A Complete Generalized Adjustment Criterion

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Abstract

Covariate adjustment is a widely used approach to estimate total causal effects from observational data. Several graphical criteria have been developed in recent years to identify valid covariates for adjustment from graphical causal models. These criteria can handle multiple causes, latent confounding, or partial knowledge of the causal structure; however, their diversity is confusing and some of them are only sufficient, but not necessary. In this paper, we present a criterion that is necessary and sufficient for four different classes of graphical causal models: directed acyclic graphs (DAGs), maximum ancestral graphs (MAGs), completed partially directed acyclic graphs (CPDAGs), and partial ancestral graphs (PAGs). Our criterion subsumes the existing ones and in this way unifies adjustment set construction for a large set of graph classes.

1 INTRODUCTION

Which covariates do we need to adjust for when estimating total causal effects from observational data? Graphical causal modeling allows to answer this question constructively, and contributed fundamental insights to the theory of adjustment in general. For instance, a simple example known as the "M-bias graph" shows that it is not always appropriate to adjust for all observed (pre-treatment) covariates (Shrier, 2008; Rubin, 2008). A few small graphs also suffice to refute the "Table 2 fallacy" (Westreich and Greenland, 2013), which is the belief that the coefficients in multiple regression models are "mutually adjusted". Thus, causal graphs had substantial impact on theory and practice of covariate adjustment (Shrier and Platt, 2008).

The practical importance of covariate adjustment has inspired a growing body of theoretical work on graphical criteria that are sufficient and/or necessary for adjustment. Pearl's back-door criterion (Pearl, 1993) is probably the most well-known, and is sufficient but not necessary for adjustment in DAGs. Shpitser et al. (2012) adapted the back-door criterion to a necessary and sufficient graphical criterion for adjustment in DAGs. Others considered graph classes other than DAGs, which can represent structural uncertainty. van der Zander et al. (2014) gave necessary and sufficient graphical criteria for MAGs that allow for unobserved variables (latent confounding). Maathuis and Colombo (2015) presented a generalized back-door criterion for DAGs, MAGs, CPDAGs and PAGs, where CPDAGs and PAGs represent Markov equivalence classes of DAGs or MAGs, respectively, and can be inferred directly from data (see, e.g., Spirtes et al., 2000; Chickering, 2003; Colombo et al., 2012; Claassen et al., 2013; Colombo and Maathuis, 2014). The generalized back-door criterion is sufficient but not necessary for adjustment.

In this paper, we extend the results of Shpitser et al. (2012), van der Zander et al. (2014) and Maathuis and Colombo (2015) to derive a single necessary and sufficient adjustment criterion that holds for all four graph classes: DAGs, CPDAGs, MAGs and PAGs.

To illustrate the use of our generalized adjustment criterion, suppose we are given the CPDAG in Figure 1a and we want to estimate the total causal effect of X on Y. Our criterion will inform us that the set $\{A, Z\}$ is an adjustment set for this CPDAG, which means that it is an adjustment set in every DAG that the CPDAG represents (Figure 1b). Hence, we can estimate the causal effect without knowledge of the full causal structure. In a similar manner, by applying our criterion to a MAG or a PAG, we find adjustment sets that are valid for all DAGs represented by this MAG or PAG. Our criterion finds such adjustment sets whenever they exist; else, our knowledge of the model structure is insufficient to compute the desired causal effect by covariate adjustment. We hope that this ability to allow for incomplete structural knowledge or latent confounding or both will help address concerns that graphical causal modelling "assumes that all [...] DAGs have been properly specified" (West and Koch, 2014).

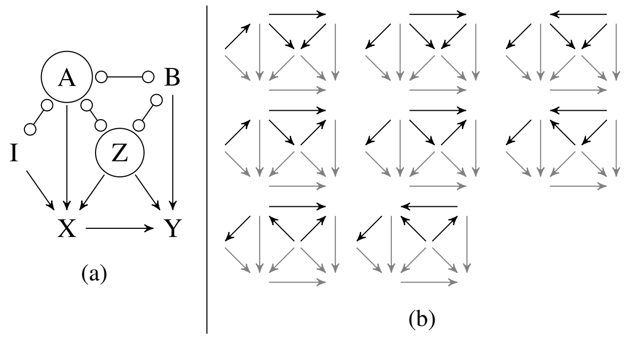

Figure 1: (a) A CPDAG in which, according to our criterion, $\{A, Z\}$ is an adjustment set for the total causal effect of X on Y. (b) The Markov equivalence class of (a), with node labels removed for simplicity and varying edges highlighted. An adjustment set for a CPDAG (PAG) is one that works in all DAGs (MAGs) of the Markov equivalence class.

We note that, although we can find all causal effects that are identifiable by covariate adjustment, we generally do not find all identifiable causal effects, since some effects may be identifiable by other means, such as Pearl's front-door criterion (Pearl, 2009, Section 3.3.2) or the ID algorithm (Tian and Pearl, 2002; Shpitser and Pearl, 2006). We also point out that MAGs and PAGs are in principle not only able to represent unobserved confounding, but can also account for unobserved selection variables. However, in this paper we assume that there are no unobserved selection variables. This restriction is mainly due to the fact that selection bias often renders it impossible to identify causal effects using just covariate adjustment. Bareinboim et al. (2014) discuss these problems and present creative approaches to work around them, e.g., by combining data from different sources. We leave the question whether adjustment could be combined with such auxiliary methods aside for future research.

2 PRELIMINARIES

Throughout the paper we denote sets in bold uppercase letters (e.g., $\mathbf{S}$), graphs in calligraphic font (e.g., $\mathcal{G}$) and nodes in a graph in uppercase letters (e.g., X)

Nodes and edges. A graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ consists of a set of nodes (variables) $\mathbf{V} = \{X_1, \dots, X_p\}$ and a set of edges $\mathbf{E}$.

There is at most one edge between any pair of nodes, and nodes are called *adjacent* if they are connected by an edge. Every edge has two edge marks that can be arrowheads, tails or circles. Edges can be *directed* $\rightarrow$, *bidirected* $\leftrightarrow$, *non-directed* $\circ\!\!-\!\!\circ$ or *partially directed* $\circ\!\!\rightarrow$. We use $\bullet$ as a stand in for any of the allowed edge marks. An edge is *into* (*out of*) a node X if the edge has an arrowhead (tail) at X. A *directed graph* contains only directed edges. A *mixed graph* may contain directed and bi-directed edges. A *partial mixed graph* may contain any of the described edges. Unless stated otherwise, all definitions apply for partial mixed graphs.

Paths. A *path* p from X to Y in $\mathcal{G}$ is a sequence of distinct nodes $\langle X, \dots, Y \rangle$ in which every pair of successive nodes is adjacent in $\mathcal{G}$. A node V *lies on a path* p if V occurs in the sequence of nodes. The *length* of a path equals the number of edges on the path. A *directed path* from X to Y is a path from X to Y in which all edges are directed towards Y, i.e., $X \rightarrow \cdots \rightarrow Y$. A directed path is also called a *causal path*. A *possibly directed path* (*possibly causal path*) from X to Y is a path from X to Y that has no arrowhead pointing to X. A path from X to Y that is not possibly causal is called a *non-causal path* from X to Y. A directed path from X to Y together with an edge $Y \rightarrow X$ ($Y \leftrightarrow X$) forms an *(almost) directed cycle*. For two disjoint subsets $\mathbf{X}$ and $\mathbf{Y}$ of $\mathbf{V}$, a path from $\mathbf{X}$ to $\mathbf{Y}$ is a path from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$. A path from $\mathbf{X}$ to $\mathbf{Y}$ is *proper* if only its first node is in $\mathbf{X}$.

