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Theory of capacities

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THEORY OF CAPACITIES (1)

by Gustave CHOQUET (2) (3).

INTRODUCTION

This work originated from the following problem, whose significance had been emphasized by M. Brelot and H. Cartan :

Is the interior Newtonian capacity of an arbitrary borelian subset X of the space R^3 equal to the exterior Newtonian capacity of X ?

For the solution of this problem, I first systematically studied the non-additive set-functions, and tried to extract from their totality certain particularly interesting classes, with a view to establishing for these a theory analogous to the classical theory of measurability.

I succeeded later in showing that the classical Newtonian capacity f belongs to one of these classes, more precisely : if A and B are arbitrary compact subsets of R^3, then

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B).$$

It followed from this that every borelian, and even every analytic set is capacitable with respect to the Newtonian capacity, a result which can, moreover, be extended to the capa-

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cities associated with the Green's function and to other classical capacities.

The above inequality, which may be called the inequality of strong sub-additivity, is equivalent to the following:

$$V_2(X;\ A,\ B) = f(X) - f(X \cup A) - f(X \cup B) + f(X \cup A \cup B) \leqq 0.$$

Now, this relation is the first of an infinite sequence of independent inequalities, each of the form $V_n(X;\ A_1, A_2, \ldots, A_n) \leqq 0$, which expresses the fact that the successive differences — in an obvious sense — of the function f are alternately positive or negative.

Thus, the Newtonian capacity is seen to be an analogue of the functions of a real variable whose successive derivatives are alternately positive and negative.

It is known from a theorem of S. Bernstein that these functions, termed completely monotone, have an integral representation in terms of functions e^{-tx}. Likewise, the set functions which are « alternating of order infinity » possess an integral representation in terms of exponentials, that is, of set functions $\psi(X)$ which satisfy

$$0 \leqq \psi \leqq 1 \quad \text{and} \quad \psi(X \cup Y) = \psi(X) \cdot \psi(Y).$$

These exponentials take values in [0, 1] only, and this makes it possible to give a remarkable probabilistic interpretation of the functions which are alternating of order infinity.

More generally, a detailed study of several other classes of functions justifies the interest in the determination of the extremal elements of convex cones of functions, and in the utilization of the corresponding integral representations.

Chapter I. — **Borelian and analytic sets in topological spaces.** — In this chapter, borelian and analytic subsets of arbitrary Hausdorff spaces are redefined and studied. In fact, a mere adaptation of the classical definitions would lead to sets of an irregular topological character for which an interesting theory of capacitability could not be constructed. Therefore, we designate as borelian and analytic sets the sets which are generated by beginning with the compact sets and using the operations of countable intersection and union, and continuous mapping (or projection) only. Thus the operations of « difference » or « complementation » are not used.

The role which the G_δ sets play in the classical theory is here played by the $K_{\sigma\delta}$ sets.

CHAPTER II. — **Newtonian and greenian capacities.** — In this chapter, the Newtonian and Greenian capacities of compact sets and, thereafter, the interior and exterior capacities of arbitrary sets, are defined. An equilibrium potential $h(X)$, and a capacity, $f(X)$, are associated with each compact subset X of a domain. The successive differences $(-1)^n\nabla_n(X; A_1, \ldots, A_n)$ are defined for each of these functions; it is shown that each ∇_n is non-positive, and the conditions for the vanishing of the ∇_n are determined.

It is shown that the sequence of these inequalities for the capacity f is complete in the sense that every inequality between the capacities of a family of compact sets obtained from p arbitrary compact sets by the operation of union is a consequence of the inequalities $\nabla_n \leqq 0$. A more penetrating analysis shows that this result remains valid for the capacities of the sets which are obtained from p arbitrary compact sets by means of the operations of union, intersection, and difference.

From the relation $\nabla_2 \leqq 0$, the following important inequality is obtained:

$$f\left(\bigcup A_i\right) - f\left(\bigcup a_i\right) \leqq \sum [f(A_i) - f(a_i)],$$

for every finite or countable family of couples of compact sets a_i and A_i satisfying the relation $a_i \subset A_i$ for each i.

Furthermore, from the relation $\nabla_2 \leqq 0$, we deduce a result concerning the capacity of certain compact sets, relative to domains which are invariant under a one-parameter group of euclidian motions.

The chapter ends with the study of the differential of $f(K)$ with respect to suitable increments ΔK of K, and with the derivation of a formula which shows that the Green's function $G(P_1, P_2)$ of a domain is a limit of the function

$$G(K_1, K_2) = \frac{f(K_1) + f(K_2) - f(K_1 \cup K_2)}{2f(K_1) . f(K_2)}.$$

Chapter III. — **Alternating and monotone functions. Capacities.** — This chapter introduces several classes of functions and certain basic concepts as follows: alternating (monotone) functions of order n or ∞ which are mappings from a commutative semi-group into an ordered commutative group and which satisfy inequalities of the form $\nabla_p \leqq 0$ ($\nabla_p \geqq 0$); the concept of conjugate set functions, connected by the relation

$$\varphi'(X') + \varphi(X) = 0, \quad \text{where} \quad X' = \complement X \text{ is the complement of } X$$

the concepts of capacity on a class of subsets of a topological space, of interior capacity (f_*) and exterior capacity (f^*), of alternating and monotone capacities, of a capacity which is the conjugate of another capacity.

Chapter IV. — **Extension and restriction of a capacity.** — The extension f_2 of a capacity f_1, defined on a class $\mathcal{E}_1$ of subsets of a space E, to a class $\mathcal{E}_2$ properly containing $\mathcal{E}_1$, by means of the equality $f_2(X) = f_1^*(X)$, can often be used as a means for regularizing the class $\mathcal{E}_1$ and also as a means of simplifying proofs of capacitability.

On the other hand, the operation of restriction will sometimes make it possible to replace the space E by a simpler space.

Furthermore, the preservation of various classes of capacities (alternating or monotone) under these operations of extension and restriction is studied; for instance, let $\mathcal{E}_1$ be the class of all compact subsets of a Hausdorff space E; then, if f_1 is alternating of order n, the same is true for every extension f_1 of f_2.

Chapter V. — **Operations on capacities and examples of capacities.** — First, several operations which leave certain classes of capacities invariant are studied: for instance, if a capacity $g(Y)$, alternating of order n, is defined on the class $\mathcal{K}(F)$ of all compact subsets of a space F, and if $Y = \varphi(X)$ denotes a mapping from $\mathcal{K}(E)$ into $\mathcal{K}(F)$ such that $\varphi(X_1 \cup X_2) \equiv \varphi(X_1) \cup \varphi(X_2)$ and which satisfies, in addition, a certain requirement of continuity on the right, then the function $f(X) = g(Y)$ is also a capacity which is alternating of order n. The projection operation is such an operation and will play an essential role in the study of capacitability.

The remainder of the chapter is devoted to the study of the following important examples of functions and capacities which are alternating of order ∞: the operation sup on a group lattice; increasing valuation on a lattice (for example, a non-negative Radon measure); functions derived from a probabilistic scheme; exponentials; energy of the restriction of a measure to a compact set; and others.

CHAPTER VI. — **Capacitability. Fundamental theorems.** — First, the alternating capacities are studied: we establish conditions, to be imposed on E, $\mathcal{E}$, and f, which will suffice to insure the preservation of capacitability under finite or denumerable union, and which will imply the validity of the relation

$$f^*\left(\bigcup A_n\right) = \lim f^*(A_n) \qquad (\text{where} \quad A_n \subset A_{n+1}).$$

It then follows from a general theorem that every $K_{\sigma\delta}$ contained in a Hausdorff space E is capacitable with respect to every alternating capacity f defined on $\mathcal{K}(E)$. In order to pass from these $K_{\sigma\delta}$ sets to arbitrary borelian and analytic sets, we use the fact that every analytic subset of E is the projection on E of a $K_{\sigma\delta}$ contained in the product space $(E \times F)$, where F is an auxiliary compact space; and we associate with f the capacity g on $\mathcal{K}(E \times F)$, where g is defined by

$$g(X) = f(pr_E X).$$

It is then easily proved that every g-capacitable subset of $(E \times F)$ has a projection on E which is f-capacitable. Now, g is alternating as well as f; hence, every $K_{\sigma\delta}$ of $(E \times F)$ is g-capacitable. From this follows the f-capacitability of all analytic subsets of E.

A number of counter-examples show that it is impossible to improve on the results obtained: in particular, by using a result of Goedel we prove that it is not possible to establish the capacitability of all complements of analytic sets.

After giving several applications of these results to measure theory, we investigate monotone capacities. Their study is reduced to that of alternating capacities by means of the concept of conjugate capacity. From the general theorems obtained in this way, special cases such as the following are derived :

If E is a complete metric space, and if f is a monotone capacity of order 2 on $\mathcal{K}(E)$, $(f(A \cup B)+f(A \cap B) \geq f(A)+f(B))$, then all borelian subsets of E and all complements of analytic sets are capacitable.

CHAPTER VII. — **Extremal elements of convex cones and integral representations. Applications.** — In this chapter, we study several convex cones whose elements are functions; we determine their extremal elements and employ them to obtain integral representations of these functions. The basic tool for these representations is the theorem of Krein and Milman concerning convex and compact subsets of locally convex spaces, and its immediate consequences. This theorem enables us to state the existence of such a representation in the case of a cone such that its base and also the set of extremal point of the base are both compact. Uniqueness of this representation implies that the cone under consideration is a lattice; but it has not been proved that this condition is sufficient to insure uniqueness.

We study in this manner the positive increasing functions defined on an ordered set, the increasing valuations on a distributive lattice, and, in particular, the simply additive measures defined on an algebra of sets; for these we use the compact spaces which Stone associates with these algebras.

The study of the cone of all positive functions which are alternating of order infinity on an ordered semi-group S illustrates the significance of the exponentials ψ defined on a semi-group $(0 \leq \psi \leq 1$, and $\psi(a \top b) = \psi(a) \cdot \psi(b))$. When $S = R_+$ or $S = R^n_+$, theorems analogous to those of S. Bernstein are obtained; when S is an additive class of subsets of E, then we find extremal elements each of which is characterized by a filter on E.

In seeking a way to study the cones whose elements are capacities on $\mathcal{K}(E)$ we are led to the introduction of a « vague topology » on the set of all positive increasing functions f defined on $\mathcal{K}(E)$: this is achieved by the use of the extension of f to the set Q_+ of all numerical functions defined on E which are non-négative, continuous, and zero outside of a compact set.

It is then proved, for instance, that if E is compact, the set of all capacities f which are positive and alternating of order ∞ on $\mathcal{K}(E)$, and which satisfy the relation $f(E) = 1$, is compact in the vague topology, as is also the set of its extremal points. This leads to a remarkable probabilistic interpretation of these capacities, and makes it possible to prove that the class of these capacities is the least functional class containing all positive Radon measures, which is stable in a certain sense, with respect to continuous mappings.

Thereafter, we take up the study of those classes of functionals on Q_+ which may be obtained from the primitive functions f_a defined by the relation $f_a(\varphi) = \varphi(a)$ by means of the following operations: superior envelope, inferior envelope, and integration $\left(g = \int f_\lambda \, d\mu(\lambda)\right)$.

The chapter ends with the study of the relations between the pseudo-norms defined on a vector lattice V and the functions f which are strongly sub-additive on V.

CHAPTER I

BORELIAN AND ANALYTIC SETS IN TOPOLOGICAL SPACES

1. Introduction. — There are difficulties in extending to an arbitrary topological space E the classical results concerning the parametric representation of borelian sets. For example, in a general setting each subset of E is the continuous and 1-1 image of an open set of a suitable compact space; for one can easily construct a compact space which contains an open set of a given cardinal and each of whose points is isolated.

In order to obtain theorems of interest, one is, therefore, led to modify the classical definitions. In particular, we shall have to eliminate the open sets and begin with the compact sets, which possess topological characteristics invariant under continuous mappings. Therefore, we shall be led to replace the sets G_δ, whose role is fundamental in the study of classical borelian and analytic sets, by the sets $K_{\sigma\delta}$ which we shall define in terms of compact sets.

1. 1. DEFINITION. — *A class* $\mathfrak{B}$ *of subsets of a set* E *which contains the intersection and the union of any denumerable family of elements of* $\mathfrak{B}$ *will be called a* borelian field *on* E.

1. 2. DEFINITION. — *If* E *is a Hausdorff topological space, the smallest borelian field on* E *which contains each compact set of* E *will be called the* K-borelian field *of* E *and denoted by* $\mathfrak{B}(K)$. *The members of* $\mathfrak{B}(K)$ *will be called* K-borelian sets.

