Zipper Unfoldings of Polyhedral Complexes

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Abstract

We explore which polyhedra and polyhedral complexes can be formed by folding up a planar polygonal region and fastening it with one zipper. We call the reverse process a *zipper unfolding*. A zipper unfolding of a polyhedron is a path cut that unfolds the polyhedron to a planar polygon; in the case of edge cuts, these are *Hamiltonian unfoldings* as introduced by Shephard in 1975. We show that all Platonic and Archimedean solids have Hamiltonian unfoldings.

We give examples of polyhedral complexes that are, and are not, zipper [edge] unfoldable. The positive examples include a polyhedral torus, and two tetrahedra joined at an edge or at a face.

1   Introduction

A common way to build a paper model of a polyhedron is to fold and glue a planar polygonal shape, called a *net* of the polyhedron. Nets of Platonic solids and other polyhedra with regular faces were first published by Dürer in 1525, and have been a source of fascination ever since—see the book by Demaine and O'Rourke [2]. The reverse process involves slicing the surface of a polyhedron to *unfold* it to a planar shape. When cuts are limited to the edges of the polyhedron these are *edge unfoldings*. It is an open problem whether every convex polyhedron has an edge unfolding [5, 2].

In this paper we replace glue by a zipper, i.e., we limit the slicing of the polyhedron's surface to a single cut path that does not self-intersect. We call these *zipper unfoldings*. When cuts are restricted to the polyhedron edges, the cut path must form a Hamiltonian path in the graph of the polyhedron. Such *Hamiltonian unfoldings* were studied by Shephard [5]. Figures 1 and 2 show Hamiltonian unfoldings of the Platonic solids.
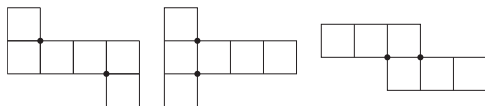

Figure 1: All Hamiltonian unfoldings of the cube.

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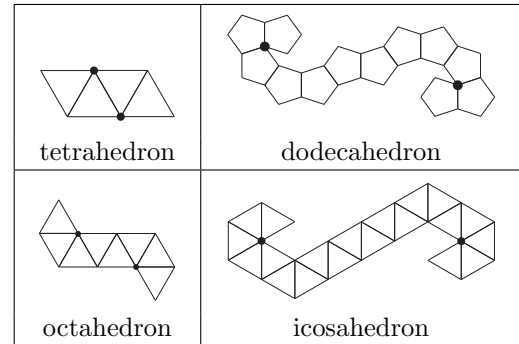[§]arlo666666@gmail.com
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Figure 2: Doubly Hamiltonian unfoldings of the other Platonic solids.

The physical model of a zipper provides a natural extension of zipper unfolding to *polyhedral complexes* which generalize polyhedra, either by allowing positive genus (these are *polyhedral manifolds*), or by allowing an edge to be incident to more than two faces, or allowing a boundary edge that is incident to only one face. We elaborate on the model of zipper unfolding in Section 1.1. Figure 3 shows a zipper edge unfolding of two tetrahedra joined at a face.
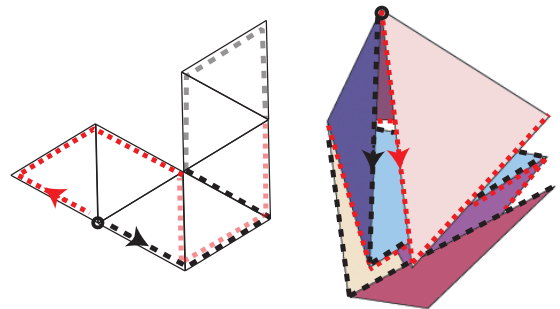

Figure 3: Zipper edge unfolding of two tetrahedra sharing a face. The zipper starts at the circle. Lightly shaded paths lie on the back of the surface.

