Geometric (pre)quantization in the polysymplectic approach to field theory

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Abstract. The prequantization map for a Poisson–Gerstenhaber algebra of the differential form valued dynamical variables in the polysymplectic formulation of the De Donder–Weyl covariant Hamiltonian field theory is presented and the corresponding prequantum Schrödinger equation is derived. This is the first step toward understanding the procedures of precanonical field quantization put forward earlier from the point of view of geometric quantization.

Keywords. De Donder–Weyl formalism, polysymplectic structure, Poisson–Gerstenhaber algebra, prequantization, geometric quantization, precanonical quantization, field quantization, prequantum Schrödinger equation.

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1. Introduction

The idea of quantization of fields based on a manifestly covariant version of the Hamiltonian formalism in field theory known in the calculus of variation of multiple integrals [6, 7, 33, 46, 54] has been proposed for several times throughout the last century dating back to M. Born [4] and H. Weyl [53]. The mathematical study of geometrical structures underlying the related aspects of the calculus of variations and classical field theory has been undertaken recently by several groups of authors [8–11, 13, 15–17, 19, 38, 42–44] including Demeter Krupka's group [36, 37] here in the Czech Republic. One of the central issues for the purposes of quantization of fields is a proper definition of Poisson brackets within the covariant Hamiltonian formalism in field theory. This has been accomplished in our earlier papers [23–25] which are based on the notion of the *polysymplectic* form as a field theoretic analogue of the symplectic form in mechanics and present a construction of Poisson brackets of differential forms leading to a Poisson–Gerstenhaber

algebra structure generalizing a Poisson algebra in mechanics. The corresponding *precanonical* quantization of field theories was developed heuristically in [26–28], its relation to the standard quantum field theory was considered in [30], and a possible application to quantum gravity was discussed in [31]. In this paper we present elements of geometric prequantization in field theory based on the above mentioned Poisson–Gerstenhaber brackets and derive the corresponding prequantum analogue of the Schrödinger equation. The main purpose of our consideration is to pave a way to a better understanding of the procedures of precanonical field quantization from the point of view of the principles of geometric quantization [1, 12, 48, 50, 55].

2. Polysymplectic structure and the Poisson–Gerstenhaber brackets

Let us briefly describe the polysymplectic structure [23, 25] which underlies the De Donder–Weyl (DW) Hamiltonian form of the field equations [7, 33, 46, 54]

$$(2.1) \qquad \partial_\mu y^a(x) = \partial H/\partial p_a^\mu, \quad \partial_\mu p_a^\mu(x) = -\partial H/\partial y^a,$$

where $p_a^\mu := \partial L/\partial y_\mu^a$, called *polymomenta*, and $H := y_\mu^a p_a^\mu - L = H(y^a, p_a^\mu, x^\mu)$, called the *DW Hamiltonian function*, are determined by the first order Lagrangian density $L = L(y^a, y_\mu^a, x^\nu)$. These equations are known to be equivalent to the Euler–Lagrange field equations if L is regular in the sense that

$$\det\left(\left\|\frac{\partial^2 L}{\partial y_\mu^a \partial y_\nu^b}\right\|\right) \neq 0.$$

Let us view classical fields $y^a = y^a(x)$ as sections in the *covariant configuration bundle* $Y \to X$ over an oriented n-dimensional space-time manifold X with the volume form ω. The local coordinates in Y are (y^a, x^μ). Let $\bigwedge_q^p(Y)$ denotes the space of p-forms on Y which are annihilated by $(q + 1)$ arbitrary vertical vectors of Y.

The space $\bigwedge_1^n(Y) \to Y$, which generalizes the cotangent bundle, is a model of a *multisymplectic phase space,* [17], which possesses the canonical structure

$$(2.2) \qquad \Theta_{MS} = p_a^\mu dy^a \wedge \omega_\mu + p\,\omega,$$

where $\omega_\mu := \partial_\mu \,\lrcorner\, \omega$ are the basis of $\bigwedge^{n-1} T^*X$. The section $p = -H(y^a, p_a^\mu, x^\nu)$ gives the multidimensional *Hamiltonian Poincaré–Cartan form* Θ_{PC}.

For the purpose of introducing the Poisson brackets which reflect the dynamical structure of DW Hamiltonian equations (1) we need a structure which is independent of p or a choice of H:

Definition 1. The *extended polymomentum phase space* is the quotient bundle $Z: \bigwedge_1^n(Y)/\bigwedge_0^n(Y) \to Y$.