2. Classification of K-borelian sets. — One can show, as in the classical theory, that the K-borelian field of E is the increasing union of a transfinite sequence of type Ω of classes

$$\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_\omega, \ldots, \mathcal{K}_\alpha, \ldots, (\alpha < \Omega)$$

where

(i) $\mathcal{K}_0$ designates the class of compact sets of E;

(ii) $\mathcal{K}_\alpha$ designates the set of denumerable intersections (unions) of elements belonging to $\mathcal{K}_\beta$ where $\beta < \alpha$ if α is even (odd), the limit numbers α being considered as even.

We shall designate in general the classes with finite indices by $\mathcal{K}$, $\mathcal{K}_\sigma$, $\mathcal{K}_{\sigma\delta}$, ..., and we shall say, for example, that a set is a $K_{\sigma\delta}$ if it belongs to the class $K_{\sigma\delta}$.

2. 1. Immediate consequences.

(i) Each finite union or intersection of sets of one class $\mathcal{K}_\alpha$ belongs to that class. Each denumerable intersection (union) of sets of $\mathcal{K}_\alpha$ belongs to $\mathcal{K}_\alpha$ if α is even (odd).

(ii) A simple argument by transfinite induction shows that each K-borelian set of E is contained in a K_σ of E. It follows that if E is a separable and complete metric space which is nowhere locally compact, not all borelian subsets of E, in the classical sense, are K-borelian; on the other hand, we shall see later that all classical borelian subsets of a separable and complete metric space are K-analytic.

If E is such that each open set G of E is a K_σ, each closed set F of E is, of course, also a K_σ; then the field of borelian sets in the classical sense is identical with the K-borelian field [6]. This is the case when, for example, E is a separable and locally compact metric space or, more generally, when E is a metric space which is a K_σ.

3. K-analytic sets. — We shall now define a class of sets analogous to the classical analytic sets.

3. 1. Definition. — *In a Hausdorff topological space, each subset which is the continuous image of a* $K_{\sigma\delta}$ *contained in a compact space will be called a* K-analytic set.

3. 2. Theorem. — *Each subset of a Hausdorff space which is the continuous image of a K-analytic set is also K-analytic. The class of K-analytic subsets of a Hausdorff space is a borelian field.*

[6] It would be interesting to see, if, conversely, this identity entails that each open set of E is a K_σ.

Proof. The first part of the theorem follows immediately from the transitivity of continuity.

In order to establish the second part of the theorem, let $A_1, A_2, \ldots, A_n, \ldots,$ be a sequence of K-analytic sets of E, where A_n is the image, under the continuous mapping f_n, of the set $B_n \subset F_n$, where B_n is a $K_{\sigma\delta}$ and F_n is a compact space.

Let us show first that $A_\sigma = \bigcup_n A_n$ is K-analytic. Let F be the compact space obtained by the compactification of the topological sum-space $\sum F_n$ by the addition of the point at infinity; let $B = \bigcup B_n$; the set B is by definition a subset of F.

We shall designate by f the mapping of B into E whose restriction to B_n is identical to f_n; this mapping is clearly continuous and we have $A_\sigma = f(B)$.

It remains only to show that B is a $K_{\sigma\delta}$. Now by definition we can set $B_n = \bigcap B_{n,i}$ $(i = 1, 2, \ldots,)$ where each $B_{n,i}$ is a K_σ of F_n. Since the F_n of F are mutually disjoint we have $B = \bigcup_n B_n = \bigcap_i \left(\bigcup_n B_{n,i}\right)$. Since $\bigcup_n B_{n,i}$ is a K_σ, B is indeed a $K_{\sigma\delta}$.

Finally, let us show that $A_\delta = \bigcap A_n$ is K-analytic. Let $F = \prod F_n$, the product space of the F_n, and designate by C the subset of F defined by $C = \prod B_n$. The set C is the intersection of the cylinders b_n of F where $b_n = B_n \times \prod_{p \neq n} F_p$; each b_n is a $K_{\sigma\delta}$, and therefore the same is true of C.

Furthermore, we shall designate by f_n the canonical extension to b_n of the given mapping of B_n into E; f_n is therefore defined at each point of C. The set of points of C at which $f_1 = f_n$ is closed relative to C, for each n, since f_1 and f_n are continuous; therefore the set of points of C at which $f_i = f_j$ for all i and j is the intersection of C and a closed subset of F. This intersection is therefore a $K_{\sigma\delta}$ which we shall designate by B.

We shall designate by f the restriction of the f_n to B. This restriction is continuous on B and since $f_n(B) \subset A_n$ for every n, we have $f(B) \subset A_\delta$. On the other hand $A_\delta \subset f(B)$. For let $y \in A_\delta$; for every n there exists an $x_n \in B_n$ such that $f_n(x_n) = y$.

The point $x = (x_n)$ of F belongs to B and we have therefore $f(x) = y$. Thus $A_\delta = f(B)$ and A_δ is the continuous image of a $K_{\sigma\delta}$.

3. 3. **Soulin's operation A.** — Suppose that A_λ is a class of subsets of a set E where λ denotes a finite sequence $(n_1, n_2, \ldots, n_i)$ of positive integers. For every infinite sequence $s = (n_1, n_2, \ldots, n_i, \ldots)$ of positive integers, set

$$A_s = \bigcap_i A_{n_1, \ldots, n_i}$$

The set $A = \bigcup_s A_s$ is called the *nucleus* associated with the class $\{A_\lambda\}$; it is also referred to as the set obtained from this class by *Souslin's operation.*

Let S denote the topological space of all sequences s, lexicographically ordered with the topology induced by that order; then it can be easily shown that A is the canonical projection on E of a set $\mathcal{A} \subset (E \times S)$, with $\mathcal{A} = \bigcap_i \mathcal{A}_i$ where each $\mathcal{A}_i (i = 1, 2, \ldots)$ is a countable union of elementary sets of the form $(A_\lambda \times \delta_\lambda)$, with δ_λ denoting an interval of S. An immediate consequence of this is the following theorem:

THEOREM. — *If a subset of a Hausdorff space* E *is obtained by Souslin's operation from a class of* K-*analytic sets, then that subset itself is* K-*analytic.*

DEFINITION. — *Every subset of a Hausdorff space* E, *obtained by Soulin's operation from a class of compact subsets of* E *is called a* K-Souslin set.

It is easily shown that, if f is a continuous mapping from a compact space E into a Hausdorff space F and if $B \subset F$, then the set $A = f^{-1}(B)$ is a K-borelian set of class K_α (respectively, K-Souslin, K-analytic), if B is of class K_α (respectively K-Souslin, K-analytic).

3. 4. DEFINITION. — *A subset* A *of a Hausdorff space is called a* set of uniqueness *if* A *is the continuous and* 1-1 *image of a* $K_{\sigma\delta}$ *of a compact space.*

3. 5. THEOREM. *Every denumerable intersection of sets of uniqueness is a set of uniqueness. Every denumerable union of* disjoint *sets of uniqueness is a set of uniqueness.*

Proof. For the first part we may refer to the end of the proof of Theorem 3. 2 and remark that if the f_n are 1-1, then there exists in B a single point $x = (x_n)$ such that $f(x) = y$. The same remark applies to the second part.

4. The K-borelian sets. — Later in this work we shall use the fact that the K-borelian sets are K-analytic. More precisely, the following theorem holds.

4. 1. THEOREM. — *Every* K-*borelian set is* K-*analytic. Furthermore, if the Hausdorff space* E *has the property that each subset of the form* $K \cap G$ *is a* K_σ, *(where* G *is open), then each* K-*borelian subset of* E *is a set of uniqueness.*

Proof. The first part is an immediate consequence of the fact that each compact set is K-analytic. The field of K-borelian sets is therefore a subfield of the field of the K-analytic sets.

We shall prove now the second part of the theorem. Assume at first that E is compact. Then each open set G of K is a K_σ by hypothesis. The borelian field generated by the open sets of E is identical with the increasing union of a transfinite sequence of type Ω of classes

$$\mathfrak{G}_0, \quad \mathfrak{G}_1, \quad \ldots\ldots, \quad \mathfrak{G}_\omega \ldots, \quad \mathfrak{G}_\alpha, \ldots\ldots, \quad (\alpha < \Omega)$$

where

(i) $\mathfrak{G}_0$ denotes the set of open sets of E;

(ii) $\mathfrak{G}_\alpha$ denotes the set of denumerable unions (intersections) of elements belonging to the $\mathfrak{G}_\beta$ where $\beta < \alpha$ if α is even (odd) with the same convention as above for the limit ordinals α.

We shall designate the classes with finite indices by $\mathfrak{G}$, $\mathfrak{G}_\delta$, $\mathfrak{G}_{\delta\sigma}$, ... Since each G is a K_σ we have $\mathfrak{G}_0 \subset \mathfrak{K}_1$. Likewise, by taking complements, we have $\mathfrak{K}_0 \subset \mathfrak{G}_1$. By transfinite induction it follows that for each $\alpha < \Omega$ we have $\mathfrak{G}_\alpha \subset \mathfrak{K}_{\alpha+1}$ and $\mathfrak{K}_\alpha \subset \mathfrak{G}_{\alpha+1}$. Moreover, $\mathfrak{G}_\alpha$ is identical with the set of complements of elements of $\mathfrak{K}_\alpha$.

Let us suppose then that for an even α we have shown that each element of $\mathfrak{G}_\alpha$ and of $\mathfrak{K}_\alpha$ is a set of uniqueness; the same is true of the elements of $\mathfrak{G}_{\alpha+1}$, because the class of sets of uniqueness is closed under the operation of denumerable intersection. Then let $K_{\alpha+1}$ be an element of $\mathfrak{K}_{\alpha+1}$. By definition we have $K_{\alpha+1} = \bigcup_n K_\alpha^n$ where $K_\alpha^n \in \mathfrak{K}_\alpha$ and we can always suppose that the K_α^n form an increasing sequence.

We have therefore $K_{\alpha+1} = K_\alpha^1 \cup \left(\bigcup_n (K_\alpha^{n+1} - K_\alpha^n)\right)$. Now $K_\alpha^{n+1} - K_\alpha^n = K_\alpha^{n+1} \cap \complement K_\alpha^n = K_\alpha^{n+1} \cap G_\alpha^n$. This set is the intersection of two elements of $\mathfrak{G}_{\alpha+1}$; hence it is a set of uniqueness. Therefore $K_{\alpha+1}$, which is a denumerable union of *disjoint* sets of uniqueness, is a set of uniqueness.

It can be shown similarly, by interchanging the roles of $\mathfrak{G}_\alpha$ and $\mathfrak{K}_\alpha$ that if, for α odd, the elements of $\mathfrak{G}_\alpha$ and $\mathfrak{K}_\alpha$ are sets of uniqueness, the same is true of the elements of $\mathfrak{G}_{\alpha+1}$ and $\mathfrak{K}_{\alpha+1}$.

Now if α is a limit number (and therefore even), and if for each $\beta < \alpha$ the elements of $\mathfrak{G}_\beta$ and of $\mathfrak{K}_\beta$ are sets of uniqueness, the same is true of the elements of $\mathfrak{G}_\alpha$ and of $\mathfrak{K}_\alpha$.

This is obvious with regard to $\mathfrak{K}_\alpha$ since it is true of denumerable intersections; for $\mathfrak{G}_\alpha$ this follows from the fact that each $\mathfrak{G}_\alpha$ can be written in the form of a denumerable union of disjoint elements of classes $\mathfrak{G}_\beta$ with $\beta < \alpha$. By transfinite induction each element of a $\mathfrak{G}_\alpha$ (or $\mathfrak{K}_\alpha$) is therefore a set of uniqueness.

Consider now the case where E is not necessarily compact. If A is a K-borelian set of E, it is contained in a $K_\sigma = \bigcup K_n$ where the K_n are compact and increasing with n. Therefore A is the union of the sets $A \cap K_1$ and $A \cap (K_{n+1} - K_n)$ for $n = 1, 2, \ldots$ Each of these sets is a K-borelian set and is contained in a compact set. They are therefore sets of uniqueness. Since they are disjoint their union is again a set of uniqueness.

4. 2. Remark. — If E is a separable complete metric space, we have already observed that a subset of E which is borelian in the classical sense is not necessarily K-borelian. On the other hand, since such a space is homeomorphic to a G_δ of a compact metric space, we can assert that a subset of E which is borelian in the classical sense is homeomorphic to a K-borelian set. Such a set is therefore always K-analytic. This remark will allow us to apply our theory of capacities to the subsets of separable complete metric spaces which are borelian or analytic in the classical sense.