Section 2 contains our results on Hamiltonian unfoldings of polyhedra: we show that all the Archimedean solids have Hamiltonian unfoldings. In Section 3 we prove that polyhedral manifolds that have zipper unfoldings have genus 0 or 1, and that the latter are quite specialized. In Section 4 we give some examples of zipper edge unfoldings of polyhedral complexes.
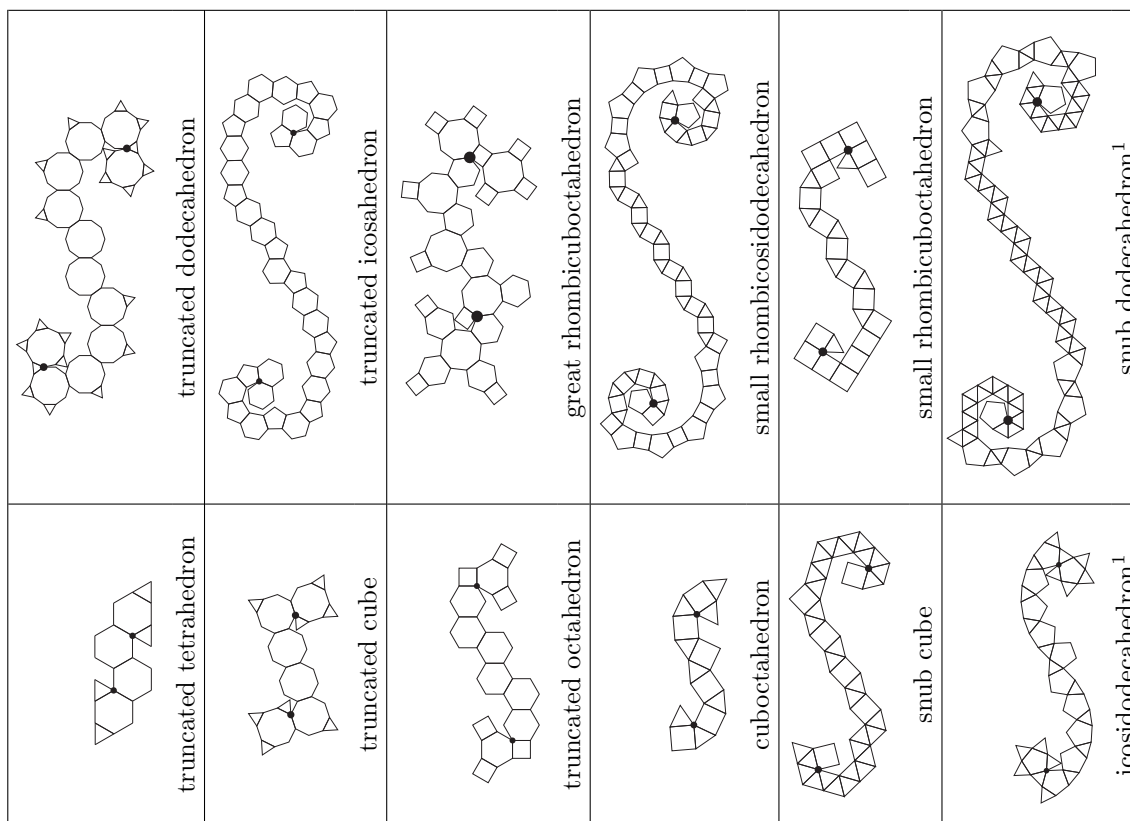
Figure 4: Hamiltonian unfoldings of the Archimedean solids. (See Figure 5 for the last one.)

1.1 Zippers

We base our model on the physical reality of cloth and zippers. Thus a zipper on the unfolded surface consists of two equal-length directed paths that do not properly cross each other or themselves. Note that we allow the paths to touch at a common point and even to run alongside each other (see Figure 3). This is similar to models of paper or linkage folding [2]. We consider our planar shapes to have a top and bottom surface; the noncrossing condition only applies to paths on the same side of the surface. *Zipping* means identifying the two paths. We impose the constraint that *unzipping* means cutting the path between the zippers—this makes a difference regarding connectivity of the unfolded shape.

We allow both *standard* zippers (e.g. for jeans) where the two paths must start at a common point of the unfolded shape, and *separating* zippers (e.g. for jackets) where the two paths may start at different points. For constructing actual models, it is also interesting (although not mathematically significant) to consider zippers with more than one pull tab. With two pull tabs the zipped portion may form two intervals (e.g. for a backpack) or one (e.g. for a rain jacket). For an interesting history of zippers, see [3].

