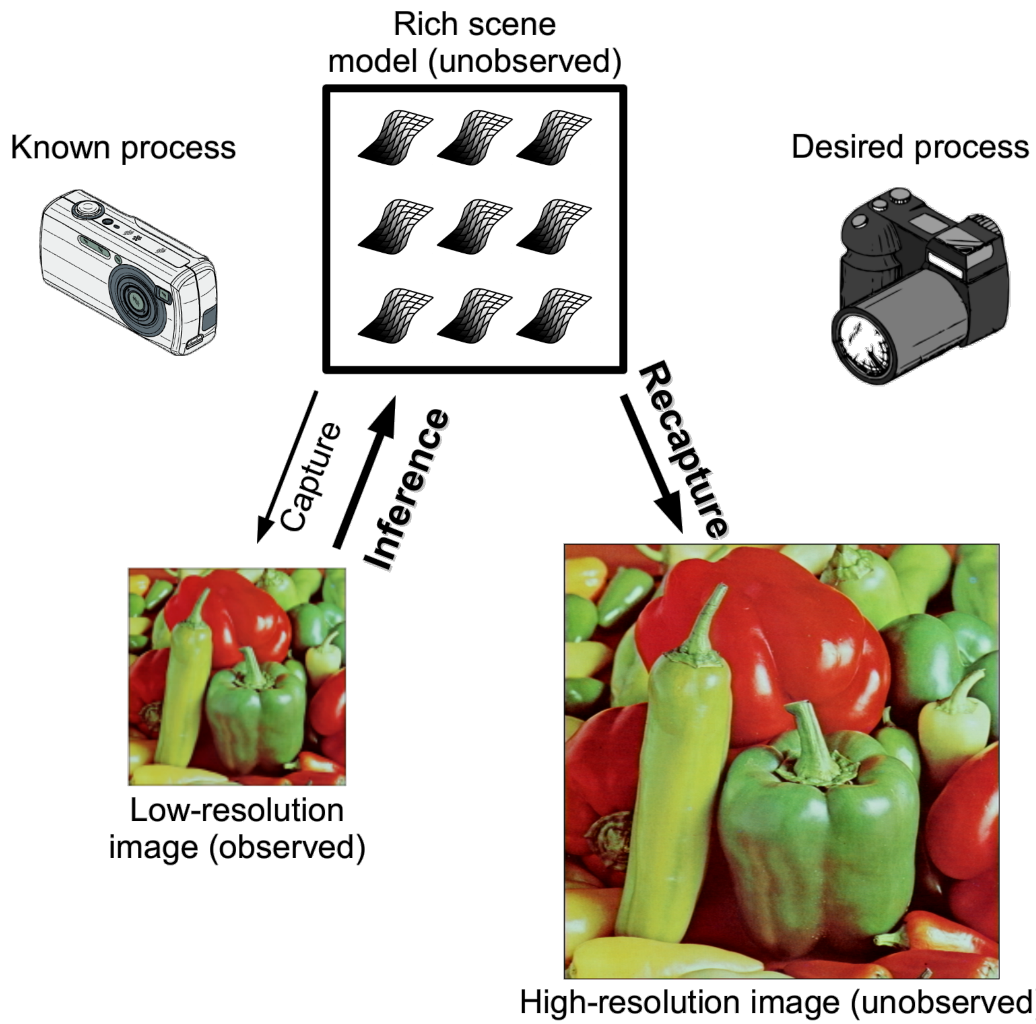
Neil Toronto

Dissertation Defense

Brigham Young University

2014/06/11

Master's Research: Super-Resolution



Toronto et al. Super-Resolution via Recapture and Bayesian Effect Modeling. CVPR 2009

Master's Research: Super-Resolution

- Model and query: Half a page of beautiful math

$$\mathbf{C}^x_{i,j} \equiv i + \tfrac{1}{2} \qquad i \in 0..m-1$$
$$\mathbf{C}^y_{i,j} \equiv j + \tfrac{1}{2} \qquad j \in 0..n-1$$

$$\mathrm{N9}(x,y) \equiv \{i \in \mathbb{Z} \mid -1 \le i - \lfloor x \rfloor \le 1\} \times \{j \in \mathbb{Z} \mid -1 \le j - \lfloor y \rfloor \le 1\}$$

$$\mathrm{dist}(x,y,\theta,d) \equiv x\cos\theta + y\sin\theta - d$$

$$\mathrm{prof}(d,\sigma,v^+,v^-) \equiv \tfrac{v^+ - v^-}{2}\,\mathrm{erf}\left(\tfrac{d}{\sqrt{2}\sigma}\right) + \tfrac{v^+ + v^-}{2}$$

$$\mathrm{edge}(x,y,\theta,d,v^+,v^-,\sigma) \equiv \mathrm{prof}(\mathrm{dist}(x,y,\theta,d),\sigma,v^+,v^-)$$

$$\mathbf{S}^{\mathrm{edge}}_{i,j}(x,y) \equiv \mathrm{edge}(x - \mathbf{C}^x_{i,j}, y - \mathbf{C}^y_{i,j}, \mathbf{S}^\theta_{i,j}, \mathbf{S}^d_{i,j}, \mathbf{S}^{v^+}_{i,j}, \mathbf{S}^{v^-}_{i,j}, \mathbf{S}^\sigma_{i,j})$$

$$\mathrm{E}[h(\mathbf{S}_{x,y})] \equiv \sum_{k,l \in \mathrm{N9}(x,y)} w(x - \mathbf{C}^x_{k,l}, y - \mathbf{C}^y_{k,l})\; h(\mathbf{S}^{\mathrm{edge}}_{k,l}(x,y))$$

$$\mathbf{S}^\theta_{i,j} \sim \mathrm{Uniform}(-\pi,\pi) \qquad \mathbf{S}^{v^+}_{i,j} \sim \mathrm{Uniform}(0,1)$$
$$\mathbf{S}^d_{i,j} \sim \mathrm{Uniform}(-3,3) \qquad \mathbf{S}^{v^-}_{i,j} \sim \mathrm{Uniform}(0,1)$$
$$\mathbf{S}^\sigma_{i,j} \sim \mathrm{Beta}(1.6,1)$$

$$\mathbf{I}_{i,j} \mid \mathbf{S}_{\mathrm{N9}(i,j)} \sim \mathrm{Normal}(\mathrm{E}[\mathbf{S}_{i,j}], \omega)$$

$$\Phi_{i,j}(\mathbf{S}_{\mathrm{N9}(i,j)}) \equiv \exp\left(-\frac{\mathrm{Var}[\mathbf{S}_{i,j}]}{2\gamma^2}\right)$$

Master's Research: Super-Resolution

- Query implementation: 600 lines of Python

Main Results: Super-Resolution

- Competitor and BEI on 4x super-resolution:

Resolution Synthesis

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- Was capable of other reconstruction tasks with few changes

Only Mostly Satisfying

Problem 1: Still not sure the program is right

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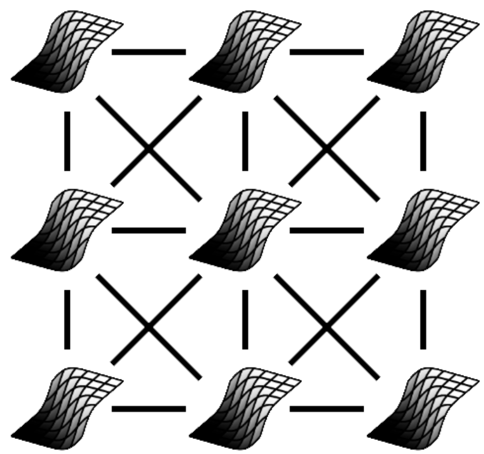
Problem 2: *smooth* edges instead of *discontinuous*



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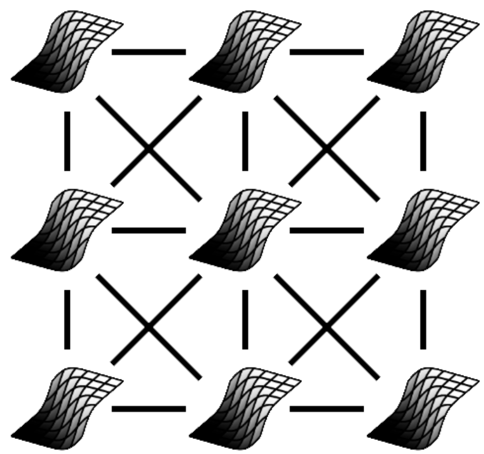

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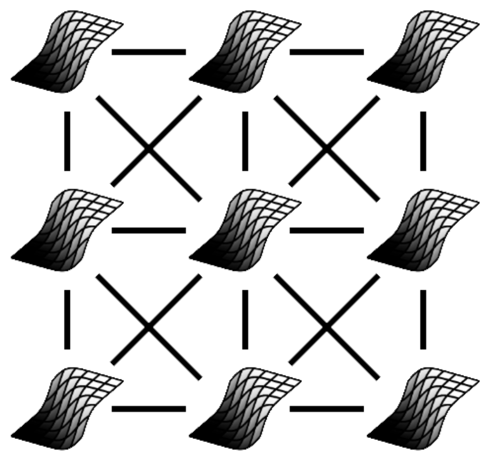
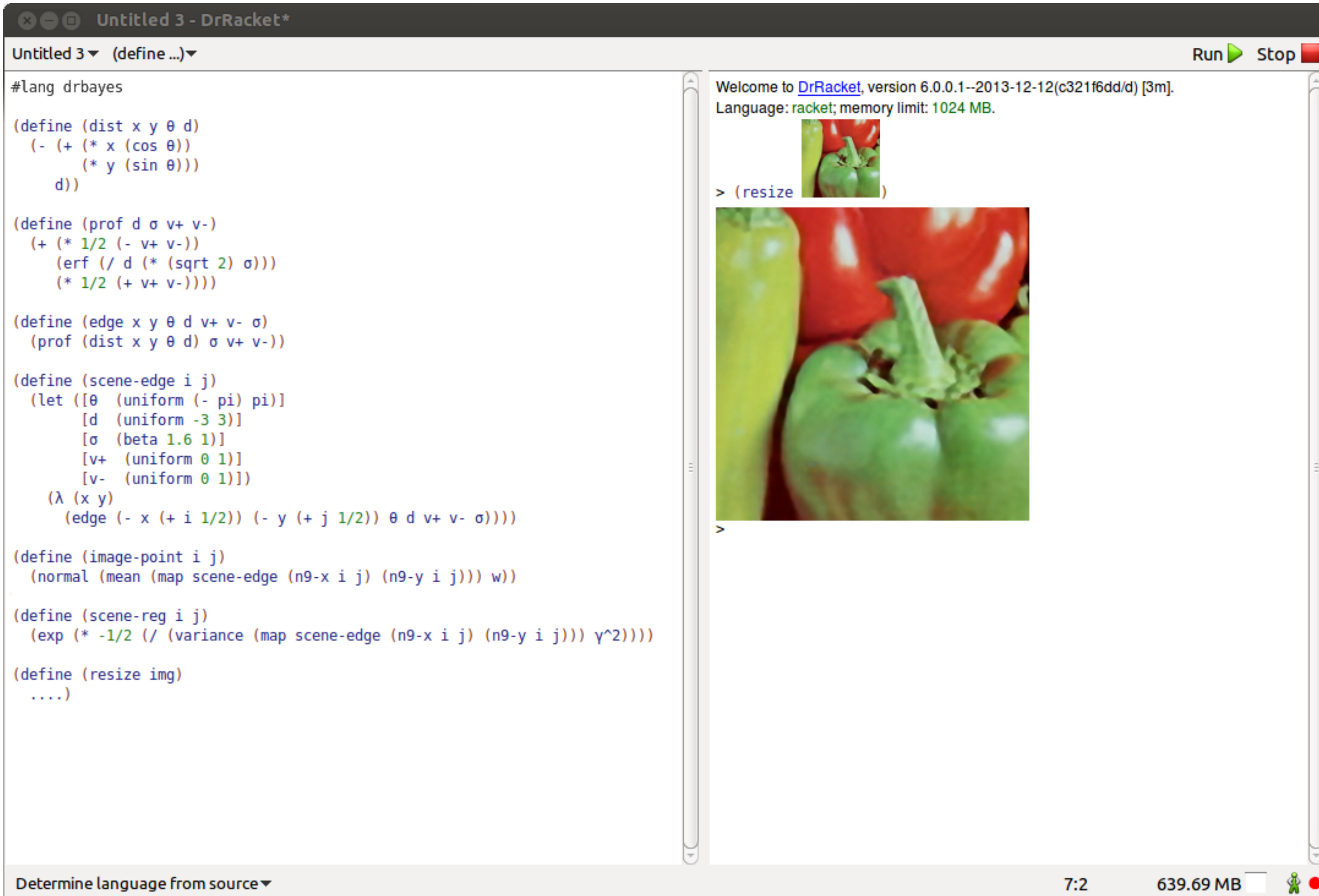


"To *approximate* blurring with a spatially varying point-spread function (PSF), we assign each facet a Gaussian PSF and convolve each analytically *before combining outputs*."

i.e. "We can't model it correctly so here's a hack."

Solution Idea: Probabilistic Language



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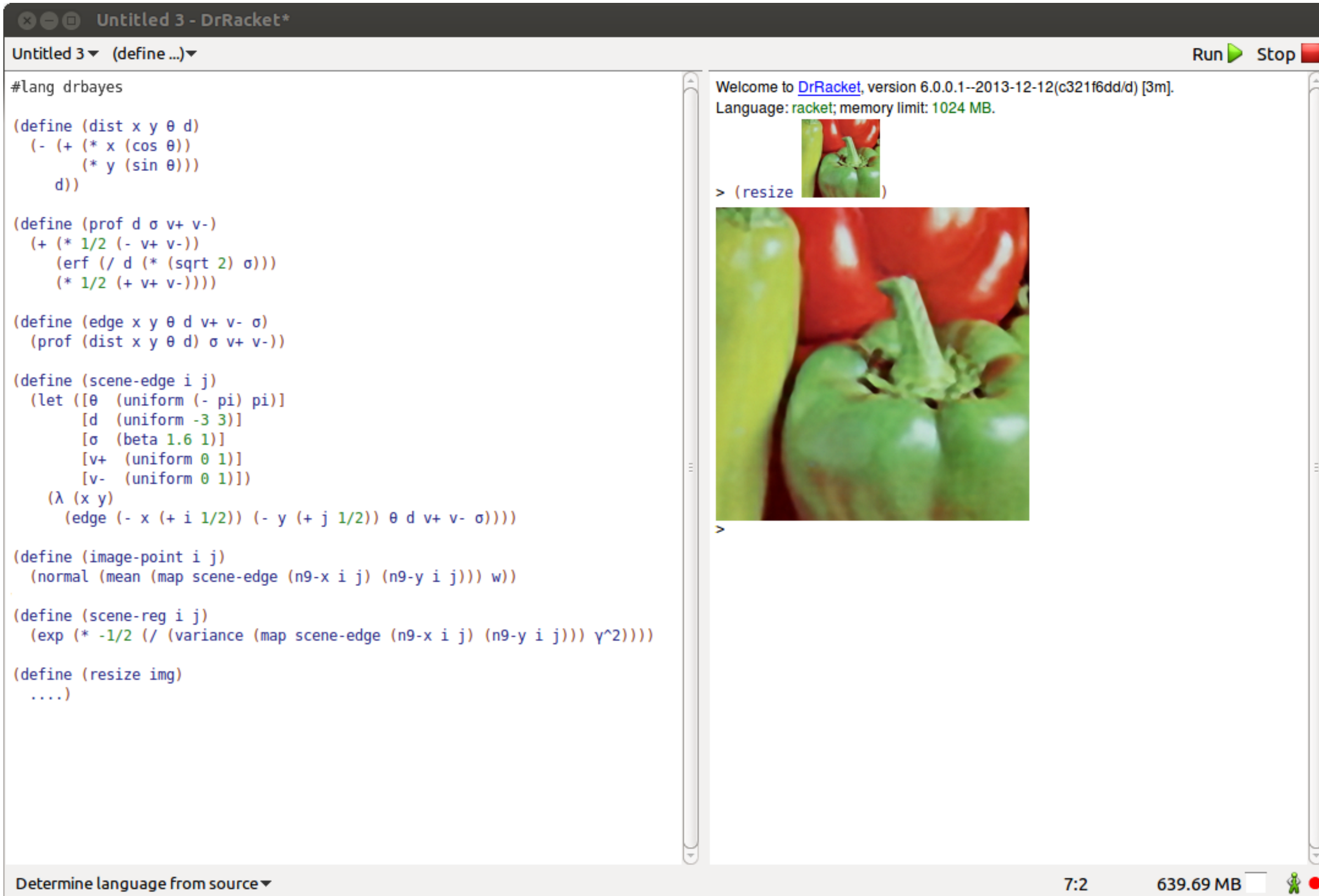

- Also somehow let me model correctly

Prior Work

Defined by an implementation

Defined by a semantics (i.e. mathematically)

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Designed for Bayesian practice

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Best of all worlds: define language using functional programming theory, make it for Bayesians, and remove limitations

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- **Trustworthy**: defined mathematically
- **Functional programming theory** has the tools to define programming languages mathematically
- **Measure-theoretic probability** is the most complete account of probability; should allow shedding common limitations

Simple Example Process

- Example process: Normal-Normal

$$X \sim \mathrm{Normal}(0, 1)$$
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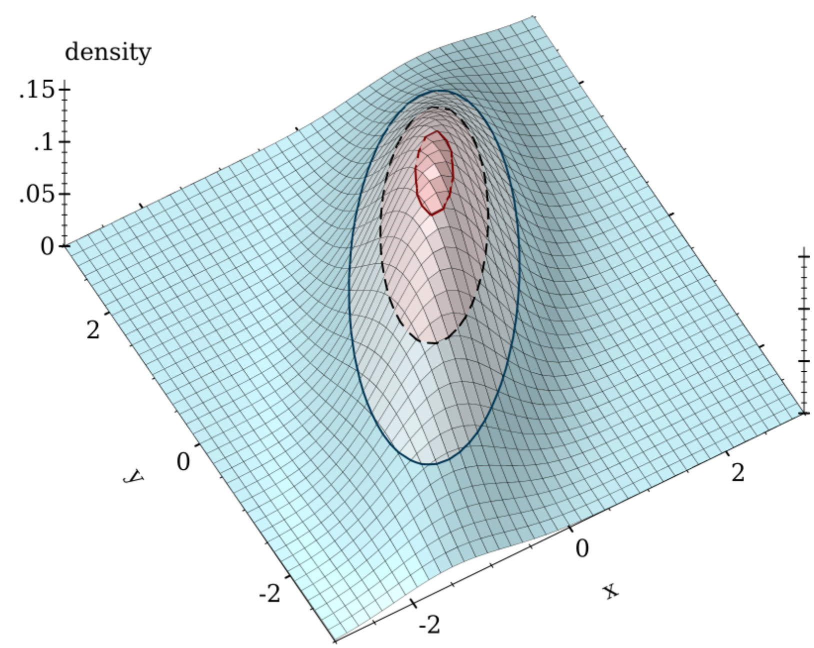
- Density model $f : \mathbb{R} \times \mathbb{R} \to [0, \infty)$:



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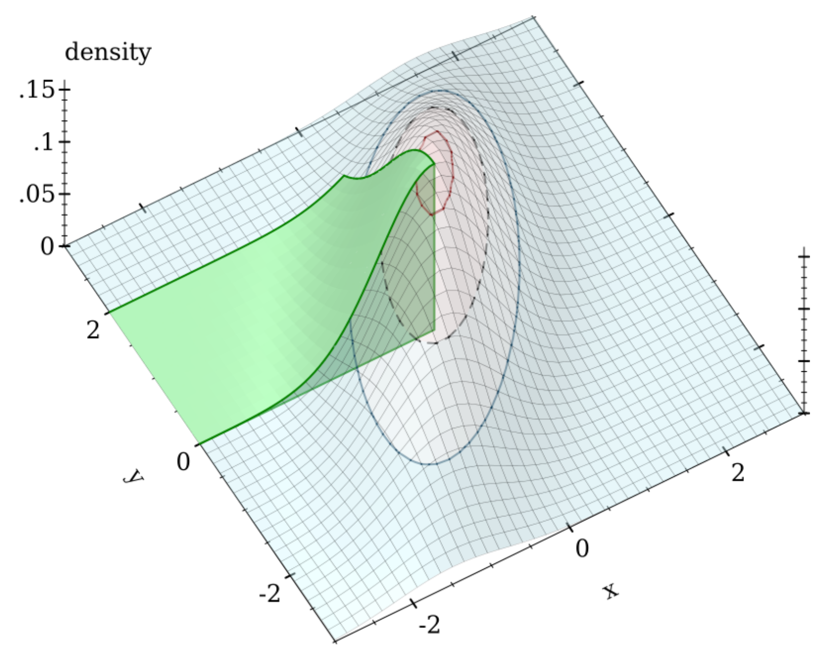

Conditional Queries

- Compute query $\Pr[X < 0 \mid Y = 1]$ using Bayes' law:

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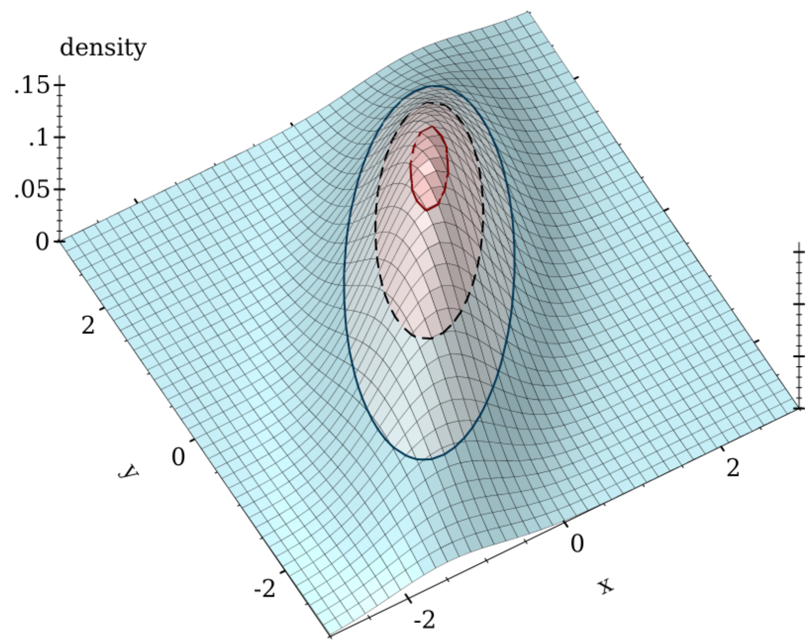

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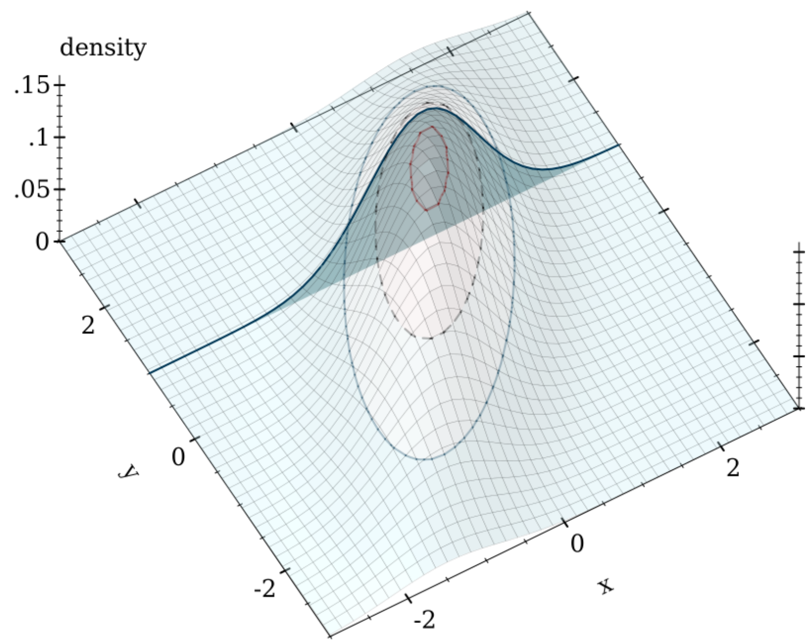

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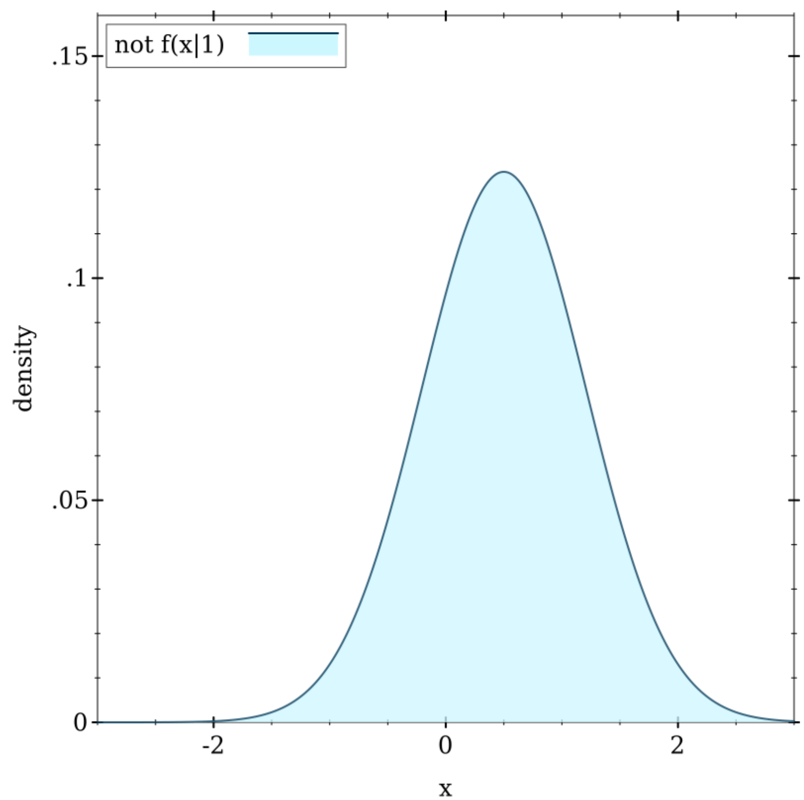

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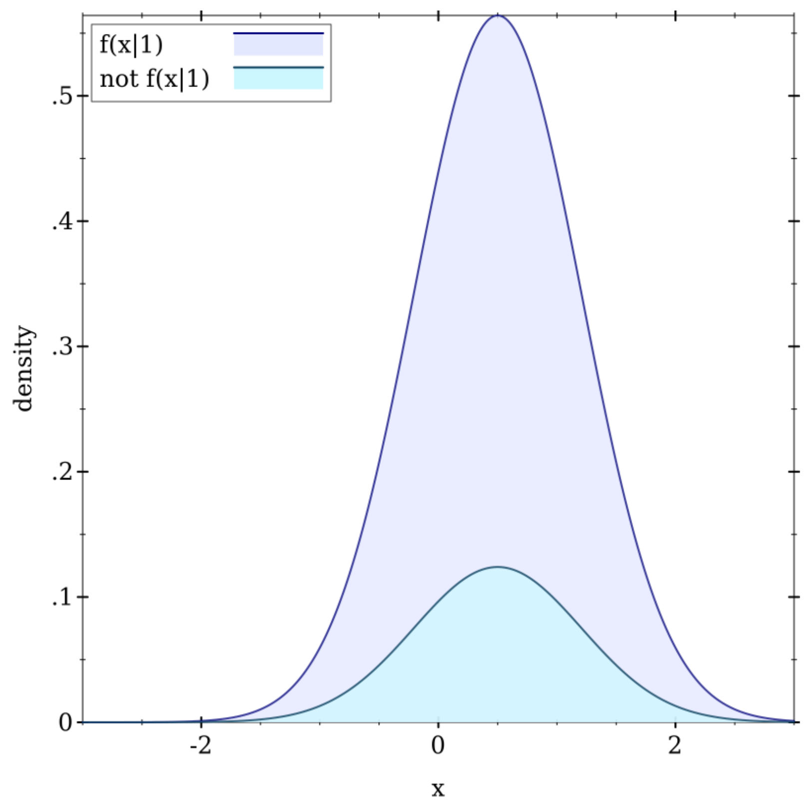

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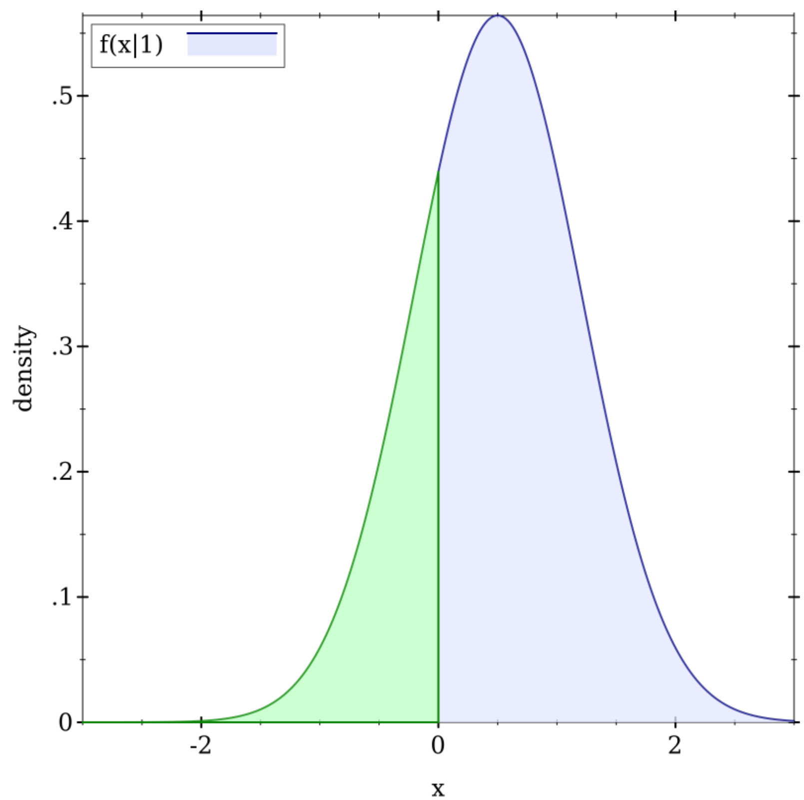

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- Tricks to get around limitations aren't general enough

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Can we avoid densities when $\Pr[e_2] = 0$?

