*MVA'94* IAPR Workshop on Machine Vision Applications Dec. 13-15, 1994, Kawasaki

# Connectivity in high dimensional images.

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## Abstract

*The aim of this research is to provide insight in the construction of the topology preserving conditions that are necessary to constitute skeletons of objects in N dimensional binary images. However, if one wants to derive these conditions, one of the first questions is, which connectivities between the elements of high dimensional images are possible and what should be chosen for foreground and background connectivity. A formula is derived as well as a best choice for the connectivity of the background.*

## 1. Introduction.

Our aim is to obtain insight in the elements that are necessary to perform topology preserving thinning or skeletonization in N dimensional binary images suitable for massively parallel implementation [1].

The motivation for this research was found first of all in skeletons from 3-D images obtained from Confocal Microscopes, CT, NMR or ultrasound sensor systems or from range sensors. Solved around 1982 [2, 3] this is already an older topic. However, the motivation for extending this to topology preserving thinning in images with a dimension higher than three can be found in the problem of finding the safest non colliding path of an object in an N dimensional space, which can be implemented with a background skeleton. These problems are frequently encountered in Printed Circuit Board and VLSI mask routing (a 3-D problem: x, y, z), planning of mutually collision free routes for multiple autonomous vehicles (a 4-D problem: x, y, ø, t ), or the planning of a simultaneously collision free path for a multi robot system (an N-D problem). Robot path finding itself, is a consequence of the robot vision problem: If a robot's movement to grasp an object, is dictated by the objects in it's field of view, it should avoid to collide with the other objects in its field of view.

If one wants to derive topology preserving conditions for high dimensional images, one of the first questions is, what are the possible connectivities?

## 2. Basic definitions.

First some basic definitions are needed for the foundation of N-D binary processing:

If $\mathbb{R}_N$ is a Euclidean space of dimension N with origin $\vec{O}$ then let $\mathbb{R}_N^{*}$ be a Euclidean space of dimension N with origin $\vec{O}$ equidistantly sampled with unit distance over each dimension.

An N-dimensional binary image $X_N$ is now defined as an N-dimensional bounded section of $\mathbb{R}_N^{*}$, with the elements of $X_N$ having the values {0,1}. See figure 1.

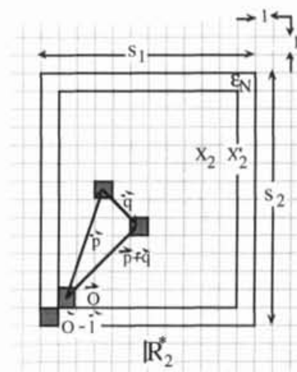


Figure 1 Binary Image $X_2$ in $\mathbb{R}_2^{*}$.

The size of the image is indicated by the vector $\vec{s}_N$: $(s_1,..,s_N)$ containing the bounds of each coordinate. Let $X'_N$ be an image of size $\vec{s}'_N$: $(s_1+2,..,s_N+2)$ having its origin in $\vec{O} - \vec{1}$, then the edge $\varepsilon_N$ of $X_N$ is defined as the elements of $X'_N \notin X_N$. The elements of $X'_N$ and $\varepsilon_N$ also have the values {0,1}. The elements of $X_2$ are referred to as pixels, the elements of $X_3$ as voxels.

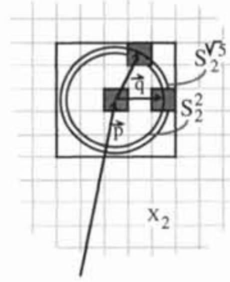


Figure 2 Neighbourhood $M_2^5$ of $\vec{p}$.

Let the position of an element in an N-dimensional image $X_N$ be denoted as $\vec{p}$, then $\vec{p}+\vec{q}$ denotes an element with position $\vec{q}$ relative to $\vec{p}$. See figure 2.

## 3. Connectivity between image elements.

We now make the following definition for the connectivity between two image elements, presupposing that each vector points to the center of an image element:

- *All elements of* $X_N$ *with the same value, on a distance* $d \leq |\vec{q}|$ *to* $\vec{p}$ *lying within the (hyper-) sphere* $S_N^d$ *with origin* $\vec{p}$ *and radius d, are elements connected to* $\vec{p}$.

It is not common to define a connectivity as a distance (see e.g.[3]), but as soon as we define the connectivity of objects, e.g. a curve in a 2-D image, we intuitively use this concept. E.g. a 4 connected curve in 2-D has only edge (4) connected pixels, whereas an 8 connected curve has both point (8) and edge (4) connected neighbours.