The local coordinates on Z are (y^a, p_a^ν, x^ν). A canonical structure on Z can be understood as an equivalence class of forms $\Theta := [p_a^\mu dy^a \wedge \omega_\mu \mod \bigwedge_0^n(Y)]$.

Definition 2. The *polysymplectic structure* on Z is an equivalence class of the forms Ω given by

$$(2.3) \qquad \Omega := [d\Theta \mod \textstyle\bigwedge_1^{n+1}(Y)] = [-dy^a \wedge dp_a^\mu \wedge \omega_\mu \mod \textstyle\bigwedge_1^{n+1}(Y)].$$

The equivalence classes are introduced as an alternative to the explicit introduction of a non-canonical connection on the multisymplectic phase space in order to define the polysymplectic structure as a "vertical part" of the multisymplectic form $d\Theta_{MS}$, see [43]. The fundamental constructions, such as the Poisson bracket below, are required to be independent of the choice of representatives in the equivalence classes, as they are expected to be independent of the choice of a connection.

Definition 3. A multivector field of degree p, $X^p \in \bigwedge^p TZ$, is called *vertical* if $X^p \lrcorner F = 0$ for any form $F \in \bigwedge_0^*(Z)$.

The polysymplectic form establishes a map of horizontal forms of degree p, $F^p \in \bigwedge_0^p(Z)$, $p = 0, 1, \ldots, (n-1)$, to vertical multivector fields of degree $(n-p)$, X_F^{n-p}, called *Hamiltonian*:

$$(2.4) \qquad X^{n-p}{}_F \lrcorner \Omega = dF^p.$$

More precisely, horizontal forms forms are mapped to the *equivalence classes* of Hamiltonian multivector fields modulo the *characteristic* multivector fields X_0^p: $X_0^p \lrcorner \Omega = 0$, $p = 2, \ldots, n$. The forms for which the map (2.4) exists are also called *Hamiltonian*. It is easy to see that the space of Hamiltonian forms is not stable with respect to the exterior product of forms. However,

Lemma 4. *The space of Hamiltonian forms is closed with respect to the graded commutative, associative* co-exterior *product*

$$(2.5) \qquad F^p \bullet F^q := *^{-1}(*F^p \wedge *F^q).$$

Proof. A straightforward proof is to solve (2.4) to see that Hamiltonian p-forms are restricted to specific $(n-p)$-polylinear forms in p_a^μ, and then to check that the $\bullet$-product preserves the space of these forms (see [24, 43]). $\square$

Note that the definition of the $\bullet$-product requires only the volume form ω on the space-time, not the metric structure. Given ω a p-form $F \in \bigwedge^p T^*X$ can be mapped to an $(n-p)$-multivector $X_F \in \bigwedge^{n-p} TX$: $X_F \lrcorner \omega = F$. Then the exterior product of multivectors ($\wedge$) induces the $\bullet$-product of forms in $\bigwedge^* T^*X$. The construction is given by the commutative diagram

$$
\begin{array}{ccc}
\bigwedge^p T^*X \otimes \bigwedge^q T^*X & \xrightarrow{\;\bullet\;} & \bigwedge^{p+q-n} T^*X \\
{\scriptstyle \omega} \downarrow & & \downarrow {\scriptstyle \omega} \\
\bigwedge^{n-p} TX \otimes \bigwedge^{n-q} TX & \xrightarrow{\;\wedge\;} & \bigwedge^{n-p+n-q} TX
\end{array}
$$

and can be lifted to forms in $\bigwedge_0^*(Z)$.

The Poisson bracket of Hamiltonian forms $\{\!\!\{\,\cdot\,,\cdot\,\}\!\!\}$ is induced by the Schouten–Nijenhuis bracket $[\![\,\cdot\,,\cdot\,]\!]$ of the corresponding Hamiltonian multivector fields:

$$(2.6) \qquad -d\{\!\!\{ F^p, F^q \}\!\!\} := [\![X^{n-p}, X^{n-q}]\!] \lrcorner\, \Omega.$$

As a consequence,

$$(2.7) \qquad \begin{aligned} \{\!\!\{ F^p{}_1, F^q{}_2 \}\!\!\} &= (-1)^{(n-p)} X^{n-p}{}_1 \lrcorner\, d F^q{}_2 \\ &= (-1)^{(n-p)} X^{n-p}{}_1 \lrcorner\, X^{n-q}{}_2 \lrcorner\, \Omega, \end{aligned}$$

whence the independence of the definition of the choice of representatives in the equivalence classes of X_F and Ω is obvious. The algebraic properties of the bracket are given by the following