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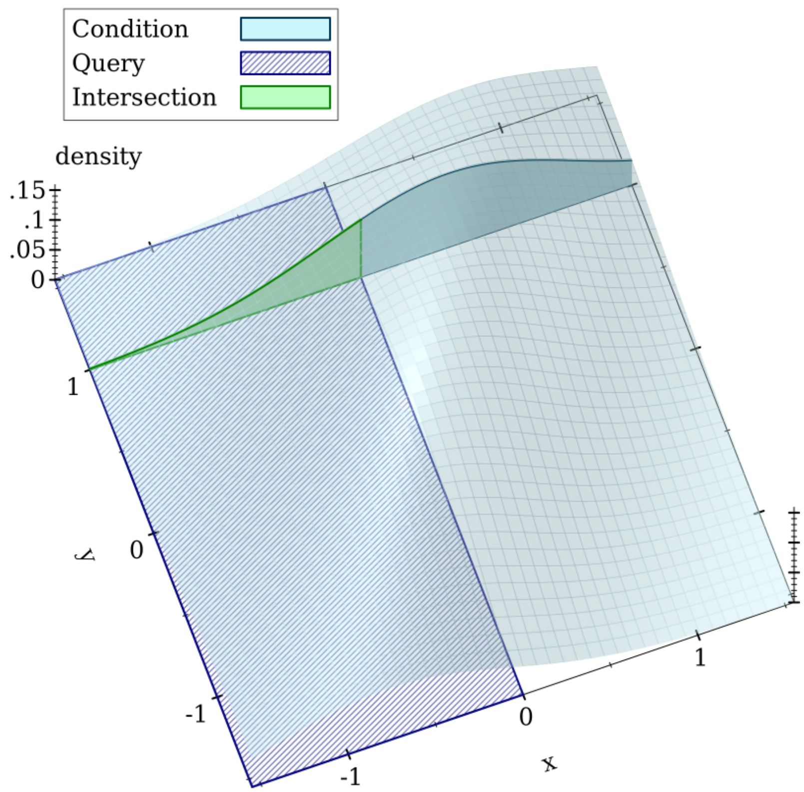

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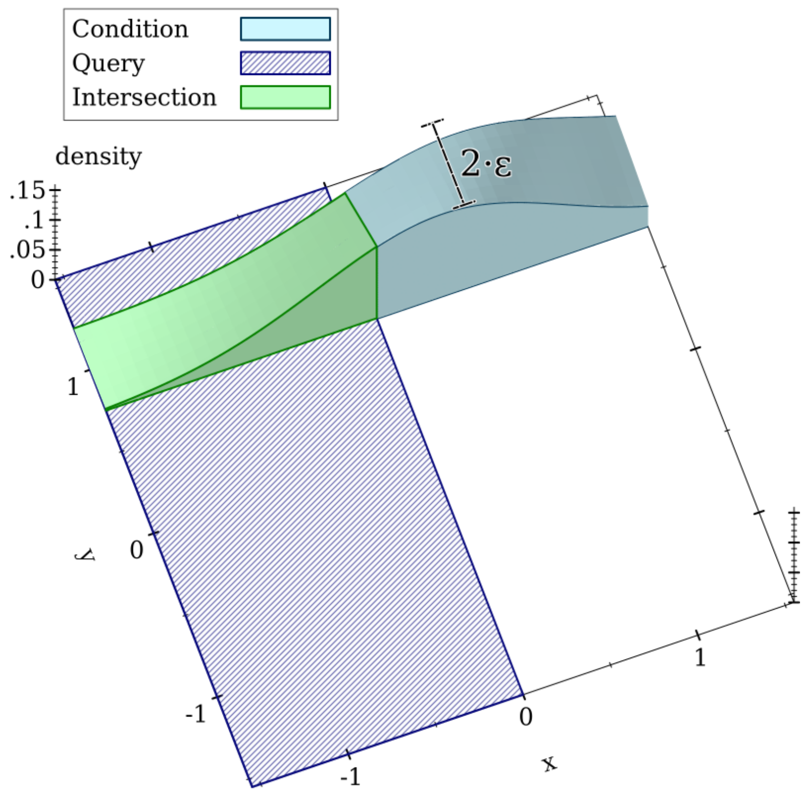

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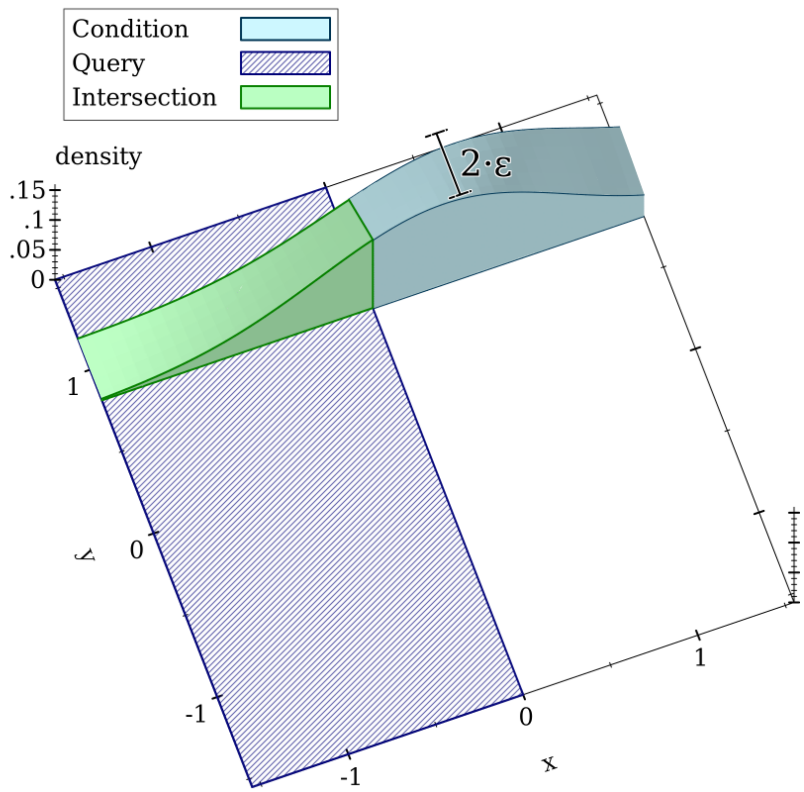

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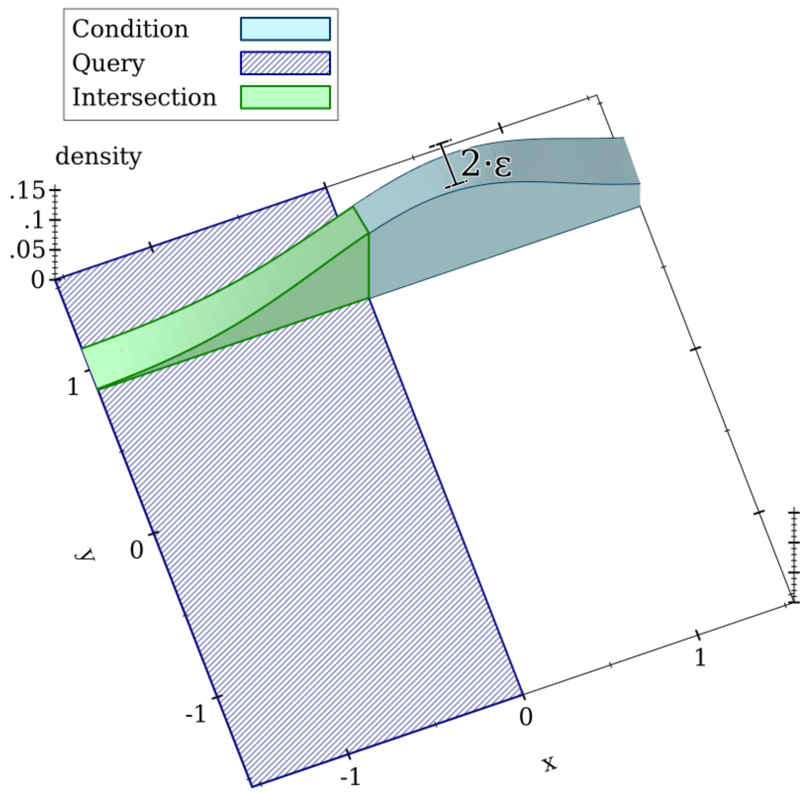

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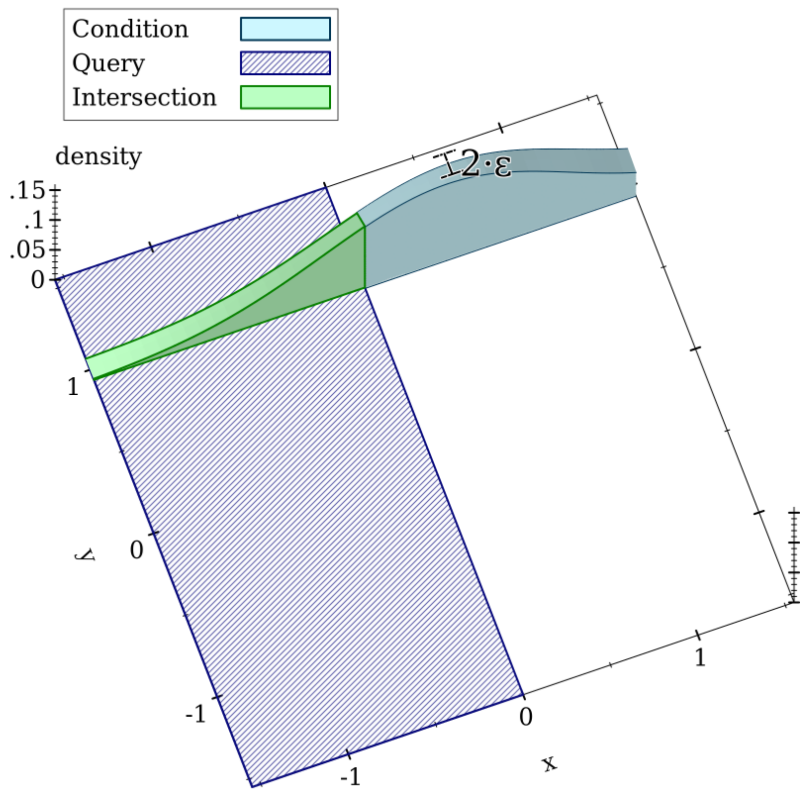

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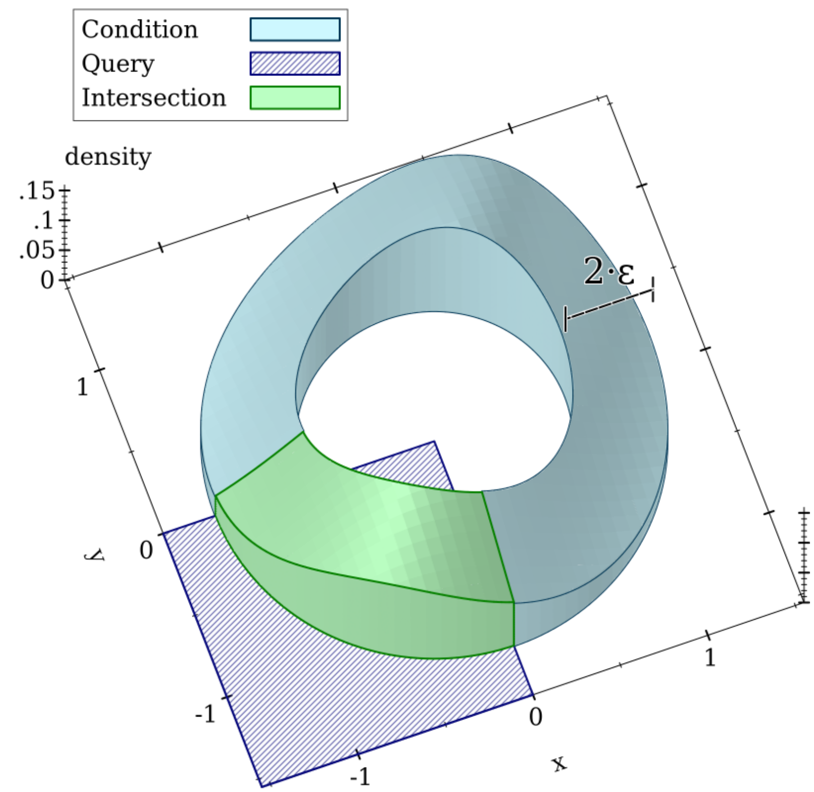

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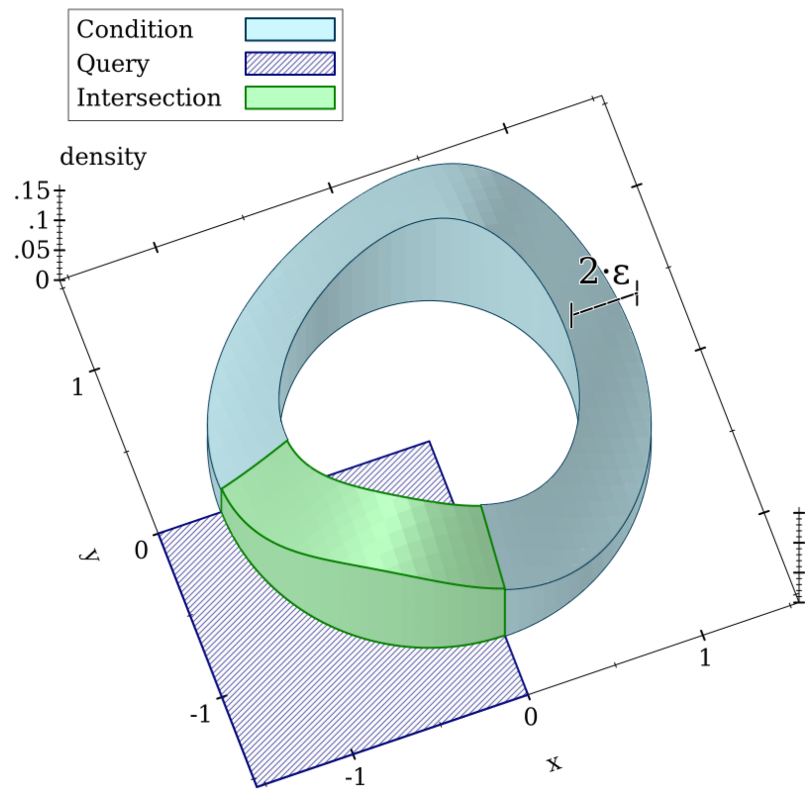

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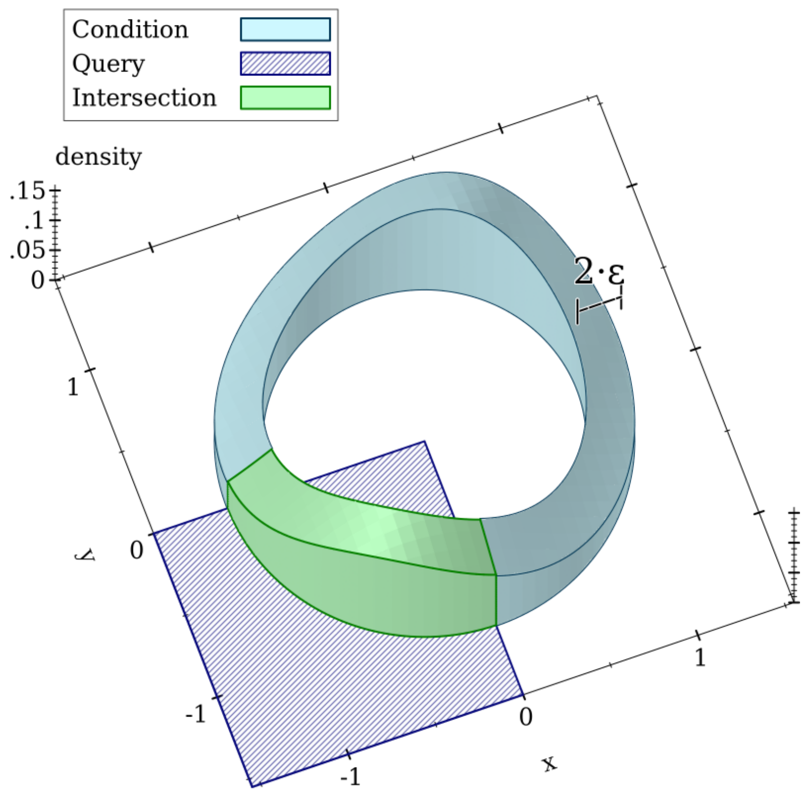

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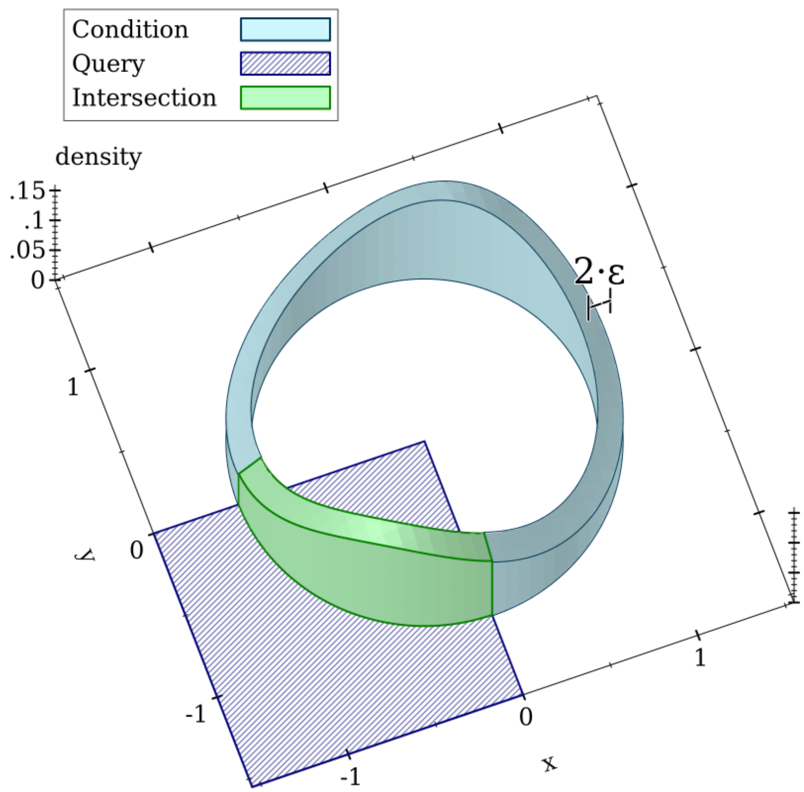

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A **uniform random source** model:

$$\Omega = [0,1] \times [0,1]$$
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- Stretches instead of integrates

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- Is this really more feasible than integrating?

Queries Using Preimages

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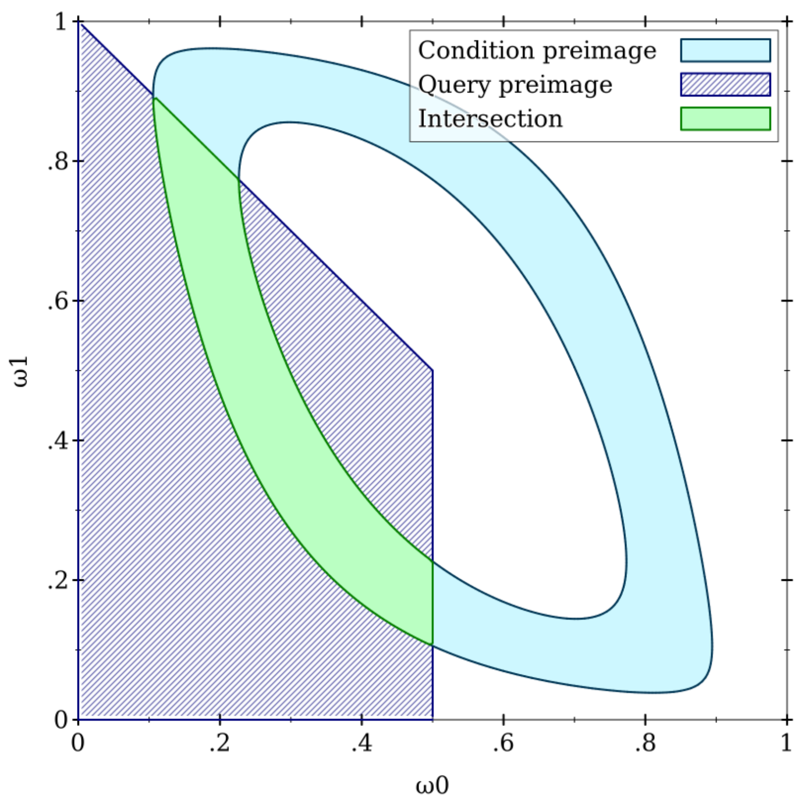

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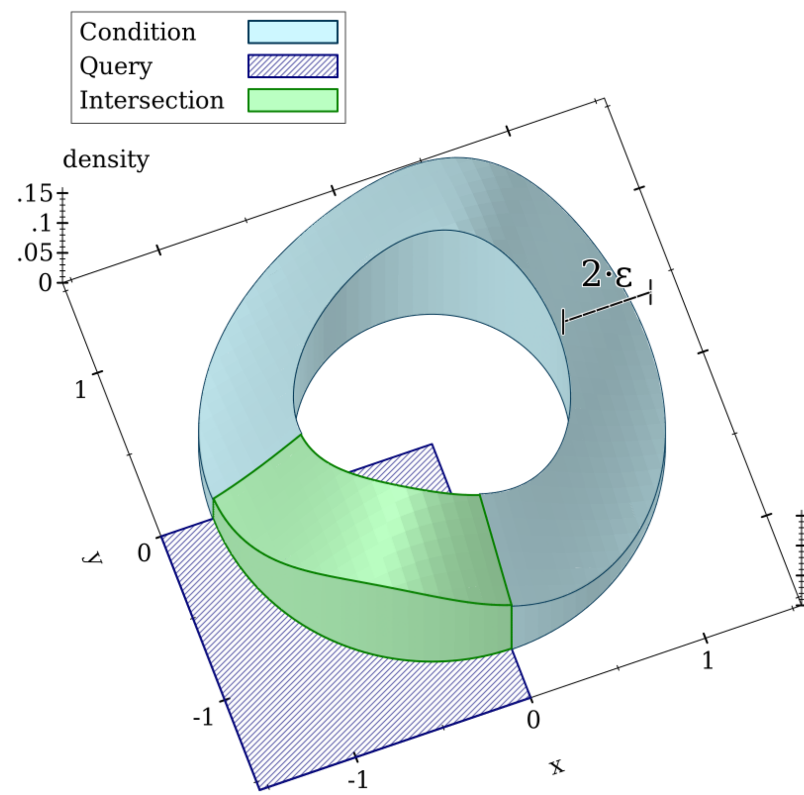

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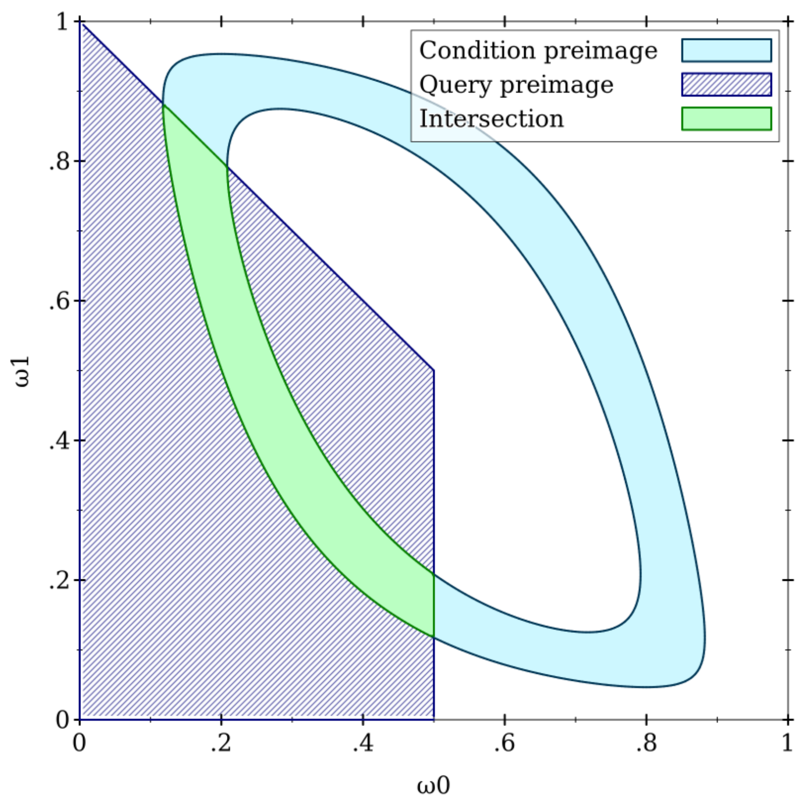

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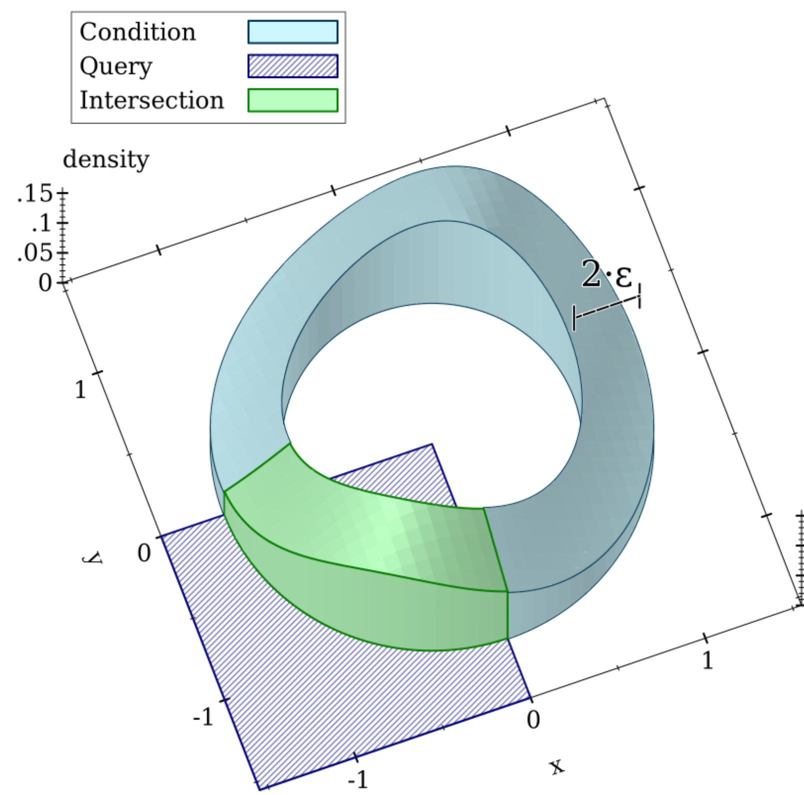