Let $M_N^n$ be an N-dimensional (hyper-)cubic neighbourhood with (odd) size n = 2k+1, having its central element in $\vec{p}$. If $\vec{p}$ is assumed to be the origin of the local coordinate system, then k is the maximum value of any component of $\vec{q}$ within $M_N^n$.

Let E be the number of elements **on** the (hyper-)sphere $S_N^d$ **within** the neighbourhood $M_N^n$. As the elements are exactly on the grid positions, E will only have non-zero values for specific values of *d*.

In order to derive an expression for E(N,k,*d*), let us consider only the elements $\vec{q}$ in the **partition** with non-negative component values, i.e., with $0 \leq q_i \leq k$ within the neighbourhood $M_N^n$. For N=2, this means considering only elements in the first **quadrant**, for N=3 only in the first **octant**, etcetera. Afterwards the number found for such a partition can be multiplied by the number of partitions, $2^N$ and compensated for the shared partition boundaries.

The number of different vectors $\vec{q}$ with the same length $|\vec{q}|$ is equal to the number of permutations among its component values. So, if all N components of $\vec{q}$ have different values, there will be N! vectors with length $|\vec{q}|$ in the partition.

Let us denote the number of times that each of the components $q_i$ of $\vec{q}$ has the value j by $n_j$, i.e.:

$$n_j = \sum_{i=1}^{N} (q_i = j) \tag{1a}$$

Then, for each distinct $0 \leq j \leq k$, the number of vectors with length $|\vec{q}|$ will be reduced by a factor $(n_j!)^{-1}$, because permutations of equal component values do not produce different vectors. Note that if only two component values are possible, e.g. 0 and 1, the result is the binomial:

$$\frac{N!}{n_j!(N-n_j)!} \tag{1b}$$

The more general case, with more than two different component values is called **multinomial:**

$$\frac{N!}{\prod_{j=0}^{k} (n_j!)} \quad \text{with} \sum_{j=0}^{k} n_j = N \tag{1c}$$

The case j = 0 is a special one, because a vector with one or more components $q_i = 0$, i.e. with $n_0 > 0$, is shared by $2^{n_0}$ partitions (quadrants, octants, etcetera). So, instead of simply multiplying afterwards by the number of partitions $2^N$, we must use the factor $2^{N - n_0}$ in order to compensate for shared vectors. So, in conclusion E(N,k,*d*) is given (for $d \leq k$) by:

$$E = \frac{N!}{\prod_{j=0}^{k} (n_j!)} 2^{(N-n_0)} \tag{1d}$$

By way of example, figure 3 shows a positive quadrant in $X_2$, in which there are two (edge-edge connected) elements lying on the circle with $d = 2$: $\vec{q} = (0, 2)$ and $\vec{q} = (2, 0)$. According to (1c), their number is indeed $\frac{N!}{n_0! \cdot n_1! \cdot n_2!} = \frac{2!}{1! \cdot 0! \cdot 1!} = 2$. According to (1d) the number of elements lying on the **complete** circle with $d = 2$ is not four (the number of quadrants) times as many, but only twice, because both elements in the first quadrant are shared with another quadrant.

Likewise, the number of (point-edge or knight's move connected) elements in the first quadrant lying on the circle with $d = \sqrt{5}$ is: $\frac{2!}{0! \cdot 1! \cdot 1!} = 2$. Because in this case none of the elements is shared among partitions (quadrants), their number on the full circle is 8, simply $2^2$ (the number of quadrants) times as many.

In 3 dimensions the number of (knight's move connected) elements on a sphere with $d = 3$ in the positive octant is: $\frac{N!}{n_0! \cdot n_1! \cdot n_2!} = \frac{3!}{0! \cdot 1! \cdot 2!} = 3$, i.e. $\vec{q} = (1, 2, 2)$, $\vec{q} = (2, 1, 2)$ and $\vec{q} = (2, 2, 1)$. Because again all the vector components are non-zero, the number of elements on the **complete** sphere with $d = 3$ will be 24, $2^3$ times as large.

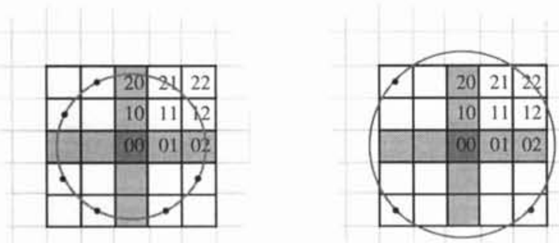


Figure 3 The vectors in the positive quadrant in $X_2$ for k = 2.