Subsequences and subpaths. A *subsequence* of a path p is a sequence of nodes obtained by deleting some nodes from p without changing the order of the remaining nodes. A subsequence of a path is not necessarily a path. For a path $p = \langle X_1, X_2, \dots, X_m \rangle$, the *subpath* from X_i to X_k ($1 \leq i \leq k \leq m$) is the path $p(X_i, X_k) = \langle X_i, X_{i+1}, \dots, X_k \rangle$. We denote the concatenation of paths by $\oplus$, so that for example $p = p(X_1, X_k) \oplus p(X_k, X_m)$. We use the convention that we remove any loops that may occur due to the concatenation, so that the result is again a path.

Ancestral relationships. If $X \rightarrow Y$, then X is a *parent* of Y. If there is a (possibly) directed path from X to Y, then X is a *(possible) ancestor* of Y, and Y is a *(possible) descendant* of X. Every node is also a descendant and an ancestor of itself. The sets of parents and (possible) descendants of X in $\mathcal{G}$ are denoted by $\mathrm{Pa}(X, \mathcal{G})$ and $(\mathrm{Poss})\mathrm{De}(X, \mathcal{G})$ respectively. For a set of nodes $\mathbf{X} \subseteq \mathbf{V}$, we have $\mathrm{Pa}(\mathbf{X}, \mathcal{G}) = \cup_{X \in \mathbf{X}} \mathrm{Pa}(X, \mathcal{G})$, with analogous definitions for $(\mathrm{Poss})\mathrm{De}(\mathbf{X}, \mathcal{G})$.

Colliders and shields. If a path p contains $X_i \bullet\!\!\rightarrow X_j \leftarrow\!\!\bullet X_k$ as a subpath, then X_j is a *collider* on p. A *collider path* is a path on which every non-endpoint node is a collider. A path of length one is a trivial collider path. A path $\langle X_i, X_j, X_k \rangle$ is an *(un)shielded triple* if X_i and X_k are (not) adjacent. A path is *unshielded* if all successive triples on the path are unshielded. Otherwise the path is *shielded*. A node X_j is a *definite non-collider* on a path p if there is at least one edge out of X_j on p, or if $X_i \bullet\!\!-\!\!\circ X_j \circ\!\!-\!\!\bullet X_k$ is a subpath of p and $\langle X_i, X_j, X_k \rangle$ is an unshielded triple. A node is of *definite status* on a path if it is a collider or a definite non-collider on the path. A path p is of definite status if every non-endpoint node on p is of definite status. An unshielded path is always of definite status, but a definite status path is not always unshielded.

m-separation and m-connection. A definite status path p between nodes X and Y is *m-connecting* given a set of nodes $\mathbf{Z}$ ($X, Y \notin \mathbf{Z}$) if every definite non-collider on p is not in $\mathbf{Z}$, and every collider on p has a descendant in $\mathbf{Z}$. Otherwise $\mathbf{Z}$ blocks p. If $\mathbf{Z}$ blocks all definite status paths between X and Y, we say that X and Y are m-separated given $\mathbf{Z}$. Otherwise, X and Y are m-connected given $\mathbf{Z}$. For pairwise disjoint subsets $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ of $\mathbf{V}$, $\mathbf{X}$ and $\mathbf{Y}$ are m-separated given $\mathbf{Z}$ if X and Y are m-separated by $\mathbf{Z}$ for any $X \in \mathbf{X}$ and $Y \in \mathbf{Y}$. Otherwise, $\mathbf{X}$ and $\mathbf{Y}$ are m-connected given $\mathbf{Z}$.

Causal Bayesian networks. A directed graph without directed cycles is a *directed acyclic graph (*DAG*)*. A Bayesian network for a set of variables $\mathbf{V} = \{X_1, \dots, X_p\}$ is a pair $(\mathcal{G}, f)$, where $\mathcal{G}$ is a DAG, and f is a joint probability density for $\mathbf{V}$ that factorizes according to the conditional independence relationships described via m-separation, that is $f(\mathbf{V}) = \prod_{i=1}^{p} f(X_i|Pa(X_i, \mathcal{G}))$ (Pearl, 2009). We call a DAG causal when every edge $X_i \to X_j$ in $\mathcal{G}$ represents a direct causal effect of X_i on X_j. A Bayesian network $(\mathcal{G}, f)$ is a *causal Bayesian network* if $\mathcal{G}$ is a causal DAG. If a causal Bayesian network is given and all variables are observed one can easily derive post-intervention densities. In particular, we consider interventions $do(\mathbf{X} = \mathbf{x})$ ($\mathbf{X} \subseteq \mathbf{V}$), which represent outside interventions that set $\mathbf{X}$ to $\mathbf{x}$ (Pearl, 2009):

$$f(\mathbf{v}|do(\mathbf{X} = \mathbf{x})) = \\ = \begin{cases} \prod_{X_i \in \mathbf{V} \setminus \mathbf{X}} f(x_i|Pa(x_i, \mathcal{G})), & \text{for values of } \mathbf{V} \\ & \text{consistent with } \mathbf{x}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Equation (1) is known as the truncated factorization formula (Pearl, 2009) or the g-formula (Robins, 1986).

Maximal ancestral graph. A mixed graph $\mathcal{G}$ without directed cycles and almost directed cycles is called *ancestral*. A *maximal ancestral graph (*MAG*)* is an ancestral graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ where every two non-adjacent nodes X and Y in $\mathcal{G}$ can be m-separated by a set $\mathbf{Z} \subseteq \mathbf{V} \setminus \{X, Y\}$. A DAG with unobserved variables can be uniquely represented by a MAG that preserves the ancestral and m-separation relationships among the observed variables (Richardson and Spirtes, 2002). The MAG of a causal DAG is a *causal* MAG.

Markov equivalence. Several DAGs can encode the same conditional independence information via m-separation. Such DAGs form a *Markov equivalence class* which can be described uniquely by a *completed partially directed acyclic graph (*CPDAG*)* . Several MAGs can also encode the same conditional independence information. Such MAGs form a Markov equivalence class which can be described uniquely by a *partial ancestral graph (*PAG*)* (Richardson and Spirtes, 2002; Ali et al., 2009). We denote all DAGs (MAGs) in the Markov equivalence class described by a CPDAG (PAG) $\mathcal{G}$ by $[\mathcal{G}]$.

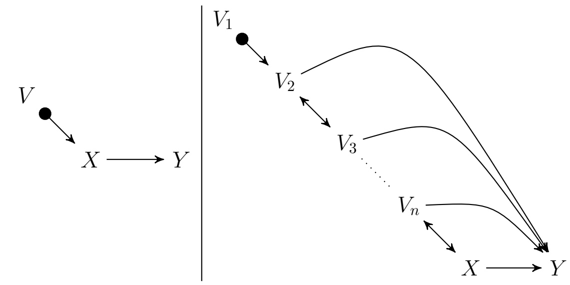

Figure 2: Two configurations where the edge $X \to Y$ is visible.

Consistent density. A density f is *consistent* with a causal DAG $\mathcal{D}$ if the pair $(\mathcal{D}, f)$ forms a causal Bayesian network. A density f is consistent with a causal MAG $\mathcal{M}$ if there exists a causal Bayesian network $(\mathcal{D}', f')$, such that $\mathcal{M}$ represents $\mathcal{D}'$ and f is the observed marginal of f'. A density f is consistent with a CPDAG (PAG) $\mathcal{G}$ if it is consistent with a DAG (MAG) in $[\mathcal{G}]$.