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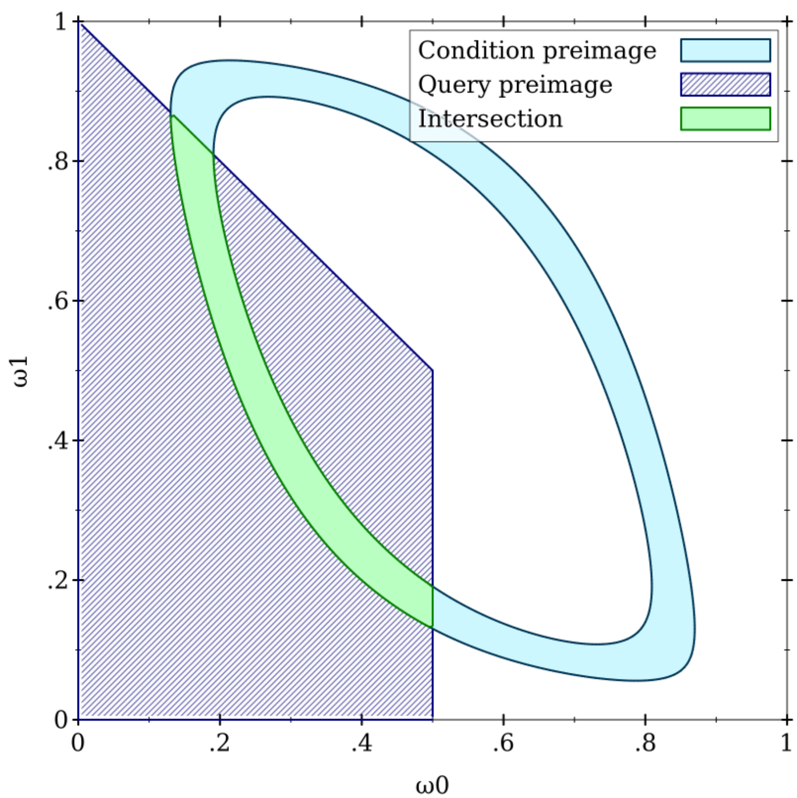

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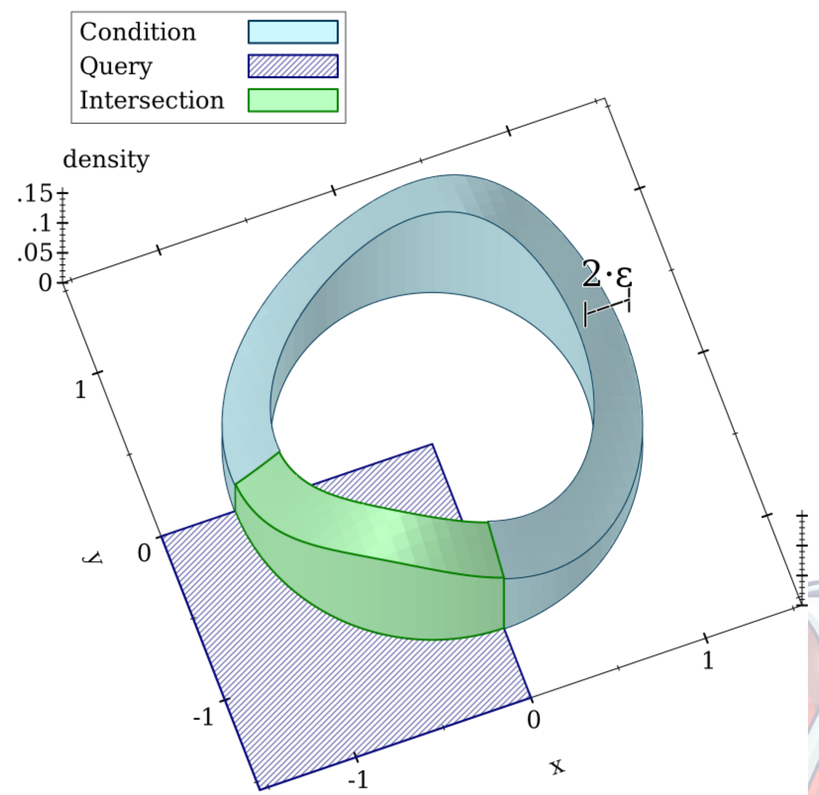

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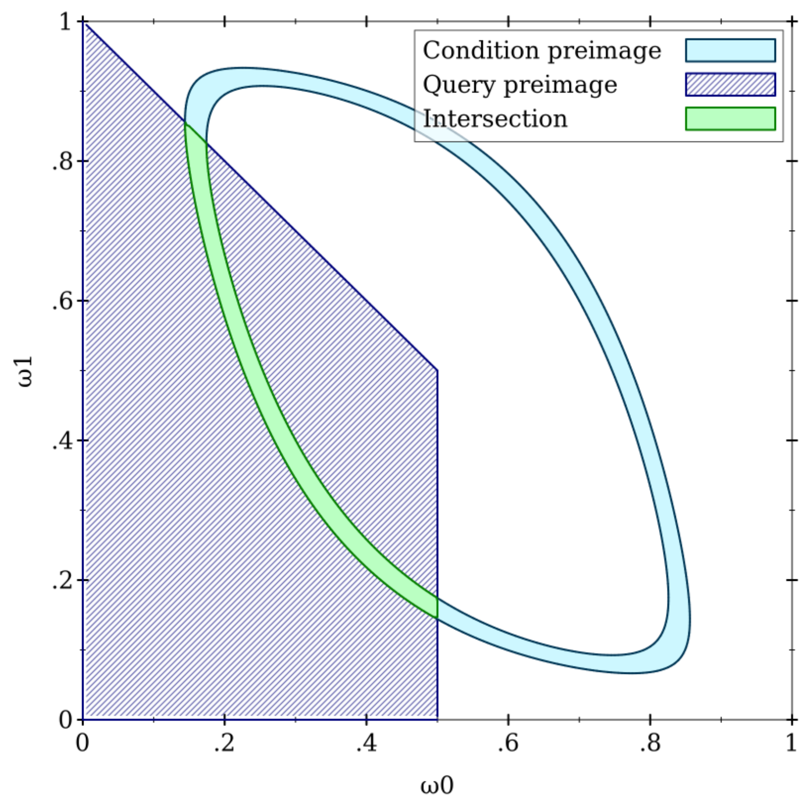

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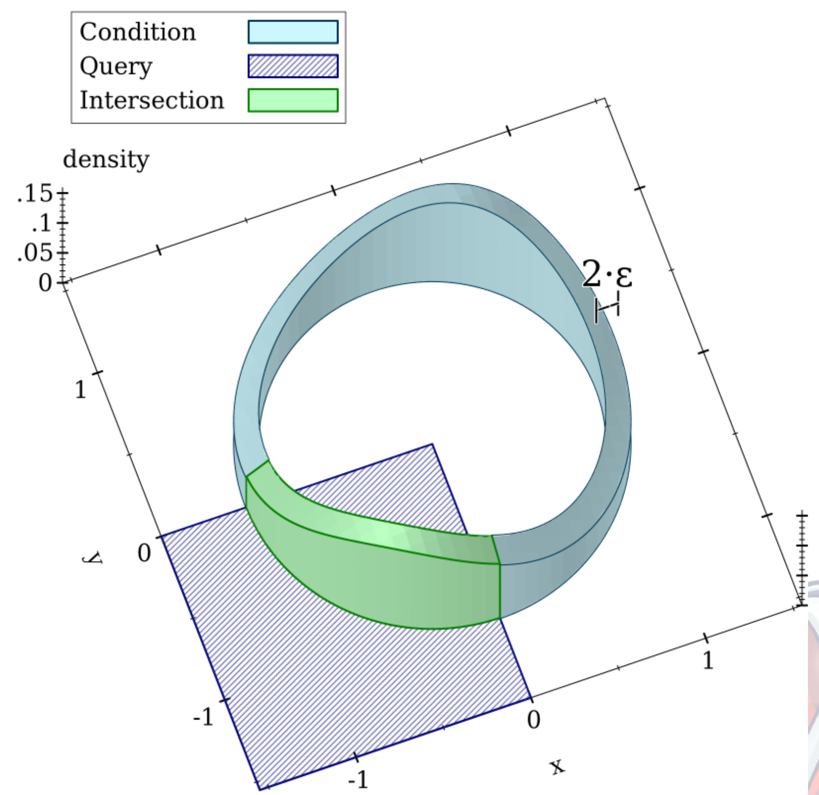

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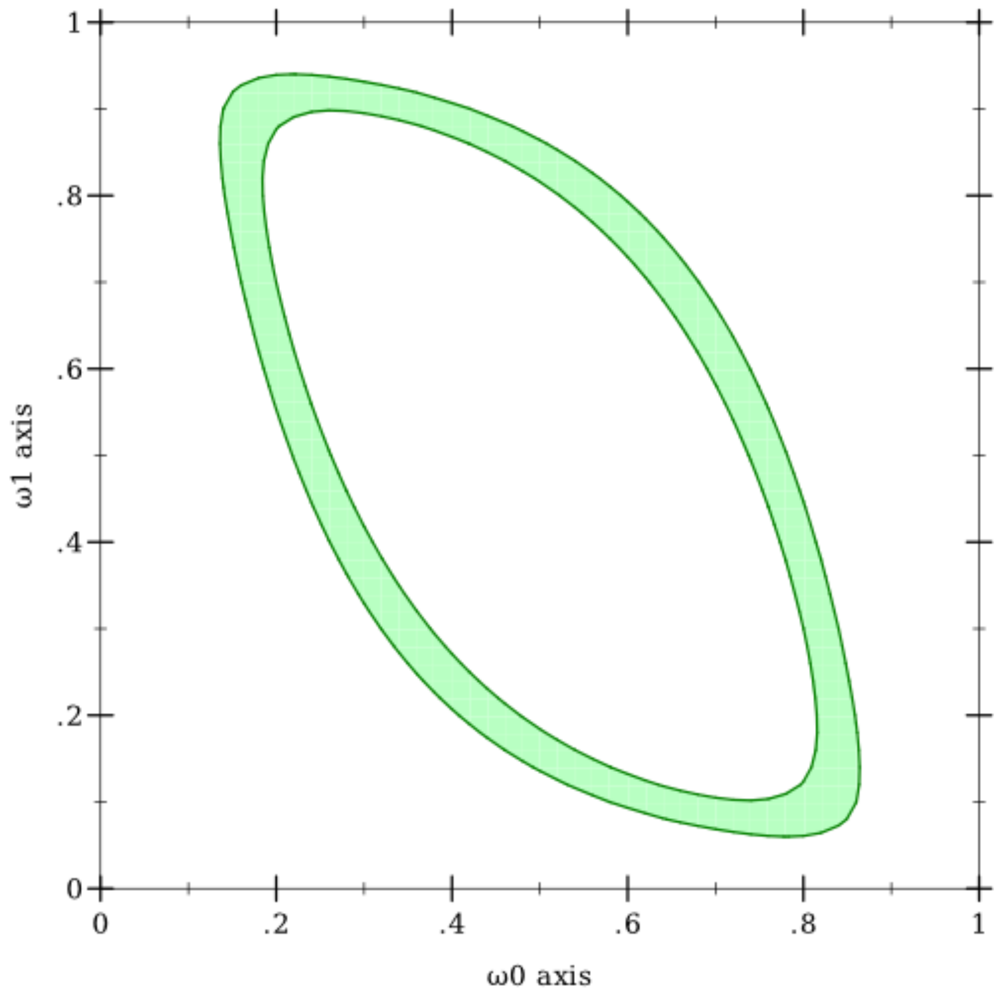
- Completely infeasible! But...

What About Approximating?

Conservative approximation with rectangles:



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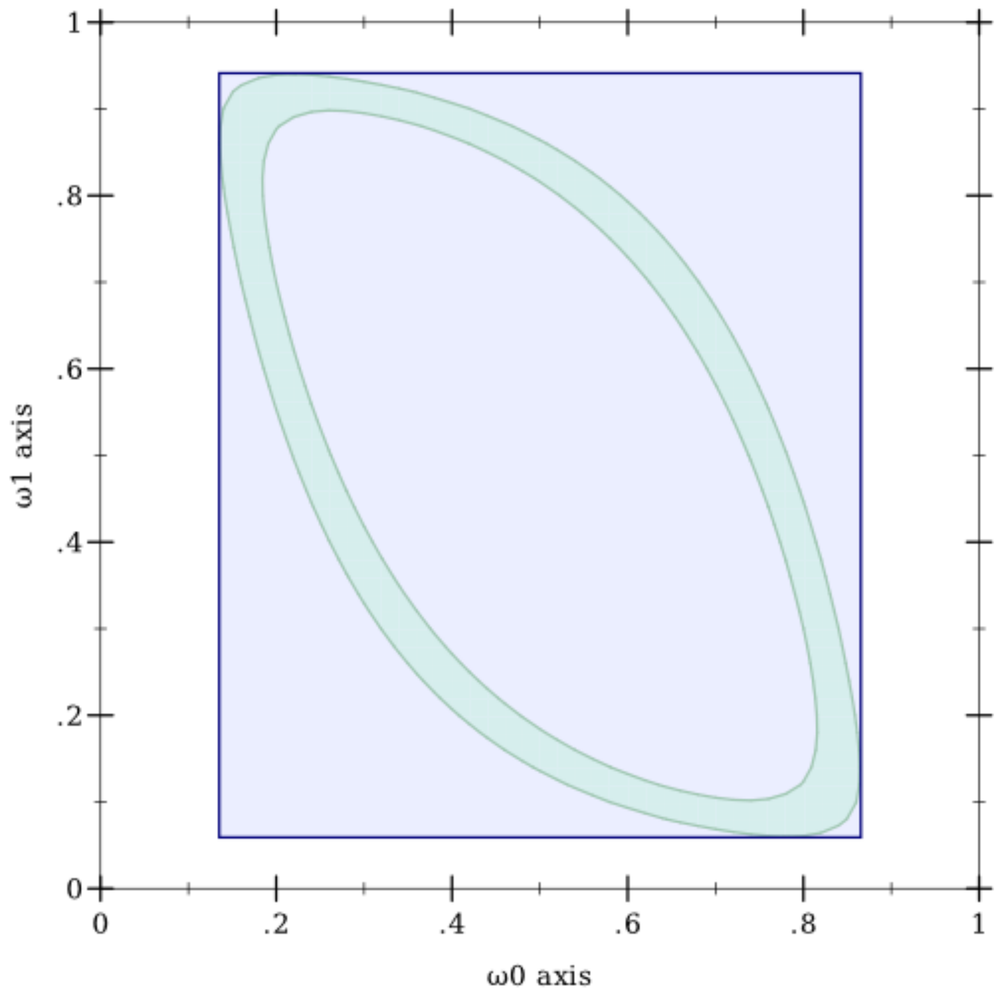

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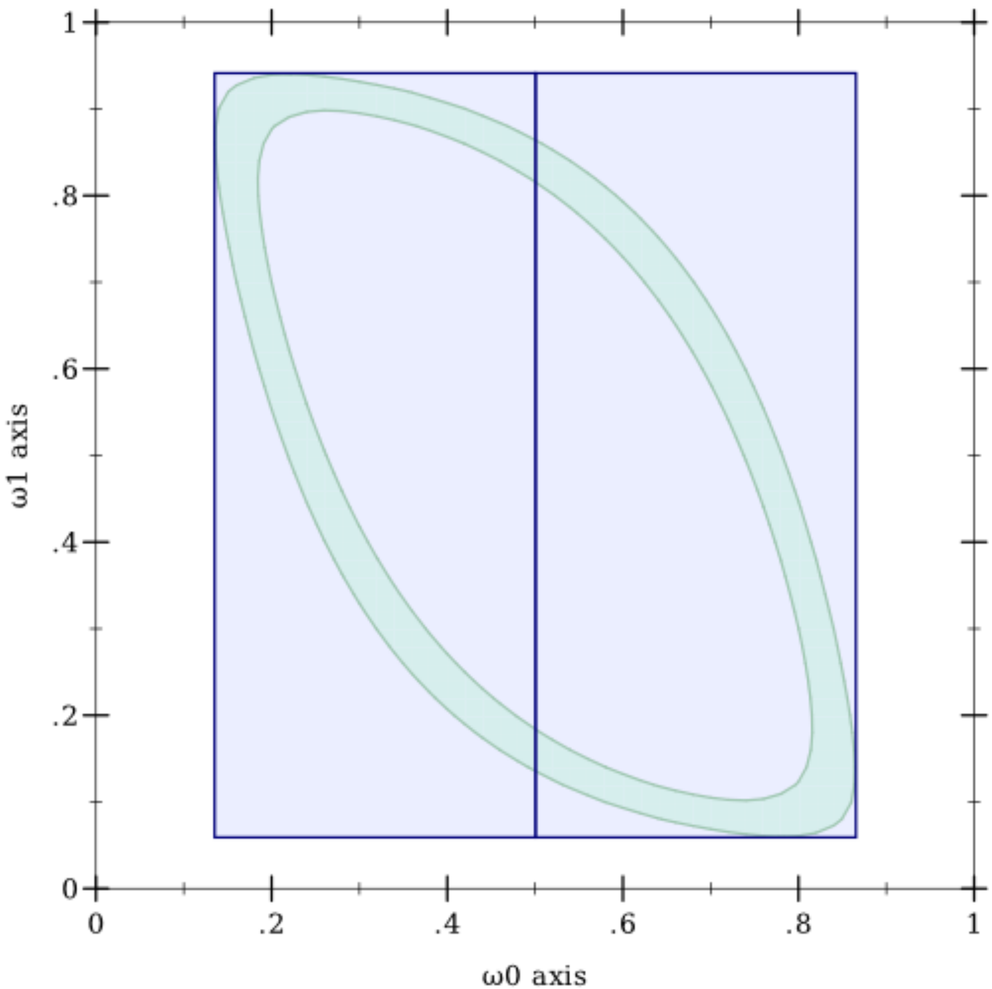

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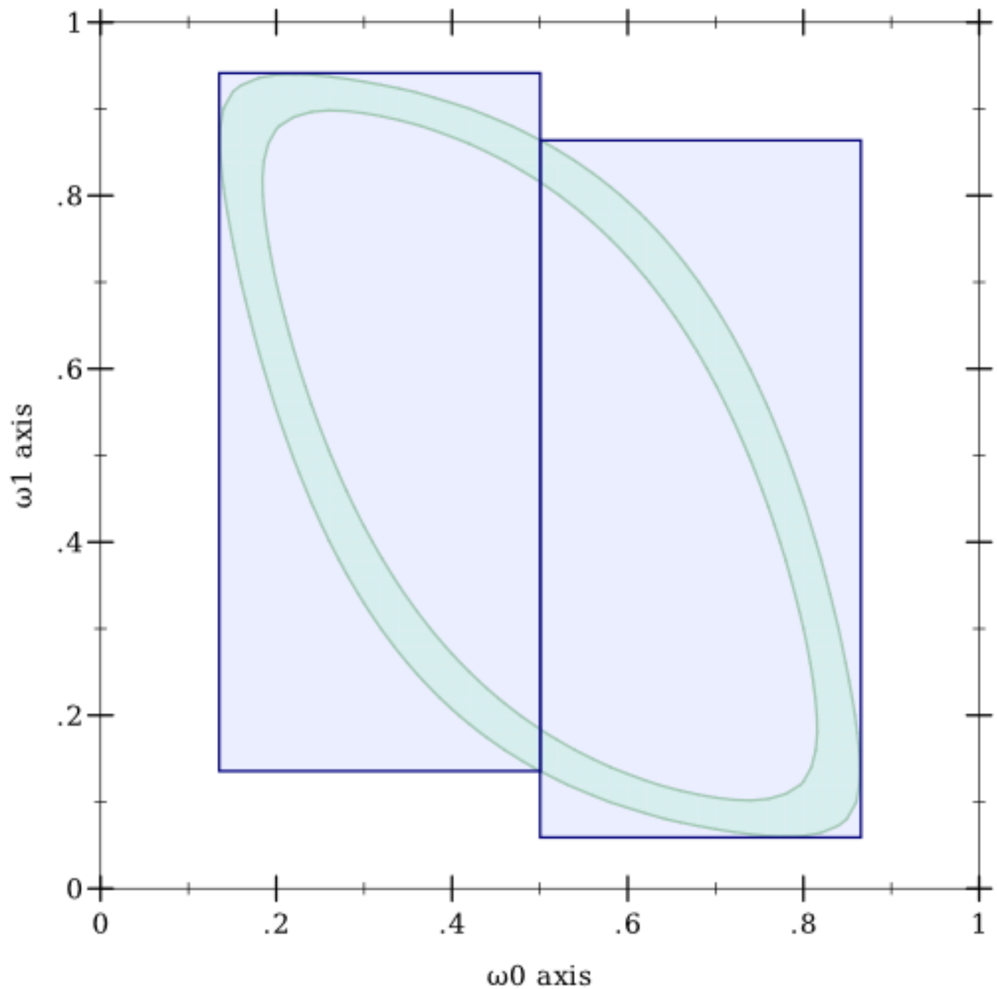

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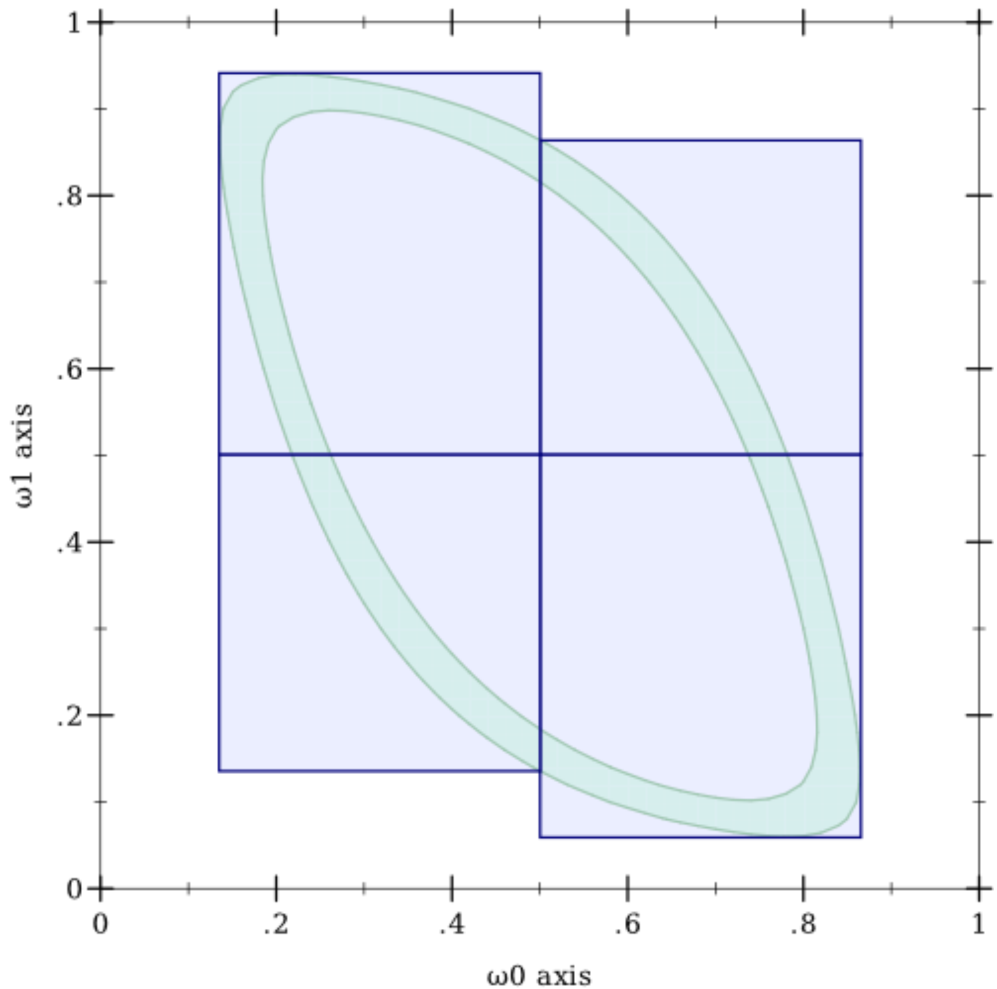

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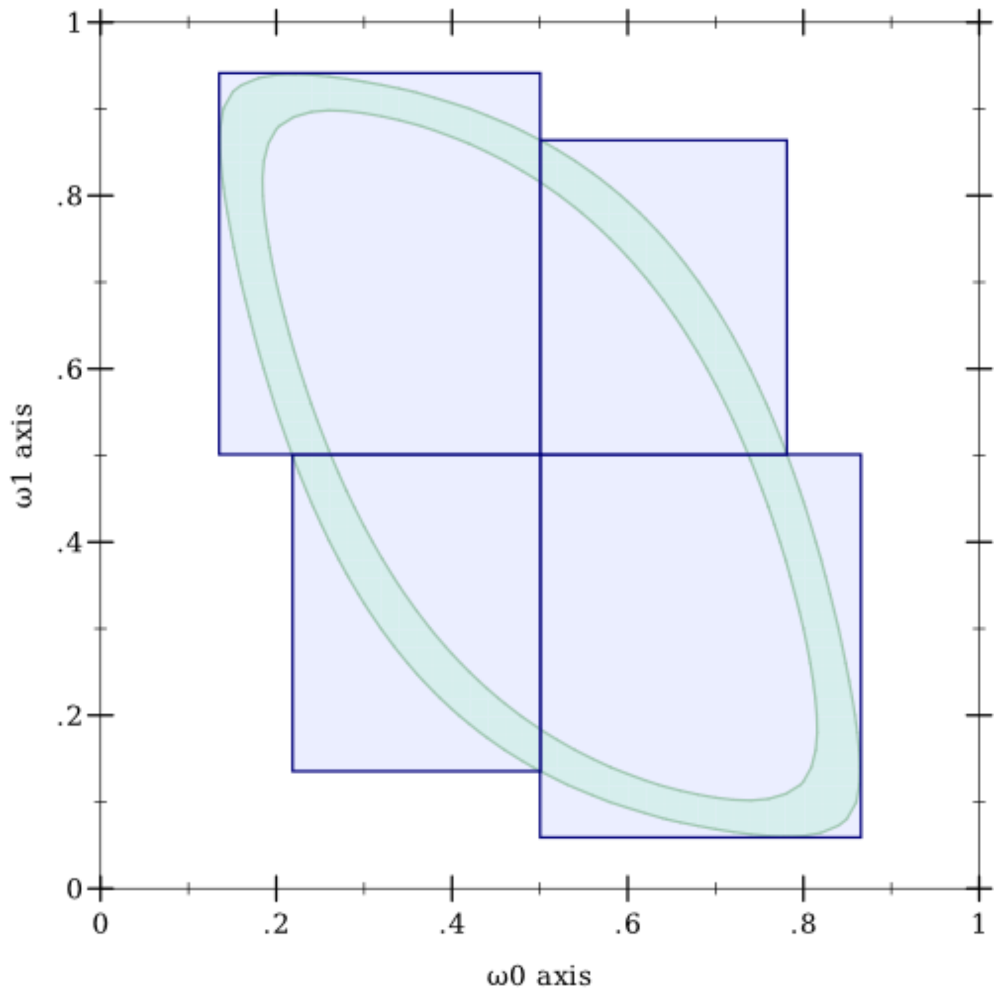