Table 1 shows E, the number of elements within $M_N^n$ and **on** $S_N^d$ for some dimensions and neighbourhood sizes, V, the number of elements within $M_N^n$ and **within** $S_N^d$, and G = V−1, commonly used to indicate the connectivity between elements. We will refer to the sphere radius *d* as the **connectivity distance *d***.

Unfortunately, for larger values of k and N, even within one partition more than one set of values $n_j$, j=0...k (cf. 1a), that results in the same *d*, and E(N,k,*d*), may occur. For N=2, this occurs with k≥5, because, e.g., |0,5| = |3,4|.

| Neighbourhood size: $M_N^n$ = | Connectivity type: | Typical $\vec{q}$ = | Sphere radius: $d$ = | Elements **on** sphere: E = | Elements **in** sphere: V = | Neighbourhood connectivity: G = |
|---|---|---|---|---|---|---|
| 1 | edge | [0] | √0 = 0 | 1•1 = 1 | 1 | 0 |
| 3 | point | [1] | √1 = 1 | 1•2 = 2 | 3 | **2** |
| 5 | point-point | [2] | √4 = 2 | 1•2 = 2 | 5 | 4 |
| 1x1 | face | (0,0) | √0 = 0 | 1•1 = 1 | 1 | 0 |
| 3x3 | edge | (0,1) | √1 = 1 | 2•2 = 4 | 5 | **4** |
| | point | (1,1) | √2 = 1.4 | 1•4 = 4 | 9 | **8** |
| 5x5 | edge-edge | (0,2) | √4 = 2 | 2•2 = 4 | 13 | 12 |
| | point-edge | (1,2) | √5 = 2.2 | 2•4 = 8 | 21 | 20 |
| | point-point | (2,2) | √8 = 2.8 | 1•4 = 4 | 25 | 24 |
| 1x1x1 | volume | (0,0,0) | √0 = 0 | 1•1 = 1 | 1 | 0 |
| 3x3x3 | face | (0,0,1) | √1 = 1 | 3•2 = 6 | 7 | **6** |
| | edge | (0,1,1) | √2 = 1.4 | 3•4 =12 | 19 | **18** |
| | point | (1,1,1) | √3 = 1.7 | 1•8 = 8 | 27 | **26** |
| 5x5x5 | face-face | (0,0,2) | √4 = 2 | 3•2 = 6 | 33 | 32 |
| | edge-face | (0,1,2) | √5 = 2.2 | 6•4 = 24 | 57 | 56 |
| | point-face | (1,1,2) | √6 = 2.5 | 3•8 = 24 | 81 | 80 |
| | edge-edge | (0,2,2) | √8 = 2.8 | 3•4 = 12 | 93 | 92 |
| | point-edge | (1,2,2) | √9 = 3 | 3•8 = 24 | 117 | 116 |
| | point-point | (2,2,2) | √12= 3.4 | 1•8 = 8 | 125 | 124 |
| 1x1x1x1 | hypervolume | (0,0,0,0) | √0 = 0 | 1•1 = 1 | 1 | 0 |
| 3x3x3x3 | volume | (0,0,0,1) | √1 = 1 | 4•2 = 8 | 9 | **8** |
| | face | (0,0,1,1) | √2 = 1.4 | 6•4 = 24 | 33 | **32** |
| | edge | (0,1,1,1) | √3 = 1.7 | 4•8 = 32 | 65 | **64** |
| | point | (1,1,1,1) | √4 = 2 | 1•16 = 16 | 81 | **80** |

Table 1 Connectivity as a function of dimension, neighbourhood size and distance.

Likewise, for N=3, with k≥3, e.g. |0,0,3| = |1,2,2|. For N≥4, it occurs in all neighbourhoods with k≥2: e.g. |0,0,0,2| = |1,1,1,1|, |0,0,0,0,2| = |0,1,1,1,1|, etc. This ambiguity for $d$ does **not** exist for k=1, because in this case $d=\sqrt{n_1}$, which is different for every distinct $n_1$-$n_0$-combination, and no other combinations than these (of vector components with values of either 0 or 1) are possible. To avoid this ambiguity, we will restrict ourselves further to k=1.

Note that there is no fundamental objection against the concept of connectivity defined in larger neighbourhoods, such as with k = 2, when a knight's move is allowed to connect pixels. (E.g. applicable in the shortest route (i.e. skeleton ?) with knight moves on a chessboard from a to b). It is just the description using only N, K and $d$, that is no longer unambiguous with N≥4.