Theorem 5. *The space of Hamiltonian forms with the operations $\{\!\!\{\,\cdot\,,\cdot\,\}\!\!\}$ and $\bullet$ is a (Poisson–)Gerstenhaber algebra, i.e.*

$$(2.8) \qquad \begin{aligned} &\{\!\!\{ F^p, F^q \}\!\!\} = -(-1)^{g_1 g_2}\{\!\!\{ F^q, F^p \}\!\!\}, \\ &(-1)^{g_1 g_3}\{\!\!\{ F^p, \{\!\!\{ F^q, F^r \}\!\!\} \}\!\!\} + (-1)^{g_1 g_2}\{\!\!\{ F^q, \{\!\!\{ F^r, F^p \}\!\!\} \}\!\!\} \\ &\qquad + (-1)^{g_2 g_3}\{\!\!\{ F^r, \{\!\!\{ F^p, F^q \}\!\!\} \}\!\!\} = 0, \\ &\{\!\!\{ F^p, F^q \bullet F^r \}\!\!\} = \{\!\!\{ F^p, F^q \}\!\!\} \bullet F^r + (-1)^{g_1(g_2+1)} F^q \bullet \{\!\!\{ F^p, F^r \}\!\!\}, \end{aligned}$$

where $g_1 = n - p - 1$, $g_2 = n - q - 1$, $g_3 = n - r - 1$.

Proof. The graded Lie algebra properties are a straightforward consequence of (2.6) and the graded Lie nature of the Schouten–Nijenhuis bracket. The graded Leibniz property can be seen as a consequence of the Frölicher–Nijenhuis theorem [14,34] adapted to the algebra of forms equipped with the co-exterior product.   □

3. Prequantization map

Having in our disposal a generalization of the symplectic structure and a Poisson algebra to the DW Hamiltonian formalism of field theory it is natural to ask if geometric quantization can be generalized to this framework. The first step in this direction would be a generalization of the *prequantization map* [1, 12, 48, 50, 55] $F \to O_F$ which maps dynamical variables F on the classical phase space to the first order (prequantum) operators O_F on (prequantum) Hilbert space and fulfils three properties:

 (Q1) the map $F \to O_F$ is linear;

 (Q2) if F is constant, then O_F is the corresponding multiplication operator;

 (Q3) the Poisson bracket of dynamical variables is related to the commutator of the corresponding operators as follows:

$$(3.1) \qquad [O_{F_1}, O_{F_2}] = -i\hbar O_{\{F_1, F_2\}}.$$

In the case of a Poisson–Gerstenhaber algebra we expect that the commutator (3.1) is replaced by the *graded* commutator $[A, B] := A \circ B - (-1)^{\deg A \deg B} B \circ A$.

Theorem 6. *The prequantum operator of a differential form dynamical variable F is given by the formula*

(3.2) $$O_F = i\hbar \pounds_{X_F} + X_F \lrcorner \Theta \bullet + F \bullet,$$

where

$$\pounds_{X_F} := [X_F, d]$$

and Θ is a (local) polysymplectic potential in the sense of (2.3).

Proof. See a straightforward calculation in [32]. $\square$

The most intriguing aspect of the representation (3.2) is that the prequantum operator O_F is non-homogeneous: for an f-form F the degree of the first term in (3.2) is $(n - f - 1)$ and the degree of the other two terms is $(n - f)$. This fact suggests that prequantum wave functions are complex non-homogeneous horizontal differential forms, i.e. sections of the complexified bundle $\bigwedge_0^*(Z)^{\mathbb{C}} \to Z$. The corresponding (graded) prequantum Hilbert space will be considered in [32] (see also [29]).

Note that formulas (3.1) and (3.2) imply that one can introduce a formal non-homogeneous "supercovariant derivative" with respect to a multivector field X:

$$\nabla_X := \pounds_X - \frac{i}{\hbar} X \lrcorner \Theta \bullet$$

with the curvature of the corresponding "superconnection" ∇ (cf. [40, 45])

(3.3) $$\Omega(X_1, X_2) := -i\hbar \big([\nabla_{X_1}, \nabla_{X_2}] - \nabla_{[\![X_1, X_2]\!]} \big)$$

coinciding with the polysymplectic form.