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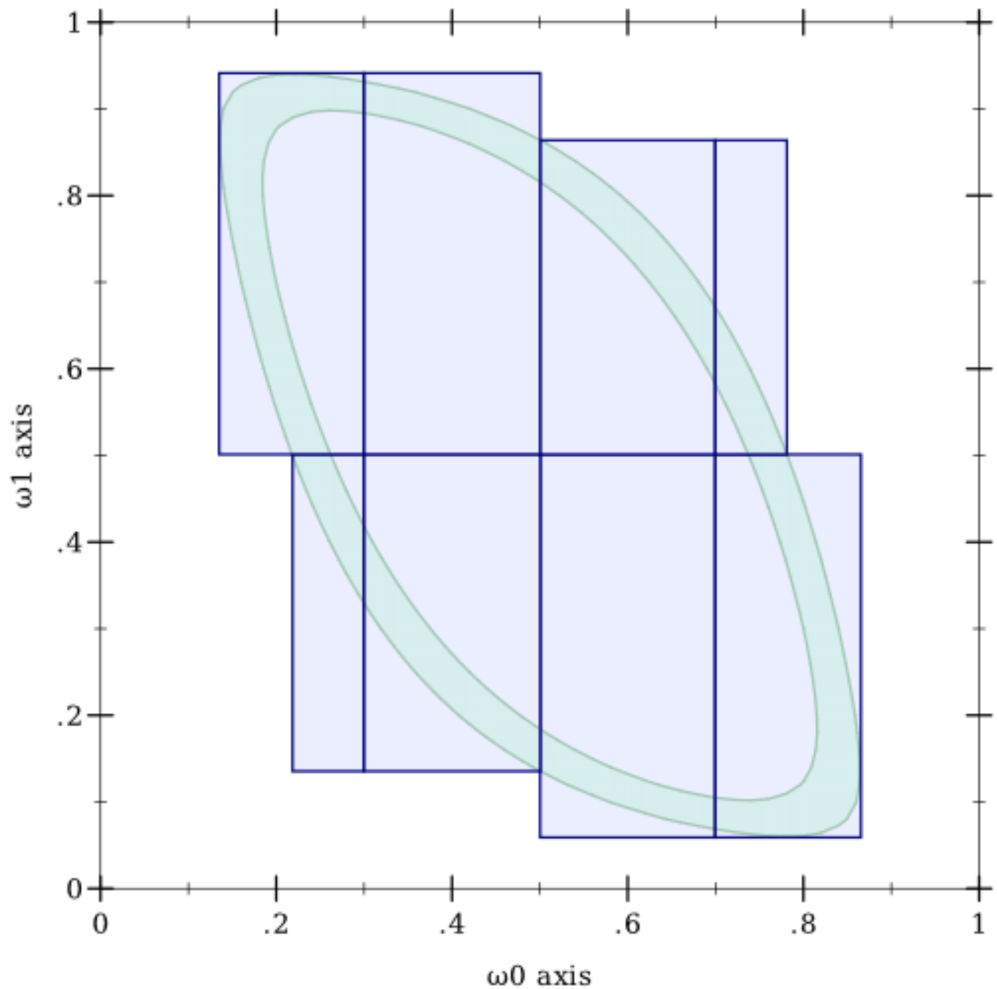

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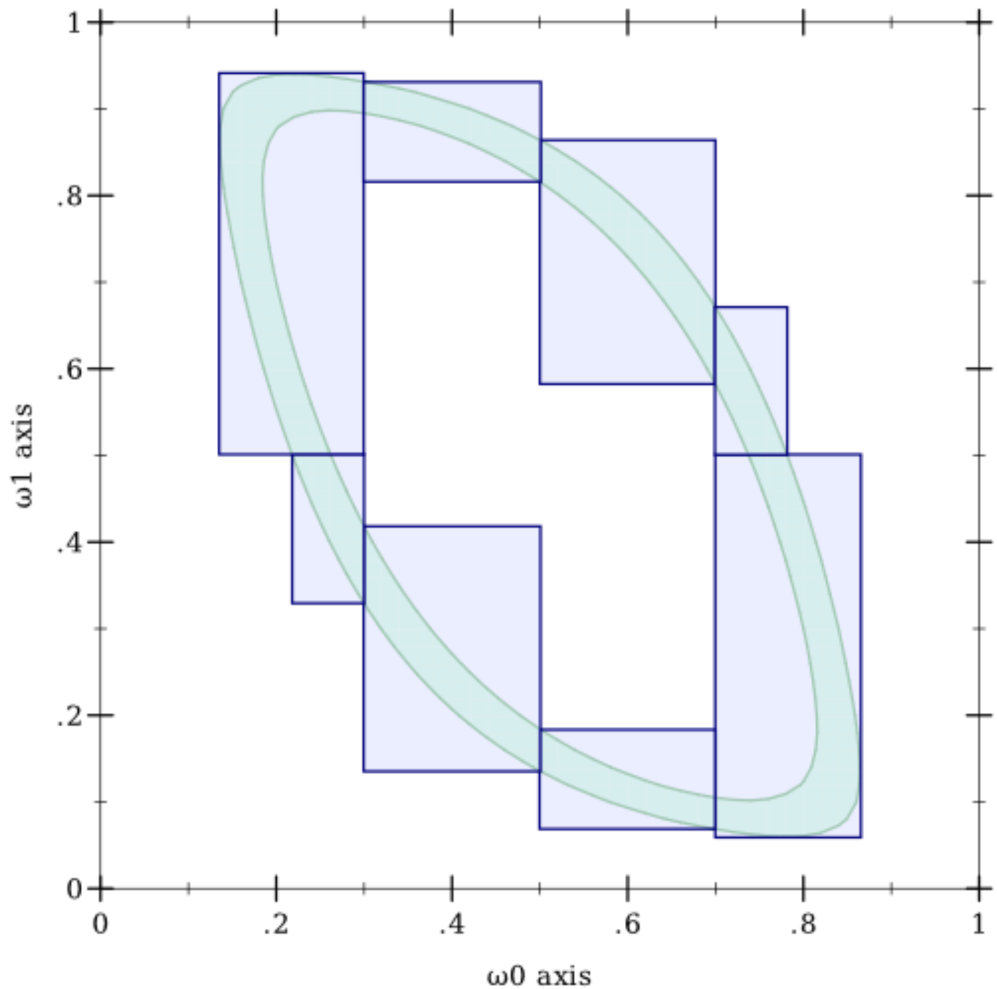

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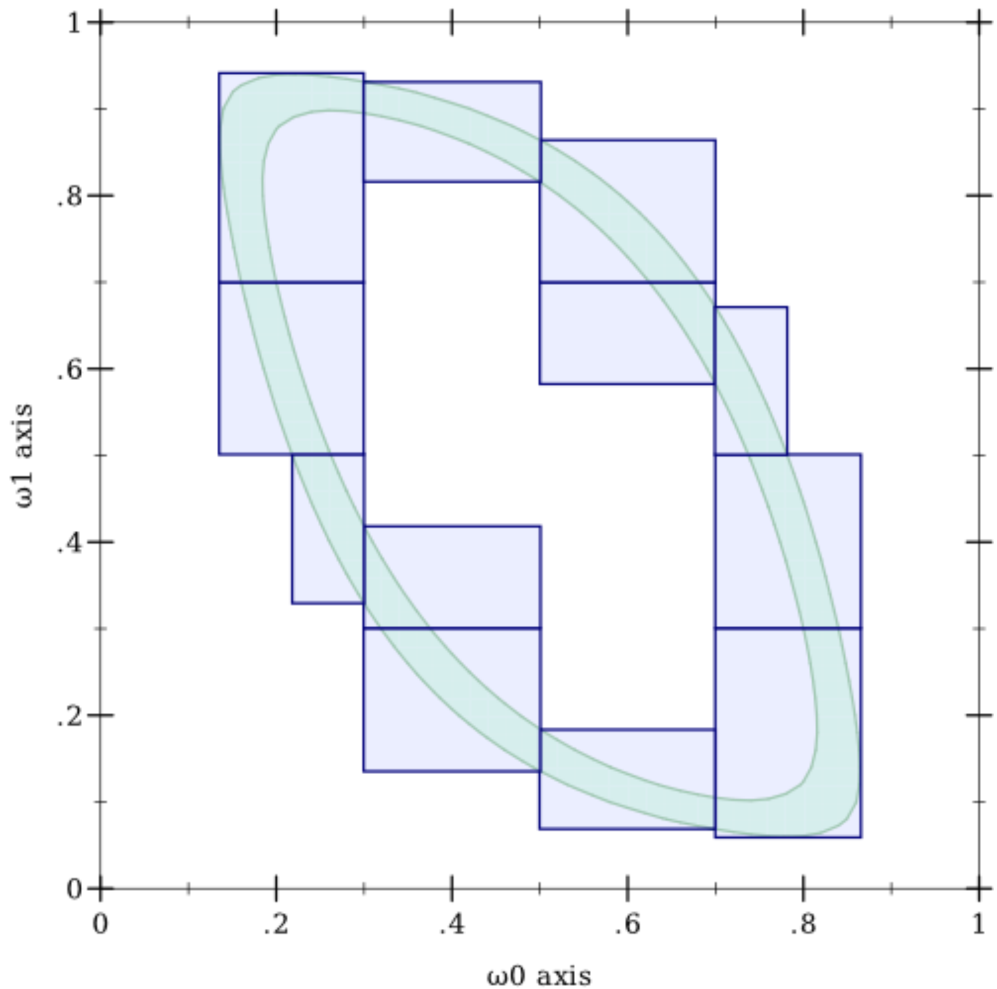

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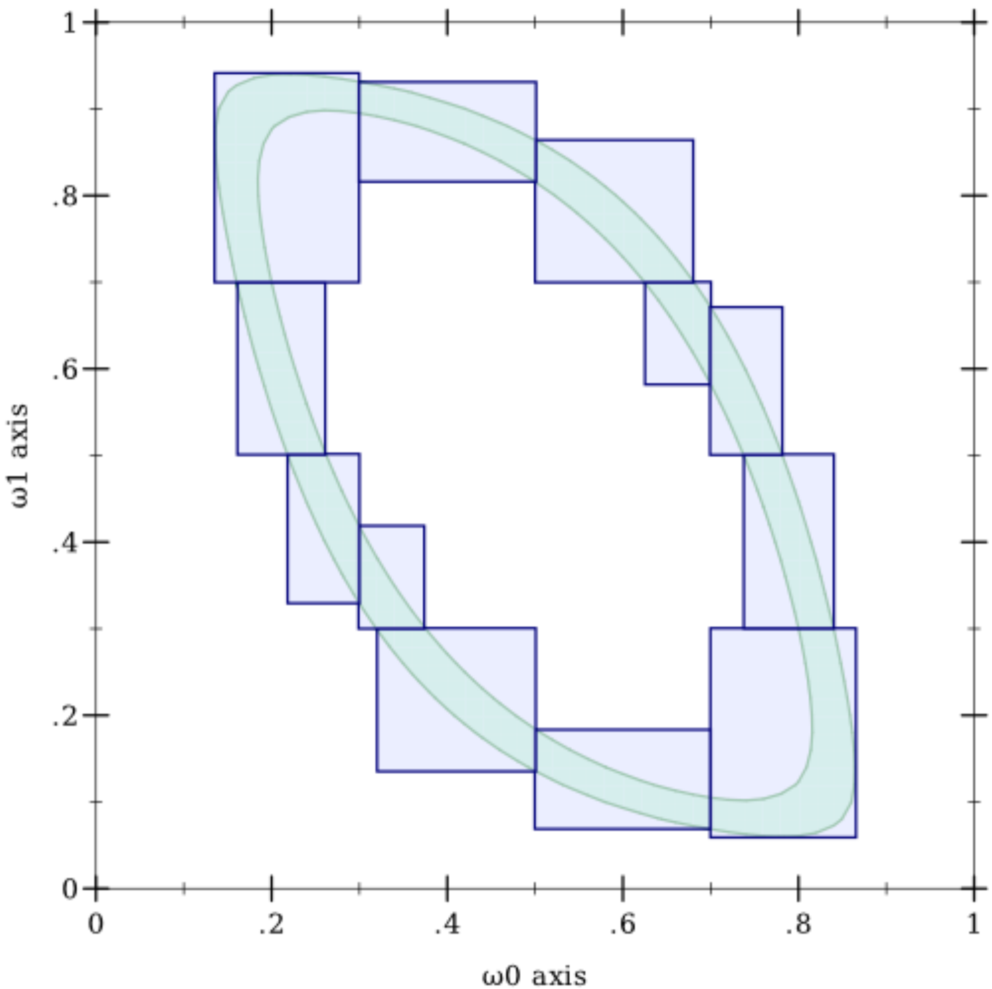

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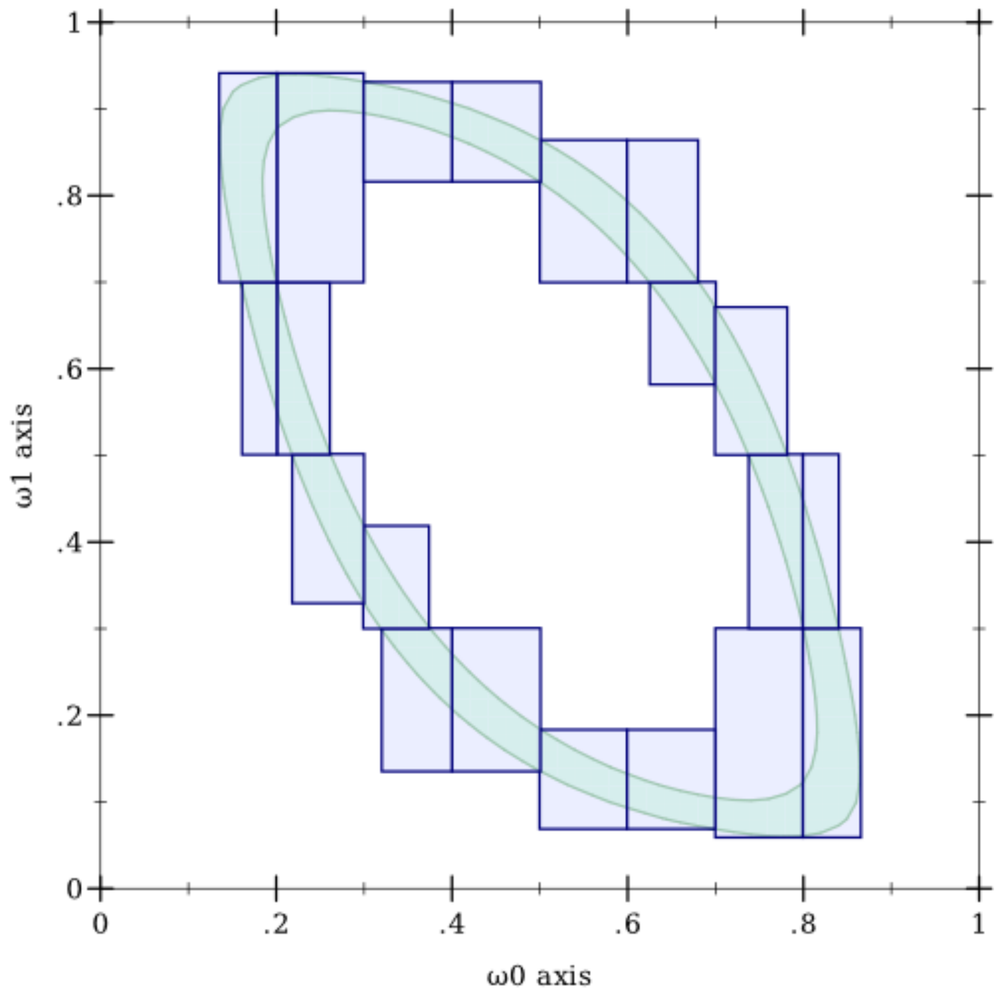

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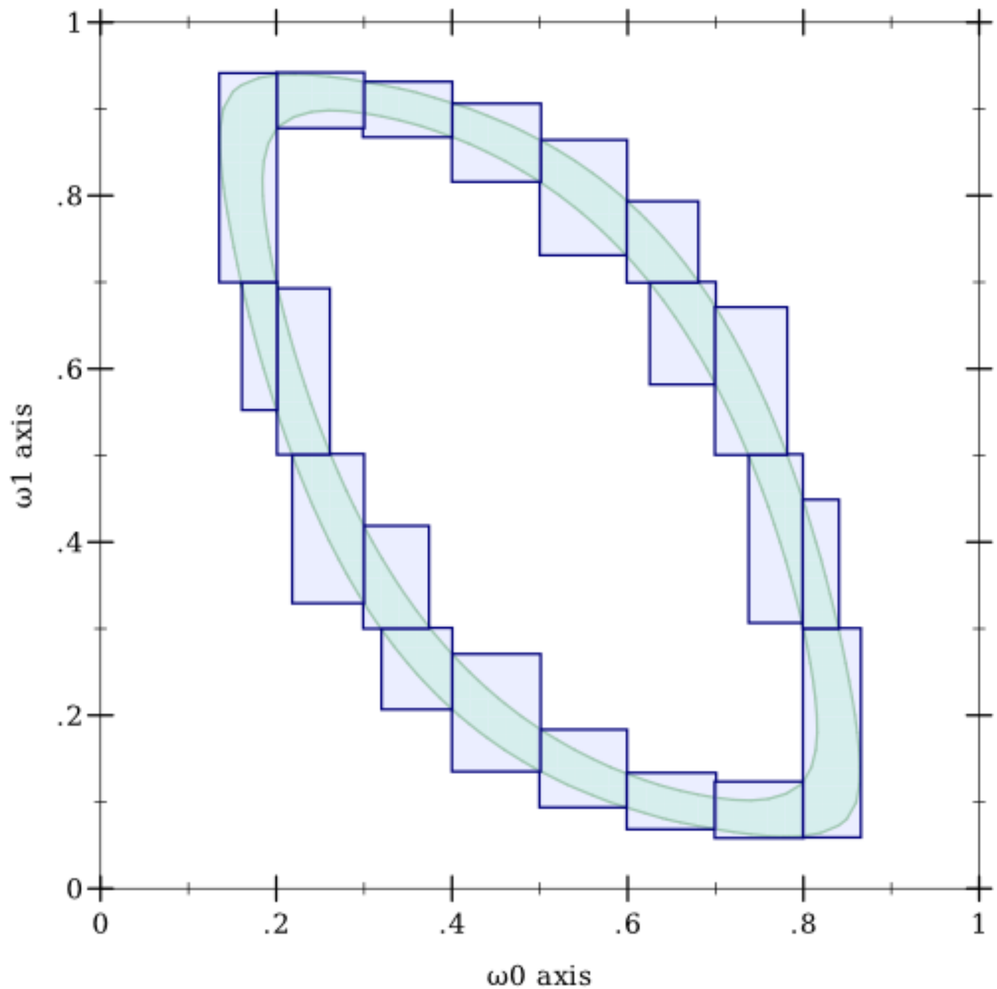

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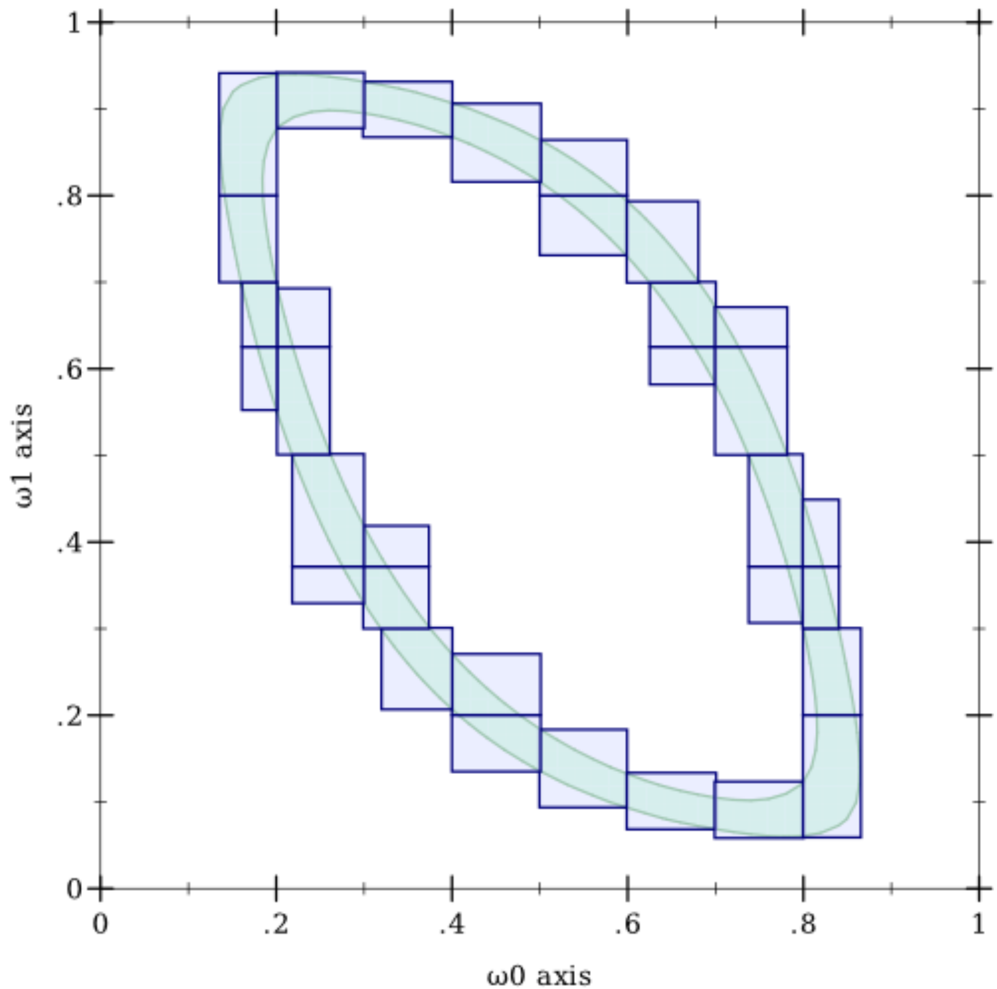

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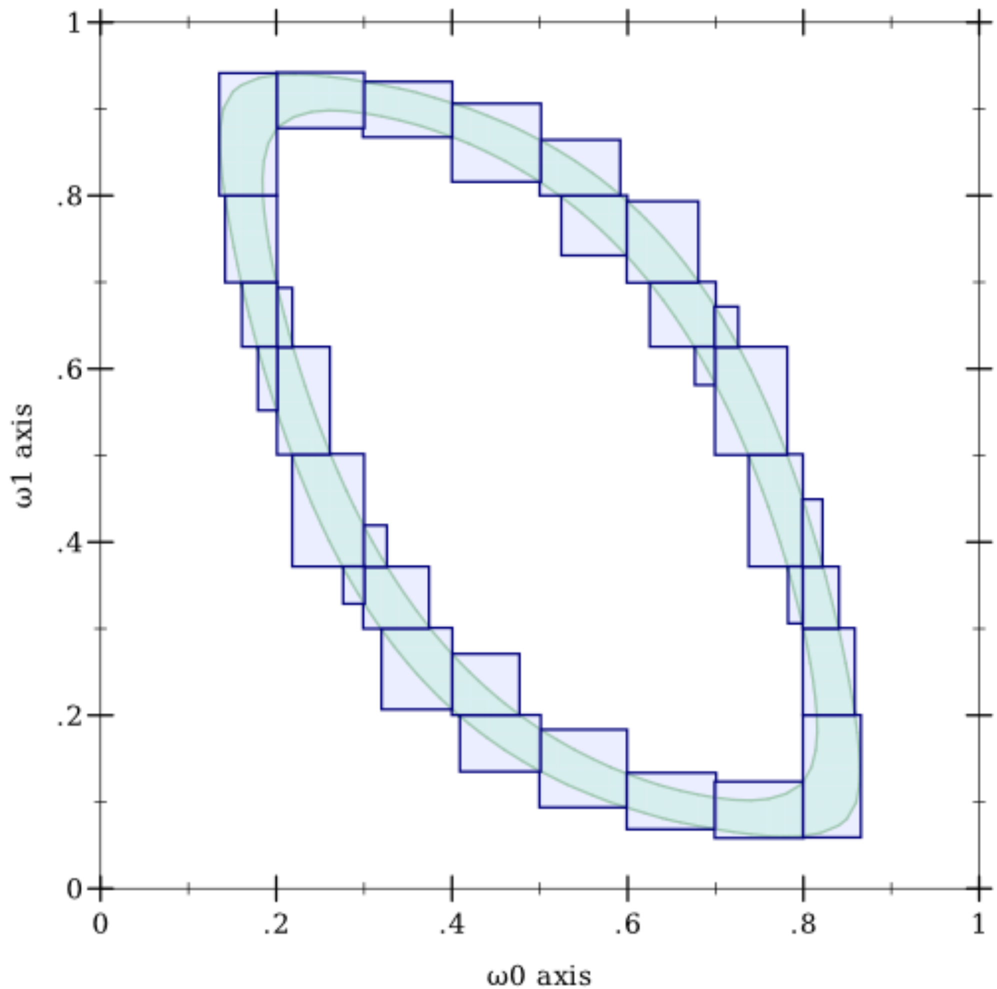

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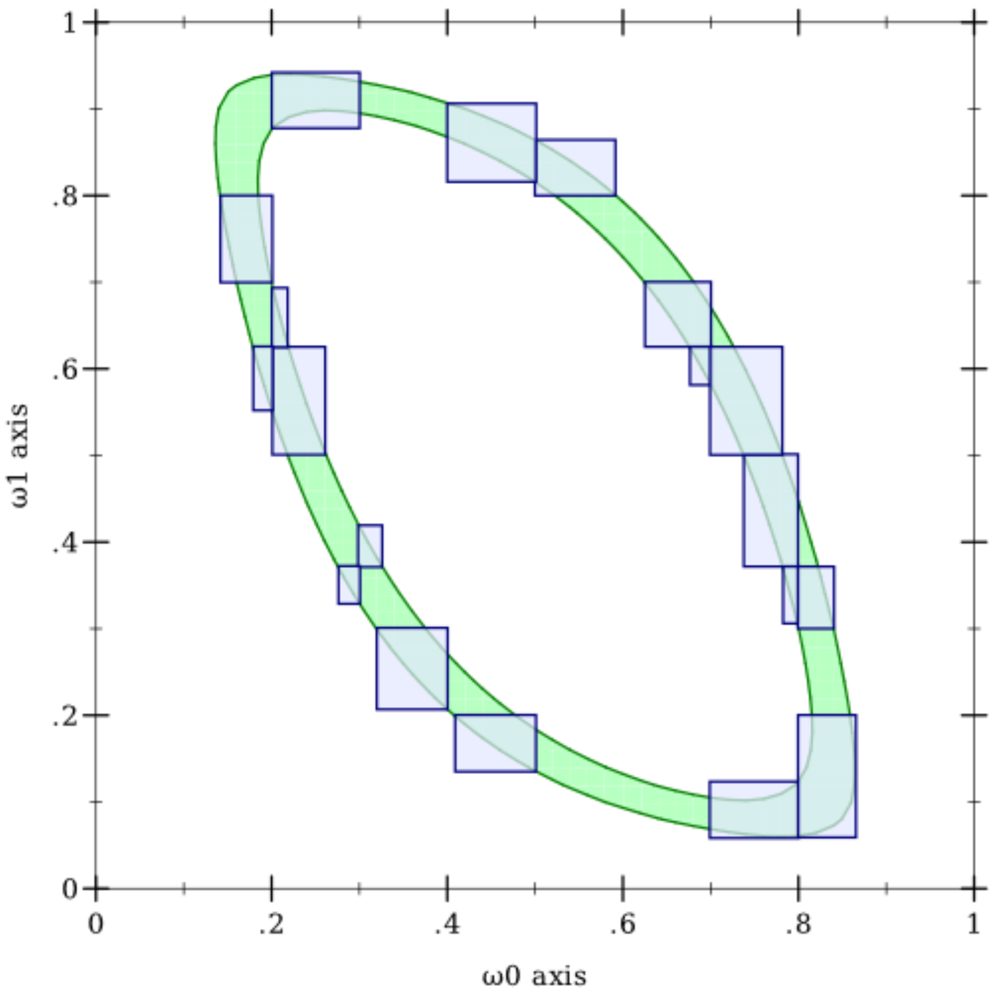

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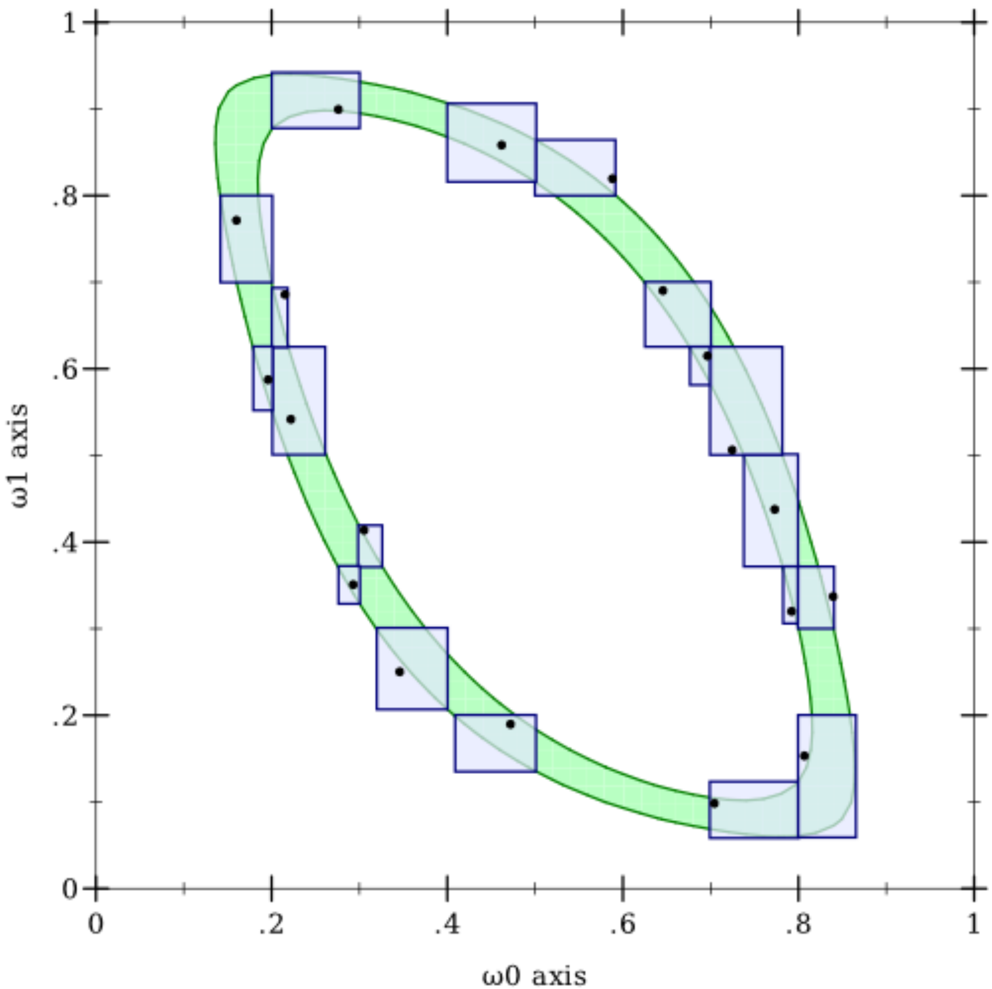

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- Standard interpretation of programs as pure functions from a random source
- Efficient way to compute preimage sets
- Efficient representation of arbitrary sets
- Efficient way to compute volumes of preimage sets

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  - Efficient way to determine whether a domain sample is actually in the preimage (just use standard interpretation)
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Standard Interpretation

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- Easy implementation in any language with lambdas

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- Can preimages be computed compositionally?