Applying the restriction that k = 1, i.e., allowing only $3^N$ neighbourhoods, the connectivity between two **image elements** will be denoted by $G_N^d$.

Consequently, we will make the following definition:

- *An element $\vec{p} + \vec{q}$ is said to be $G_N^d$-connected to $\vec{p}$ if both have the same value, $|\vec{q}| \leq d$ and $\max(q_i) = 1$.*

For example the element (1, 1, 1) is said to be $G_3^{\sqrt{3}}$-(or 26, or point-point) connected to (0, 0, 0).

## 4. Background connectivity.

In $X_N$ the enclosing background's task is to separate objects. Consequently a separating unit layer of background should be thick enough to prevent the touching of foreground objects. This thickness depends on the connectivity and the smallest objects to measure this thickness on are N dimensional "tiles"; $2^N$ connected objects in a $2^N$ neighbourhood.. Assume we would like to perforate one (probed) 2-D tile with another (perforating) 2-D tile, see figure 4, then:

- *The layer thickness $D_N^d$ of a probed tile is the length of the center line segment of the perforating tile.*

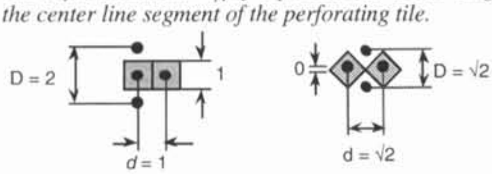


Figure 4 Layer thickness and connectivity distance in $X_2$.

The layer thickness $\{D_N^d \mid (d > 1)\}$ is always $d$, the connectivity distance of the probed tile, due to the fact that on any grid position the center line segment of the perforating tile and the center of the probed tile are in not in line but intersect halfway ($x = \frac{1}{2}d$), where the surface-to-surface distance is zero [1].

- *A $G_N^d$-connected tile can perforate a tile $T_{N,N}^d$ with layer thickness $D_N^d$, if the connectivity distance d of the $G_N^d$-connected tile is $\geq$ the layer thickness $D_N^d$ of $T_{N,\tilde{N}}^d$.*

This leads to the conclusion that $G_N^1$-connected objects are the only objects that cannot be perforated by, nor can perforate any other $G_N^d$-connected object, and that all other $\{G_N^d \mid (d > 1)\}$-connected objects can perforate any other $\{G_N^d \mid (d > 1)\}$-connected object.

Note that $G_N^1$ has this property because the (hyper-square) image element has a layer thickness unequal to zero in all dimensions. For all higher connectivities there is at least in one of the dimensions a layer thickness zero. Consequently:

- *A reasonable choice for the background connectivity in image $X_N$ is the lowest possible connectivity in the image ($G_N^1$), as it prevents leakage of foreground via background and it is not able to perforate the foreground.*

The latter property is useful in propagation (labeling) operations, where propagation from the image edge over the background should stop at object borders.

## 5. Conclusions.

We have derived an expression for the connectivity $G_N^d$ in an N dimensional image, based on the number of image elements within a hypercubic $3^N$ neighbourhood and within a hypersphere with radius $d$. For low dimensional images ($N \leq 3$) this leads to the more commonly known notations from table 1, e.g. point-connected or 26-connected.

We concluded that in any dimension the lowest connectivity is the best choice for the background of the object, as it prevents leaking of foreground and it is not able to perforate the foreground.

## 6. References.

[1] P.P. Jonker, "Morphological Image Processing: Architecture and VLSI design. : Kluwer Deventer / Dordrecht / Boston, 1992, ISBN 90-201-2766-7

[2] S. Lobregt, P.W. Verbeek and F.C.A. Groen, "Three dimensional skeletonization: Principle and algorithm" IEEE Trans. Patt. Anal. Machine Intell. vol. 2, pp. 75-77, 1980.

[3] J. Toriwaki, S. Yokoi, T. Yonekura and F. Fukumura "Topological properties and topological-preserving transformation of a three dimensional binary picture" in Proc. Int. Conf. Patt. Recogn., Munich, 1982, pp. 414-419.

[4] R.C. Gonzales, R.E. Woods "Digital Image Processing" Addison Wesley Publ. Cie, 1992, ISBN 0-201-50803-6

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[1] Note that these connectivity problems do not exist in images sampled on a hexagonal grid or equivalent in higher dimensions. On the intersection point the surface-to-surface distance is minimal but never zero.