4. 3. Remark. — We have not obtained in the preceding all the results parallel to those concerning the borelian sets in the classical sense. We shall state here a few of these in the form of problems.

4. 4. Problem. — If a subset A of a compact space E is homeomorphic to a K-borelian set of class $\mathcal{K}_\alpha$ (respectively K-Souslin), is A a K-borelian set, and if it is, is A of the class $\mathcal{K}_\alpha$ (respectively K-Souslin)?

The results of Šneĭder [1 and 2] ([4]) show that the answer to this question is affirmative whenever E is such that the union of every class of open subsets of E is union of a countable subclass of that class of open subsets.

4. 5. Problem. — If E is compact, is each subset of uniqueness (or more generally, each continuous image $(\aleph_0 - 1)$([5]) of a $K_{\sigma\delta}$ of a compact space F) a K-borelian set?

4. 6. Problem. — If E is compact, is every K-analytic subset of E also a K-Souslin set?

5. The operation of projection. — In the classical theory of analytic sets one shows that each analytic subset of a Euclidian space R^n is the orthogonal projection of some G_δ of a space R^{n+1} containing R^n. We shall need later the following analogous theorem.

5. 1. Theorem. — *If* E *is a Hausdorff space, then each* K-*borelian subset of* E *(and more generally each* K-*analytic set which is a subset of a* K_σ*) is the canonical projection on* E *of a* $K_{\sigma\delta}$ *of the product space of* E *and a compact auxiliary space.*

Proof. The proof will be given first under the assumption that E is compact. If the set $A \subset E$ is the continuous image under the mapping f of a set B, which is a $K_{\sigma\delta}$ in the compact space F, the set A is the projection on E of the graph $\Gamma \subset (E \times F)$ of the function $y = f(x)$ defined on B.

Now the continuity of f implies that Γ is identical with the intersection of $\overline{\Gamma}$ and the product $E \times B$, that is to say, Γ is the

([4]) Numbers in square brackets refer to the bibliography given at the end of this report.

([5]) An application is $(\aleph_0 - 1)$ if the inverse image of every point is at most enumerable.

intersection of a compact set with a $K_{\sigma\delta}$. Therefore Γ is a $K_{\sigma\delta}$, which proves the theorem.

More generally, if E is a Hausdorff space and if A is K-analytic and contained in the union $\bigcup K_n$ of compact sets K_n of E, then A is the continuous image, by means of the function f, of some B, which is a $K_{\sigma\delta}$ contained in the sum $\sum F_n$ of compact spaces F_n, such that $f(B \cap F_n) \subset K_n$. If we take for F the compact space obtained by the Alexandroff compactification of $\sum F_n$, then the graph of f in $E \times F$ is still a $K_{\sigma\delta}$ and its projection on E is identical with A.

CHAPTER II.

NEWTONIAN AND GREENIAN CAPACITIES

6. Newtonian and Greenian capacities. — Let D be a domain in R^{ν} which possesses a Green's function. (For $\nu \geqq 3$ any domain D possesses a Green's function, but for $\nu = 2$ there are domains D which are not « Greenian »).

Let G(P, Q) be this Green's function, and let μ be a Radon measure on a compact subset $K \subset D$. The potential of μ for this kernel G(P, Q) is by definition

$$U^{\mu}(Q) = \int G(P, Q)\, d\mu(P).$$

If μ is positive, this potential is positive and superharmonic on D; it is harmonic on (D — K) and tends to 0 whenever Q tends toward a point on the boundary of D, with the exception of the so-called irregular frontier points, which form a rare set in a sense defined in modern potential theory (see, for example, M. Brelot [1]).

Let us say that a positive measure μ on K is *admissible* if $U^{\mu}(Q) \leqq 1$ everywhere on D. The total mass of μ is the integral $\int d\mu$. The supremum of the total masses of admissible measures on K is called the *capacity* of K (relative to D). For example, the capacity of K is zero if the potential of each non-zero positive measure on K is unbounded on D.

For a fixed domain D, this capacity is denoted by $f(K)$. The following properties of $f(K)$ are well known.

6. 1. $f(K) \geqq 0$ and is an increasing function of K, that is,

$$f(K_1) \leqq f(K_2) \quad \text{if } K_1 \subset K_2.$$

6. 2. $f(K)$ is subadditive, that is,

$$f(K_1 \cup K_2) \leqq f(K_1) + f(K_2).$$

For let μ be an admissible measure on $(K_1 \cup K_2)$ whose total mass m differs from $f(K_1 \cup K_2)$ by less than ε. If μ_1 and μ_2 are the restrictions of μ to K_1 and K_2 respectively, and if m_1 and m_2 are their total masses, then $m \leqq m_1 + m_2$ and μ_1 and μ_2 are admissible. Then $m \leqq m_1 + m_2 \leqq f(K_1) + f(K_2)$, and the inequality stated above follows.

We shall soon see, in fact, that $f(K)$ satisfies much sharper inequalities which, in a certain sense, cannot be improved.

6. 3. $f(K)$ is continuous on the right.

This means that for any compact set K and any number $\varepsilon > 0$, there exists a neighborhood V of K such that for every compact set K′ satisfying the relation $K \subset K' \subset V$, we have $0 \leqq f(K') - f(K) \leqq \varepsilon$. The proof of this property will be omitted.

6. 4. Interior and exterior capacities. Capacitability. — We shall associate with every subset A of D an interior capacity and an exterior capacity.

We define the *interior capacity* of A to be sup $f(K)$ for $K \subset A$ and denote it by $f_*(A)$. In particular, the interior capacity of every open set $G \subset D$ is defined. This fact enables us, then, to define the *exterior capacity* of A to be inf $f_*(G)$ for $A \subset G$; the exterior capacity of A is denoted by $f^*(A)$. Thus, for every open set G we have $f_*(G) = f^*(G)$. More generally we shall say that the set A is *capacitable* if $f_*(A) = f^*(A)$, and we shall designate the common value of the two capacities by $f(A)$; the notation $f(A)$ will not lead to confusion since, as will be shown later in the general theory of capacities, $f_*(A) = f^*(A) = f(A)$ whenever A is a compact set. (This result follows easily from the continuity on the right of f.)

We say that a property holds *quasi everywhere* (*nearly everywhere*) if it holds at each point of D except at the points of a set of exterior capacity (interior capacity) zero.

When the set of exceptional points is capacitable, the two notions coincide; we shall see in the following chapters that this situation occurs when the set of exceptional points is borelian or analytic.

We now prove the following property, which will soon be needed : The union of a finite number of sets of exterior capacity zero is a set of exterior capacity zero.

For if $f^*(A_1) = f^*(A_2) = 0$, there exists, for every $\varepsilon > 0$, open sets G_1 and G_2 containing A_1 and A_2 respectively whose capacities are less than ε. But $f(G_1 \cup G_2) \leqq f(G_1) + f(G_2) \leqq 2\varepsilon$; indeed, each compact set K contained in $G_1 \cup G_2$ is the union of two compact sets K_1 and K_2 such that $K_1 \subset G_1$ and $K_2 \subset G_2$ (7). Then $f(K) \leqq f(K_1) + f(K_2) \leqq f(G_1) + f(G_2)$; since $f(G_1 \cup G_2) - f(K)$ can be made arbitrarily small, the subadditivity for open sets follows. Since $f(G_1 \cup G_2)$ can be made arbitrarily small, we have $f(A_1 \cup A_2) = 0$. The proof is complete.

6. 5. **Equilibrium potential.** — It is shown in potential theory that for every compact set $K \subset D$ there exists one and only one admissible measure μ defined on K such that its potential U^μ is quasi everywhere in K equal to 1. Its total mass is equal to the capacity $f(K)$ of K. This measure is the *equilibrium distribution* of K and its potential is the *equilibrium potential* of K. The equilibrium distribution is the only admissible measure on K whose total mass is equal to $f(K)$.

6. 6. **Fundamental principles.** — We recall the following two assertions which we shall need presently.

Let U^μ be the potential of a Radon measure μ defined on a compact set $K \subset D$ and such that U^μ is bounded on D.

6. 7. *If* $U^\mu \geqq 0$ *quasi everywhere on* K, *then the same inequality holds everywhere on* D.

The property stated in 6. 7 is an immediate consequence of the *general maximum principle.* We shall not state this principle however, because it involves the notion of energy which we shall not use.

6. 8. *If* $U^\mu \geqq 0$ *everywhere on* D, *then the total mass of* μ *is positive*; *it is zero only when* $U^\mu \equiv 0$.

It follows readily from these two properties that, if $U^\mu \geqq 0$ quasi everywhere on K, then the total mass of μ is positive; it is zero only when $U^\mu \equiv 0$.

(7) For a proof of this fact see 17.4, Chapter IV.

7. Successive differences. — If $\varphi(x)$ is a real function of the real variable $x \geqq 0$ the fact that φ if increasing may be expressed by stating that $\Delta_1(x, a) = \varphi(x+a) - \varphi(x) \geqq 0$ for all $a > 0$. Similarly, the fact that it is convex may be expressed by stating that

$$\begin{aligned}\Delta_2(x; a, b) &= \Delta_1(x+b, a) - \Delta_1(x, a)\\ &= \varphi(x+a+b) - \varphi(x+a) - \varphi(x+b) + \varphi(x) \geqq 0\end{aligned}$$

for all $a, b \geqq 0$. More generally, if φ has a derivative of order n and if this derivative has constant sign, then this fact may be stated by saying that the difference Δ_n of order n always has this same sign.

The successive differences of φ then furnish a means of studying the nature of the increase of φ. This method is of interest because it can be extended to the study of functions not necessarily of a number x, but of a set, or more generally of elements of a commutative semi-group, addition being replaced by the semi-group operation.

It will be shown presently that the successive differences relative to the capacities $f(K)$ are alternately positive and negative; therefore, it will be convenient to so alter the sign that the final expressions all have the same sign.

7. 1. Successive differences relative to equilibrium potentials and to capacities. — For every compact set $K \subset D$ we designate by $h(K)$ the equilibrium potential of K, and by $f(K)$ the capacity of K. If X, A_1, A_2, ..., are compact subsets of D, we define

$$\nabla_1(X; A_1)_h = h(X) - h(X \cup A_1)$$

and, in general,

$$\begin{aligned}\nabla_{n+1}(X; A_1, \ldots, A_{n+1})_h = \nabla_n(X; A_1, \ldots, A_n)_h\\ - \nabla_n(X \cup A_{n+1}; A_1, \ldots, A_n)_h.\end{aligned}$$

The differences $\nabla_n(X; A_1, \ldots, A_n)_f$ are defined in the same way.

The index f or h will be omitted when no ambiguity is possible.

Functional properties of the differences ∇_n.

7. 2. $\nabla_n(X; A_1, \ldots, A_n)$ is a symmetric function of the variables A_i. This property is a consequence of the following development of ∇_n:

$$\nabla_n = h(X) - \Sigma h(X \cup A_i) + \Sigma h(X \cup A_i \cup A_j) - \cdots + (-1)^n h(X \cup A_1 \cup \cdots \cup A_n).$$

This symmetry permits ∇_n to be written in the form $\nabla_n(X; \{A_i\})$. The index n may as well be omitted since it is determined when the family $\{A_i\}$ is determined.

7. 3. $\nabla_n(X; \{A_i\}) = \nabla_n(X; \{A'_i\})$ if $X \cup A_i = X \cup A'$ for all i. This follows from the fact that the A_i always occur in the development of ∇_n in a union with X. In particular, $\nabla_n = 0$ if $A_i \subset X$ for all i.

7. 4. $\nabla_n(X; \{A_i\}) = \nabla_n(\emptyset; \{A_i\}) - \nabla_{n+1}(\emptyset; \{A_i, X\})$ where the expression $\{A_i, X\}$ denotes the family of sets consisting of X and the A_i. This formula is easily derived from the expression that defines $\nabla_{n+1}(\emptyset; \{A_i, X\})$ in terms of the ∇_n. It shows that ∇_n is the sum of two functions, each of which is a function symmetric in all its variables.

7. 5. $\nabla_n(X; A_1, \ldots, A_{n-1}, A_n \cup a_n) - \nabla_n(X; A_1, \ldots, A_n)$

$$= \nabla_n(X \cup A_n; A_1, \ldots, A_{n-1}, a_n).$$

In order to verify this relation it is sufficient to express each of the ∇_n in terms of ∇_{n-1}. The six terms thus obtained cancel pairwise.

Fundamental properties of $h(X)$ and $f(X)$.