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[1]corrected from printed proceedings

2 Polyhedra

In this section we give Hamiltonian unfoldings of all the Platonic and Archimedean solids. Our results are presented in Figures 1, 2, 4 and 5. We note that prisms and antiprisms also have Hamiltonian unfoldings, so we can extend to the class of "convex uniform polyhedra". Our unfoldings of the Platonic solids have the additional properties that they are S-shaped, and symmetric, and the faces form a chain in the unfolded state. We say that a Hamiltonian unfolding is *doubly Hamiltonian* if the faces form a chain in the unfolded state.

Our unfoldings of Archimedean solids are generally S-shaped (except for the great rhombicosidodecahedron) but not always symmetric or doubly Hamiltonian. In some cases a doubly Hamiltonian unfolding does not exist, e.g., the cuboctahedron has 6 square faces and 8 triangular faces, with squares only adjacent to triangles and vice versa, so the faces cannot form a chain.

Open Problem 1 *Which convex polyhedra have doubly Hamiltonian unfoldings?*

As noted by Shephard [5], a polyhedron has a Hamiltonian unfolding only if its graph has a Hamiltonian path, and therefore the rhombic dodecahedron has no
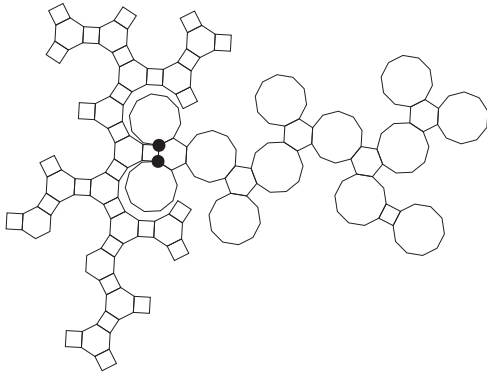
Figure 5: Hamiltonian unfolding of great rhombicosido-decahedron.

Hamiltonian unfolding because its graph has no Hamiltonian path (see also Coxeter [1]).

It is an open question whether every convex polyhedron has an edge unfolding, but when the condition of cutting along edges is relaxed every convex polyhedron can be unfolded (see [2]), which leads to the following:

Open Problem 2 *Does every convex polyhedron have a zipper unfolding?*

3 Polyhedral Manifolds

A *polyhedral manifold* is a finite union of planar polygons in 3D Euclidean space such that every point has a neighbourhood homeomorphic to a disk. We consider only embedded manifolds. Polyhedra are the special case of polyhedral manifolds with genus 0, i.e. homeomorphic to a sphere. Polyhedral manifolds of higher genus are called *toroidal polyhedra*, and include, for example, the Császár polyhedron, and the toroids from Bonnie Stewart's book, "Adventures Among the Toroids." Figure 6 shows a zipper edge unfolding of a toroidal polyhedron of genus 1 (a *torus polyhedron*) to an *annulus* (a polygonal region with one hole).
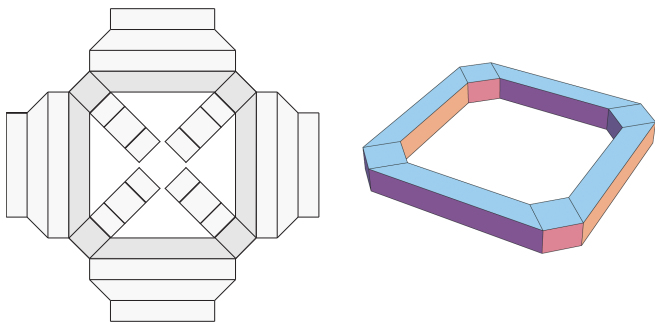

Figure 6: A zipper edge unfolding of a torus polyhedron using a separating zipper.

Theorem 1 *If P is a polyhedral manifold that has a zipper unfolding to a planar polygonal region F, then either P is a polyhedron and F is a polygon, or—in the case of a separating zipper—P is a torus polyhedron and F is an annulus whose outer perimeter is equal to its inner perimeter.*

Proof. Because a polyhedral manifold has no boundary, the zipper paths must include all boundary points of F. We claim that the zipper paths *only* contain boundary points: If a zipper path contains a point interior to F, then zipping would attach more surface to this point, creating a non-disc-like neighbourhood. (No extra surface is attached in case the two sides of the zipper travel together across the interior of F, but, by assumption, unzipping would then cut the surface, so we would be on the boundary of F not in the interior.)