One of the important questions is what is the dynamical equation for the wave functions. Let us consider how geometric prequantization can help us to find an answer.

4. Prequantum Schrödinger equation

The origin of the Schrödinger equation in quantum mechanics from the point of view of geometric (pre)quantization can be understood as follows. The classical equations of motions are incorporated in the vector field X_* which annihilates the exterior differential of the (Hamiltonian) Poincaré–Cartan form

(4.1) $$\Theta = p \, dq - H(p, q) \, dt,$$

i.e.

(4.2) $$X_* \lrcorner d\Theta = 0.$$

The classical trajectories in the phase space are known to be the integral curves of X_*.

Now, if we think of geometric prequantization based on the presymplectic structure given by $d\Theta$ we notice that the zero "observable" has a non-trivial (presymplectic) prequantum operator:

$$(4.3) \qquad 0 \rightarrow O_0 = i\hbar\pounds_{X_*} + X_* \lrcorner \Theta,$$

where $X_* = X^t\partial_t + X^q\partial_q + X^p\partial_p$ and

$$(4.4) \qquad X^q = \partial_p H, \quad X^p = -\partial_q H,$$

as it follows from (4.2) under the assumption $X^t = 1$ (which is just a choice of time parametrization). The obvious consistency requirement then is that O_0 vanishes on prequantum wave functions $\Psi = \Psi(p, q, t)$, i.e.,

$$(4.5) \qquad O_0(\Psi) = 0.$$

Using the explicit form of the operator O_0 derived from (4.3), (4.4):

$$(4.6) \qquad O_0 = i\hbar\partial_t + i\hbar(\partial_p H\partial_q - \partial_q H\partial_p) + p\partial_p H - H(p, q)$$

one can write (4.5) in the form of the *prequantum Schrödinger equation*

$$(4.7) \qquad i\hbar\partial_t\Psi = O_H\Psi,$$

where O_H is the (symplectic) prequantum operator of the Hamilton canonical function:

$$(4.8) \qquad O_H = -i\hbar(\partial_p H\partial_q - \partial_q H\partial_p) - p\partial_p H + H(p, q).$$

The above consideration demonstrates the origin of the Schrödinger equation in the classical relation (4.2). The subsequent steps of quantization just reduce the Hilbert space of the wave functions (by choosing a *polarization*) and construct a proper operator of H on this Hilbert space, the form of the Schrödinger equation (4.7) remaining intact. This observation motivates our consideration of the field theoretic prequantum Schrödinger equation in the following section: having obtained it on the level of prequantization one may have a better idea as to what is the covariant Schrödinger equation for quantum fields within the approach based on DW Hamiltonian formulation (2.1).

There has been a little discussion of the prequantum Schrödinger equation in the literature (cf. [5, 35, 39, 47]) for the reason that it works on a wrong Hilbert space of functions over the phase space, thus contradicting the uncertainty principle. It can serve, therefore, only as an intermediate step toward the true quantum mechanical Schrödinger equation.

Let us note that equations (4.7), (4.8) recently appeared within the hypothetical framework of "subquantum mechanics" proposed by J. Souček [49] whose starting point was quite different from geometric quantization. A possible connection between the "subquantum mechanics" and geometric prequantization could be an interesting subject to study, particularly in connection with the question recently revisited by G. Tuynman [51, 52] as to "were there is the border between classical and quantum mechanics in geometric quantization?"

5. Prequantum Schrödinger equation in field theory

In this section we present a field theoretic generalization of the above derivation of the prequantum Schrödinger equation.

It is known [8–11,13,25,44] that the classical field equations in the form (2.1) are encoded in the multivector field of degree n, $X^n_* \in \bigwedge^n TZ$, which annihilates the exterior differential of the multidimensional Hamiltonian Poincaré–Cartan n-form

$$(5.1) \qquad \Theta_{PC} = p^\mu_a dy^a \wedge \omega_\mu - H(y^a, p^\mu_a)\omega,$$

i.e.

$$(5.2) \qquad X^n{}_* \,\lrcorner\, d\Theta_{PC} = 0.$$

Let us extend the geometric prequantization map (3.2) to the case of the "pre-polysymplectic" form $d\Theta_{PC}$, usually called *multisymplectic*. Again, the feature of this extension is that there is a non-trivial prequantum operator corresponding to the zero function on the polymomentum phase space:

$$(5.3) \qquad O_0 = i\hbar[X^n_*, d] + X^n{}_* \,\lrcorner\, \Theta_{PC} \bullet .$$

Then the consistency requires the prequantum wave function $\Psi = \Psi(y^a, p^\mu_a, x^\mu)$ to obey the condition

$$(5.4) \qquad O_0(\Psi) = 0$$

which is expected to yield the field theoretic prequantum Schrödinger equation.