7. 6. **Theorem.**

(i) *For every* X *and* $\{A_i\}$ *it is true that* $0 \leqq -\nabla_n(X; \{A_i\})_k \leqq 1$.

The potential $(-\mathsf{V}_n)_h$ *is equal to* 0 *quasi everywhere on* X, *and it is an increasing function of each of the* A_i.

(ii) *This potential is a decreasing function of* X, *and moreover it is a decreasing function of* n *in the sense that*

$$-\mathsf{V}(X: \{A_i\}_{i\in I}) \leqq -\mathsf{V}(X; \{A_i\}_{i\in J}) \quad \textit{whenever} \quad I \supset J.$$

Proof. This theorem is proved by induction on n and by using the functional properties of the V_n. To simplify the notation, let $\mathsf{V}'_n = -\mathsf{V}_n$.

Consider first (i) in the case $n = 1$. The function $\mathsf{V}'(X; A_1)$ is the potential of a measure defined on $X \cup A_1$ since $\mathsf{V}'_n(X, A_1) = h(X \cup A_1) - h(X)$. Now $0 \leqq h(X \cup A_1) \leqq 1$ and $0 \leqq h(X) \leqq 1$, with $h(X) = 1$ quasi everywhere on X and $h(X \cup A_1) = 1$ quasi everywhere on $(X \cup A_1)$.

Thus $\mathsf{V}'_1 \leqq 1$; $\mathsf{V}'_1 = 0$ quasi everywhere on X and $\mathsf{V}'_1 \geqq 0$ quasi everywhere on A_1.

Hence $\mathsf{V}'_1 \geqq 0$ quasi everywhere on $X \cup A_1$; and, by virtue of the fundamental principle 6. 7, $\mathsf{V}'_1 \geqq 0$ everywhere. Moreover, $\mathsf{V}'_1(X; A_1)$ is an increasing function of A_1. This fact is an immediate consequence of the functional property 7. 5:

$$\mathsf{V}'_1(X; A_1 \cup a) - \mathsf{V}'_1(X; A_1) = \mathsf{V}'_1(X \cup A_1; a) \geqq 0.$$

Consider next (i) in the general case. We suppose the first part of the theorem to be true for all $p \leqq n$ and show that it is true for $p = n + 1$. Since

$$\mathsf{V}'_{n+1}(X; A_1, \ldots, A_{n+1}) = \mathsf{V}'_n(X; A_1, \ldots, A_n) - \mathsf{V}'_n(X \cup A_{n+1}; A_1, \ldots, A_n),$$

V'_n is the (bounded) potential of a measure defined on

$$X \cup \left(\bigcup_1^{n+1} A_i\right).$$

For each V'_n of the second member, $0 \leqq \mathsf{V}'_n \leqq 1$ everywhere, so that $\mathsf{V}'_{n+1} \leqq 1$ everywhere. Each of the V'_n is zero quasi everywhere on X, and therefore similarly for V'_{n+1}. On A_{n+1}

the first V'_n is positive and the second is quasi everywhere zero; thus V'_{n+1} is quasi everywhere positive there. Because of the symmetry of V'_{n+1} with respect to the variables A_i, the preceding result holds for all A_i. Then, since the potential V'_{n+1} is quasi everywhere positive on the union of X and the A_i, $V'_{n+1} \geqq 0$ everywhere. Also, we have on X:

$$\begin{aligned} V'_{n+1}(X; A_1, \ldots, A_{n+1}) &= V'_n(X; A_1, \ldots, A_n) - V'_n(X \cup A_{n+1}; A_1, \ldots, A_n) \\ &= 0 - 0 = 0 \text{ quasi everywhere on X.} \end{aligned}$$

This completes the proof of our assertion that $(-V_n)_h$ is equal to 0 quasi everywhere on X for every n.

That V'_{n+1} is an increasing function of each of the A_i is an immediate consequence of property 7. 5, just as in the case of V_1.

Consider next the proof of (ii). Clearly

$$V'_n(X \cup a; \{A_i\}) - V'_n(X; \{A_i\}) = -V'_{n+1}(X; \{a, A_i\}) \leqq 0,$$

which shows that V'_n is a decreasing function of X.

From this same relation we see that

$$V'_{n+1}(X; \{a, A_i\}) \leqq V'_n(X, \{A_i\}),$$

that is, V' decreases whenever an element is adjoined to the family of the A_i; therefore whenever any number of elements is adjoined.

Complement of theorem 7. 6.

7. 7. Definition. — *The* essential envelope $\tilde{K}$ *of a compact set* $K \subset D$ *is the closure of the set of points of* D *on which* $h(K) = 1$.

The set $\tilde{K}$ is compact and $(K - \tilde{K})$ is a set of exterior capacity zero; the relationship of K to $\tilde{K}$ is expressed by saying that K is quasi contained in $\tilde{K}$. Since $h(K) = h(\tilde{K})$, we have $\tilde{\tilde{K}} = \tilde{K}$. Similarly, $(\tilde{K}_1 \subset \tilde{K}_2)$ implies $\tilde{K}_1 \subset \tilde{K}_2$; and, moreover, $\widetilde{K_1 \cup K_2} = (\widetilde{\tilde{K}_1 \cup \tilde{K}_2})$ for any choice of K_1 and K_2.

7. 8. **Restrictive hypothesis on D.** — We suppose that for all $K \subset D$ the open set $(D - \tilde{K})$ is connected; this will be the case if the frontier of D is connected (this frontier contains the point at infinity if the dimension of D is greater than 2 and if D is unbounded).

When the condition « $(D - K)$ *connected for every* $K \subset D$ » *is satisfied, we will say that* D *is* simple.

7. 9. **Statement of the complement of theorem 7. 6.** — *When* D *is simple, a necessary and sufficient condition that* $\mathsf{V}(X;\ \{A_i\})_h \equiv 0$ *on* D *is that there exists an* i_0 *such that* $\tilde{A}_{i_0} \subset \tilde{X}$. *When* $\mathsf{V}(X;\ \{A_i\}) \not\equiv 0$, *the set of points of* D *where* $\mathsf{V} = 0$ *is contained in* $\tilde{X}$ *and differs from* $\tilde{X}$ *by a set of exterior capacity zero.*

Proof. We shall use the following fact: if $\tilde{A} \subset \tilde{B}$ and if $A \neq B$, then at every point of $\complement \tilde{A}$ we have $h(A) < h(B)$. For let $m \in \complement \tilde{A}$. There exists a point $m_0 \in (\tilde{B} - \tilde{A})$ such that all spheres S with center m_0 intersect $(\tilde{B} - \tilde{A})$ in a compact set b of non-zero capacity. If S is taken sufficiently small so that $D - (\tilde{B} \cup \tilde{b})$ is connected and $m \notin b$, then $h[\tilde{A} \cup \tilde{b}] - h[\tilde{A}]$ is harmonic and strictly positive on $\complement (\tilde{A} \cup \tilde{B})$ We have, a fortiori, $h(\tilde{B}) > h(\tilde{A})$ since $\tilde{B} \supset \tilde{A} \cup \tilde{b}$.

Consider first the case $n = 1$. $\mathsf{V}_1(X;\ A) = h(X) - h(X \cup A)$ is identically zero if $\widetilde{A \cup X} = \tilde{X}$, which is equivalent to $\tilde{A} \subset \tilde{X}$; otherwise $\mathsf{V}_1 \neq 0$ at each point of $\complement \tilde{X}$. Moreover, we know that $\mathsf{V}_1 = 0$ quasi everywhere on X.

Consider next the general case. We now assume the assertion true for $p \leq n$ and prove that it holds also for $p = n + 1$. If one of the A_i is such that $\tilde{A}_{i_0} \subset \tilde{X}$, then $\mathsf{V}_{n+1} \equiv 0$. Otherwise, consider

$$\mathsf{V}'_{n+1}(X;\ A_1,\ \ldots,\ A_{n+1}) = \mathsf{V}'_n(X;\ A_1,\ \ldots,\ A_n) - \mathsf{V}'_n(X \cup A_{n+1};\ A_1,\ \ldots,\ A_n).$$

The first term of the difference is greater than 0 on $(\tilde{A}_{n+1} - \tilde{X})$, and the second term of the difference is zero quasi everywhere; then the difference is greater than 0 quasi everywhere on that set. At every point of $\complement\left(\widetilde{\left(\bigcup A_i\right)\cup X}\right)$, the difference $V'_{n+1}(X)$ is positive and harmonic; it is therefore greater than 0. Thus V'_{n+1} is quasi everywhere greater than 0 on $\complement\tilde{X}$. In fact, this strict inequality holds everywhere on $\complement\tilde{X}$. The proof is entirely analogous to the above. We replace each $\tilde{A}_i$ with $\tilde{X}\cup\tilde{b}_i$, where each b_i is compact and small enough so that we may conclude that a certain harmonic function is greater than 0. Finally, the theorem follows from the fact that, as we have already seen, $V'_{n+1}=0$ quasi everywhere on X.

7. 10. Corollary of theorem 7. 6. — *If* $(V_n)_f$ *designates the differences associated with the capacity f, we have* $(V_n)_f\leqq 0$ and $(V_n)_f$ *possesses the same monotonic properties as* $(V_n)_h$.

Proof. The potential $(V_n)_h$ is a linear combination of potentials $h(k)$, and the total mass of the measure which generates it is the sum of the total masses of the equilibrium distributions of the compact sets K, with the same coefficients, $+1$ or -1, as the corresponding potentials $h(K)$. Moreover, according to the second fundamental property 6. 8 of potentials, since $(V_n)_h\leqq 0$ everywhere, the total mass of the measure which generates it is negative. Thus $(V_n)_f\leqq 0$.

The monotony properties of $(V_n)_f$ follow, as in the case of the $(V_n)_h$, from the functional properties of the V_n and from the fact that all the $(V_n)_f$ are negative.

7. 11. Complement of corollary 7. 10. — *We deduce immediately from the complement* 7. 9 *of Theorem* 7. 6 *that, under the hypothesis that* D *is* simple, *a necessary and sufficient condition that* $V(X;\{A_i\})=0$ *is that for some* $i=i_0$, *we have* $\tilde{A}_{i_0}\subset\tilde{X}$.

7. 12. Remark — Whenever a function $\varphi(x)$ of a real variable x satisfies inequalities analogous to those shown for

the capacity f, it is increasing, concave,... and possesses derivatives of all orders, alternately positive and negative. The opposite $-\varphi$ of such a function is said to be completely monotonous although the term is not especially descriptive. It is known that such a function is analytic. The capacity thus appears as an analytic set function, with *derivatives* alternately positive and negative. We shall say that a capacity is a set function which is *alternating of infinite order.*

8. **The inequality** $(\nabla_2)_f \leqq 0$. — This inequality can be written as follows:

8. 1. $f(X) - f(X \cup A_1) - f(X \cup A_2) + f(X \cup A_1 \cup A_2) \leqq 0.$

If A and B are any two compact sets, let $X = A \cap B$, $A_1 = A$ and $A_2 = B$. Then the inequality 8. 1 implies

8. 2. $$f(A \cup B) + f(A \cap B) \leqq f(A) + f(B).$$

Since $f \geqq 0$, the capacity satisfies an inequality stronger than subadditivity. This inequality plays an important role in the following.

We remark here that ordinary subadditivity is sometimes wrongly called convexity. In fact, the preceding inequality, which is stronger than subadditivity, is, as we have seen above, analogous to a condition of concavity.

We shall proceed to give another form to the condition $\nabla_2 \leqq 0$. Let a, k, A, be three compact sets with $a \subset A$. Setting $X = a$, $A_1 = k$, $A_2 = A$, it follows that

8. 3. $$f(A \cup k) - f(A) \leqq f(a \cup k) - f(a).$$

In other words, when a fixed compact set, k, is adjoined to a compact set X, the smaller the X, the greater the increase in the capacity of X.

Application. — *Let* (A_i) *and* (a_i), $i = 1, 2, \ldots, n$, *be two families of compact sets such that* $a_i \subset A_i$ *for all* i. *Then*

8. 4. $$f\left(\bigcup A_i\right) - f\left(\bigcup a_i\right) \leqq \Sigma\left(f(A_i) - f(a_i)\right).$$

Proof. According to the inequality 8. 3. above, we may write

$$f(A_1 \cup A_2) - f(a_1 \cup A_2) \leqq f(A_1) - f(a_1),$$
$$f(A_2 \cup a_1) - f(a_1 \cup a_2) \leqq f(A_2) - f(a_2),$$

from which, adding termwise;

8. 5. $$f(A_1 \cup A_2) - f(a_1 \cup a_2) \leqq \sum_{i=1,2} (f(A_i) - f(a_i)).$$

If the inequality 8. 4. is satisfied for order n, it is sufficient to apply inequality 8. 5. to obtain 8. 4. also for order $n + 1$.