Visible and invisible edges. All directed edges in DAGs and CPDAGs are said to be visible. Given a MAG $\mathcal{M}$ or a PAG $\mathcal{G}$, a directed edge $X \to Y$ is *visible* if there is a node V not adjacent to Y such that there is an edge between V and X that is into X, or if there is a collider path from V to X that is into X and every non-endpoint node on the path is a parent of Y. Otherwise, $X \to Y$ is said to be *invisible* (Zhang, 2006; Maathuis and Colombo, 2015).

A directed visible edge $X \to Y$ means that there are no latent confounders between X and Y.

3 MAIN RESULT

Throughout, let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ represent a DAG, CPDAG, MAG or PAG, and let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be pairwise disjoint subsets of $\mathbf{V}$, with $\mathbf{X} \neq \emptyset$ and $\mathbf{Y} \neq \emptyset$. Here $\mathbf{X}$ represents the intervention variables and $\mathbf{Y}$ represents the set of response variables, i.e., we are interested in the causal effect of $\mathbf{X}$ on $\mathbf{Y}$.

We define sound and complete graphical conditions for adjustment sets relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. Thus, if a set $\mathbf{Z}$ satisfies our conditions relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 3.3), then it is a valid adjustment set for calculating the causal effect of $\mathbf{X}$ on $\mathbf{Y}$ (Definition 3.1), and every existing valid adjustment set satisfies our conditions (see Theorem 3.4). First, we define what we mean by an adjustment set.

Definition 3.1. ***(Adjustment set;*** *Maathuis and Colombo, 2015) Let $\mathcal{G}$ represent a* DAG, CPDAG, MAG *or* PAG. *Then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$*

in $\mathcal{G}$ if for any density f consistent with $\mathcal{G}$ we have

$$f(\mathbf{y}|do(\mathbf{x})) = \begin{cases} f(\mathbf{y}|\mathbf{x}) & \textit{if } \mathbf{Z} = \emptyset, \\ \int_{\mathbf{Z}} f(\mathbf{y}|\mathbf{x},\mathbf{z})f(\mathbf{z})d\mathbf{z} = E_{\mathbf{Z}}\{f(\mathbf{y}|\mathbf{z},\mathbf{x})\} & \textit{otherwise.} \end{cases}$$

If $\mathbf{X} = \{X\}$ and $\mathbf{Y} = \{Y\}$, we call $\mathbf{Z}$ an adjustment set relative to (X, Y) in the given graph.

To define our generalized adjustment criterion, we introduce the concept of *amenability*:

Definition 3.2. ***(Amenability for** $DAGs$**,** $CPDAGs$**,** $MAGs$ **and** $PAGs$**)*** *A* DAG, CPDAG, MAG *or* PAG *$\mathcal{G}$ is said to be adjustment amenable, relative to $(\mathbf{X}, \mathbf{Y})$ if every possibly directed proper path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a visible edge out of $\mathbf{X}$.*

For conciseness, we will also write "amenable" instead of "adjustment amenable". The intuition behind the concept of amenability is the following. In MAGs and PAGs, directed edges $X \to Y$ can represent causal effects, but also mixtures of causal effects and latent confounding; in CPDAGs and PAGs, there are edges with unknown direction. This complicates adjustment because paths containing such edges can correspond to causal paths in some represented DAGs and to non-causal paths in others. For instance, when the graph $X \to Y$ is interpreted as a DAG, the empty set is a valid adjustment set with respect to (X, Y) because there is only one path from X to Y, which is causal. When the same graph is however interpreted as a MAG, it can still represent the DAG $X \to Y$, but also for example the DAG $X \to Y$ with a non-causal path $X \leftarrow L \to Y$ where L is latent. A similar problem arises in the CPDAG $X \circ\!\!-\!\!\circ Y$.

We will show that for a graph $\mathcal{G}$ that is not amenable relative to $(\mathbf{X}, \mathbf{Y})$, there is no adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in the sense of Definition 3.1 (see Lemma 5.2). Note that every DAG is amenable, since all edges in a DAG are visible and directed. For MAGs, our notion of amenability reduces to the one defined by van der Zander et al. (2014).

We now introduce our Generalized Adjustment Criterion (GAC) for DAGs, $CPDAG$s, MAGs and PAGs.

Definition 3.3. ***(Generalized Adjustment Criterion (GAC))*** *Let $\mathcal{G}$ represent a* DAG, CPDAG, MAG *or* PAG*. Then $\mathbf{Z}$ satisfies the generalized adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if the following three conditions hold:*

(0) $\mathcal{G}$ is adjustment amenable relative to $(\mathbf{X}, \mathbf{Y})$, and

(1) no element in $\mathbf{Z}$ is a possible descendant in $\mathcal{G}$ of any $W \in \mathbf{V} \setminus \mathbf{X}$ which lies on a proper possibly causal path from $\mathbf{X}$ to $\mathbf{Y}$, and

(2) all proper definite status non-causal paths in $\mathcal{G}$ from $\mathbf{X}$ to $\mathbf{Y}$ are blocked by $\mathbf{Z}$.

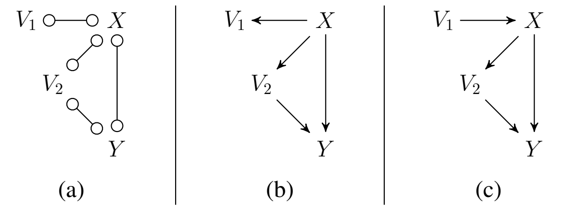

Figure 3: (a) PAG $\mathcal{P}$, (b) MAG $\mathcal{M}_1$, (c) MAG $\mathcal{M}_2$ used in Example 4.2.

Note that condition (0) does not depend on $\mathbf{Z}$. In other words, if condition (0) is violated, then there is no set $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ that satisfies the generalized adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$.

Condition (1) defines a set of nodes that cannot be used in an adjustment set. Denoting this set of forbidden nodes by

$$\mathbf{F}_{\mathcal{G}}(\mathbf{X}, \mathbf{Y}) = \{W' \in \mathbf{V} : W' \in \mathrm{PossDe}(W, \mathcal{G}) \text{ for some } W \notin \mathbf{X} \text{ which lies on a proper possibly causal path from } \mathbf{X} \text{ to } \mathbf{Y}\}, \quad (2)$$

condition (1) can be stated as: $\mathbf{Z} \cap \mathbf{F}_{\mathcal{G}}(\mathbf{X}, \mathbf{Y}) = \emptyset$. We will sometimes use this notation in examples and proofs.

We now give the main theorem of this paper.

Theorem 3.4. *Let $\mathcal{G}$ represent a* DAG, CPDAG, MAG *or* PAG*. Then $\mathbf{Z}$ is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 3.1) if and only if $\mathbf{Z}$ satisfies the generalized adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ (Definition 3.3).*

4 EXAMPLES

We now provide some examples that illustrate how the generalized adjustment criterion can be applied.