If the zipper is nonseparating then the zipper paths start at a common boundary point of F and must remain on that connected component of the boundary. Thus F is a polygon, one zipper path traverses half the boundary clockwise and the other path traverses the other half counterclockwise. Then P is a polyhedron.

If the zipper is a separating zipper, then the two zipper paths may start at different points. They may traverse at most two connected components of the boundary of F, so F is a polygon or an annulus. In the latter case, one zipper path must traverse the outer boundary (say clockwise) and the other path must traverse the inner boundary. This traversal must also be clockwise otherwise we create a Möbius band, which is impossible on an embedded polyhedral manifold. Then P is a torus polyhedron. Finally, if F is a polygon then the two zipper paths must traverse the boundary in opposite directions; otherwise we create a Möbius band. Therefore the two paths must start at the same boundary point, and P is a polyhedron. $\square$

The remainder of this section is about zipper edge unfoldings of torus polyhedra.

Lemma 2 *The dual graph of a zipper edge unfolding of a torus polyhedron consists of trees attached to a cycle.*

A cycle of faces that unfolds to a planar annulus will be called a *planar cycle of faces*. An example is darkly shaded in Figure 6; in this example the dual graph of the unfolding has 4 outer and 4 inner trees attached to the cycle, and each tree is a path of length 3.

Figure 7 shows some torus polyhedra that have no zipper edge unfoldings: the "picture frame" with triangular cross section, because it has no planar cycle of faces; and the "picture frame" with square cross section, because, although it has a planar cycle of faces, the remaining faces cannot be joined to it without causing overlaps, or making the outer perimeter larger than the inner one.
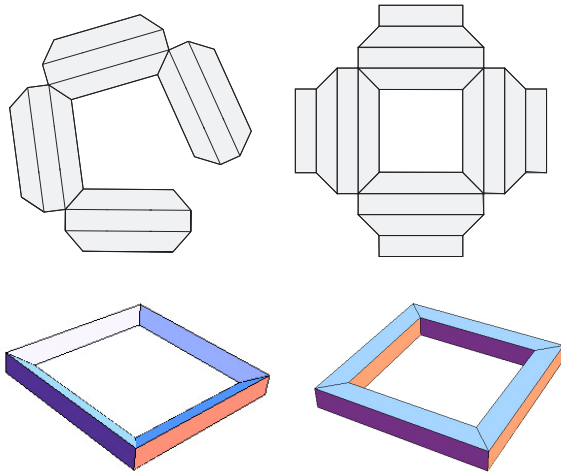
Figure 7: Torus polyhedra with no zipper edge unfoldings: "picture frames" with triangular and square cross sections.

Torus polyhedra with zipper edge unfoldings are very specialized. More precisely, we claim that any perturbation of a torus polyhedron with a zipper edge unfolding creates a torus polyhedron with no zipper edge unfolding. We also conjecture that recognition is NP-hard.

Open Problem 3 *Is the problem of recognizing torus polyhedra with zipper edge unfoldings NP-hard?*

4 Polyhedral Complexes

We define a *polyhedral complex* as a finite union of planar polygons in 3D Euclidean space whose only intersections are edge-to-edge joins and the resulting vertex coincidences. We distinguish polyhedral complexes *with* or *without boundary*.

There are several possible models of "cutting" a polyhedral complex that are consistent with the familiar cutting of polyhedra. One such model is used by Liu and Tai [4] in their algorithm to explore all the possible unfoldings of polyhedral complexes. As discussed in Section 1.1 we use the physical constraints of cloth and zippers as the basis for our model, which we define in terms of the zipper paths on the unfolded shape. We do not attempt to translate this to conditions on how polyhedral complexes are cut. We note that a cut must join two surfaces, and that, for example, an edge where k faces join must be cut between $\lceil \frac{k}{2} \rceil$ and $k - 1$ times.

We give several examples of polyhedral complexes that have zipper edge unfoldings. The first examples are chains of tetrahedra joined at edges, where each tetrahedron's edges of attachment are either incident (Figure 8) or opposite (Figure 9).