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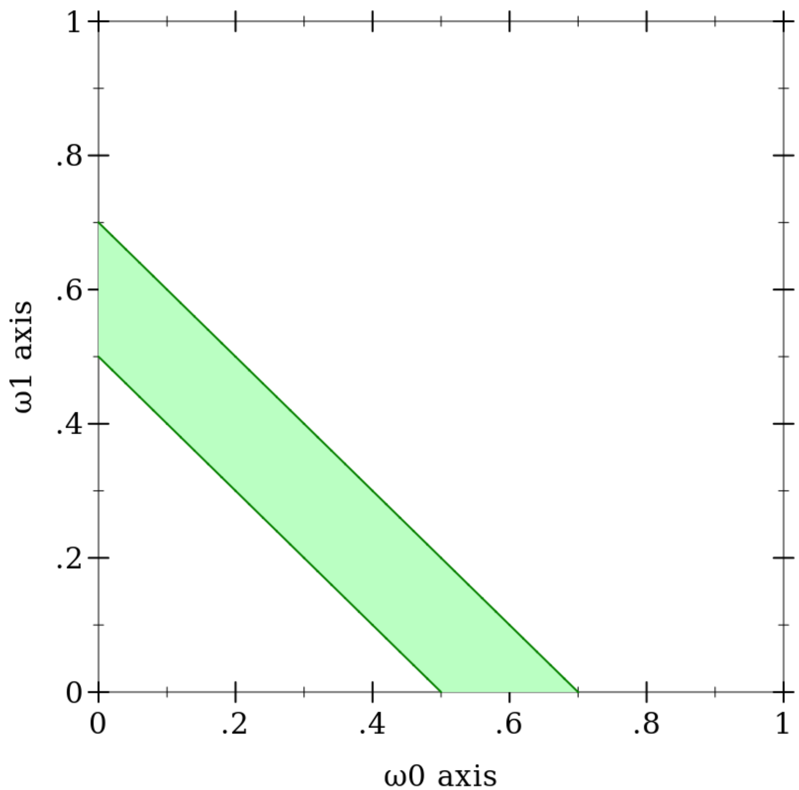

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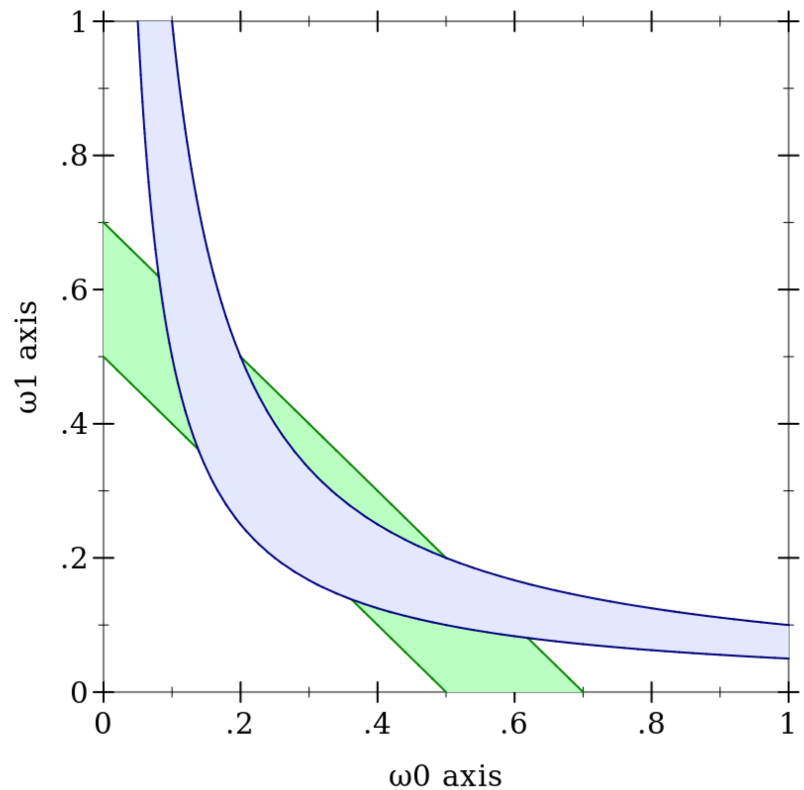
$f_1^{-1}([0.5, 0.7])$ and $f_2^{-1}([0.05, 0.1])$:



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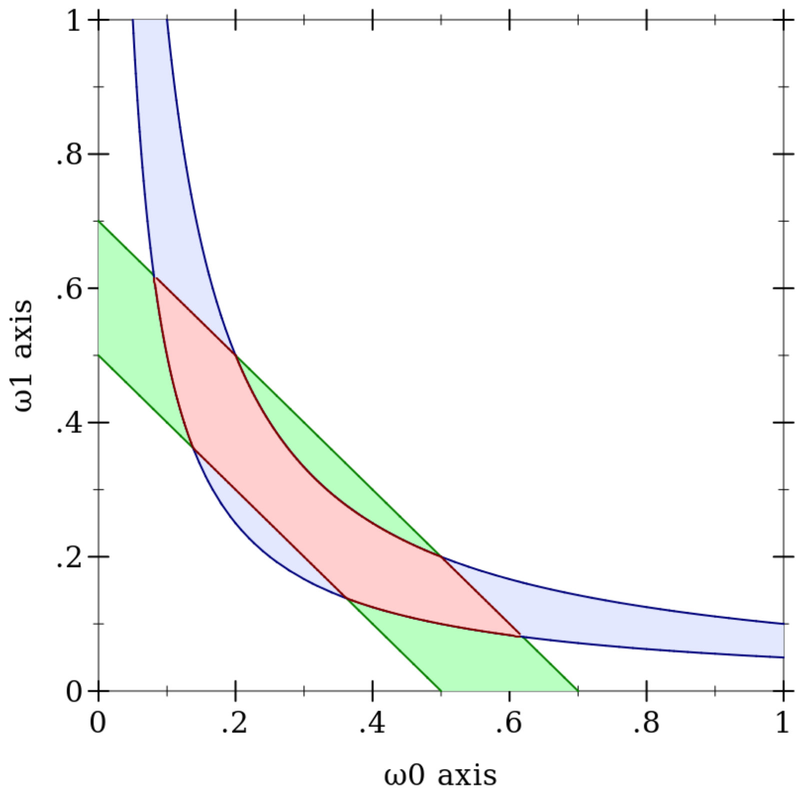
$f^{-1}([0.5, 0.7] \times [0.05, 0.1])$:



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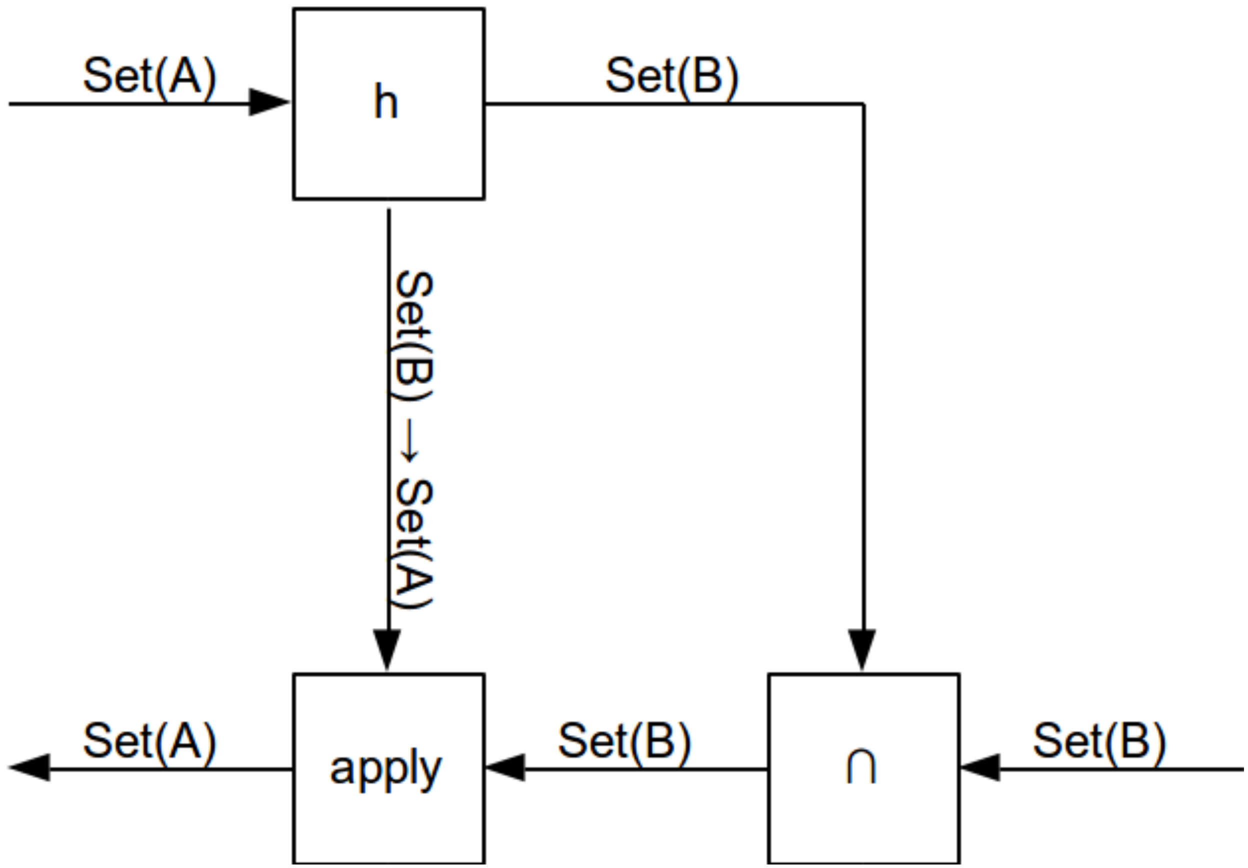

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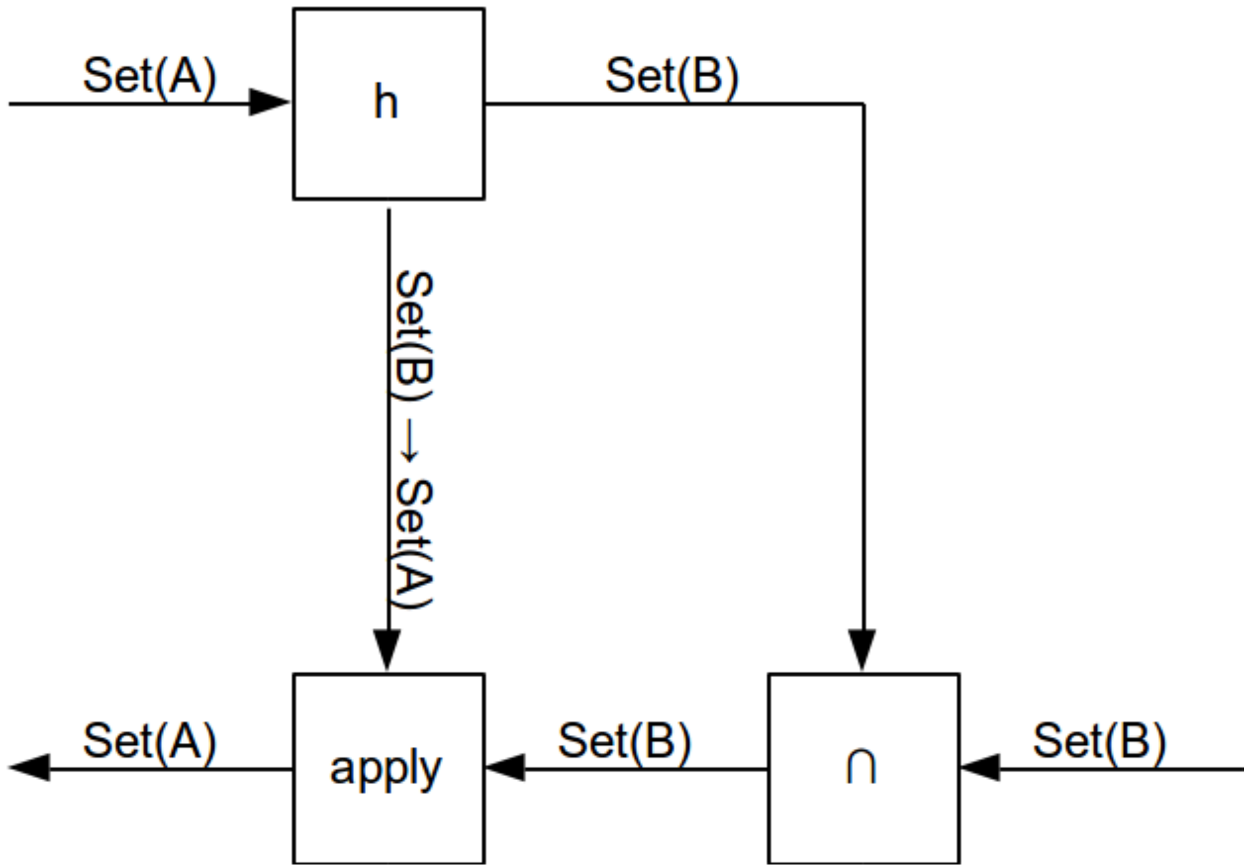

$$A \underset{\text{pre}}{\rightsquigarrow} B ::= \mathrm{Set}(A) \to \langle \mathrm{Set}(B), \mathrm{Set}(B) \to \mathrm{Set}(A) \rangle$$

Nonstandard Interpretation: Preimages Under Pairing

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Proof sketch. Preimages distribute over cartesian products.

- Similar theorems for every kind of term

Nonstandard Interpretation: Correctness

Theorem. For all programs p, $[\![p]\!]_{\mathrm{pre}}$ computes preimages under $[\![p]\!]$.

Proof. By structural induction on program terms.

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- Q. Why interpret programs as uncomputable functions, then?
  - A. So we know exactly what to approximate.
- Q. Where did you get a λ-calculus that could operate on arbitrary, possibly infinite sets, anyway?
  - A. Well...

Lambda-ZFC

λ calculus

Lambda-ZFC

+

λ calculus

Infinite sets and operations

Lambda-ZFC

+

=

λ calculus

Infinite sets and operations

λ_{ZFC}

Lambda-ZFC

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- Contemporary math, but with lambdas and general recursion; or functional programming, but with infinite sets

Lambda-ZFC

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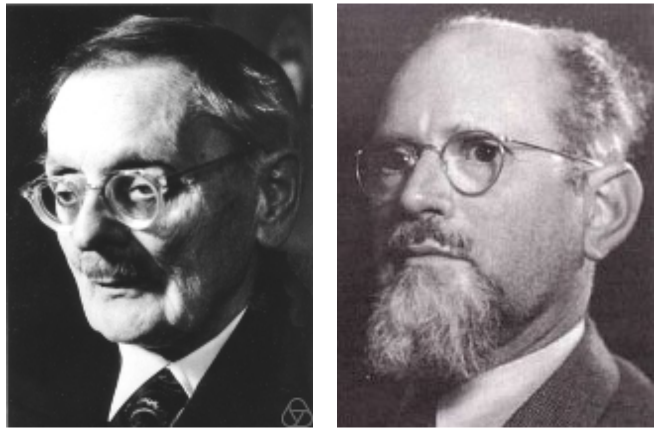

Infinite sets and operations

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Infinite sets and operations

λ_{ZFC}

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- Can express uncountably infinite operations, can't solve its own halting problem
- Can use contemporary mathematical theorems directly

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  - An interval or union of intervals
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- Derive $[\![\cdot]\!]'_{\text{pre}} : p \to (\Omega \underset{\text{pre}}{\rightsquigarrow}' B)$

In Theory...

Theorem (sound). $[\![\cdot]\!]'_{\mathrm{pre}}$ computes overapproximations of the preimages computed by $[\![\cdot]\!]_{\mathrm{pre}}$.

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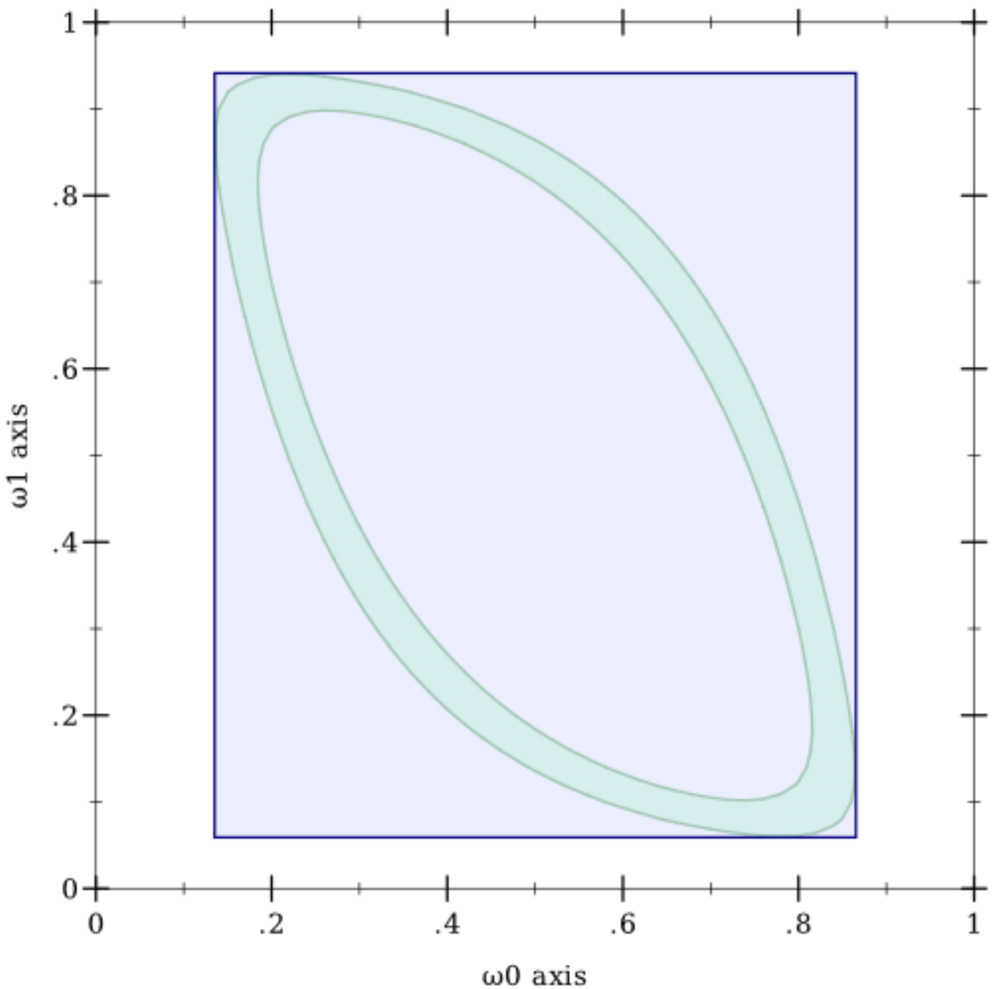
Theorem (decreasing). $[\![\cdot]\!]'_{\text{pre}}$ never returns preimages larger than the given subdomain.

- Consequence: Refining preimage partitions never explodes

In Practice...

Theorems prove this always works:



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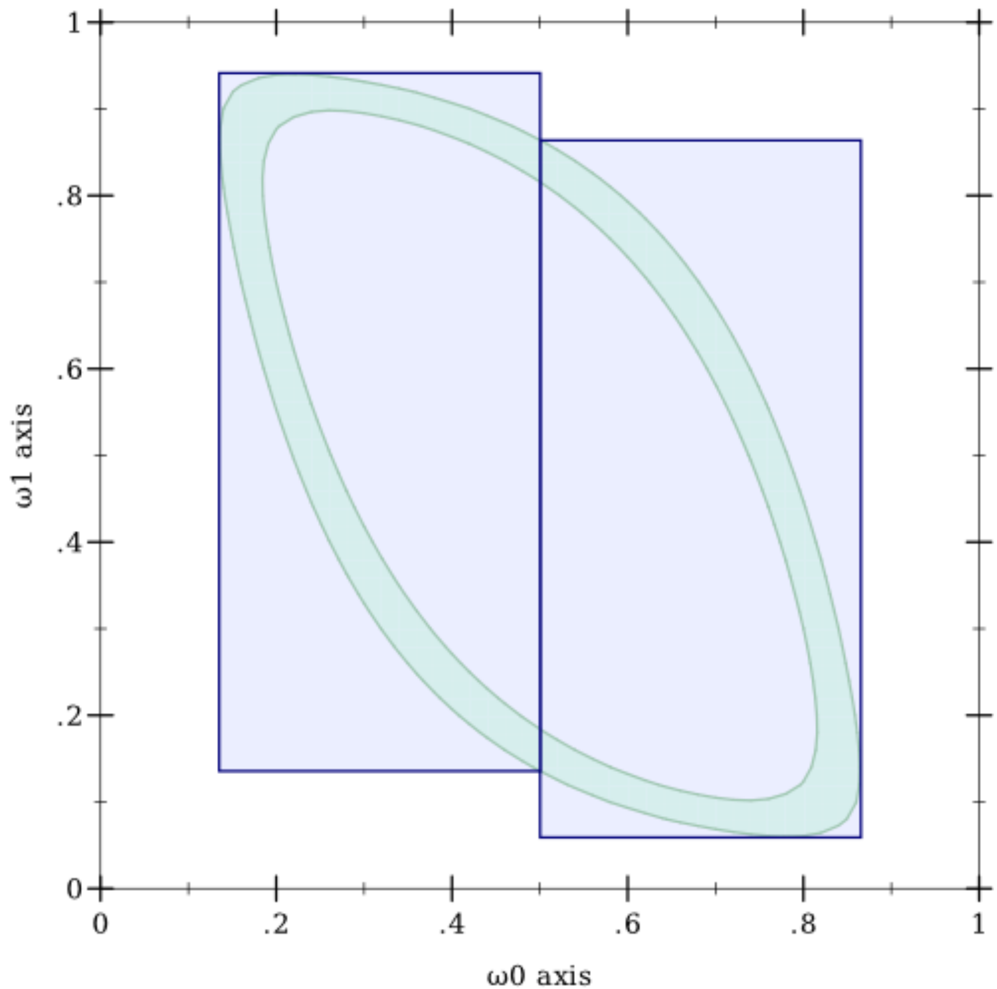

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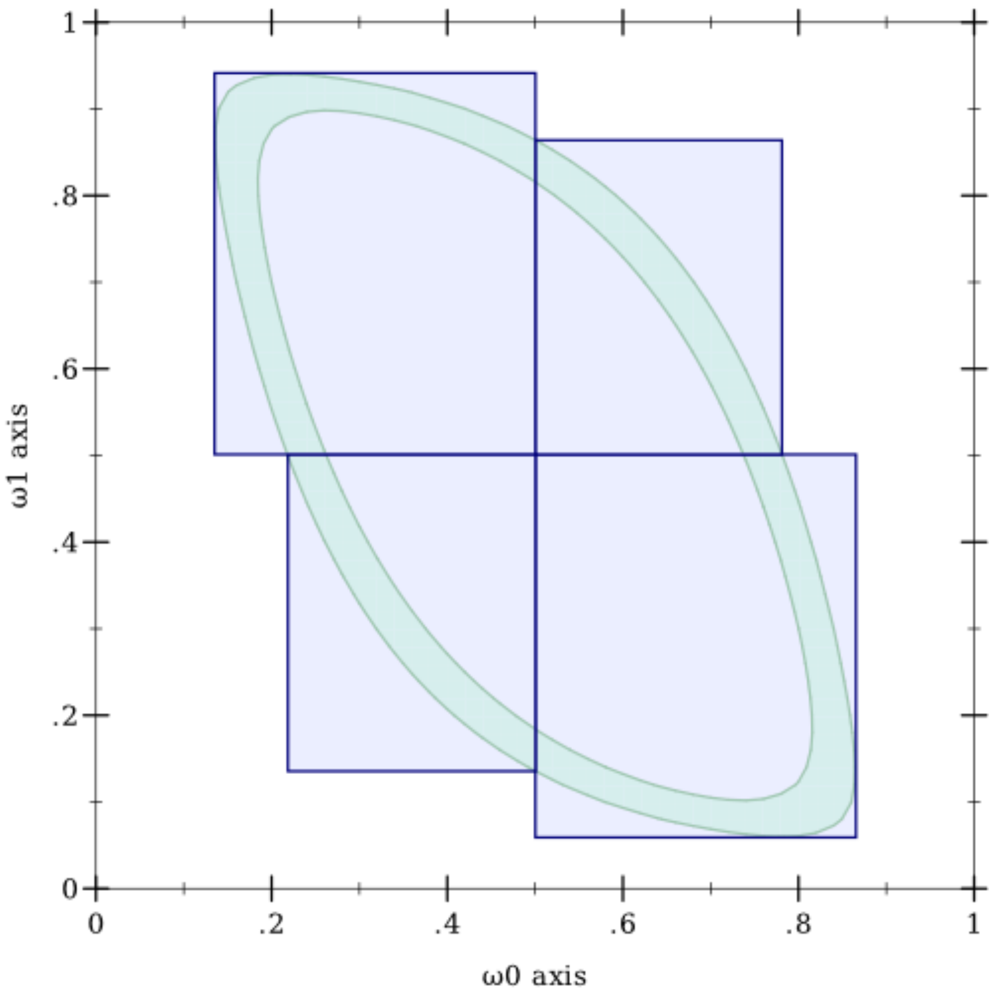