Example 4.1. *We first return to the example of the Introduction. Consider the* CPDAG *$\mathcal{C}$ in Figure 1a. Note that $\mathcal{C}$ is amenable relative to (X, Y) and that $\mathbf{F}_{\mathcal{C}}(X, Y) = \{Y\}$. Hence, any node other than X and Y can be used in an adjustment set. Note that every definite status non-causal path p from X to Y has one of the following paths as a subsequence: $p_1 = \langle X, Z, Y \rangle$ and $p_2 = \langle X, A, B, Y \rangle$, and nodes on p that are not on p_1 or p_2 are non-colliders on p. Hence, if we block p_1 and p_2, then we block all definite status non-causal paths from X to Y. This implies that any superset of $\{Z, A\}$ and $\{Z, B\}$ is an adjustment set relative to (X, Y) in $\mathcal{C}$, and all adjustment sets are given by: $\{Z, A\}$, $\{Z, B\}, \{Z, A, I\}, \{Z, B, I\}, \{Z, A, B\}$ and $\{Z, A, B, I\}$.*

Example 4.2. *To illustrate the concept of amenability, consider Figure 3 with a* PAG *$\mathcal{P}$ in (a), and two* MAG*s $\mathcal{M}_1$ and $\mathcal{M}_2$ in $[\mathcal{P}]$ in (b) and (c). Note that $\mathcal{P}$ and $\mathcal{M}_1$ are not amenable relative to (X, Y). For $\mathcal{P}$ this is due to the*

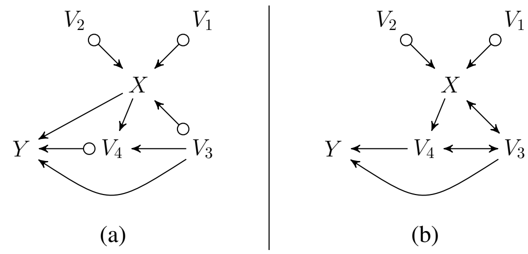

Figure 4: (a) PAG $\mathcal{P}_1$, (b) PAG $\mathcal{P}_2$ used in Example 4.3.

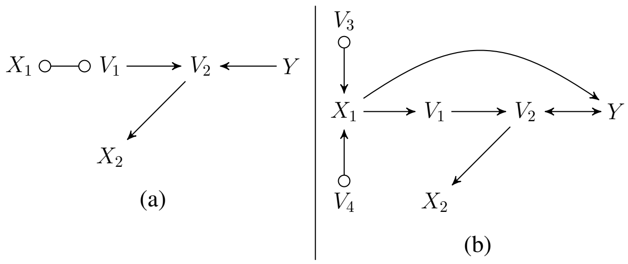

Figure 5: (a) CPDAG $\mathcal{C}$, (b) PAG $\mathcal{P}$ used in Example 4.4.

path $X \circ\!\!-\!\!\circ Y$, and for $\mathcal{M}_1$ this is due to the invisible edge $X \to Y$. On the other hand, $\mathcal{M}_2$ is amenable relative to (X, Y), since the edges $X \to Y$ and $X \to V_2$ are visible due to the edge $V_1 \to X$ and the fact that V_1 is not adjacent to Y or V_2. Since there are no proper definite status non-causal paths from X to Y in $\mathcal{M}_2$, it follows that the empty set satisfies the generalized adjustment criterion relative to (X, Y) in $\mathcal{M}_2$. Finally, note that $\mathcal{M}_1$ could also be interpreted as a DAG. *In that case it would be amenable relative to (X, Y). This shows that amenability depends crucially on the interpretation of the graph.*

Example 4.3. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the* PAG*s in Figure 4(a) and Figure 4(b), respectively. Both* PAG*s are amenable relative to (X, Y). We will show that there is an adjustment set relative to (X, Y) in $\mathcal{P}_1$ but not in $\mathcal{P}_2$. This illustrates that amenability is not a sufficient criterion for the existence of an adjustment set.*

We first consider $\mathcal{P}_1$. Note that $\mathbf{F}_{\mathcal{P}_1}(X, Y) = \{V_4, Y\}$ is the set of nodes that cannot be used for adjustment. There are two proper definite status non-causal paths from X to Y in $\mathcal{P}_1$: $X \leftarrow\!\!\circ V_3 \to Y$ and $X \to V_4 \leftarrow V_3 \to Y$. These are blocked by any set containing V_3. Hence, all sets satisfying the GAC relative to (X, Y) in $\mathcal{P}_1$ are: $\{V_3\}$, $\{V_1, V_3\}$, $\{V_2, V_3\}$ and $\{V_1, V_2, V_3\}$.

We now consider $\mathcal{P}_2$. Note that $\mathbf{F}_{\mathcal{P}_2}(X, Y) = \mathbf{F}_{\mathcal{P}_1}(X, Y) = \{V_4, Y\}$. There are three proper definite status non-causal paths from X to Y in $\mathcal{P}_2$: $p_1 = X \leftrightarrow V_3 \to Y$, $p_2 = X \leftrightarrow V_3 \leftrightarrow V_4 \to Y$ and $p_3 = X \to V_4 \leftrightarrow V_3 \to Y$. To block p_1, we must also use V_3. This implies that we must use V_4 to block p_2. But $V_4 \in \mathbf{F}_{\mathcal{P}_2}(X, Y)$. Hence, there is no set $\mathbf{Z}$ that satisfies the GAC relative to (X, Y) in $\mathcal{P}_2$.

Example 4.4. *Let $\mathbf{X} = \{X_1, X_2\}$ and $\mathbf{Y} = \{Y\}$ and consider the* CPDAG *$\mathcal{C}$ and the* PAG *$\mathcal{P}$ in Figures 5(a) and 5(b). We will show that for both graphs there is no set that satisfies the generalized back-door criterion of Maathuis and Colombo (2015) relative to $(\mathbf{X}, \mathbf{Y})$, but there are sets that satisfy the generalized adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in these graphs.*

Recall that a set $\mathbf{Z}$ satisfies the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ and a CPDAG *(*PAG*) $\mathcal{G}$ if $\mathbf{Z}$ contains no possible descendants of $\mathbf{X}$ in $\mathcal{G}$ and if for every $X \in \mathbf{X}$ the set $\mathbf{Z} \cup \mathbf{X} \setminus \{X\}$ blocks every definite status path from X to every $Y \in \mathbf{Y}$ in $\mathcal{G}$ that does not start with a visible edge out of X.*

We first consider the CPDAG *$\mathcal{C}$. To block the path $X_2 \leftarrow V_2 \leftarrow Y$, we must use node V_2, but $V_2 \in \mathrm{PossDe}(X_1, \mathcal{C})$. Hence, no set $\mathbf{Z}$ can satisfy the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{C}$. However, $\{V_1, V_2\}$ satisfies the generalized adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{C}$.*

We now consider $\mathcal{P}$. To block the path $X_2 \leftarrow V_2 \leftrightarrow Y$, we must use node V_2. But, $V_2 \in \mathrm{De}(X_1, \mathcal{P})$ and thus there is no set satisfying the generalized back-door criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{P}$. However, sets $\{V_1, V_2\}$, $\{V_1, V_2, V_3\}$, $\{V_1, V_2, V_4\}$, $\{V_1, V_2, V_3, V_4\}$ all satisfy the generalized adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{P}$.

5 PROOF OF THEOREM 3.4

For DAGs and MAGs, our generalized adjustment criterion reduces to the following adjustment criterion:

Definition 5.1. *(**Adjustment Criterion (AC)**) Let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ represent a* DAG *or* MAG *. Then $\mathbf{Z}$ satisfies the adjustment criterion relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if the following three conditions hold:*

(0) $\mathcal{G}$ is adjustment amenable with respect to $(\mathbf{X}, \mathbf{Y})$, and*

(a) no element in $\mathbf{Z}$ is a descendant in $\mathcal{G}$ of any $W \in \mathbf{V} \setminus \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$, and

(b) all proper non-causal paths in $\mathcal{G}$ from $\mathbf{X}$ to $\mathbf{Y}$ are blocked by $\mathbf{Z}$.