8. 6. Geometric application of the inequality $V_2 \leqq 0$.

We shall suppose D to be invariant under a one parameter continuous group of motions T_λ, where the parameter λ is chosen so that $T_{\lambda_1} . T_{\lambda_2} = T_{\lambda_1 + \lambda_2}$. For every compact set K_0 in D and all pairs of values α, β, of λ let $K_{\alpha\beta} = \bigcup_{\alpha \leqq \lambda \leqq \beta} T_\lambda(K_0)$ and $K_\beta = K_{0\beta}$.

In other words, $K_{\alpha\beta}$ is generated by the motions of K_0 for λ varying between α and β.

Because of the invariance of D with respect to the T_λ, clearly $f(K_{\alpha\beta}) = f(K_{(\beta-\alpha)})$. Let $f(K_\beta) = \varphi(\beta)$ and

$K_{0\alpha_1} = A_1$; $K_{x,(\alpha_1+x)} = X$; $K_{(\alpha_1+x)(\alpha_1+\alpha_2+x)} = A_2$, where α_1, α_2, x are positive numbers. Then the inequality $V_2 \leqq 0$ becomes

$$\varphi(x) - \varphi(x + \alpha_1) - \varphi(x + \alpha_2) + \varphi(x + \alpha_1 + \alpha_2) \leqq 0.$$

Hence, the second differences relative to $\varphi(x)$ are negative, that is, $\varphi(x)$ is a concave function.

Thus *the capacity of* K_β *is an increasing concave function of* β.

This property can be easily verified for the solids of R^3 whose capacity can be explicitly calculated.

EXAMPLE. — If the T_λ represent translations in $D = R^3$ the K_β are unions of parallel segments of length β.

9. Complete system of inequalities. — We have obtained a system of inequalities $V_n \leqq 0$ which are satisfied by the function $f(K)$. We shall show that in a certain natural sense there are no others, that is, that every inequality identically satisfied by f is a consequence of $\bigvee_n \leqq 0$.

Let $\{A_i\}_{i \in I}$ $(I = \{1, 2, \ldots, n\})$ be a family of n compact subsets of D. For each $J \subset I$, let

$$B_J = \bigcup_{i \in J} A_i \text{ and } x_J = f(B_J) \text{ for } J \neq \emptyset,$$

There are $N = 2^n - 1$ subsets J of I. We may then associate with each family $\{A_i\}_{i \in I}$ the point of the Euclidean space R^N, whose coordinates are $(x_J)_{J \subset I}$. Our object is to characterize the locus of this point in R^N when D and I remain fixed but the family $\{A_i\}_{i \in I}$ is allowed to vary.

9. 1. Definition. — *We denote by C_n the set consisting of the points $(x_J)_{J \subset I}$ of R^N when the family $\{A_i\}_{i \in I}$ varies, I and D remaining fixed. We denote by L_n the set oonsisting of the points of R^N defined by the following N inequalities:*

$$-\lambda_H = \bigvee (B_{I-H};\ \{A_i\}_{i \in H}) \leqq 0 \quad \textit{where} \quad H \subset I \quad \textit{and} \quad H \neq \emptyset.$$

We have omitted, in this definition, the index of $\bigvee$ which is obviously equal to $\overline{\overline{H}}$.

The second part of this definition requires an explanation. Each $\bigvee$ is a linear combination (with coefficients $+1$ or -1) of terms of the form $f(B_J)$; if we then replace each $f(B_J)$ by x_J we have a form which is linear with respect to the x_J. The set of points of R^N for which $\bigvee \leqq 0$ is then a closed half-space in R^N. More explicitly,

$$-\lambda_H = \sum_{J \supset (I-H)} (-1)^{p_J} x_J \qquad \text{where} \qquad p_J = \overline{\overline{J - (I - H)}}.$$

9. 2. Theorem. — (i) *The set L_n is a convex cone of dimension N; it can be represented parametrically in the form*

$$\overrightarrow{OM} = \sum_{H \subset I} \lambda_H \vec{V}_H \quad (\lambda_H \geqq 0)$$

where the vector $\vec{V}_H$ of R^N has the components x_J^H defined as follows:

$$x_J^H = 0 \quad \textit{if} \quad H \cap J = \emptyset, \quad \textit{and} \quad x_J^H = 1 \quad \textit{if} \quad H \cap J \neq \emptyset.$$

(ii) $\qquad C_n \subset L_n \quad$ *and* $\quad \mathring{C}_n = \mathring{L}_n$ [8].

[8] The notation $\mathring{A}$ means the interior of A.

9. 3. *Proof of* (i). We shall use the expression of $-\lambda_H$ as a function of the x_J obtained above and calculate

$$-\sum_{H \cap J_0 \neq \emptyset} \lambda_H \qquad \text{for a} \qquad J_0 \subset I.$$

The coefficient of x_{J_0} is

$$\sum_H (-1)^{\overline{\overline{J_0-(I-H)}}} \quad \text{where} \quad (I-H) \subset J_0 \quad \text{and} \quad H \cap J_0 \neq \emptyset.$$

It follows that this coefficient is

$$[-1+(1-1)^{\overline{\overline{J_0}}}] = -1.$$

Similarly the cœfficient of x_J for $J \neq J_0$ is

$$\sum_H (-1)^{\overline{\overline{J-(I-H)}}} \quad \text{where} \quad (I-H) \subset J \quad \text{and} \quad H \cap J_0 \neq \emptyset.$$

By examining first the case where $J_0 \subset J$ and then the case where $J_0 \not\subset J$, we find that the coefficient of x_J is always 0. Thus,

$$x_{J_0} = \sum_{H \cap J_0 \neq \emptyset} \lambda_H.$$

This gives the solution of the system of equations

$$-\lambda_H = \sum_{J \supset I-H} (-1)^{p_J} x_J.$$

The second members of these equations are thus linearly independent forms, and the vectors $\vec{V}_H$ are also linearly independent.

The formula $\overrightarrow{OM} = \Sigma \lambda_H \vec{V}_H$ follows immediately from the expression of the x_J as functions of the λ_H.

9. 4. *Proof of* (ii). The relation $C_n \subset L_n$ is an immediate consequence of the fact that, for every point of C_n the V associated with this point are all negative, according to corollary 7. 10. The relation $\mathring{C}_n = \mathring{L}_n$, which expresses the identity of the interior of C_n and the interior of the cone L_n, is much less obvious.

We present here a general outline of the proof. Let us suppose for a while that for every system of numbers $\lambda_H \geq 0$ (with

$H \subset I$ and $H \neq \emptyset$) there exists a family of compact sets K_H with $f(K_H) = \lambda_H$, which are additive in the sense that for every subfamily $\{K_{H_p}\}$ of this family, we have

$$f\left(\bigcup K_{H_p}\right) = \sum f(K_{H_p}) = \sum \lambda_{H_p}.$$

For this family of compact sets and for each $i \in I$, let $A_i = \bigcup_{H \ni i} K_H$. In the space R^N, the point M representative of the system of sets A_i is then defined by $\overrightarrow{OM} = \sum \lambda_H \vec{V}_H$. For we have here, with the notation already introduced, $f(B_J) = f\left(\bigcup_{i \in J} A_i\right)$. Now

$$f\left(\bigcup_{i \in J} A_i\right) = f\left(\bigcup_{H \cap J \neq 0} K_H\right) = \sum_{H \cap J \neq 0} \lambda_H.$$

We have then, $x_J = \sum_{H \cap J \neq 0} \lambda_H$. Thus, under the initial hypothesis of additivity we see that every point of the cone L_n is a point of C_n.

As a matter of fact, this hypothesis is realized only approximately, in a sense which we shall make precise, for the capacities considered here.

We shall use a hypothesis a little different, and, in fact, weaker than that of additivity, and attempt to show that it is realized for our capacities.

We shall suppose that for any given number $\varepsilon > 0$, there exists a family of compact sets $K_H \subset D$ ($H \subset I$ and $H \neq \emptyset$) such that

9. 5. for each of these we have $f(K_H) = 1$;

9. 6. $f\left(\bigcup K_H\right) = \sum f(K_H) - \eta = N - \eta$ where $0 \leqq \eta \leqq \varepsilon$;

9. 7. for every λ such that $0 \leqq \lambda \leqq 1$ and for every H, there exists a compact $K_H(\lambda)$ such that

(a) $f(K_H(\lambda)) = \lambda$,

(b) $K_H(\lambda') \subset K_H(\lambda)$ if $\lambda' < \lambda$,

(c) $K_H(1) = K_H$.

For every system of numbers $\lambda_H \geq 0$ such that $\Sigma\lambda_H \leq 1$ and for every $i \in I$, let

$$A_i = \bigcup_{H \ni i} K_H(\lambda_H).$$

We designate by m the point of L_n defined by

$$\overrightarrow{Om} = \sum \lambda_H \vec{V}_H.$$

The set of these points, under the condition $\sum \lambda_H \leq 1$, is a simplex S of dimension N. The definitions above of A_i associate with each m the point $M = \Phi(m)$ representing in R^N the family of the A_i. If the $K_H(\lambda)$ formed an additive family, the mapping Φ would be an identity. We shall see that with our hypothesis, Φ is a continuous mapping which differs arbitrarily little from an identity if ε is taken sufficiently small.

9. 8. **Φ is continuous.** — It is sufficient to show that each $f(B_J)$ is a uniformly continuous function of m; since $f(B_J)$ is an increasing function of λ_H, it is enough to give to the λ_H positive increments $\Delta\lambda_H$. From the inequality

$$f\left(\bigcup K_p\right) - f\left(\bigcup k_p\right) \leq \sum f(K_p) - f(k_p),$$

it follows, since the $K_H(\lambda)$ increase with λ, that

$$f(B'_J) - f(B_J) \leq \sum \Delta\lambda_H,$$

where the B_J and B'_J are associated respectively with the points $m = (\lambda_H)$ and $m + \Delta m = (\lambda_H + \Delta\lambda_H)$. This inequality proves the required continuity.

9. 9. **Φ differs arbitrarily little from an identity.** — It is sufficient to show that each $f(B_J)$ differs arbitrarily little from $\sum_{H \cap J \neq \emptyset} \lambda_H$. More generally, given any family $(K_p)_{p \in P}$ of compact sets such that $\sum_{p \in P} f(K_p) - f\left(\bigcup K_p\right) < \varepsilon$, the same inequality holds when we replace each K_p by a compact subset of K_p. This follows from the inequality used above by writing it in

the form $\left(\sum f(k_p)\right) - f\left(\bigcup k_p\right) \leq \left(\sum f(K_p)\right) - f\left(\bigcup K_p\right)$ where $k_p \subset K_p$.

Now it is a well known fact that, if $M = \Phi(m)$ is a continuous mapping of the N dimensional simplex S of R^N into R^N such that $Mm \leq \eta$ for all m, the image $\Phi(S)$ of S contains all points of S at a distance $\geq \eta$ from the boundary of S. Since η tends to 0 with ε, it follows that each interior point of S is a point M which represents a family (A_i) of compact subsets of D.

Finally, if we notice that in our second hypothesis the constant which occurs in the definition of S, that is, in the condition $\sum \lambda_H \leq 1$, can be replaced by an arbitrary positive constant a, we get immediately $\mathring{C}_n = \mathring{L}_n$.

9. 10. **Proof of the second hypothesis.** — We shall prove this hypothesis for the case in which the constant a has the value 1.

It is sufficient to show that for every integer N and for every $\varepsilon > 0$ there exists a family of compact *regular* sets K_i, $(i = 1, 2, \ldots, N)$, such that $f(K_i) = 1$ for every i and $f\left(\bigcup K_i\right) = N - \eta$, with $0 \leq \eta \leq \varepsilon$, where a compact subset of D is called *regular* when it is the union of a finite number of cubes. For if C is a cube and if $C_{(\rho)}$ denotes the cube concentric with C and obtained from C by a homothety of ratio $\rho \geq 0$, then $f(C_{(\rho)})$ is a continuous increasing function of ρ. More generally, let $K = \bigcup C_n$, where each C_n is a cube and let $K_{(\rho)} = \bigcup C_{n(\rho)}$. Then, recalling the inequality (8. 4), it follows that $f(K_{(\rho)})$ is an increasing and continuous function of ρ with $f(K_{(0)}) = 0$ and $f(K_{(1)}) = 1$. The third part of the second hypothesis is thus satisfied whenever the compact sets K_i are regular.