It is easy to see that the operator (5.3) is non-homogeneous: the first term has the degree $-(n-1)$ while the last one has the degree $-n$. Therefore, the prequantum wave function in (4) is a horizontal non-homogeneous form

$$\Psi = \psi\omega + \psi^\nu\omega_\nu,$$

a section of the bundle $(\bigwedge^{n-1}_0(Z) \oplus \bigwedge^n_0(Z))^{\mathbb{C}} \to Z$ which generalizes the complex line bundle over the symplectic phase space used in the usual geometric quantization.

Assuming $X^n_* \,\lrcorner\, \omega = $ const and introducing the vertical Hamiltonian multivector associated with H:

$$X^n{}_H \,\lrcorner\, d\Theta = dH,$$

where Θ is a potential of the polysymplectic form Ω, the vertical part of X^n_* takes the form:

$$X^n{}_*^V = (-1)^n(X^n_* \,\lrcorner\, \omega)X^n_H.$$

Then (5.4) yields the prequantum Schrödinger equation in the form:

$$(5.5) \qquad \begin{aligned} &i\hbar(\partial_\mu\psi^\mu - (-1)^n\partial_\mu\psi\, dx^\mu) \\ &\quad = -(-1)^n(i\hbar X^n{}_H \,\lrcorner\, d\Psi + X^n{}_H \,\lrcorner\, \Theta\bullet\Psi) + H\bullet\Psi. \end{aligned}$$

For *odd* n the right-hand side of (5.5) is identified with the (polysymplectic)

prequantum operator of H (see (3.2)), and (5.5) takes a particularly appealing form (cf. [22])

$$(5.6) \qquad i\sigma\hbar d \bullet \Psi = O_H(\Psi),$$

where $\sigma = \pm 1$ for Euclidean/Lorentzian spacetimes (in our conventions $*\omega = \sigma$), and $d\bullet$ is the *co-exterior* differential [28] which is non-vanishing only on the subspace of $(n-1)$- and n-forms:

$$d \bullet (\psi^\nu \omega_\nu) = \partial_\mu \psi^\nu dx^\mu \bullet \omega_\nu = \sigma \partial_\nu \psi^\nu, \quad d \bullet (\psi\omega) = \sigma \partial_\mu \psi dx^\mu.$$

For *even n* the right-hand side of (5.5) is not O_H because of the wrong sign in front of the first two terms. The left hand side is also different from the one in (5.6). A distinction between even and odd space-time dimensions is a problematic feature of the present derivation based on a specific prequantization formula (3.2). (See Note added in proofs.)

The meaning of our discussion in this section is that it provides a hint to the actual form of the covariant Schrödinger equation in field theory which one can expect within the approach to field quantization based on the covariant DW Hamiltonian formalism.

6. Discussion

We presented a formula of prequantum operators corresponding to Hamiltonian forms. It realizes a representation of the Poisson–Gerstenhaber algebra of Hamiltonian forms by operators acting on prequantum wave functions given by nonhomogeneous forms Ψ, the sections of $\bigwedge_0^*(Z)^{\mathbb{C}} \to Z$. We also argued that these wave functions fulfill the prequantum Schrödinger equation (5.5).

The next step in geometric quantization would be to reduce the prequantum Hilbert space by introducing a *polarization* in the polymomentum phase space. A generalization of the *vertical* polarization reduces the space of wave functions to the functions depending on field variables and space-time variables: $\Psi(y^a, x^\mu)$. A construction of quantum operators on the new Hilbert space of quantum wave functions requires further generalization of the techniques of geometric quantization, such as the notion of the metaplectic correction and the Blattner-Kostant-Sternberg pairing, which is not developed yet.