This adjustment criterion is a slightly reformulated but equivalent version of the adjustment criterion of Shpitser et al. (2012) for DAGs and of van der Zander et al. (2014) for MAGs, with amenability directly included in the criterion. This adjustment criterion was shown to be sound and complete for DAGs (Shpitser et al., 2012; Shpitser, 2012) and MAGs (van der Zander et al., 2014). We therefore only need to prove Theorem 3.4 for CPDAGs and PAGs.

To this end, we need three main lemmas, given below. Throughout, we let $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ represent a CPDAG or

$$
\begin{tikzcd}
 & \text{Lemma 5.5} \arrow[r] & \text{Lemma 5.2} \arrow[d] \\
 & \text{Lemma 5.3} \arrow[r] & \textbf{Theorem 3.4} \\
\text{Lemma 5.7} \arrow[r] & \text{Lemma 5.8} \arrow[r] & \text{Lemma 5.4} \arrow[u]
\end{tikzcd}
$$

Figure 6: Proof structure of Theorem 3.4.

a PAG, and we let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be pairwise disjoint subsets of $\mathbf{V}$, with $\mathbf{X} \neq \emptyset$ and $\mathbf{Y} \neq \emptyset$. We use GAC and AC to refer to the generalized adjustment criterion (Definition 3.3) and adjustment criterion (Definition 5.1), respectively.

Lemma 5.2 is about condition (0) of the GAC:

Lemma 5.2. *If a* CPDAG *(*PAG*)* $\mathcal{G}$ *satisfies condition (0) of the GAC relative to* $(\mathbf{X}, \mathbf{Y})$*, then every* DAG *(*MAG*) in* $[\mathcal{G}]$ *satisfies condition (0*) of the AC relative to* $(\mathbf{X}, \mathbf{Y})$*. On the other hand, if* $\mathcal{G}$ *violates condition (0) of the GAC relative to* $(\mathbf{X}, \mathbf{Y})$*, then there exists no set* $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ *that is an adjustment set relative to* $(\mathbf{X}, \mathbf{Y})$ *in* $\mathcal{G}$ *(see Definition 3.1).*

Next, we assume that $\mathcal{G}$ satisfies condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$. Under this assumption, we show that $\mathbf{Z}$ satisfies conditions (1) and (2) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if and only if $\mathbf{Z}$ satisfies conditions (a) and (b) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in every DAG (MAG) in $[\mathcal{G}]$. This is shown in two separate lemmas:

Lemma 5.3. *Let condition (0) of the GAC be satisfied relative to* $(\mathbf{X}, \mathbf{Y})$ *in a* CPDAG *(*PAG*)* $\mathcal{G}$*. Then the following two statements are equivalent:*

- $\mathbf{Z}$ *satisfies condition (1) of the GAC relative to* $(\mathbf{X}, \mathbf{Y})$ *in* $\mathcal{G}$*.*
- $\mathbf{Z}$ *satisfies condition (a) of the AC relative to* $(\mathbf{X}, \mathbf{Y})$ *in every* DAG *(*MAG*) in* $[\mathcal{G}]$*.*

Lemma 5.4. *Let condition (0) of the GAC be satisfied relative to* $(\mathbf{X}, \mathbf{Y})$ *in a* CPDAG *(*PAG*)* $\mathcal{G}$*, and let* $\mathbf{Z}$ *satisfy condition (1) of the GAC relative to* $(\mathbf{X}, \mathbf{Y})$ *in* $\mathcal{G}$*. Then the following two statements are equivalent:*

- $\mathbf{Z}$ *satisfies condition (2) of the GAC relative to* $(\mathbf{X}, \mathbf{Y})$ *in* $\mathcal{G}$*.*
- $\mathbf{Z}$ *satisfies condition (b) of the AC relative to* $(\mathbf{X}, \mathbf{Y})$ *in every* DAG *(*MAG*) in* $[\mathcal{G}]$*.*

The proofs of Lemmas 5.2, 5.3 and 5.4 are discussed in Sections 5.1, 5.2 and 5.3, respectively. Some proofs require additional lemmas that can be found in the supplement. The proof of Lemma 5.4 is the most technical, and builds on the work of Zhang (2006).

Figure 6 shows how all lemmas fit together to prove Theorem 3.4.

Proof of Theorem 3.4: First, suppose that the CPDAG (PAG) $\mathcal{G}$ and the sets $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ satisfy all conditions of the GAC. By applying Lemmas 5.2, 5.3 and 5.4 in turn, it directly follows that all conditions of the AC are satisfied by $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ and any DAG (MAG) in $[\mathcal{G}]$.

To prove the other direction, suppose that the tuple $\mathcal{G}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ does not satisfy all conditions of the GAC. First, suppose that $\mathcal{G}$ violates condition (0) relative to $(\mathbf{X}, \mathbf{Y})$. Then by Lemma 5.2, there is no adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$, and hence $\mathbf{Z}$ is certainly not an adjustment set.

Otherwise, $\mathbf{Z}$ must violate condition (1) or (2) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$. By applying Lemmas 5.3 and 5.4 in turn, this implies that there is a DAG $\mathcal{D}$ (MAG $\mathcal{M}$) in $[\mathcal{G}]$ such that $\mathbf{Z}$ violates conditions (a) or (b) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ $(\mathcal{M})$. Since the AC is sound and complete for DAGs and MAGs, this implies that $\mathbf{Z}$ is not an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ $(\mathcal{M})$, so that $\mathbf{Z}$ is certainly not an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$.

$\square$

5.1 PROOF OF LEMMA 5.2

The proof of Lemma 5.2 is based on the following lemma:

Lemma 5.5. *Let* X *and* Y *be nodes in a* PAG $\mathcal{P}$*, such that there is a possibly directed path* p^* *from* X *to* Y *in* $\mathcal{P}$ *that does not start with a visible edge out of* X*. Then there is a* MAG $\mathcal{M}$ *in* $[\mathcal{P}]$ *such that the path* p *in* $\mathcal{M}$*, consisting of the same sequence of nodes as* p^* *in* $\mathcal{P}$*, contains a subsequence that is a directed path from* X *to* Y *starting with an invisible edge in* $\mathcal{M}$*.*

The proof of Lemma 5.5 is given in the supplement.

Proof of Lemma 5.2: First suppose that $\mathcal{G}$ satisfies condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$, meaning that every proper possibly directed path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$ starts with a visible edge out of $\mathbf{X}$. Any visible edge in $\mathcal{G}$ is visible in all DAGs (MAGs) in $[\mathcal{G}]$, and any proper directed path in a DAG (MAG) in $[\mathcal{G}]$ corresponds to a proper possibly directed path in $\mathcal{G}$. Hence, any proper directed path from $\mathbf{X}$ to $\mathbf{Y}$ in any DAG (MAG) in $[\mathcal{G}]$ starts with a visible edge out of $\mathbf{X}$. This shows that all DAGs (MAGs) in $[\mathcal{G}]$ satisfy condition (0*) of the AC relative to $(\mathbf{X}, \mathbf{Y})$.

Next, suppose that $\mathcal{G}$ violates condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$. We will show that this implies that there is no set $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ that is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. We give separate proofs for CPDAGs and PAGs.

Thus, let $\mathcal{G}$ represent a CPDAG and suppose that there is a proper possibly directed path p from a node $X \in \mathbf{X}$ to a node $Y \in \mathbf{Y}$ that starts with a non-directed edge ($\circ\!\!-\!\!\circ$).