Let $G(P_0, Q)$ be the Green's function for D with pole P_0. If $S(P_0, \rho)$ denotes the open Green's sphere defined by $G(P_0, Q) \geq \rho$, it is well known that its capacity is $\frac{1}{\rho}$.

The procedure will now be as follows; we shall suppose the N points P_i, $i = 1, 2, \ldots, N$, so chosen that the restriction of $G(P_i, Q)$ to $S(P_j, 1/2)$ is $\leq \delta$ for all couples i, j with $i \neq j$ (δ will be determined later as a function of ε). Since for each

i we have $f(S(P_i, 1/2)) = 2$, we can find a regular compact set of capacity $> 3/2$ in the open set $S(P_i, 1/2)$. Starting with this compact set, we can construct a compact regular subset K_i of $S(P_i, 1/2)$ with capacity $= 1$ by a procedure already used.

Now, the equilibrium potential $h(K_i)$ satisfies the relation $h(K_i) \leqq \inf[1, 2G(P_i, Q)]$ everywhere on D since

$$\inf[1, 2G(P_i, Q)]$$

is the equilibrium potential of $S(P_i, 1/2)$ and $K_i \subset S(P_i, 1/2)$. For every pair i, j with $i \neq j$, the restriction of $h(K_i)$ to K_j is $< 2\delta$; then $\sum h(K_j)$ is, on each K_i, less than $(1 + 2\delta N)$. Then $\dfrac{\sum h(K_i)}{1 + 2\delta N}$ is on D the potential of a positive admissible measure (see the beginning of this chapter) of total mass $\dfrac{N}{1 + 2\delta N}$. Thus $\dfrac{N}{1 + 2\delta N} \leqq f\left(\bigcup K_i\right) \leqq N$; hence, $N - f\left(\bigcup K_i\right) \leqq \dfrac{2\delta N^2}{1 + \delta N}$. For given ε and N, δ can always be chosen small enough so that this quantity does not exceed ε.

It remains to choose, for every $\delta > 0$, the N points P_i so that the restrictions described above are satisfied. When D has a boundary D which is sufficiently regular, we designate by $\{\pi_i\}$ a family of N distinct points of D^* and by $\{V_i\}$ mutually disjoint neighborhoods of these points. For every i there exists a neighborhood W_i of π_i such that for every $P_i \in W_i$ and every $Q \in V_i$, $G(P_i, Q) < \delta$. If moreover $\delta < 1/2$, then $S(P_i, 1/2) \subset V_i$. Thus $G(P_i, Q) \leqq \delta$ on $S(P_j, 1/2)$ for $i \neq j$.

In the general case, a proof has been kindly given by M. Brelot (9) at my request.

9. 11. **Study of the frontier of C_n.** — We have proved the relations $C_n \subset L_n$ and $\mathring{C}_n = \mathring{L}_n$, but it remains to determine which frontier points of the cone L_n belong to C_n. This depends essentially on the topological nature of D and probably on its homology group. We shall give a complete determination of

(9) M. Brelot. Existence theorem of n capacities, in these Annals, tome 5.

C_n only when D is simple (see 7. 8). We shall not give a proof here; it would be analogous to the proof of the second part of Theorem 9. 2. and would follow essentially of the result stated in 7. 11.

We shall call the set of points of L_n defined by a set of equalities of the form $(\lambda_{H_p} = 0)$ a *face* of L_n. If this set contains r equalities, the dimension of the face is $N - r$.

The first essential fact is that if a point interior to a face of L_n belongs to C_n, then each point interior to this face also belongs to C_n. Such an open face will be called an *open face* of C_n.

9. 12. **Determination of the open faces of C_n.**

RULE. — *Let* $\{\lambda_{H_p} = 0\}$ *be the set of equalities which determine a face of* L_n. *Its interior is an open face of* C_n *if and only if* $\{\lambda_{H_p} = 0\}$ *is* hereditary *in the following sense: if it contains an equality* $\lambda_H = 0$, *it must also contain all its* descendents *relative to at least one index* $i_0 \in H$.

For a better understanding of this rule, recall that we had set $\lambda_H = V'(B_{I-H}; \{A_i\}_{i \in H})$ for any system of sets (A_i).

With the hypotheses made on D, if $\lambda_H = 0$, there exists an $i_0 \in H$ such that $\tilde{A}_{i_0} \subset \tilde{B}_{I-H}$. It follows that the equality $\lambda_H = 0$ implies that $\lambda_{H'} = 0$ for every H' such that $i_0 \in H'$ and $H' \subset H$. It is this fact that leads to the definition and the preceding result. More precisely, a set $\mathcal{E}$ of equalities $\lambda_{H_p} = 0$ defines an open face of L_n if for every p there exists an $i_p \in H_p$ such that for every H' which satisfies $i_p \in H' \subset H_p$, the equality $\lambda_{H'} = 0$ belongs to $\mathcal{E}$; these h' are the *descendents* of H_p relative to i_p.

9. 13. EXAMPLE. Let $I = 1, 2, 3$ so that $N = 2^3 - 1 = 7$.
The open face $\lambda_{1,2} = 0$, $\lambda_1 = 0$ belongs to C_n.
The open face $\lambda_{1,2} = 0$, $\lambda_{2,3} = 0$, $\lambda_2 = 0$ belongs to C_n.
The open face $\lambda_{1,2} = 0$, $\lambda_{2,3} = 0$, $\lambda_1 = 0$ does not belong to C_n.

9. 14. **Canonical parametrization of the set of open faces.** — *Every open face is characterized by a set of independent relations of the form* $\tilde{A}_i \subset \tilde{B}_{I-H}$ *where* $i \in H$. *Conversely, to each set of such relations corresponds a face whose equations are all the* $\lambda_{H'} = 0$ *where* $i \in H' \subset H$ *and* i *and* H *are indices relative to one of the given relations.*

As an example, we shall now give the set of all the systems of relations which define the faces of C_3. For brevity of notation we indicate the relation $\tilde{A}_i \subset \tilde{B}_{I-H}$ by writing $(i|(I-H))$ It is necessary to add to the systems listed those which follow from them by permutations of the indices.

Systems including a single relation.

$\{(1|2\}$ $\{(1|2, 3)\}$

Systems including two relations.

$\{(1|2), (1|3)\}$ $\{(1|2), (2|1)\}$ $\}(1|2), (2|3)\}$
$\{(1|2), (3|2)\}$ $\{(1|2), (2|3, 1)\}$ $\{(1|2, 3), (2|3, 1)\}$

Systems including three relations.

$\{(1|2), (1|3), (2|1)\}$ $\{(1|2), (2|3), (3|1)\}$
$\{(1|2), (3|2), (2|3, 1)\}$ $\{(1|2, 3), (2|3, 1), (3|1, 2)\}$.

There is no system of four relations.

Observe, for example, that the system

$\{(1|2, 3), (2|3, 1), (3|1, 2)\}$

determines the face $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

9. 15. Conséquences of theorem 9. 2.

Corollary. — *If an equality of the form* $\sum_{H \subset I} \alpha_H f(B_H) \geq 0$ *holds for every family* $\{A_i\}_{i \in I}$ *of compact subsets of* D, *there exist* N *constants* $\beta_J \geq 0$ *such that the linear form* $\sum \alpha_H x_H$ *is a linear combination with coefficients* β_J *of the linear forms* λ_J *as follows:*

$$\sum_{H \subset I} \alpha_H x_H \equiv \sum_{J \subset I} \beta_J \lambda_J.$$

In fact, the linear form $\sum \alpha_H x_H$ is positive on the cone L_n. Then it is a linear combination with positive coefficients of the linear forms which define L_n.

An equivalent statement is obtained by replacing the capacities $f(B_H)$ by the potentials $h(B_H)$ and the linear forms λ_j of x_H by the corresponding linear combinations of $h(B_H)$.

10. Inequalities concerning all operations of the algebra of sets. — The inequalities $\nabla(B_{I-H}; \{A_i\}_{i \in H}) \leqq 0$ concern only unions of the sets A_i. We have already seen that it is possible to deduce other inequalities from them which involve intersections; the following is an example:

$$f(A \cup B) + f(A \cap B) \leqq f(A) + f(B).$$

If n compact sets $\{A_i\}$ are given, we can derive from them a certain family of sets, in general distinct, by using the operations of union, intersection, and difference of sets. More precisely, we first form the N « atoms » as follows:

$$E_J = \left(\bigcap_{i \in J} A_i\right) \cap \left(\bigcap_{i \in I-J} \complement A_i\right)$$

Then we take unions F_p of any number of atoms. Thus we obtain $\mathfrak{N} = (2^N - 1) = (2^{(2n-1)} - 1)$ sets.

Let $\varphi(f\{F_p\}) \geqq 0$ be an identically true relation whatever $A_i \subset D$ may be, where $\varphi(\{Y_p\})$ designates any continuous function of the positive variables Y_p ($p = 1, 2, \ldots, \mathfrak{N}$).

We assert that this relation is a consequence of the inequalities $\nabla \leqq 0$.

This is equivalent to saying that, if we consider in the space $R^{\mathfrak{N}}$ of dimension $\mathfrak{N}$ the set $\mathcal{C}_n$ of the points of coordinates $x_p = f(F_p)$ when the system of compact sets $\{A_i\}$ varies arbitrarily, the closure $\mathcal{C}_n$ of $\mathcal{C}_n$ is identical to the cone $\mathfrak{L}_n$ defined by the relations $\nabla \leqq 0$ in which we understand now as variables not the A_i, but the E_J defined above.

More precisely, it can be shown that the interior of $\mathcal{C}_n$ is identical to the interior of $\mathfrak{L}_n$.

The fact that the F_p are not compact sets is not disturbing. In fact, each of these sets is a K_σ; it is capacitable, and we can from now on apply to these sets the inequalities $\nabla \leqq 0$.

The only difficulty arises from the fact that the E_J (which replace the A_i) are not arbitrary capacitable sets, but are mutually disjoint.

We have evidently $\mathcal{C}_n \subset \mathfrak{L}_n$. In order to show that $\mathring{\mathcal{C}}_n = \mathring{\mathfrak{L}}_n$, we show that for every point $(x_p) \in \mathring{\mathfrak{L}}_n$ there exists a family of mutually disjoint compact sets (E_J) to which is assigned in $R^{\mathcal{N}}$ a representative point identical with (x_p).

In the procedure used to compare $\mathring{C}_n$ with $\mathring{L}_n$, we used a variable base of sets A_i which were not mutually disjoint. With the notation thus used we had

$$A_i = \bigcup_{H \ni i} K_H(\lambda_H).$$

But we remark that, instead of considering the compact sets K_H and their subsets $K_H(\lambda_H)$, we could just as well have used the compact sets with two indices $K_{H,i}(\lambda_H)$ with $K_{H,i} \cap K_{H,j} = \emptyset$ for $i \neq j$, and set $A_i = \bigcup_{H \ni i} K_{H,i}(\lambda_H)$ with the condition that the capacities of $K_{H,i}(\lambda_H)$ and of $\bigcup_i K_{H,i}(\lambda_H)$ should be still equal to λ_H.

In fact, it is usually not possible to subdivide a compact set $K_H(\lambda_H)$ into n compact sets of the same capacity, but this subdivision can be approximated as we shall now show.

It is sufficient to change as follows the construction of the K_H. Instead of taking $K_H(\lambda_H)$ = a union of cubes, we shall let $K_H(\lambda_H)$ = the boundary of this union of cubes. Then assuming an $\eta \geqq 0$ given, we shall set $K_{H,i}(\lambda_H) = K\ [(1 + i\eta)\lambda_H]$, $i = 1, 2, \ldots, n$. In order to show that $\mathring{\mathcal{C}}_n = \mathring{\mathcal{L}}_n$, it is essentially this idea that could be used. We shall not show the details of the proof but give only the result.

10. 1. Theorem. — *Let $\varphi(\{x_p\})$ be a continuous function of the positive variables $(x_p\ (p = 1, 2, \ldots, \mathcal{N})$. Let us suppose that for every index p, $f(F_p)$ designates the capacity of a set F_p defined in terms of compact sets A_i $(i = 1, 2, \ldots, n)$ by a given sequence of operations $\cup$, $\cap$, «difference ».*

If the relation $\varphi(\{f(F_p)\}) \geqq 0$ is satisfied by any family (A_i), the relation $\varphi(\{x_p\}) \geqq 0$ is a consequence of the $\mathcal{N}$ relations $V \leqq 0$ in which each V is considered as a linear form of the variables x_p.