It is also possible to make a chain of tetrahedra joined at faces. See Figure 3 for the case of two tetrahedra.
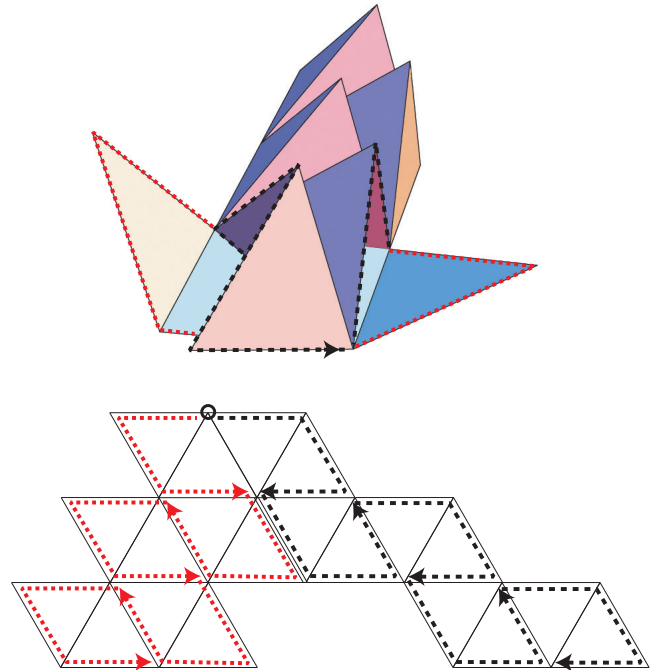

Figure 8: A zipper edge unfolding of a chain of 5 tetrahedra joined at incident edges. The zipper starts at the circle. The last two tetrahedra are not yet completely zipped up.
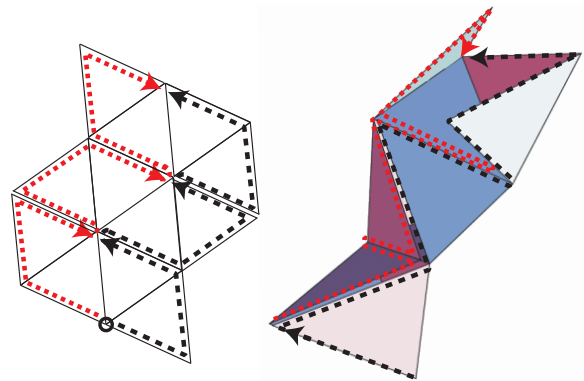

Figure 9: A zipper edge unfolding of a chain of 3 tetrahedra joined at opposite edges. The third tetrahedron is not completely zipped up.

This can be extended to a chain of five regular tetrahedra joined at faces, but not beyond that; otherwise the tetrahedra intersect in 3D. We conjecture that there is no zipper edge unfolding of two cubes joined at a face. However, there is a zipper edge unfolding if the cubes are "squashed"—see Figure 10.
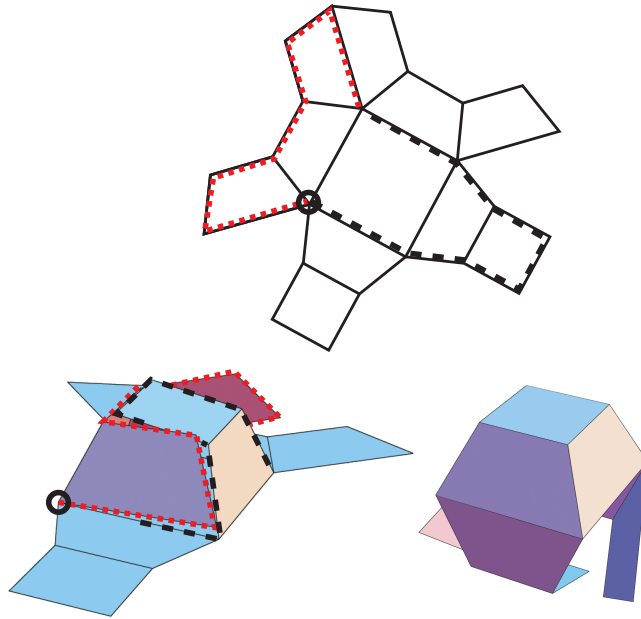

Figure 10: A zipper edge unfolding of two "squashed" cubes sharing a face: (*above*) the unfolding; (*right*) folding of top cube following zipper paths shown above; (*left*) folding of bottom cube.

Open Problem 4 *Which polyhedral complexes have zipper [edge] unfoldings?*

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