Let $p' = \langle X, V_1, \dots, Y\rangle$ (where $V_1 = Y$ is allowed) be a shortest subsequence of p such that p' is also a proper possibly directed path from X to Y starting with a non-directed edge in $\mathcal{G}$. We first show that p' is a definite status path, by contradiction. Thus, suppose that p' is not a definite status path. Then the length of p' is at least 2, and we write $p' = \langle X, V_1, \dots, V_k = Y\rangle$ for $k \geq 2$. Since the subpath $p'(V_1, Y)$ is a definite status path (otherwise we can choose a shorter path), this means that V_1 is not of a definite status on p'. This implies the existence of an edge between X and V_2. This edge must be of the form $X \to V_2$, since $X \circ\!\!-\!\!\circ V_2$ implies that we can choose a shorter path, and $X \leftarrow V_2$ together with $X \circ\!\!-\!\!\circ V_1$ implies $V_1 \leftarrow V_2$ by Lemma 1 from Meek (1995) (see Section 1 of the supplement), so that p' is not possibly directed from X to Y. But the edge $X \to V_2$ implies that $V_1 \to V_2$, since otherwise Lemma 1 from Meek (1995) implies $X \to V_1$. But then V_1 is a definite non-collider on p', which contradicts that V_1 is not of definite status.

Hence, p' is a proper possibly directed definite status path from X to Y. By Lemma 7.6 from Maathuis and Colombo (2015) (see Section 1 of the supplement), there is a DAG $\mathcal{D}_1$ in $[\mathcal{G}]$ such that there are no additional arrowheads into X, as well as a DAG $\mathcal{D}_2$ in $[\mathcal{G}]$ such that there are no additional arrowheads into V_1. This means that the paths corresponding to p' are oriented as $p'_1 = X \to V_1 \to \dots \to Y$ and $p'_2 = X \leftarrow V_1 \to \dots \to Y$ in $\mathcal{D}_1$ and $\mathcal{D}_2$. An adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}_2$ must block the non-causal path p'_2, by using at least one of the non-endpoints nodes on this path. But all these nodes are in $\mathbf{F}_{\mathcal{D}_1}(\mathbf{X}, \mathbf{Y})$ (see (2)). Hence, there is no set $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ that satisfies the AC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}_1$ and $\mathcal{D}_2$ simultaneously. Since the AC is sound and complete for DAGs, this implies that there is no $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ that is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$.

Finally, let $\mathcal{G}$ represent a PAG and suppose that there is a proper possibly directed path p from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$ that does not start with a visible edge out of X in $\mathcal{G}$.

By Lemma 5.5, there is a subsequence p' of p such that there is a MAG $\mathcal{M}$ in $[\mathcal{G}]$ where the corresponding path is directed from X to Y and starts with an invisible edge. Then $\mathcal{M}$ is not amenable relative to $(\mathbf{X}, \mathbf{Y})$. By Lemma 5.7 from van der Zander et al. (2014) (see Section 1 of the supplement) this means that there is no set $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ that is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{M}$. Hence, there is no set $\mathbf{Z}' \subseteq \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y})$ that is an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. □

5.2 PROOF OF LEMMA 5.3

Proof of Lemma 5.3: First, suppose that $\mathbf{Z}$ satisfies condition (1) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. Then $\mathbf{Z} \cap \mathbf{F}_{\mathcal{G}}(\mathbf{X}, \mathbf{Y}) = \emptyset$. Since $\mathbf{F}_{\mathcal{D}}(\mathbf{X}, \mathbf{Y}) \subseteq \mathbf{F}_{\mathcal{G}}(\mathbf{X}, \mathbf{Y})$ ($\mathbf{F}_{\mathcal{M}}(\mathbf{X}, \mathbf{Y}) \subseteq \mathbf{F}_{\mathcal{G}}(\mathbf{X}, \mathbf{Y})$) for any DAG $\mathcal{D}$ (MAG $\mathcal{M}$) in $[\mathcal{G}]$, it follows directly that $\mathbf{Z}$ satisfies condition (a) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in all DAGs (MAGs) in $[\mathcal{G}]$.

To prove the other direction, suppose that $\mathcal{G}$ satisfies condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$, but that $\mathbf{Z}$ does not satisfy condition (1) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. Then there is a node $V \in \mathbf{Z} \cap \mathbf{F}_{\mathcal{G}}(\mathbf{X}, \mathbf{Y})$, i.e., $V \in \mathbf{Z}$ and V is a possible descendant of a node W on a proper possibly directed path from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$ in $\mathcal{G}$. We denote this path by $p = \langle X, V_1, \dots, V_k, Y\rangle$, where $k \geq 1$ and $W \in \{V_1, \dots, V_k\}$. Then the subpaths $q = p(X, W)$ and $r = p(W, Y)$ are also proper possibly directed paths. Moreover, there is a possibly directed path s from W to V, where this path is allowed to be of zero length (if $W = V$). We will show that the existence of these paths implies that there is a DAG $\mathcal{D}$ (MAG $\mathcal{M}$) in $[\mathcal{G}]$ such that $\mathbf{Z}$ violates condition (a) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ ($\mathcal{M}$).

By Lemma B.1 from Zhang (2008) (see Section 1 of the supplement), there are subsequences q', r' and s' of q, r and s that are unshielded proper possibly directed paths (again s' is allowed to be a path of zero length). Moreover, q' must start with a directed (visible) edge, since otherwise the concatenated path $q' \oplus r'$, which is again a proper possibly directed path from X to Y, would violate condition (0) of the GAC.

Lemma B.1 from Zhang (2008) then implies that q' is a directed path from X to W in $\mathcal{G}$. Hence, the path corresponding to q' is a directed path from X to W in any DAG (MAG) in $[\mathcal{G}]$.

By Lemma 7.6 from Maathuis and Colombo (2015), there is at least one DAG $\mathcal{D}$ (MAG $\mathcal{M}$) in $[\mathcal{G}]$ that has no additional arrowheads into W. In this graph $\mathcal{D}$ ($\mathcal{M}$), the path corresponding to r' is a directed path from W to Y, and the path corresponding to s' is a directed path W to V. Hence, $V \in \mathbf{F}_{\mathcal{D}}(\mathbf{X}, \mathbf{Y})$ ($V \in \mathbf{F}_{\mathcal{M}}(\mathbf{X}, \mathbf{Y})$), so that $\mathbf{Z}$ does not satisfy condition (a) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{D}$ ($\mathcal{M}$). □

5.3 PROOF OF LEMMA 5.4

We first define a distance between a path and a set in Definition 5.6. We then give the proof of Lemma 5.4. This proof relies on Lemma 5.7 and Lemma 5.8 which are given later in this section.

Definition 5.6. *(**Distance-from-$\mathbf{Z}$**; Zhang, 2006) Given a path p from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\mathbf{Z}$ in a* DAG *or* MAG, *for every collider Q on p, there is a directed path (possibly of zero length) from Q to a member of $\mathbf{Z}$. Define the* distance-from-$\mathbf{Z}$ *of Q to be the length of a shortest directed path (possibly of length 0) from Q to $\mathbf{Z}$, and define the* distance-from-$\mathbf{Z}$ *of p to be the sum of the distances from $\mathbf{Z}$ of the colliders on p.*

Proof of Lemma 5.4: Let $\mathcal{G}$ represent an amenable

CPDAG (PAG) that satisfies condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$, and let $\mathbf{Z}$ satisfy condition (1) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$.