More precisely, with the notations already introduced we have: $\mathcal{C}_n \subset \mathcal{L}_n$ and $\mathring{\mathcal{C}}_n = \mathring{\mathcal{L}}_n$.

11. Possibilities of extension of the preceding theorems. — All the preceding results apply without modification to potentials and capacities relative to Greenian spaces (see Choquet and Brelot [1]). They apply equally well to plane domains which are not Greenian and, more generally, to Riemann surfaces, taking in the definition of the capacity the following precautions : If we study the capacities of the compact sets contained in a circle of diameter equal to or less than d, take for kernel Log d/r and for capacity of a compact set the supremum of the total masses of the admissible measures on this compact set.

More generally, Theorem 7. 6. relative to the successive differences of the potentials of equilibrium and its corollary relative to the successive differences of the capacities are extended without any difficulty in the proof, to every capacity associated with a theory of potential in which the two fundamental principles 6. 4 and 6. 5 are satisfied. Such potentials can be defined on a space which is not necessarily either R^n or even a group ; exemples can be constructed by replacing the domain D by any locally compact space.

Differentiability of capacity. — Let D be a Greenian domain in a Euclidean space, or more generally, let D be a Greenian space. Let K be a compact subset of D such that $f(K) \neq 0$ and let $m \in D - K$. Let ΔK be any compact subset of D contained in the sphere $B(m,\rho)$ and such that $f(\Delta K) \neq 0$.

11. 1. THEOREM. — *If $h_m(K)$ denotes the value at m of the equilibrium potential $h(K)$ of K, then*

$$\lim_{\rho \to 0} \frac{f(K \cup \Delta K) - f(K)}{f(\Delta K)} = [1 - h_m(K)]^2.$$

Proof. We shall consider the restriction of the potential $h(K \cup \Delta K) - h(K)$ on K and on ΔK. This is quasi everywhere 0 on K and quasi everywhere $[(1 - h(K)]$ on ΔK, that is, equal to $[1 - h_m(K)]$ within ε (where $\varepsilon \to 0$ with ρ). Then

$$\frac{f(K \cup \Delta K) - f(K)}{1 - h_m(K)}$$

is equivalent to the total mass of the measure on $(K \cup \Delta K)$ whose potential is 1 on ΔK and 0 quasi everywhere on K.

Now this last potential is $[h(\Delta K) - b(\Delta K, K)]$ where $b(\Delta K, K)$ equals 0 on ΔK and $h(\Delta K)$ on K. The potential $b(\Delta K, K)$ is equivalent (for $\rho \to 0$) on K to $[f(K) . b(m, K)]$, where $b(m, K)$ denotes the potential of the measure obtained by the sweeping out process (balayage) on K of the unit mass at m. The total mass of this measure on K is a function of m which is harmonic on $(D - K)$, and which is 0 on the boundary of D and 1 on the boundary of K and is thus identical with $h(K)$. Thus the total mass of the measure which generates $[h(\Delta K) - b(\Delta K, K)]$ is equivalent to $f(\Delta K)(1 - h_m(K))$; this fact proves the theorem.

11. 2. **Extension of the Green's function.** — Let P_1 and P_2 be two distinct points of D and K_1, K_2 two compact sets of positive capacity contained in $B(P_1, \rho)$ and $b(P_2, \rho)$ respectively. We shall study the behavior of

$$f(K_1) + f(K_2) - f(K_1 \cup K_2)$$

when $\rho \to 0$.

We could use the preceding result, but it is quicker to prove that this potential is the sum of two potentials U_1 and U_2 of measures μ_1 and μ_2 each defined on $(K_1 \cup K_2)$, where the restrictions of U_1 on K_1 and K_2 are respectively 0 and $h(K_1)$ and the restrictions of U_2 on K_1 and K_2 are respectively $h(K_2)$ and 0. The restriction of $h(K_1)$ on K_2 can be approximated by $G(P_1,P_2).f(K_1)$. It follows easily that the total mass of μ_1 is equivalent to $G(P_1, P_2).f(K_1).f(K_2)$; the same is true of the total mass of μ_2. Thus

$$\frac{f(K_1) + f(K_2) - f(K_1 \cup K_2)}{2f(K_1)f(K_2)} \to G(P_1, P_2) \qquad \text{when } \rho \to 0,$$

and the convergence is uniform when P_1 and P_2 belong to two disjoint compact sets.

Thus the ratio,

$$G(K_1, K_2) = \frac{f(K_1) + f(K_2) - f(K_1 \cup K_2)}{2f(K_1)f(K_2)}$$

defined on the set of pairs of compact sets of non-zero capacity is a natural extension of the Green's function. It is a positive and symmetric function of K_1 *and* K_2 *and can be extended by continuity to the set of pairs of points of* D, *and is there identical with* $G(P_1, P_2)$.

CHAPTER III

ALTERNATING AND MONOTONE FUNCTIONS. CAPACITIES.

12. Successive différences of a function. — Let $\mathcal{E}$ be a commutative semi-group [10] and $\mathcal{F}$ a commutative group. The operation in $\mathcal{E}$ will be denoted by $\top$ and in $\mathcal{F}$ by $+$. Let $y = \varphi(x)$ be a mapping from $\mathcal{E}$ into $\mathcal{F}$.

The successive differences of $\varphi(x)$ with respect to the parameters $a_1, a_2, \ldots,$ are defined as follows.

$$\nabla_1(x; a_1)_\varphi = \varphi(x) - \varphi(x \top a_1), \quad \text{and in general,}$$
$$\nabla_{n+1}(x; a_1, \ldots, a_n, a_{n+1})_\varphi = \nabla_n(x; a_1, \ldots, a_n)_\varphi - \nabla_n(x \top a_{n+1}; a_1, \ldots, a_n)_\varphi$$

In the above definition the element x and the elements a_i are, of course, assumed to be elements of $\mathcal{E}$.

As in the special case treated in the preceding chapter, the following properties of the function ∇_n can be verified immediately.

12. 1. $\nabla_n(x; a_1, \ldots, a_n)$ is a symmetric function of the a_i; it is therefore possible to write this function in the concise form $\nabla_n(x; \{a_i\}_{i \in I})$, or, if I is a given fixed set, $\nabla(x; \{a_i\})$.

12. 2. $\nabla(x; \{a_i\}) = \nabla(x; \{a'_i\})$ whenever $x \top a_i = x \top a'_i$ for each i; moreover, $\nabla(x; \{a_i\}) = 0$ if, for at least one i_0, we have $(x \top a_{i_0}) = x$; (this case occurs when $\mathcal{E}$ contains a zero element and when a_{i_0} is this zero element).

12. 3. If $\mathcal{E}$ contains a zero element 0, we have

$$\nabla_n(x; \{a_i\}) = -\nabla_{n+1}(0; \{a_i, x\}) + \nabla_n(0; \{a_i\}).$$

(10) This means that a mapping of the form $c = a \top b$ is defined from $\mathcal{E}^2$ into $\mathcal{E}$, with the operation $\top$ assumed commutative and associative.

12. 4. $\nabla_n(x;\, a_1, \ldots, a_{n-1}, a_n \top \alpha_n) - \nabla_n(x;\, a_1, \ldots, a_n)$
$$= \nabla_n(x \top a_n;\, a_1, \ldots, a_{n-1}, \alpha_n).$$

13. **Alternating functions.** — We now make the additional assumption that $\mathcal{E}$ and $\mathcal{F}$ possess an ordering compatible with their algebraic structure, and that $\mathcal{E}$ contains a zero element. The relations defining the two orderings are denoted by $\prec$ and $\leqq$ respectively.

13. 1. DEFINITION. — *A mapping φ from $\mathcal{E}$ into $\mathcal{F}$ will be called* alternating of order n, *where n is an integer $\geqq 1$, if $\nabla_p(x;\, \{a_i\}) \leqq 0$ for each $p \leqq n$ and for every finite family $\{a_i\}$ which is positive, (that is, for which $0 \prec a_i$ for each i).*

The mapping φ will be called alternating of order ∞, *if it is alternating of order n for each $n \geqq 1$.*

It is an immediate consequence that if $\mathcal{E}$ is an idempotent semi-group ($a \top a = a$ for every a), then φ is alternating of order n if and only if $\nabla_n(x;\, \{a_i\}) \leqq 0$ for every positive family $\{a_i\}$. In fact, $\nabla_n(x;\, a_1, \ldots, a_n) = \nabla_{n-1}(x;\, a_1, \ldots, a_{n-1})$ whenever $a_n = a_{n-1}$, since the equation $[(x \top a_n) \top a_{n-1} = (x \top a_n)]$ implies the equation $[\nabla_{n-1}(x \top a_n;\, a_1, \ldots, a_{n-1}) = 0]$.

13. 2. **Immediate properties.** — If φ is alternating of order n, then every function $\nabla_p(x;\, \{a_i\})$ (where $p < n$) is alternating of order $(n - p)$.

When $\mathcal{E}$ is such that $a \prec b$ implies $b = a \top c$ where $c \succ 0$, every $\nabla_p (p < n)$ is an increasing function of x, and every ∇_p $(p \leqq n)$ is a decreasing function of each a_i. Finally, ∇_p is an increasing function of p in the sense that

$$\nabla(x;\, \{a_i\}_{i \in J}) \leqq \nabla(x;\, \{a_i\}_{i \in I})$$

for $J \subset I$ and $\bar{\bar{I}} \leqq n$.

The verification of these properties is analogous to that of the same properties in the case of the Greenian capacities.

13. 3. **Examples of alternating functions.**

(i) If $\mathcal{E}$ is the positive half of the real axis (*i. e.* all points $x > 0$) and $\mathcal{F}$ is the real axis, then the statement that the function $y = \varphi(x)$ is alternating of order ∞ is equivalent to the statement that $\varphi(x)$ possesses derivatives of all orders and that $(-1)^n \varphi^{(n)} \leqq 0$ for each $n \geqq 1$.

(ii) If $\mathscr{E}$ is the class of all compact subsets of a Greenian domain, and if the operation $\top$ is the union, then the capacity $f(x)$ of the element x is alternating of order ∞ ; the same is true for the equilibrium potential $h(x)$. (In the latter case, $\mathscr{F}$ is the vector space of all real-valued functions defined on D, with the classical order structure.)

14. **Set functions.** — We shall not continue here the general study of alternating functions, but shall restrict our remarks to the case where $\mathscr{E}$ is a class of subsets of a set E, where the operation $\top$ is either union or intersection, and where $\mathscr{F}$ is the real axis. It should be remarked, however, that some of the definitions and theorems could be easily extended to the case where $\mathscr{F}$ is an ordered, commutative, topological group.

14. 1. We shall continue to use the term « alternating » for mappings φ when $\top$ is union ($\cup$); but when $\top$ is intersection ($\cap$), we shall use the term « monotone » for the function $(-\varphi)$.

More precisely, let $\mathscr{E}$ be a class of subsets of a set E and $\varphi(X)$ a mapping from $\mathscr{E}$ into the extended set of real numbers (containing $+\infty$ and $-\infty$)(11). $\mathscr{E}$ will then be called *additive* (*multiplicative*), if from $A_1 \in \mathscr{E}$ and $A_2 \in \mathscr{E}$ it follows that $(A_1 \cup A_2) \in \mathscr{E}$ $((A_1 \cap A_2) \in \mathscr{E})$. For additive $\mathscr{E}$, the differences ∇ with respect to φ will be denoted by $\vee$; for multiplicative $\mathscr{E}$ by $\wedge$ (these symbols are designed to recall the symbols $\cup$ and $\cap$).

A function φ defined on an additive class $\mathscr{E}$ is called *alternating of order n* if its differences $\vee$ of orders $p \leqq n$ are nonpositive ($\leqq 0$).

A function φ defined on a multiplicative class $\mathscr{E}$ is called *monotone of order n* if its differences $\wedge$ of orders $p \leqq n$ are nonnegative ($\geqq 0$).

If we call a function φ defined on $\mathscr{E}$ *increasing* whenever $(A_1 \subset A_2) \rightarrow \varphi(A_1) \leqq \varphi(A_2)$, it follows immediately from the definition that every increasing function which is defined on additive $\mathscr{E}$ is alternating of order 1, and conversely. Analogously,

(11) With the understanding that the expressions $[(+\infty) - (+\infty)]$ and $[(-\infty) - (-\infty)]$ may take arbitrary values.

every increasing function which is defined on multiplicative $\mathcal{E}$ is monotone of order 1 and conversely.