However, the quantum operator $\widehat{H}$ is already known from the heuristic procedure of "precanonical quantization" [26–28, 30, 31] based on quantization of a small Heisenberg-like subalgebra of brackets of differential forms generalizing the canonical variables. Within precanonical quantization it was found suitable to work in terms of the space-time Clifford algebra valued operators and wave functions, rather than in terms of non-homogeneous forms and the graded endomorphism valued operators acting on them. In general, a relation between the two formulations is given by the "Chevalley quantization" map from the *co*-exterior algebra to the Clifford algebra: $\omega_\mu \bullet \to -(1/\varkappa)\gamma_\mu$, where the constant κ is introduced to match the physical dimensions ($1/\varkappa \sim \text{length}^{n-1}$). The corresponding Clifford product of

forms is given by (cf. [2, 3, 18, 20, 21])

$$\omega_\mu \vee \omega_\nu = \omega_\mu \bullet \omega_\nu + \varkappa^{-2}\eta_{\mu\nu}.$$

Note that the appearence of the metric $\eta_{\mu\nu}$ at this stage is related to the fact that a definition of the scalar product of wave functions represented by non-homogeneous forms, i.e. their probabilistic interpretation, requires a space-time metric.

Under the above "Cliffordization" and the vertical polarization the wave function becomes Clifford valued: $\Psi = \Psi(y^a, x^\mu)$ and the left hand side of (5.5), (5.6) can be expressed in terms of the Dirac operator acting on $\Psi = \psi + \psi_\nu \gamma^\nu$; in particular, $d\bullet \sim \gamma \gamma^\nu \partial_\nu$, where $\gamma \sim \gamma_1 \gamma_2 \cdots \gamma_n$ corresponds to the Hodge duality operator $*$. Similarly, the operator of $H\bullet$ is represented as $\sim \gamma \widehat{H}$. The coefficients not specified here are fixed by the requirement that the resulting Dirac-like equation is causal and consistent, thus leading to the covariant Schrödinger equation for quantum fields in the form

(6.1) $i\hbar\varkappa\gamma^\mu\partial_\mu\Psi = \widehat{H}\Psi.$

A similar reasoning leads to the representation of polymomenta, which is of the form:

$$\hat{p}_a^\mu = -i\hbar\varkappa\gamma^\mu \frac{\partial}{\partial y^a}.$$

These results have been anticipated within precanonical field quantization earlier [26–28] (see also [41] where similar relations were postulated). This approach also allows us to derive the explicit form of $\widehat{H}$. For example, in the case of interacting scalar fields y^a one can show that [28]

$$\widehat{H} = -\tfrac{1}{2}\hbar^2\varkappa^2 \triangle + V(y),$$

where $\triangle$ is the Laplace operator in the space of field variables.

What we have arrived at is a multidimensional hypercomplex generalization of the Schrödinger equation from quantum mechanics to field theory, where the space-time Clifford algebra, which arose from quantization of differential forms, generalizes the algebra of the complex numbers in quantum mechanics, and the notion of the unitary time evolution is replaced by the space-time propagation governed by the Dirac operator. In [30] we discussed how this description of quantum fields can be related to the standard description in the functional Schrödinger representation. In doing so the Schrödinger wave functional arises as a specific composition of amplitudes given by Clifford-valued wave functions of the precanonical approach, and the parameter $\varkappa$ appears to be related to the ultra-violet cutoff.

Obviously, in this presentation we have left untouched a lot of important issues both on the level of prequantization and on the level of quantization. A development of the present version of geometric quantization in field theory would further clarify the mathematical foundations of precanonical quantization of fields and also advance its understanding and applications. The whole field appears to us as appealing, mathematically rich and unexplored as the field of the geometric quantization approach to quantum mechanics was 25–30 years ago.

Note added in proofs

A distinction between odd and even n in Section 5 can be avoided by noticing that the prequantization map (3.2) can be modified as follows:

$$(6.2) \qquad O'_F = (-1)^{(n-f-1)}(i\hbar \pounds_{X_F} + X_F \lrcorner \Theta\bullet) + F\bullet,$$

where $(n - f)$ is the co-exterior degree of F. Then the right-hand side of (5.5) is identified with $O'_H(\Psi)$ for any n. The left-hand side of (5.5) also can be written in a universal form for any n using the reversion anti-automorphism β in a co-exterior Grassmann algebra: $\beta(F) := (-1)^{\frac{1}{2}(n-f)(n-f-1)}F$. Then the prequantum Schrödinger equation (5.5) can be written as follows:

$$(6.3) \qquad i\sigma\hbar(-1)^{\frac{1}{2}n(n-1)}\beta(d \bullet \Psi) = O'_H(\Psi).$$

Note that a choice between two representations (3.2) and (6.2) can be made once the scalar product is specified.

Acknowledgments

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