We first prove that if $\mathbf{Z}$ does not satisfy condition (2) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$, then $\mathbf{Z}$ does not satisfy condition (b) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in any DAG (MAG) in $[\mathcal{G}]$. Thus, assume that there is a proper definite status non-causal path p from $X \in \mathbf{X}$ to $Y \in \mathbf{Y}$ that is m-connecting given $\mathbf{Z}$ in $\mathcal{G}$. Consider any DAG $\mathcal{D}$ (MAG $\mathcal{M}$) in $[\mathcal{G}]$. Then the path corresponding to p in $\mathcal{D}$ ($\mathcal{M}$) is a proper non-causal m-connecting path from $\mathbf{X}$ to $\mathbf{Y}$ given $\mathbf{Z}$. Hence, $\mathbf{Z}$ violates condition (b) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ and $\mathcal{D}$ ($\mathcal{M}$).

Next, we prove that if $\mathbf{Z}$ violates condition (b) of the AC relative to $(\mathbf{X}, \mathbf{Y})$ in some DAG (MAG) in $[\mathcal{G}]$, then $\mathbf{Z}$ violates condition (2) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. Thus, assume that there is a DAG $\mathcal{D}$ (MAG $\mathcal{M}$) in $[\mathcal{G}]$ such that there is a proper non-causal m-connecting path from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{D}$ ($\mathcal{M}$) given $\mathbf{Z}$. We choose a shortest such path p, such that no equally short proper non-causal m-connecting path has a shorter distance-from-$\mathbf{Z}$ than p. By Lemma 5.8 below, the corresponding path p^* in $\mathcal{G}$ is an m-connecting proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ given $\mathbf{Z}$. Hence $\mathbf{Z}$ violates condition (b) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$. $\square$

Lemma 5.7. *Let $\mathcal{M}$ represent a MAG (DAG) and let $\mathcal{P}$ be the PAG (CPDAG) of $\mathcal{M}$. Let $\mathcal{P}$ satisfy condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$, and let $\mathbf{Z}$ satisfy condition (1) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{P}$. Let p be a shortest proper non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\mathbf{Z}$ in $\mathcal{M}$ and let p^* denote the corresponding path constituted by the same sequence of variables in $\mathcal{P}$. Then p^* is a proper definite status non-causal path in $\mathcal{P}$.*

Lemma 5.7 is related to Lemma 1 from Zhang (2006). The proof of Lemma 5.7 is given in the supplement.

Lemma 5.8. *Let $\mathcal{M}$ represent a MAG (DAG) and let $\mathcal{P}$ be the PAG (CPDAG) of $\mathcal{M}$. Let $\mathcal{P}$ satisfy condition (0) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$, and let $\mathbf{Z}$ satisfy condition (1) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{P}$. Let p be a shortest proper non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\mathbf{Z}$ in $\mathcal{M}$, such that no equally short such path has a shorter* distance-from-$\mathbf{Z}$ *than p. Let p^* denote the corresponding path constituted by the same sequence of variables in $\mathcal{P}$. Then p^* is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\mathbf{Z}$ in $\mathcal{P}$.*

Lemma 5.8 is is related to Lemma 2 from Zhang (2006).

Proof of Lemma 5.8. By Lemma 5.7, p^* is a proper definite status non-causal path in $\mathcal{P}$. It is only left to prove that p^* is m-connecting given $\mathbf{Z}$ in $\mathcal{P}$.

Every definite non-collider on p^* in $\mathcal{P}$ corresponds to a non-collider on p in $\mathcal{M}$, and every collider on p^* is also a collider on p. Since p is m-connecting given $\mathbf{Z}$, no non-collider is in $\mathbf{Z}$ and every collider has a descendant in $\mathbf{Z}$. Let Q be an arbitrary collider (if there is one). Then there is a directed path (possibly of zero length) from Q to a node in $\mathbf{Z}$ in $\mathcal{M}$. Let d be a shortest such path from Q to a node $Z \in \mathbf{Z}$. Let d^* denote the corresponding path in $\mathcal{P}$, constituted by the same sequence of variables. Then d^* is an unshielded possibly directed path from Q to Z in $\mathcal{P}$ (Lemma B.1 from Zhang (2008)).

It is only left to show that d^* is a directed path. If d^* is of zero length, this is trivially true. Otherwise, suppose for contradiction that there is a circle mark on d^*. Then d^* must start with a circle mark at Q (cf. Lemma B.2 from Zhang, 2008 and Lemma 7.2 from Maathuis and Colombo, 2015; see Section 1 of the supplement).

Let S be the first node on d after Q. If S is not a node on p, then following the proof of Lemma 2 from Zhang (2006) there is a path $p' = p(X, W) \oplus W \bullet\!\!\rightarrow S \leftarrow\!\!\bullet V \oplus p(V, Y)$, where W and V are nodes distinct from Q on $p(X, Q)$ and $p(Q, Y)$ respectively and p' is m-connecting given $\mathbf{Z}$ in $\mathcal{M}$. Since p' is non-causal and shorter than p, or as long as p but with a shorter distance-from-$\mathbf{Z}$ than p, the path p' must be non-proper, i.e. $S \in \mathbf{X}$. But, in that case the path $\langle S, V \rangle \oplus p(V, Y)$ is a proper non-causal m-connecting path from $\mathbf{X}$ to $\mathbf{Y}$ given $\mathbf{Z}$ that is shorter than p in $\mathcal{M}$. This contradicts our assumption about p.

If S is a node on p, then it lies either on $p(X, Q)$ or $p(Q, Y)$. Assume without loss of generality that S is on $p(Q, Y)$. Following the proof of Lemma 2 from Zhang (2006), there exists a path $p' = p(X, W) \oplus W \bullet\!\!\rightarrow S \oplus p(S, Y)$ in $\mathcal{M}$, where W is a node on $p(X, Q)$ distinct from Q that is m-connecting given $\mathbf{Z}$ in $\mathcal{M}$. Since p' is proper, and shorter than p, or as long as p but with a shorter distance-from-$\mathbf{Z}$ than p, the path p' must be causal in $\mathcal{M}$. Let p'^* denote the corresponding path constituted by the same sequence of variables in $\mathcal{P}$. Then p'^* is a possibly causal path and $Z \in \mathrm{PossDe}(S, \mathcal{P})$, so $Z \in \mathbf{F}_{\mathcal{P}}(\mathbf{X}, \mathbf{Y}) \cap \mathbf{Z}$. This is in contradiction with our assumption of $\mathbf{Z}$ satisfying condition (1) of the GAC relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{P}$.

Thus, the path d^* is directed and Q is an ancestor of $\mathbf{Z}$ in $\mathcal{P}$. This proves that p^* is a proper definite status non-causal path from $\mathbf{X}$ to $\mathbf{Y}$ that is m-connecting given $\mathbf{Z}$ in $\mathcal{M}$.

$\square$

6 DISCUSSION

We have derived a generalized adjustment criterion that is necessary and sufficient for adjustment in DAGs, MAGs, CPDAGs and PAGs. Our criterion unifies existing criteria for DAGs and MAGs, and provides a new result for CPDAGs and PAGs, where only a sufficient criterion existed until now. This is relevant in practice, in particular in combination with algorithms that can learn CPDAGs or PAGs from observational data.