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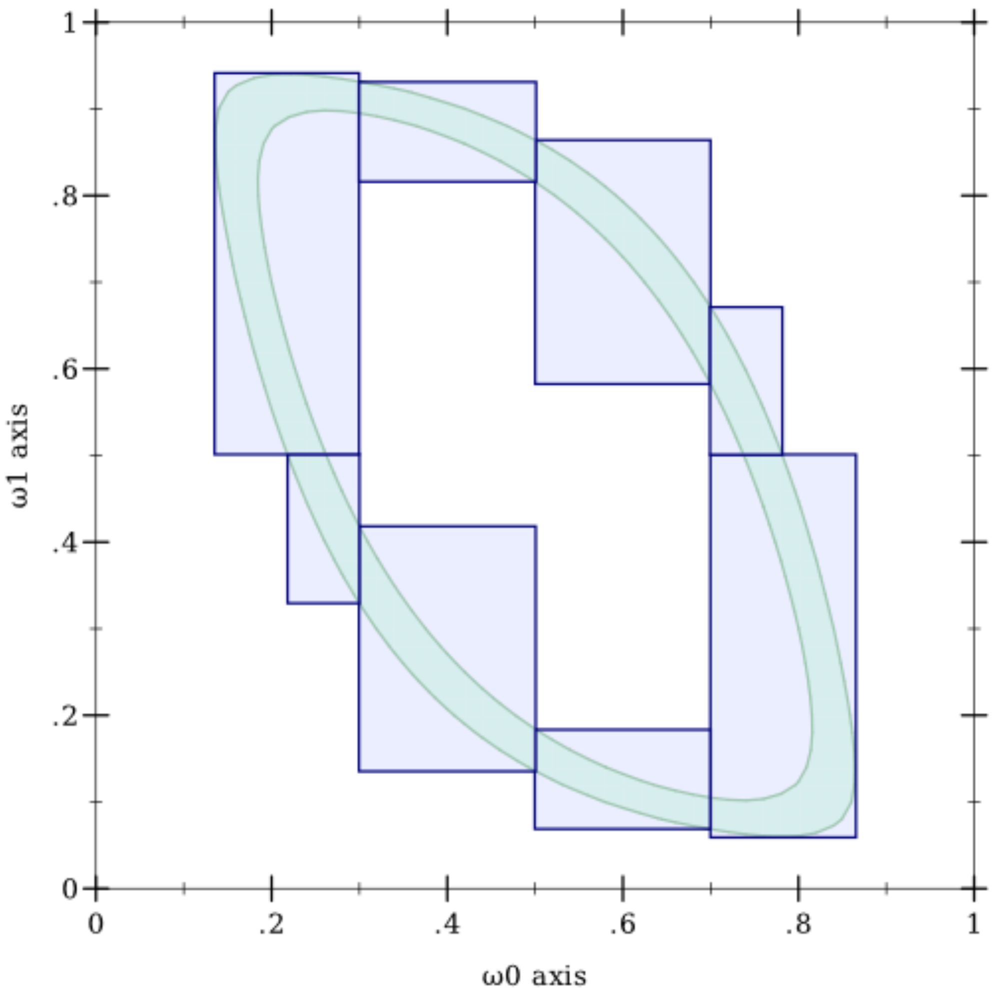

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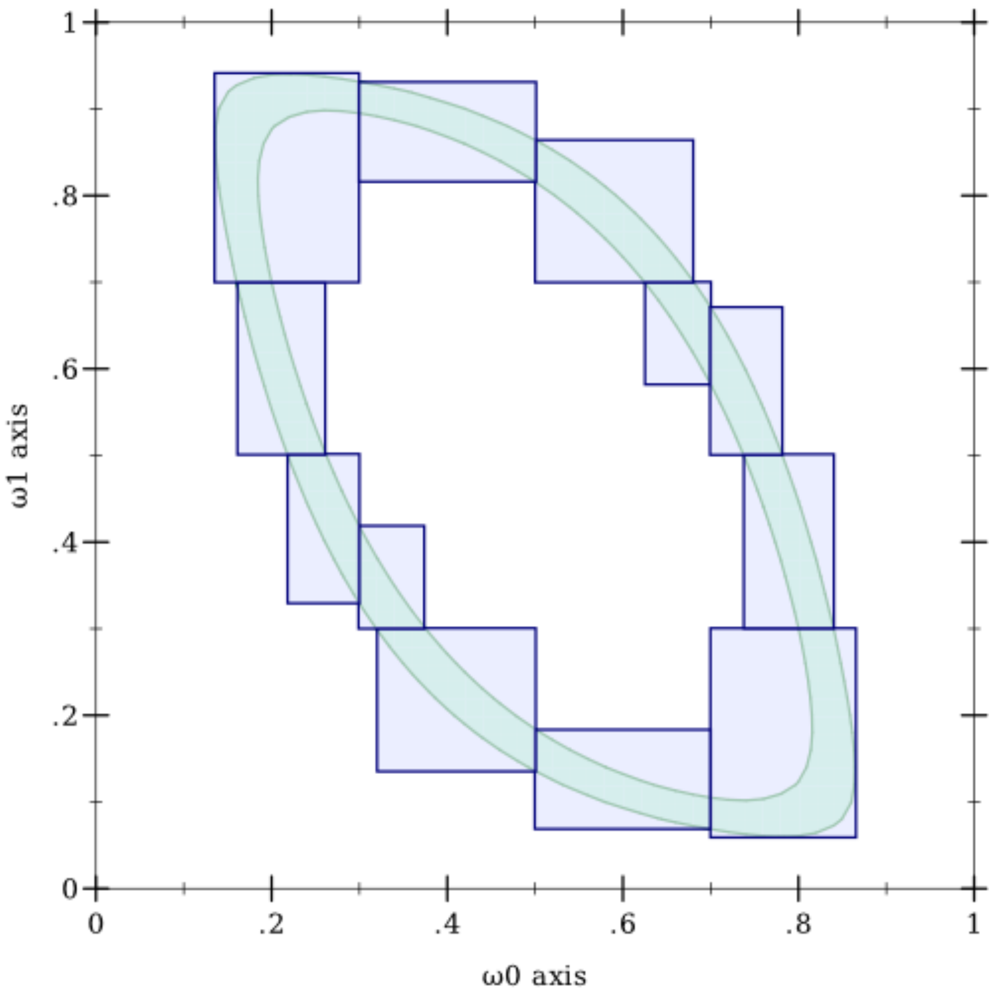

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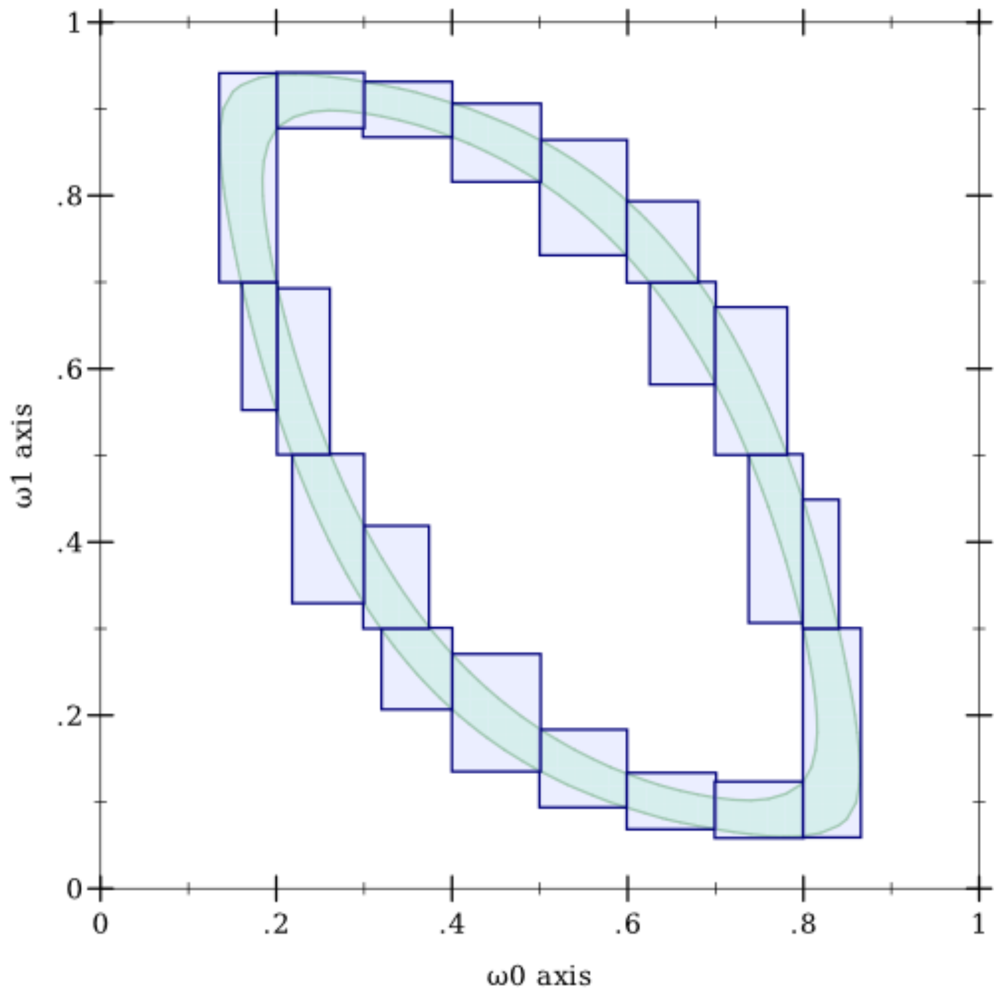

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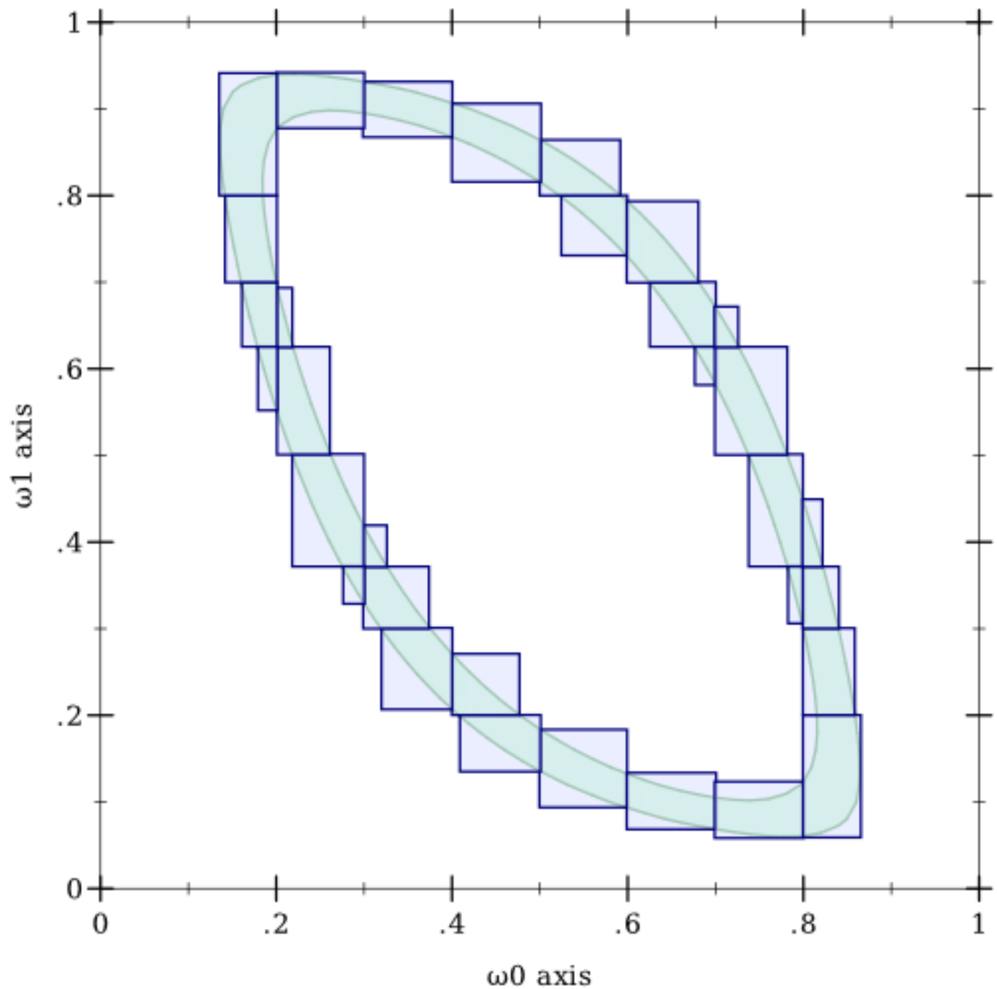

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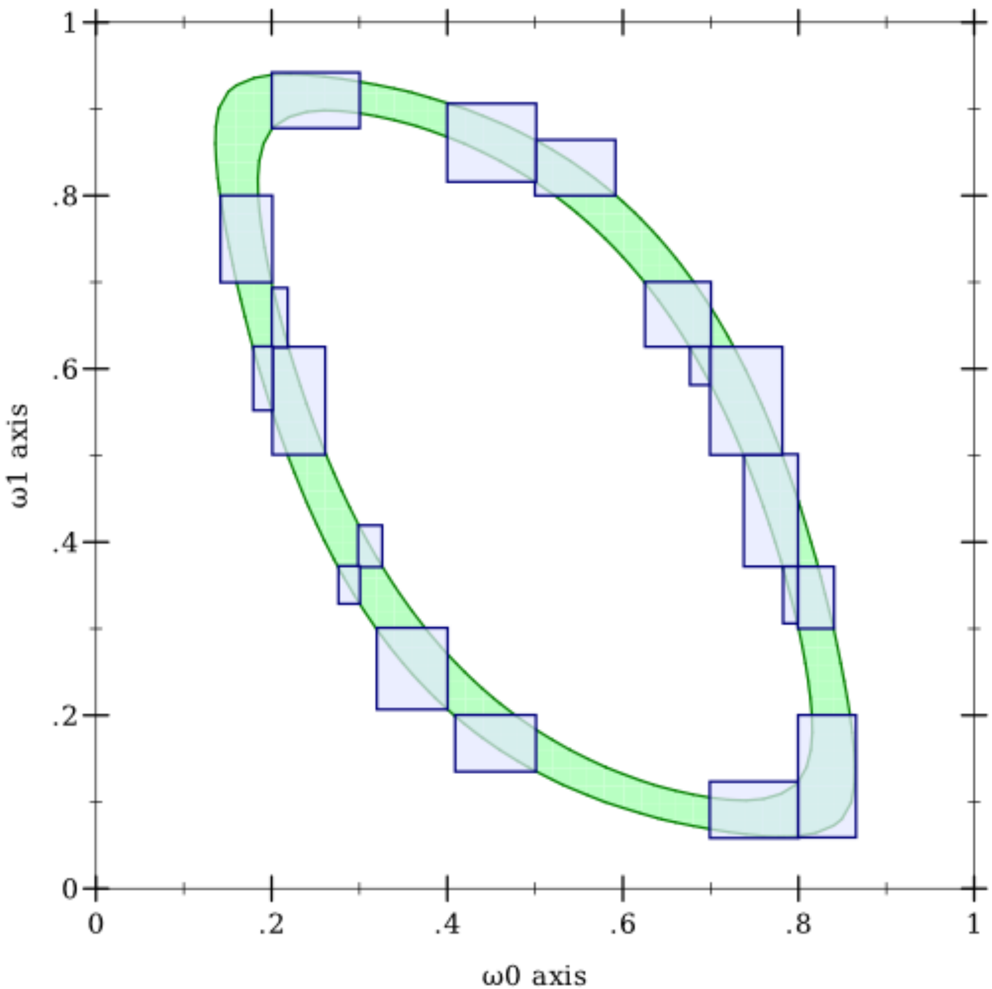

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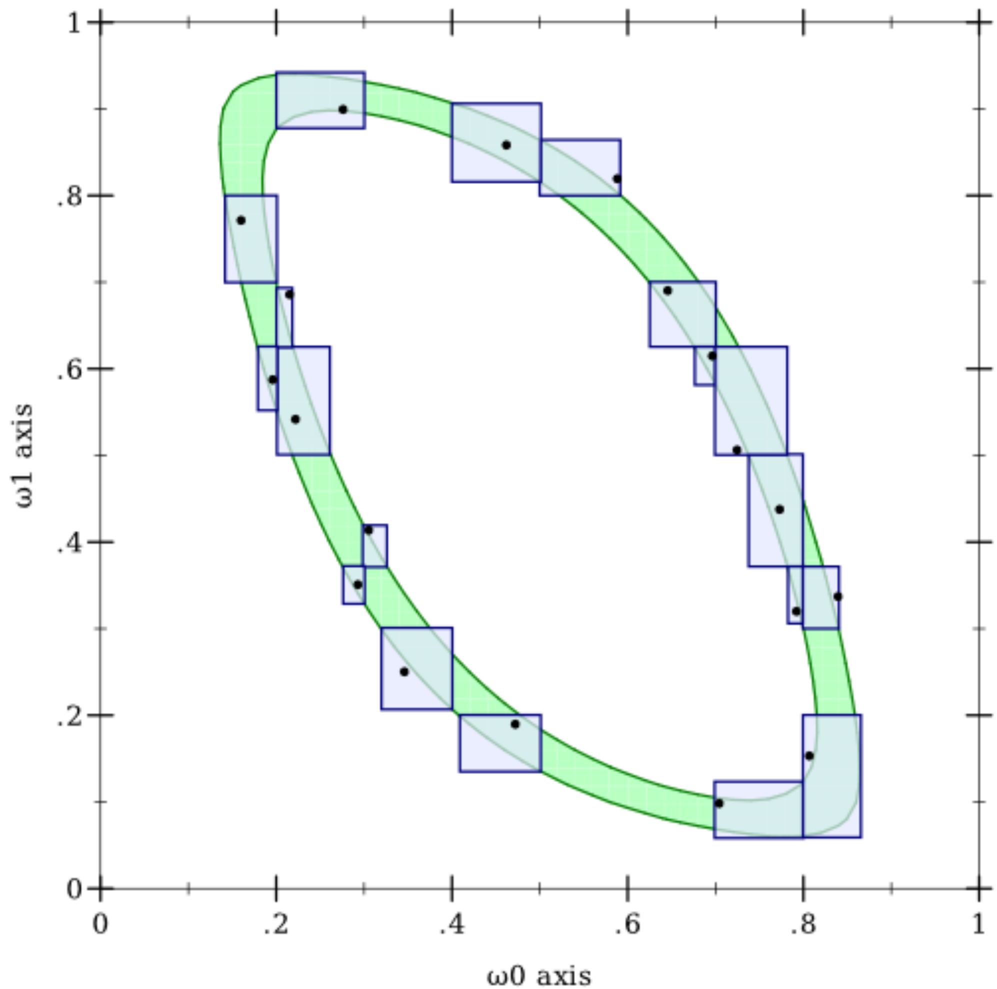

Importance Sampling

- Alternative to arbitrarily low-rate rejection sampling:

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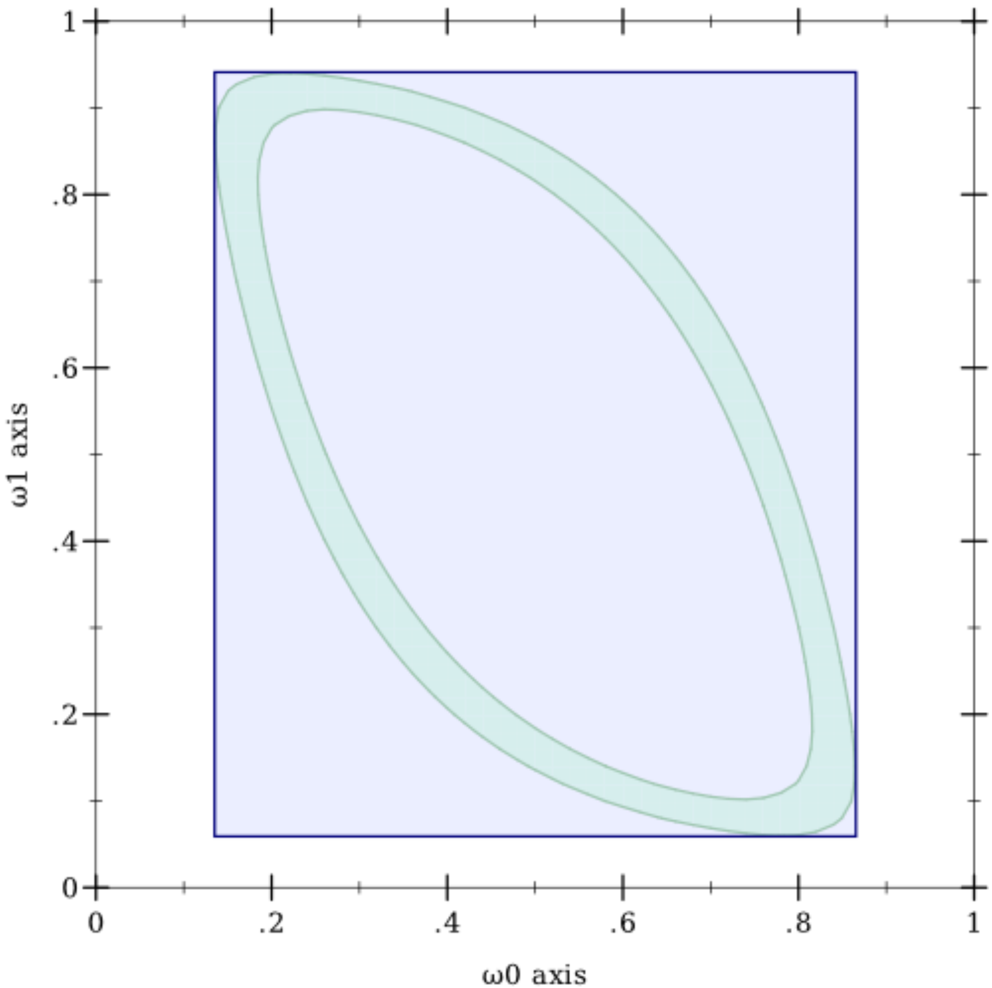
First, refine using preimage computation:



Importance Sampling

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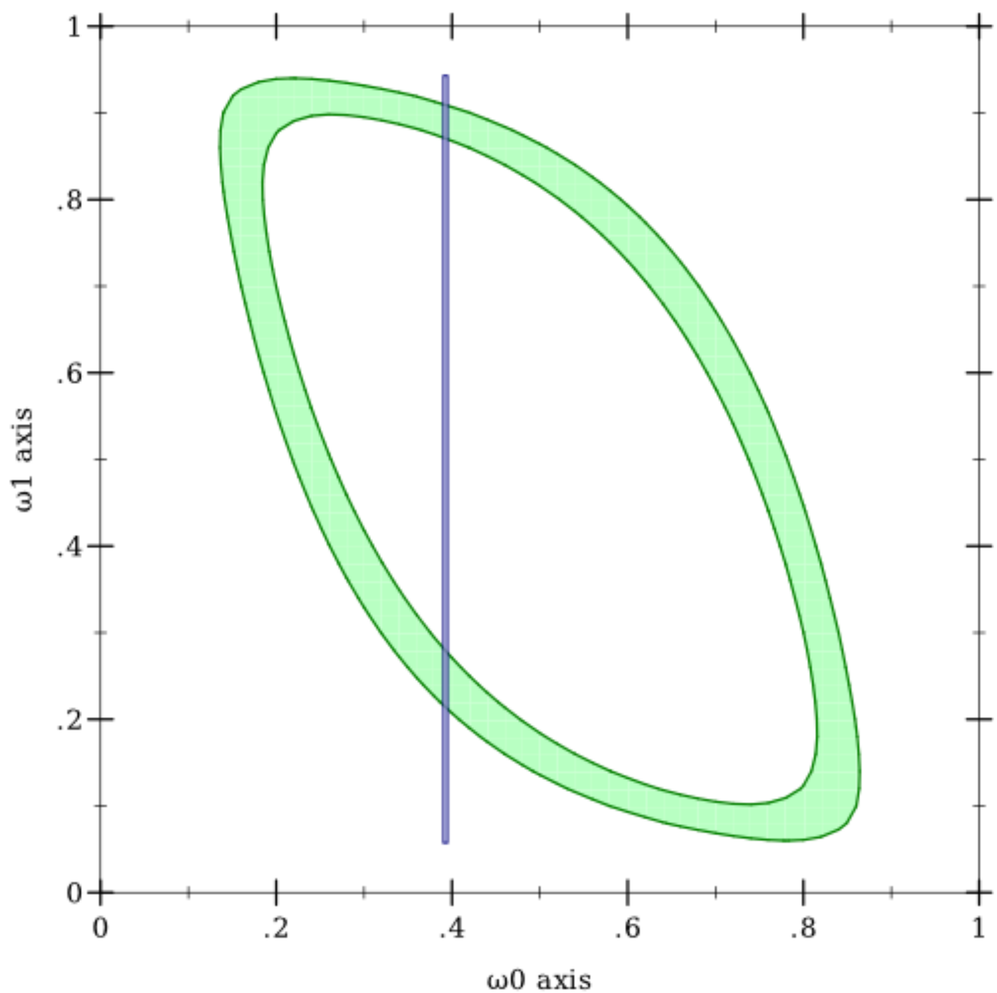
Second, randomly choose from arbitrarily fine partition:



Importance Sampling

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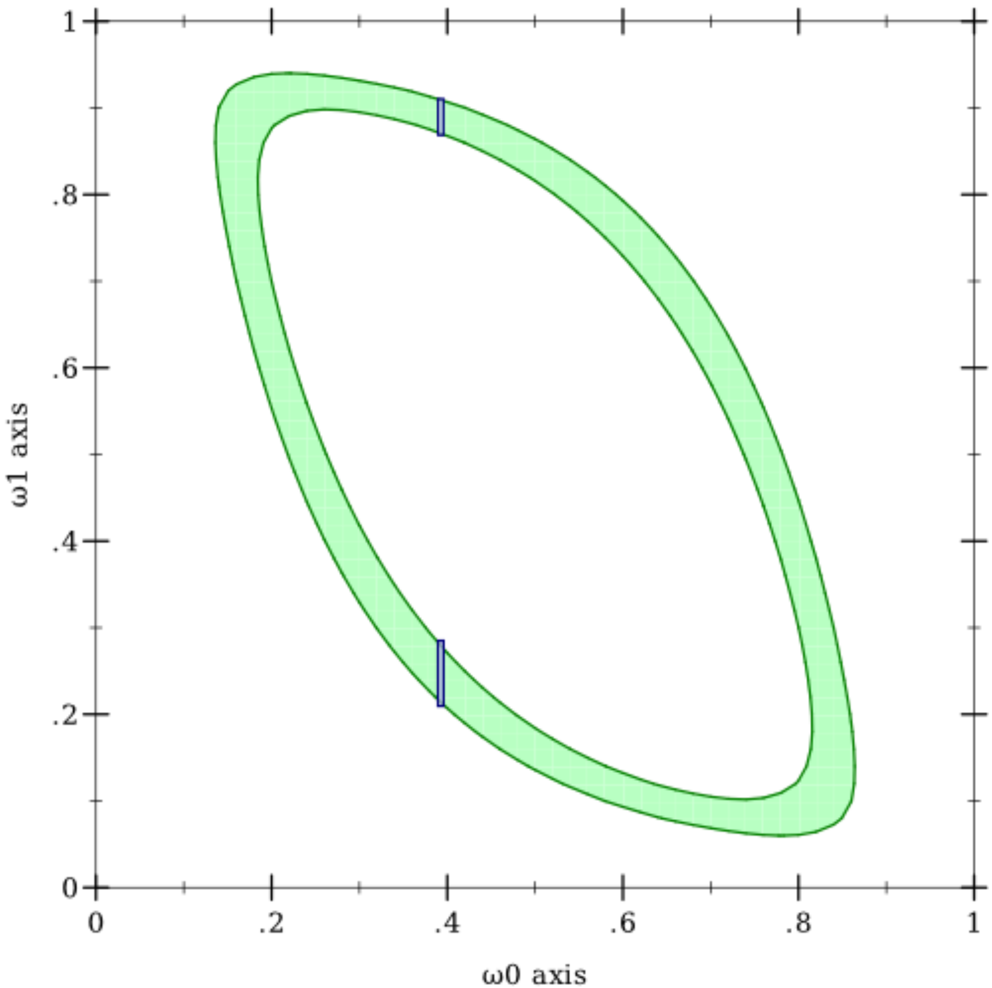
Third, refine again:



Importance Sampling

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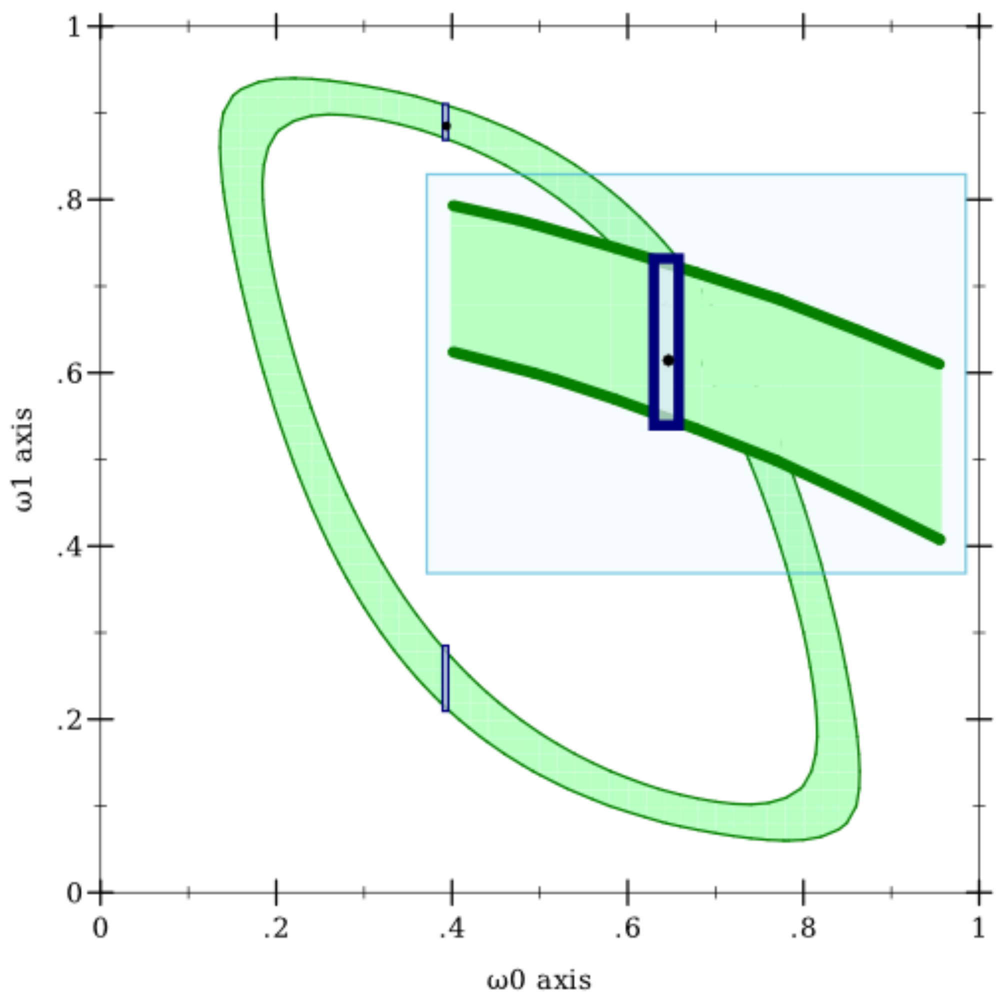
Fourth, sample uniformly:



Importance Sampling

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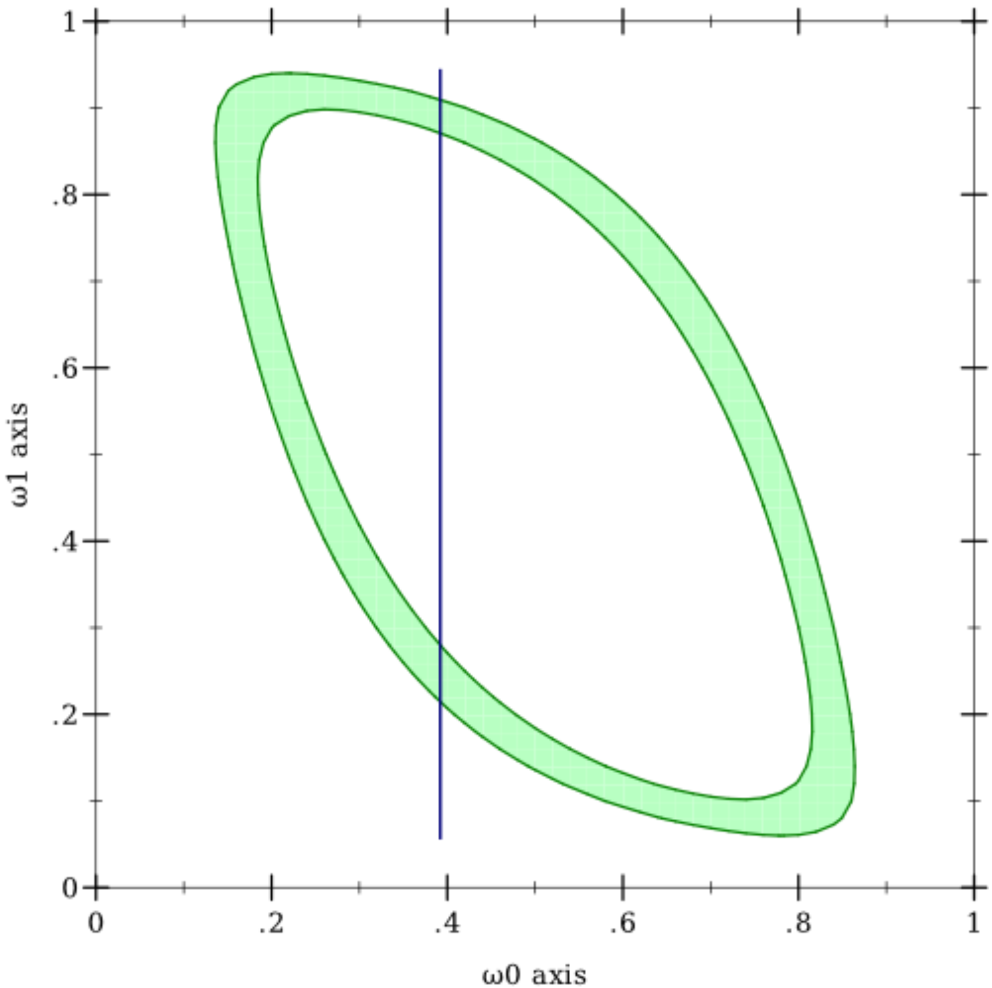
Do process "in the limit"; i.e. choose $[\omega_0, \omega_0] \times \Omega_1$:



What About Recursion?

- General recursion, programs that halt with probability 1; e.g.

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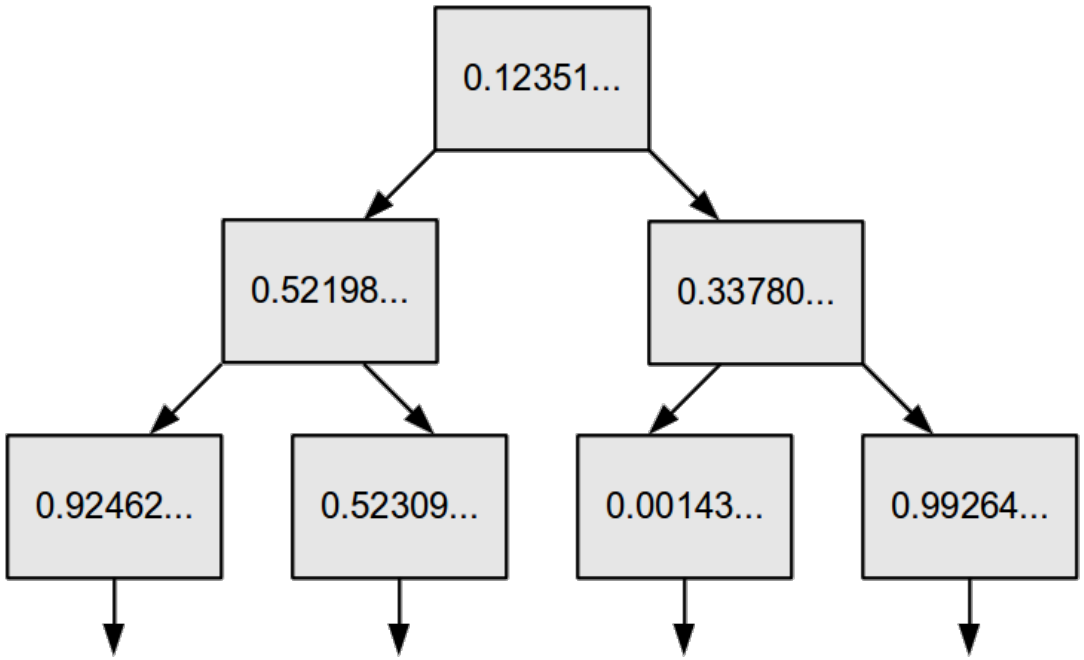
- Random domain needs to be big enough and the right shape

Program Domain Values

- Values $\omega \in \Omega$ are infinite binary trees:



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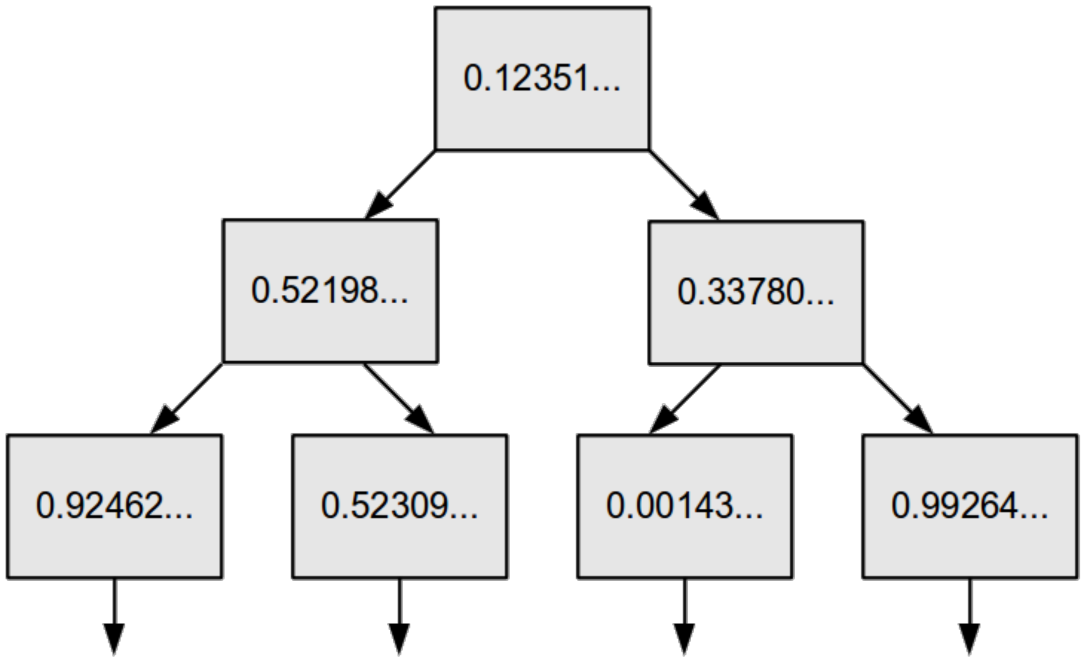

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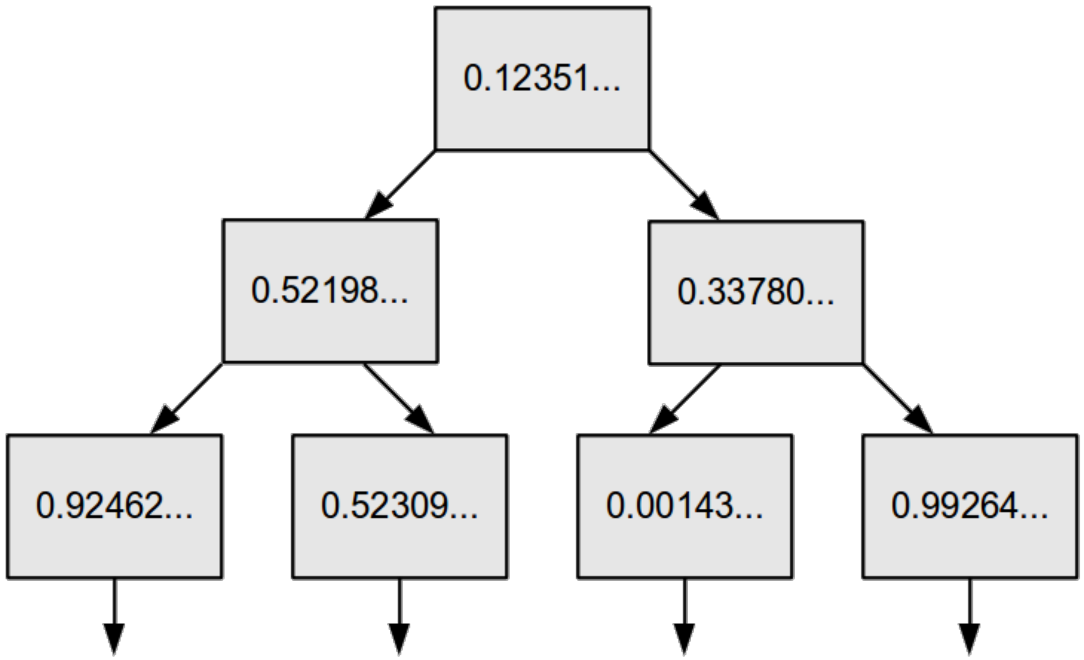

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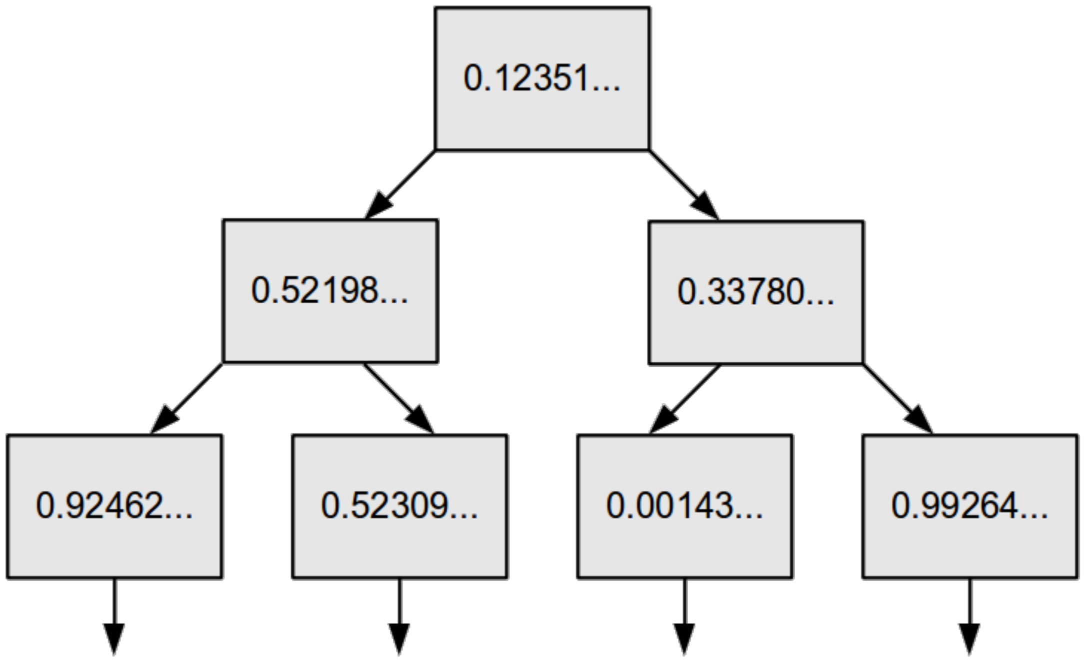

- Every expression in a program is assigned a node
- Implemented using lazy trees of random values
- No probability *density* for domain, but there is a *measure*

Demo: Normal-Normal With Circular Condition

- Normal-Normal process:

$$X \sim \mathrm{Normal}(0, 1)$$
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- Goal: Sample in the preimage of

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(set-list reals reals (interval (- 1 ε) (+ 1 ε)))
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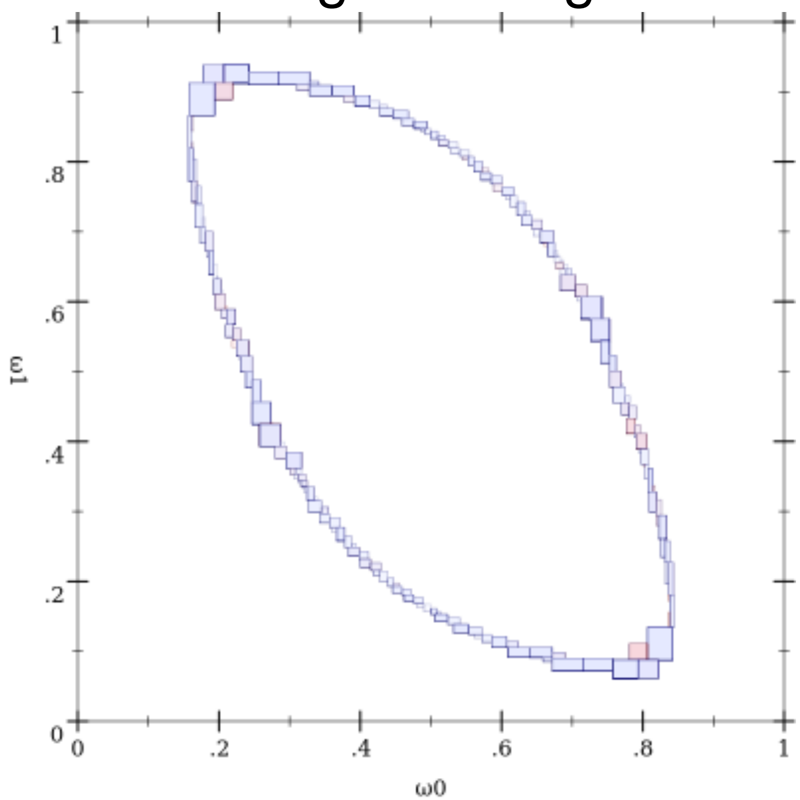
Demo: Normal-Normal With Circular Condition

For ε = 0.01:

Preimage rectangles



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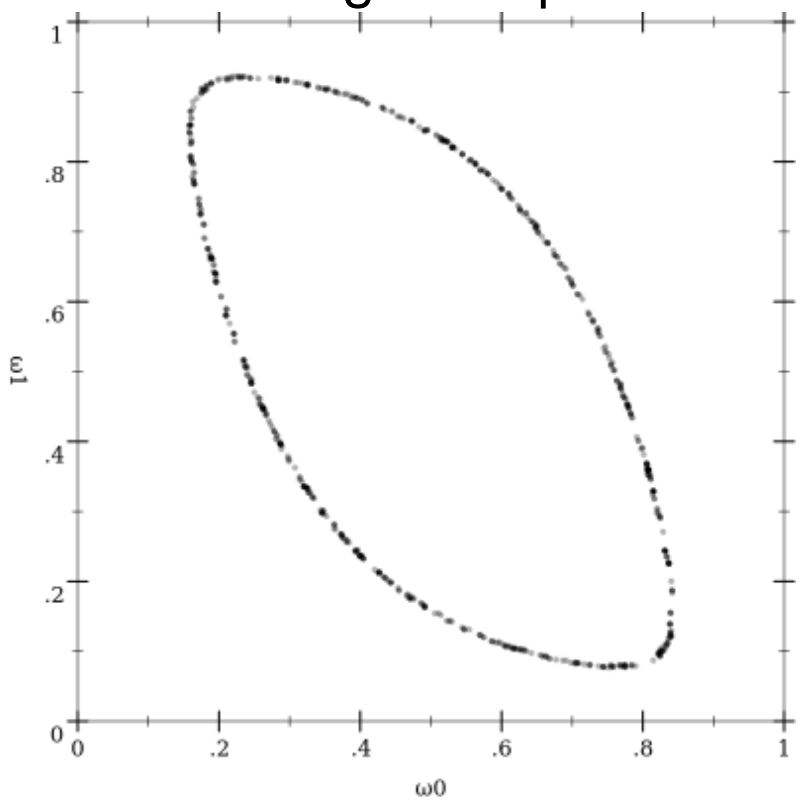

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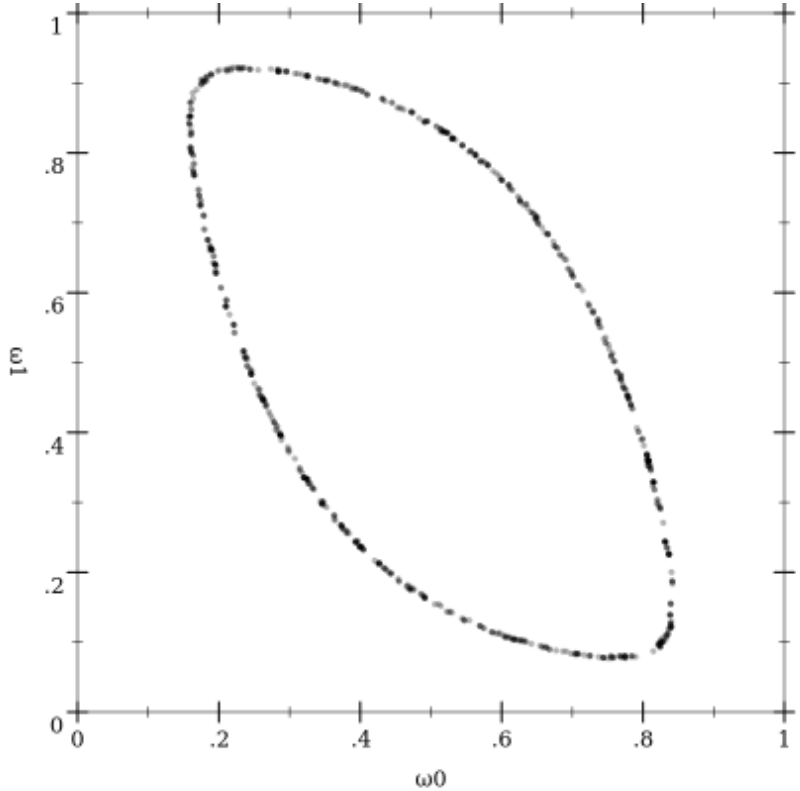
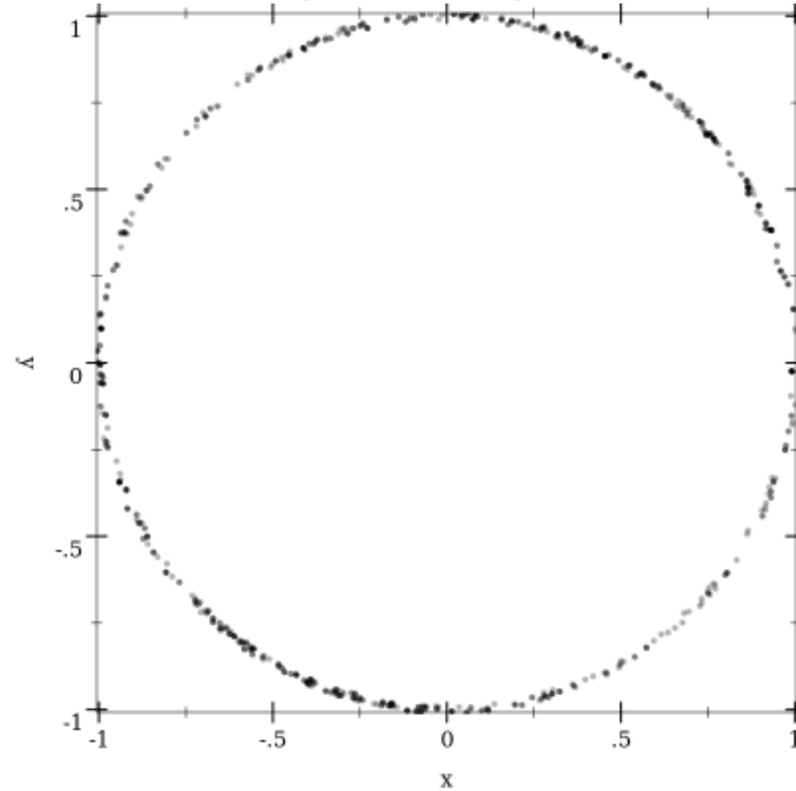


Output samples



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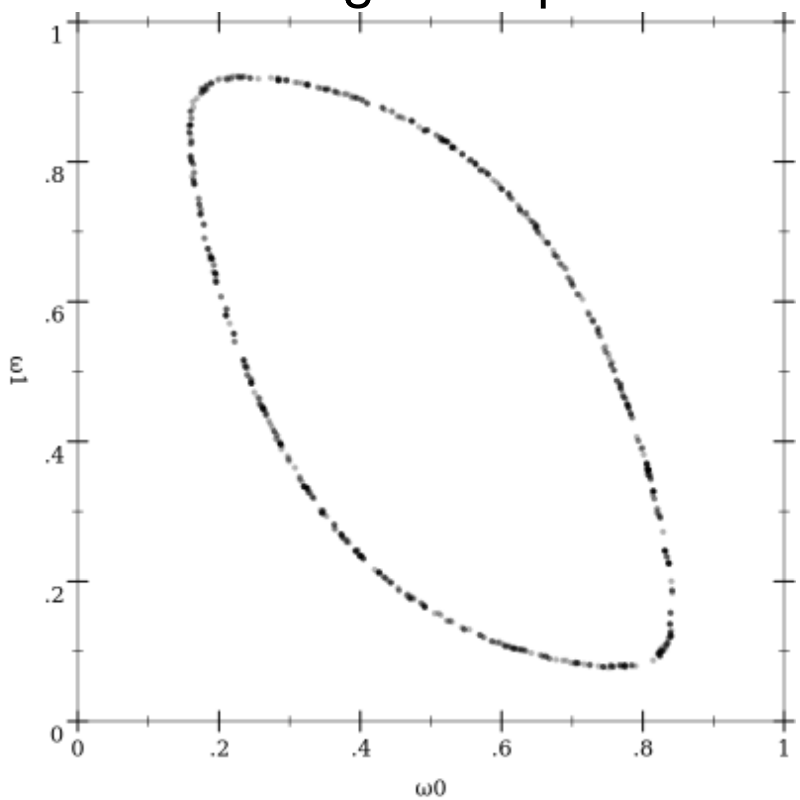

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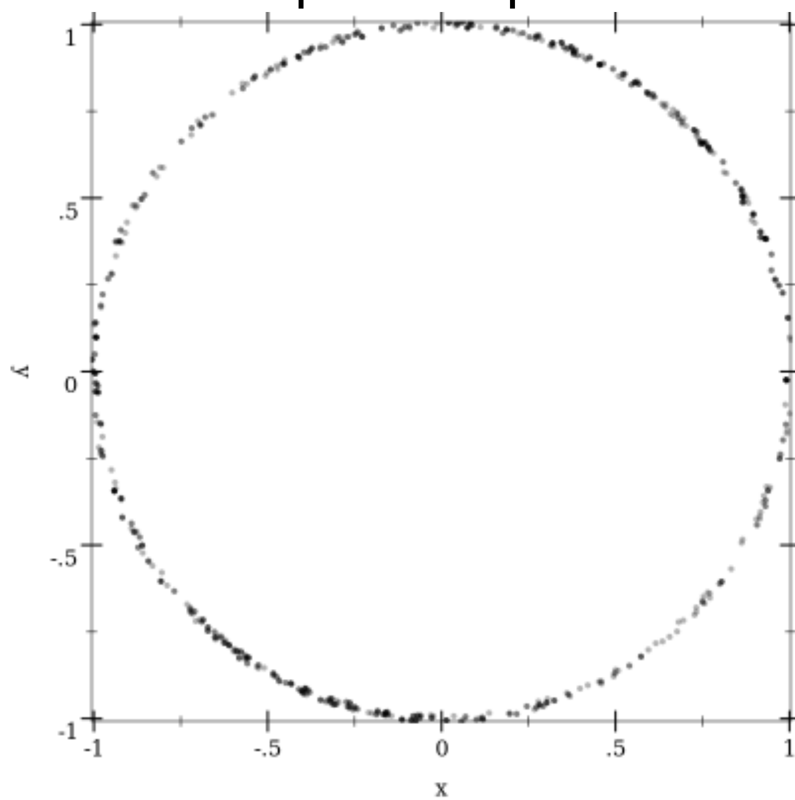

- Works fine with much smaller ε

Demo: Thermometer

- Normal-Normal thermometer process:

$$
\begin{aligned}
X &\sim \mathrm{Normal}(90, 10) \\
Y &\sim \mathrm{Normal}(X, 1) \\
Y' &= \min(100, Y)
\end{aligned}
$$

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- Implementation:

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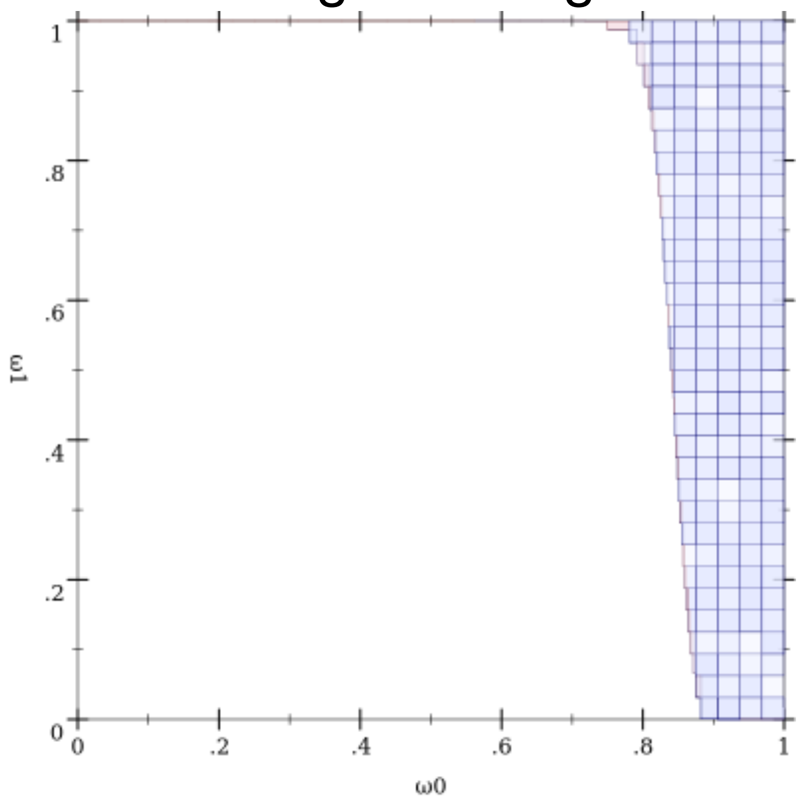

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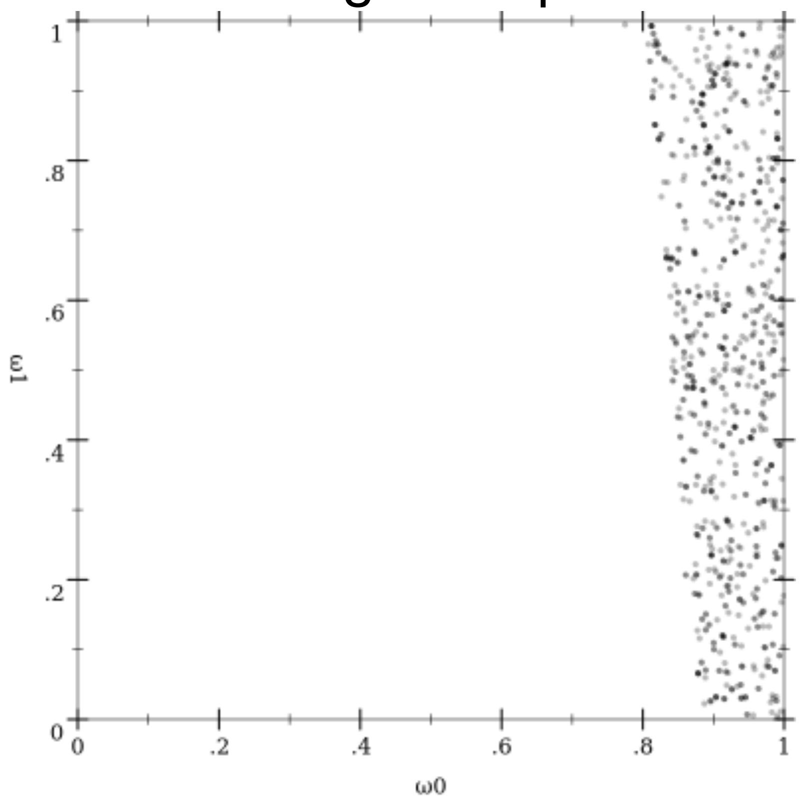

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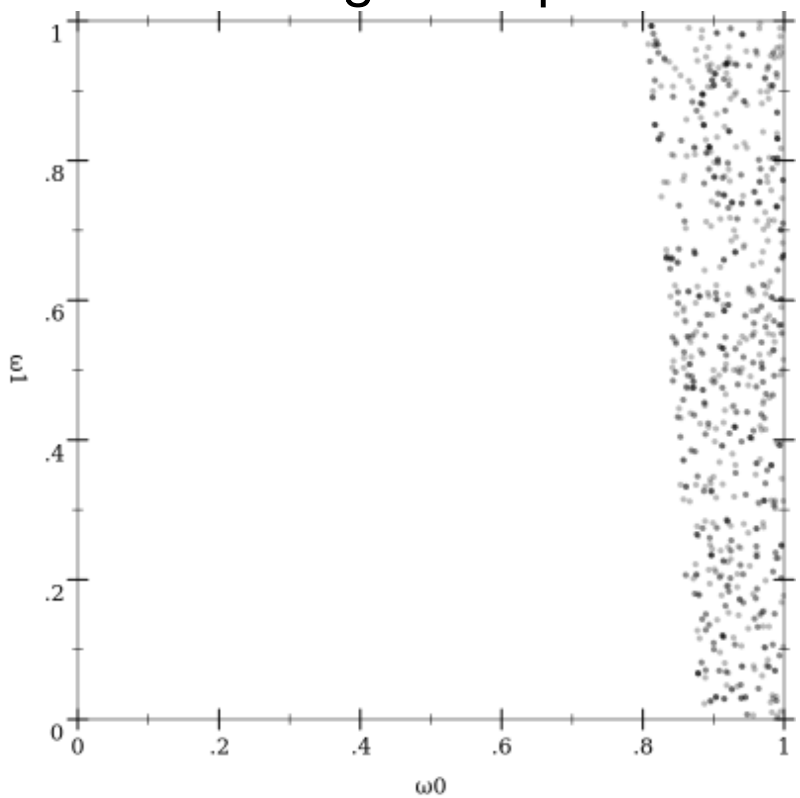
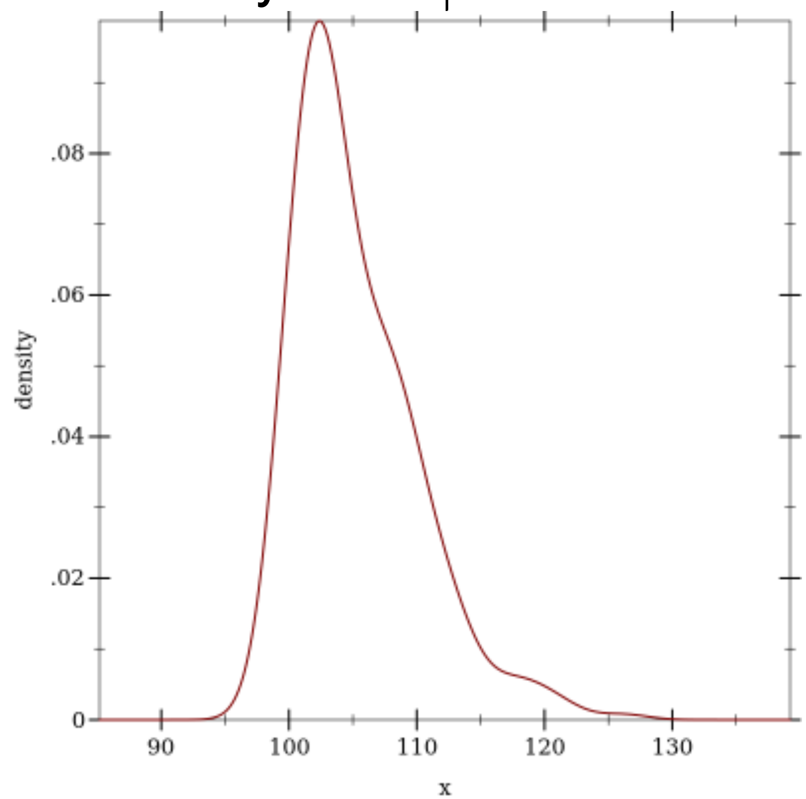
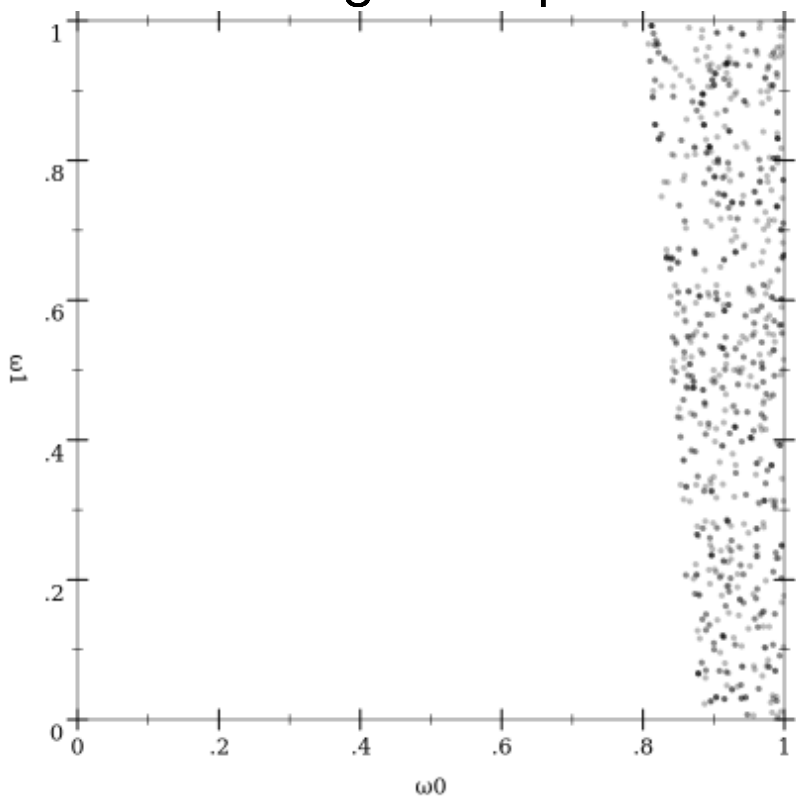


Density of $X \mid Y' = 100$

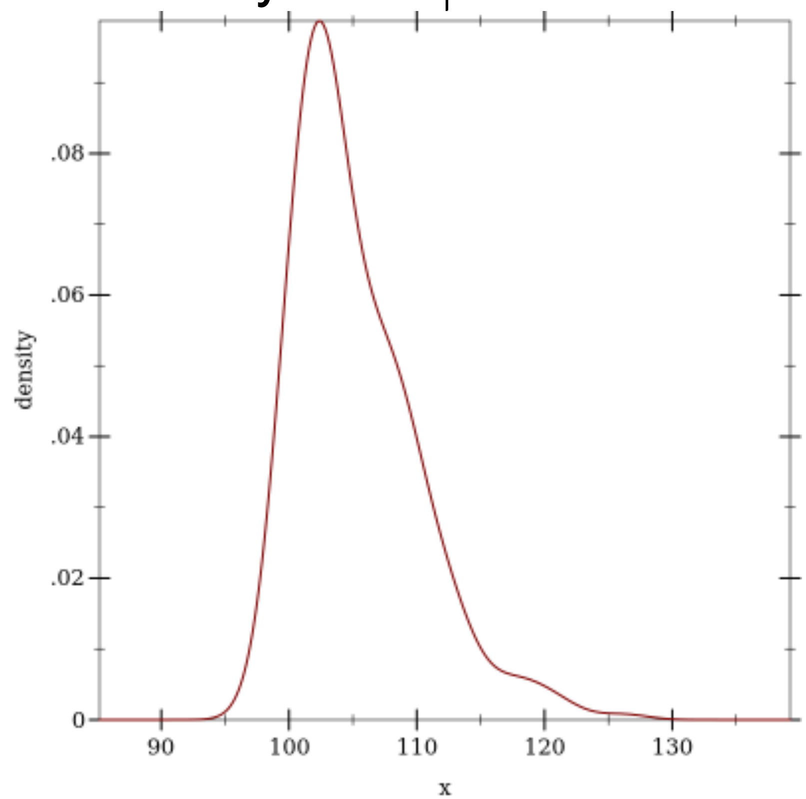


Demo: Thermometer

Preimage samples



Density of $X \mid Y' = 100$



Calculated from samples: mean 105.1, stddev 4.6

Demo: Stochastic Ray Tracing

- Idea: Model light transmission and reflection, condition on paths that pass through aperture

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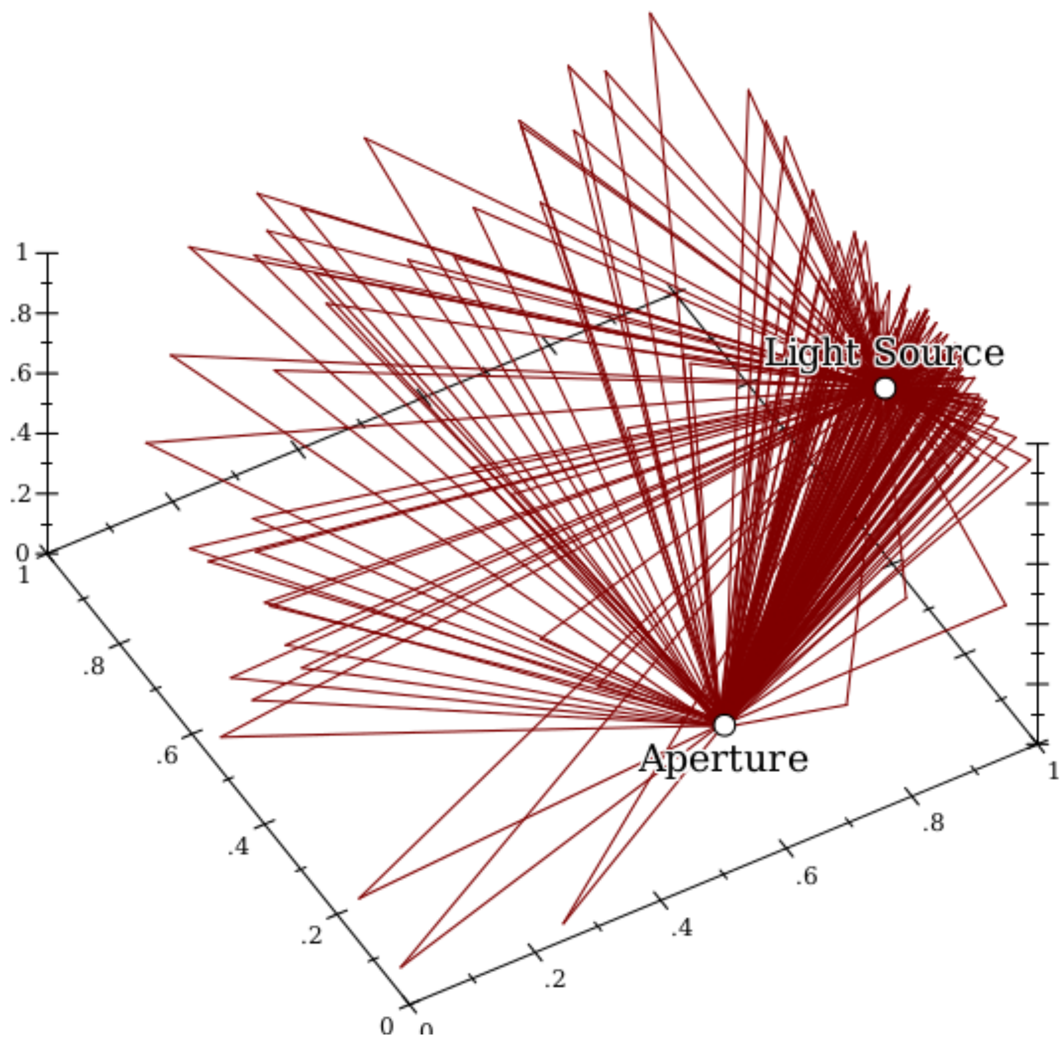

Demo: Stochastic Ray Tracing

- Part of the implementation (totals ~50 lines):

```
(define/drbayes (ray-plane-intersect p0 v n d)
  (let ([denom  (- (vec-dot v n))])
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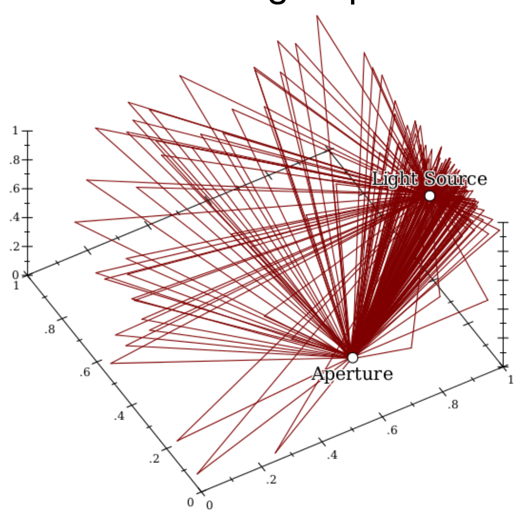
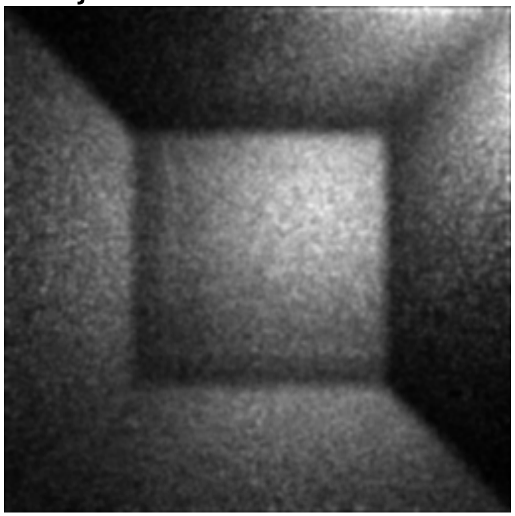
- Constrained light path outputs:

Paths Through Aperture



Projected and Accumulated



Other Inference Tasks

- Typical
  - Hierarchical models
  - Bayesian regression
  - Model selection

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  - Bayesian regression
  - Model selection
- Atypical
  - Programs that halt with probability < 1, or never halt
  - Probabilistic program verification (sample in preimage of error condition)

Thesis Statement

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- Was it falsifiable?

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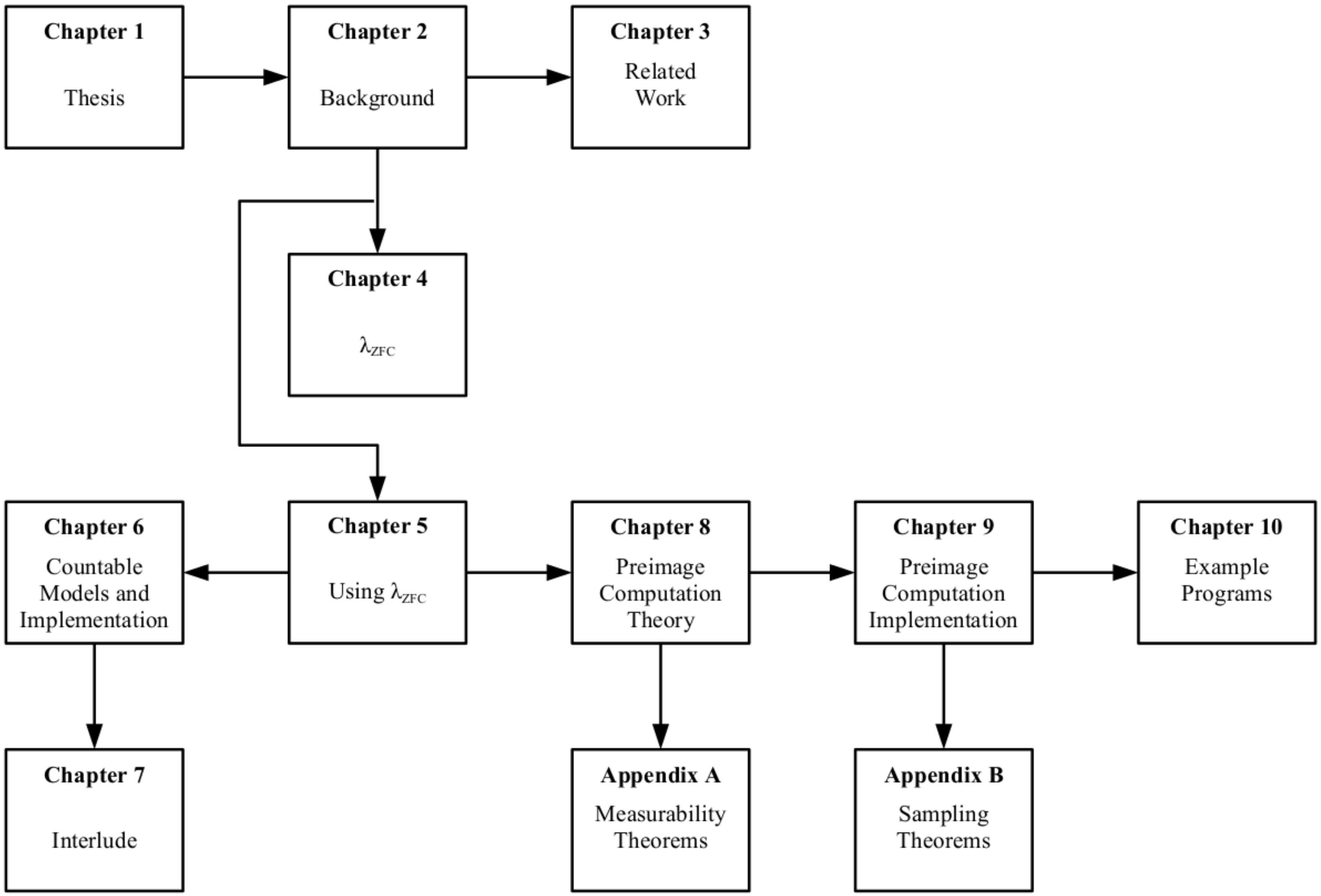
- Primitives include uncomputable operations like limits
- Applies to all probabilistic programming languages

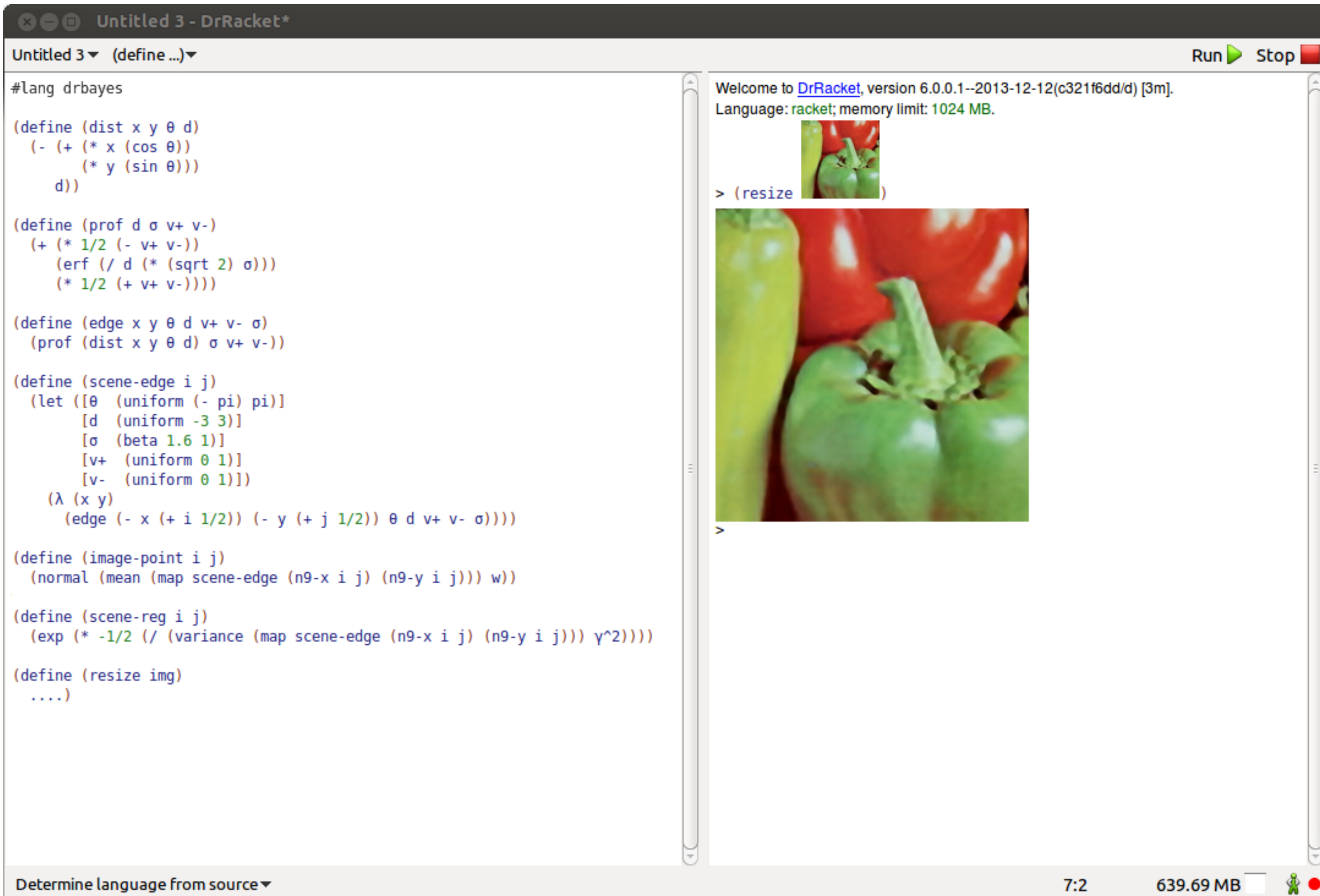
What I Did



Chapter 1
Thesis
Chapter 2
Background
Chapter 3
Related Work
Chapter 4
λ_{ZFC}
Chapter 6
Countable Models and Implementation
Chapter 5
Using λ_{ZFC}
Chapter 8
Preimage Computation Theory
Chapter 9
Preimage Computation Implementation
Chapter 10
Example Programs
Chapter 7
Interlude
Appendix A
Measurability Theorems
Appendix B
Sampling Theorems

What I Did

The core calculus for this:



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