Our generalized adjustment criterion is stated in terms of paths that need to be blocked, which is intuitively appealing. A logical next step for future research would be to transform our criterion into an algorithmically constructive version that could be used to efficiently perform tasks like enumeration of all minimal adjustment sets for a given graph. This has already been done for DAGs and MAGs by van der Zander et al. (2014), and we strongly suspect that their results can be extended to CPDAGs and PAGs as well. In a similar spirit, it would be desirable to have an easily checkable condition to determine if there exists any adjustment set at all, as done for the generalized back-door criterion for single interventions by Maathuis and Colombo (2015). In turn, these results could then be used to characterize distances between graphs, as done by Peters and Bühlmann (2015). Future work might also explore under which circumstances our restriction to not allow for latent selection variables might be relaxed, or whether our criterion could be combined with methods to recover from selection bias (Bareinboim et al., 2014).

As pointed out in Section 4, our criterion sometimes has to interpret PAGs or MAGs differently than DAGs or CPDAGs. This is the case precisely when the first edge on some proper possibly causal path in a MAG or PAG is not visible. However, this difference in interpretation is irrelevant for DAGs or CPDAGs that would be amenable when viewed as a MAG or PAG. For instance, if we are given a DAG $\mathcal{D}$ that is amenable when interpreted as a MAG $\mathcal{M}$, then its adjustment sets also work for every DAG that the MAG $\mathcal{M}$ represents, many of which could contain latent confounding variables. Reading a DAG as a MAG (or a CPDAG as a PAG) can thus allow computing adjustment sets that are to some extent invariant to confounding.

We note that an adjustment set relative to $(\mathbf{X}, \mathbf{Y})$ in a given graph can only exist if the total causal effect of $\mathbf{X}$ on $\mathbf{Y}$ is identifiable in the graph. If the effect of $\mathbf{X}$ on $\mathbf{Y}$ is not identifiable, one may be interested in computing all possible total causal effects of $\mathbf{X}$ on $\mathbf{Y}$ for DAGs represented by the given graph. Such an approach is used in the IDA algorithm of Maathuis et al. (2009, 2010), by considering all DAGs represented by a CPDAG and applying back-door adjustment to each of these DAGs. Similar ideas could be used for MAGs and PAGs, but listing all relevant DAGs described by a MAG or PAG seems rather non-trivial.

There is also an interesting connection between amenability and instrumental variables: a MAG or PAG $\mathcal{G}$ with $\mathbf{X} = \{X\}$ is amenable with respect to $(\mathbf{X}, \mathbf{Y})$ whenever it contains an *instrument* I, i.e. there exists a variable that is a parent of X but not a parent of any child of X (e.g., I in Figure 1a). Thus, instruments are useful to find adjustment sets in nonparametric graphical models that allow for latent confounding. This connection is perhaps surprising given that the notion of instruments originates from causal effect identifications in linear models (Angrist et al., 1996).

In summary, our generalized adjustment criterion exhaustively characterizes the options to identify total causal effects by covariate adjustment in DAGs, MAGs, CPDAGs, and PAGs. Our results entail several existing, less general or less powerful ones (Pearl, 1993; Shpitser et al., 2012; Textor and Liśkiewicz, 2011; van der Zander et al., 2014; Maathuis and Colombo, 2015) as special cases.

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References

Ali, R. A., Richardson, T. S., and Spirtes, P. (2009). Markov equivalence for ancestral graphs. *Ann. Stat.*, 37:2808–2837.

Angrist, J. D., Imbens, G. W., and Rubin, D. B. (1996). Identification of causal effects using instrumental variables. *J. Am. Stat. Assoc.*, 91(434):444–455.

Bareinboim, E., Tian, J., and Pearl, J. (2014). Recovering from selection bias in causal and statistical inference. In *Proceedings of AAAI 2014*, pages 2410–2416.

Chickering, D. M. (2003). Optimal structure identification with greedy search. *J. Mach. Learn. Res.*, 3:507–554.

Claassen, T., Mooij, J., and Heskes, T. (2013). Learning sparse causal models is not NP-hard. In *Proceedings of UAI 2013*, pages 172–181.

Colombo, D. and Maathuis, M. H. (2014). Order-independent constraint-based causal structure learning. *J. Mach. Learn. Res.*, 15:3741–3782.

Colombo, D., Maathuis, M. H., Kalisch, M., and Richardson, T. S. (2012). Learning high-dimensional directed acyclic graphs with latent and selection variables. *Ann. Stat.*, 40:294–321.

Maathuis, M. H. and Colombo, D. (2015). A generalized back-door criterion. *Ann. Stat.*, 43:1060–1088.

Maathuis, M. H., Colombo, D., Kalisch, M., and Bühlmann, P. (2010). Predicting causal effects in large-scale systems from observational data. *Nat. Methods*, 7:247–248.

Maathuis, M. H., Kalisch, M., and Bühlmann, P. (2009). Estimating high-dimensional intervention effects from observational data. *Ann. Stat.*, 37:3133–3164.

Meek, C. (1995). Causal inference and causal explanation with background knowledge. In *Proceedings of UAI 1995*, pages 403–410.

Pearl, J. (1993). Comment: Graphical models, causality and intervention. *Stat. Sci.*, 8:266–269.

Pearl, J. (2009). *Causality*. Cambridge University Press, Cambridge, second edition.

Peters, J. and Bühlmann, P. (2015). Structural intervention distance (SID) for evaluating causal graphs. *Neural Comput.*, 27:771–799.

Richardson, T. and Spirtes, P. (2002). Ancestral graph Markov models. *Ann. Stat.*, 30:962–1030.

Robins, J. (1986). A new approach to causal inference in mortality studies with a sustained exposure period-application to control of the healthy worker survivor effect. *Math. Mod.*, 7:1393–1512.

Rubin, D. (2008). Author's reply. *Stat. Med.*, 27:2741–2742.

Shpitser, I. (2012). Appendum to "On the validity of covariate adjustment for estimating causal effects". Unpublished manuscript.

Shpitser, I. and Pearl, J. (2006). Identification of joint interventional distributions in recursive semi-markovian causal models. In *Proceedings of AAAI 2006*, pages 1219–1226.

Shpitser, I., VanderWeele, T., and Robins, J. M. (2012). On the validity of covariate adjustment for estimating causal effects. In *Proceedings of UAI 2010*, pages 907–916.

Shrier, I. (2008). Letter to the editor. *Stat. Med.*, 27:2740–2741.

Shrier, I. and Platt, R. W. (2008). Reducing bias through directed acyclic graphs. *BMC Med. Res. Methodol.*, 8(70).

Spirtes, P., Glymour, C., and Scheines, R. (2000). *Causation, Prediction, and Search*. MIT Press, Cambridge, second edition.

Textor, J. and Liśkiewicz, M. (2011). Adjustment criteria in causal diagrams: An algorithmic perspective. In *Proceedings of UAI 2011*, pages 681–688.

Tian, J. and Pearl, J. (2002). A general identification condition for causal effects. In *Proceedings of AAAI 2002*, pages 567–573.

van der Zander, B., Liśkiewicz, M., and Textor, J. (2014). Constructing separators and adjustment sets in ancestral graphs. In *Proceedings of UAI 2014*, pages 907–916.

West, S. G. and Koch, T. (2014). Restoring causal analysis to structural equation modeling. *Struct. Equ. Modeling*, 21:161–166.

Westreich, D. and Greenland, S. (2013). The table 2 fallacy: presenting and interpreting confounder and modifier coefficients. *Am. J. Epidemiol.*, 177:292–298.

Zhang, J. (2006). *Causal Inference and Reasoning in Causally Insufficient Systems*. PhD thesis, Carnegie Mellon University.

Zhang, J. (2008). On the completeness of orientation rules for causal discovery in the presence of latent confounders and selection bias. *Artif. Intell.*, 172:1873–1896.