14. 2. **Conjugate functions.** — If φ is a function defined on a class $\mathcal{E}$ of subsets of E, we shall denote by φ' the function which is defined on the class $\mathcal{E}'$ of the complements $X' = (E - X)$ of all elements X of $\mathcal{E}$ by the relation,

$$\varphi'(X') + \varphi(X) = 0.$$

We have, obviously, $(\varphi')' = \varphi$, and $(\mathcal{E}')' = \mathcal{E}$. The two functions φ and φ' are called *conjugate functions.*

It follows immediately that if φ is alternating of order n on additive $\mathcal{E}$, then $\mathcal{E}'$ is multiplicative and φ' is monotone of order n on $\mathcal{E}'$.

14. 3. **Alternating functions of order 2.** — If φ is alternating of order 2 on additive $\mathcal{E}$, then φ is also increasing and we have,

$$\varphi(A \cup k) - \varphi(a \cup k) \leqq \varphi(A) - \varphi(a)$$

whenever $a \subset A$ and a, A, k, $\in \mathcal{E}$. From this inequality it follows that.

$$\varphi\left(\bigcup A_i\right) - \varphi\left(\bigcup a_i\right) \leqq \sum(\varphi(A_i) - \varphi(a_i))$$

whenever $a_i \subset A_i$ for every i.

If $\mathcal{E}$ is additive and multiplicative, the two statements below are equivalent.

(i) φ is alternating of order 2.

(ii) φ is increasing and satisties.

$$\varphi(A_1 \cup A_2) + \varphi(A_1 \cap A_2) \leqq \varphi(A_1) + \varphi(A_2).$$

If φ is alternating of order 2 on $\mathcal{E}$, and if $\varphi \geqq 0$, then φ is also sub-additive, that is $\varphi(A_1 \cup A_2) \leqq \varphi(A_1) + \varphi(A_2)$. We shall not prove these elementary properties which have in large measure been proved in the preceding chapter.

14. 4. **Monotone functions of order 2.** — If φ is monotone of order 2 on multiplicative $\mathcal{E}$, then from the properties of its conjugate, φ', the corresponding properties for φ can be deduced. We find that φ is increasing, and

$$\varphi(A \cap k) - \varphi(a \cap k) \leqq \varphi(A) - \varphi(a).$$

whenever $a \subset A$, and

$$\varphi\left(\bigcap A_i\right) - \varphi\left(\bigcap a_i\right) \leq \sum (\varphi(A_i) - \varphi(a_i)),$$

whenever $a_i \subset A_i$ for all i.

When $\mathcal{E}$ is both additive and multiplicative, the following two statements are equivalent:

(i) φ is monotone of order 2.

(ii) φ is increasing and satisfies

$$\varphi(A_1 \cup A_2) + \varphi(A_1 \cap A_2 \geq \varphi(A_1) + \varphi(A_2).$$

If φ is monotone of order 2, and if $\varphi(\emptyset) = 0$, then φ is supra-additive in the sense that $\varphi(A_1 \cup A_2) \geq \varphi(A_1) + \varphi(A_2)$ whenever $(A_1 \cap A_2) = \emptyset$.

14. 5. Alternating and monotone functions of order 2.

THEOREM. — *If $\mathcal{E}$ is both additive and multiplicative, then every function $\varphi(X)$ defined on $\mathcal{E}$, which is both alternating and monotone of order 2, is increasing and satisfies*

$$\varphi(A_1 \cup A_2) + \varphi(A_1 \cap A_2) = \varphi(A_1) + \varphi(A_2).$$

Conversely, if a function φ defined on $\mathcal{E}$ is increasing and satisfies the above relation, then, for every $n \geq 1$,

$$\nabla_n(X; \{A_i\}) = \varphi(X \cap a) - \varphi(a) \leq 0, \quad \text{where} \quad a = \bigcap A_i,$$

$$\Lambda_n(X; \{A_i\}) = \varphi(X \cup A) - \varphi(A) \geq 0, \quad \text{where} \quad A = \bigcup A_i.$$

The function φ, which is thus seen to be alternating and monotone of all orders, is called additive.

Proof. If φ is both alternating and monotone of order 2, then we obtain simultaneously,

$$\varphi(A_1 \cup A_2) + \varphi(A_1 \cap A_2) \leq \quad \text{and} \quad \geq \varphi(A_1) + \varphi(A_2),$$

whence the equality of the two members. Let us assume now that φ is increasing and that the above mentioned equality holds. Clearly, this equality implies

$$\varphi(X) - \varphi(X \cup A_1) = \varphi(X \cap A_1) - \varphi(A_1).$$

and hence $\nabla_1(X;\ A_1) = \varphi(X \cap a) - \varphi(a)$, where $a = A_1$.

We now assume that the relation $\bigvee_p = \varphi(X \cap a) - \varphi(a)$ holds for all orders $p \leq n$ and we prove it for $p = n+1$.

If $a = \bigcap_{i \leq n} A_i$, and $a' = \bigcap_{i \leq n+1} A_i$, then

$$\begin{aligned} \bigvee_{n+1}(X;\, A_1, \ldots, A_{n+1}) \\ &= [\varphi(X \cap a) - \varphi(a)] - [\varphi((X \cup A_{n+1}) \cap a) - \varphi(a)] \\ &= \varphi(X \cap a) - \varphi((X \cap a) \cup a'). \end{aligned}$$

From the fundamental equality, the last expression is equal to $\varphi(X \cap a') - \varphi(a')$, which is indeed the desired quantity; it is obviously non-positive.

The second relation for the $\bigwedge_n$ is deduced by duality from that for the $\bigvee_n$.

15. Capacities. — Let E be a topological space, $\mathcal{E}$ a class of subsets of E, and φ a mapping from $\mathcal{E}$ into the extended real line $[-\infty, +\infty]$.

15. 1. Continuity on the right. — *We shall say that* φ *is continuous on the right at* A ($A \in \mathcal{E}$), *if for every neighborhood* W *of* $\varphi(A)$ *there exists a neighborhood* V *of* A *in* E, *such that*

$$\varphi(X) \in W \quad \text{whenever} \quad X \in \mathcal{E} \quad \text{and} \quad A \subset X \subset V$$

Obviously this definition may be applied also to the case where $\varphi(A) = +\infty$ or $\varphi(A) = -\infty$

If φ is continuous on the right at every $A \in \mathcal{E}$, we shall say that φ is continuous on the right on $\mathcal{E}$.

15. 2. Capacity on a class $\mathcal{E}$ of sets. — *A function* φ *defined on* $\mathcal{E}$ *is called a capacity on* $\mathcal{E}$ *if* φ *is increasing and continuous on the right on* $\mathcal{E}$.

We shall now define the following functions of subsets A of E.

Interior capacity of $A = \varphi_*(A) = \sup \varphi(X)$ (for $X \in \mathcal{E}$ and $X \subset A$). When there exists no element of $\mathcal{E}$ contained in A, we set $\varphi_*(A) = \inf \varphi(X)$ (for all $X \in \mathcal{E}$).

In particular, $\varphi_*(\omega)$ is thereby defined for every open set ω, and we can now define for any A :

Exterior capacity of $A = \varphi^*(A) = \inf \varphi_*(\omega)$ (ω open and $A \subset \omega$).

We have always, $\varphi_* \leq \varphi^*$ and φ_*, φ^* are increasing functions.

A set A is called *capacitable* if $\varphi_*(A) = \varphi^*(A)$. It is a trivial conclusion that every open set is capacitable. We shall consider only capacities for which every element A of $\mathcal{E}$ is capacitable. This will occur in particular when $\mathcal{E}$ is *absorbing*: A class $\mathcal{E}$ of subsets of E is called absorbing if for every open subset ω of E and for every pair (A_1, A_2) of elements of $\mathcal{E}$ such that $A_i \subset \omega$ $(i = 1, 2)$, there exists an element A_3 of $\mathcal{E}$, satisfying $(A_1 \cup A_2) \subset A_3 \subset \omega$. For instance, $\mathcal{E}$ is absorbing when it is additive.

For simplicity, let us assume that $\varphi(A)$ is finite. For every $\varepsilon > 0$ there exists, by virtue of the continuity on the right of φ, an open set ω such that $A \subset \omega$, and $0 \leqq \varphi(A') - \varphi(A) \leqq \varepsilon$ for every A' satisfying $A \subset A' \subset \omega$.

Moreover, since $\mathcal{E}$ is absorbing, to every $B \in \mathcal{E}$ and contained in ω, there corresponds a $C \in \mathcal{E}$ such that $(A \cup B) \subset C \subset \omega$. Hence $\varphi(B) \leqq \varphi(C) \leqq \varphi(A) + \varepsilon$, from which we deduce $\varphi_*(\omega) \leqq \varphi(A) + \varepsilon$, and therefore, $\varphi^*(A) \leqq \varphi(A)$. Clearly, since moreover $\varphi^*(A) = \varphi(A)$, the element A of $\mathcal{E}$ is capacitable. There is therefore no contradiction, when for arbitrary capacitable sets A we define

$$\varphi(A) = \varphi_*(A) = \varphi^*(A).$$

15. 3. **Alternating capacities.** — We shall introduce a scale of classes of capacities.

A capacity φ on $\mathcal{E}$ is called alternating *of order $\mathcal{A}_\alpha$ if $\mathcal{E}$ is additive, (a restriction which is not essential for $\mathcal{A}_{1,a}$) and if φ satisfies one of the following conditions $\mathcal{A}_\alpha$:*

$\mathcal{A}_{1,a}$: *If $\{A_n\}$ is any* increasing *sequence of subsets of* E, *then $\varphi^*(A_n) \to \varphi^*(A)$, where $A = \bigcup A_n$.*

$\mathcal{A}_{1,b}$: *Given $\varepsilon > 0$, there exists an $\eta > 0$ such that the inequality*

$\varphi(A_i) - \varphi(a_i) < \eta$ $(a_i \subset A_i$, a_i and $A_i \in \mathcal{E}$ with $i = 1, 2)$ implies the inequality

$$\varphi(A_1 \cup A_2) - \varphi(a_1 \cup a_2) < \varepsilon.$$

$\mathcal{A}_n$: *The function φ is alternating of order n $(n = 2, 3, \ldots,)$.*
$\mathcal{A}_\infty$: *The function is alternating of order ∞.*

15. 4. **Monotone capacities.** — *A capacity φ defined on $\mathcal{E}$ is called* monotone *of order $\mathcal{M}_\alpha$ if $\mathcal{E}$ is multiplicative, (a restriction which is not essential for $\mathcal{M}_{1,a}$), and if φ satisfies one of the following conditions $\mathcal{M}_\alpha$:*

$\mathcal{M}_{1,a}$: *If* $\{A_n\}$ *is any* decreasing *sequence of subsets of* E, *then* $\varphi^*(a_n) \to \varphi_*(a)$, *where* $a = \bigcap A_n$.

$\mathcal{M}_{1,b}$: *Given* $\varepsilon > 0$, *there exists an* $\eta > 0$ *such that the inequality*

$$\varphi(A_i) - \varphi(a_i) < \eta \ (a_i \subset A_i,\ a_i \textit{ and } A_i \in \mathcal{E},\ \textit{with } i = 1, 2)$$

implies

$$\varphi(A_1 \cap A_2) - \varphi(a_1 \cap a_2) < \varepsilon.$$

$\mathcal{M}_n$: *The function* φ *is monotone of order* n $(n = 2, 3\ldots,)$.

$\mathcal{M}_\infty$: *The function* φ *is monotone of order* ∞.

15. 5. Immediate consequences of the definitions.

$$\mathcal{A}_{n+1} \Rightarrow \mathcal{A}_n \quad \text{and} \quad \mathcal{M}_{n+1} \Rightarrow \mathcal{M}_n \quad \text{for} \quad n \geq 2$$

$$\mathcal{A}_2 \Rightarrow \mathcal{A}_{1,b} \quad \text{and} \quad \mathcal{M}_2 \Rightarrow \mathcal{M}_{1,b}.$$

The above relations are an immediate consequence of the properties of functions which are alternating or monotone of order $n \geq 2$ (studied at the beginning of this chapter). For example, the relation $\mathcal{M}_2 \Rightarrow \mathcal{M}_{1,b}$ derives from the inequality

$$\varphi\left(\bigcap A_i\right) - \varphi\left(\bigcap a_i\right) \leq \sum \varphi(A_i) - \varphi(a_i).$$

An important theorem which will be proved in the sequel, states that in very general cases the following relations hold :

$$\mathcal{A}_{1,b} \Rightarrow \mathcal{A}_{1,a} \quad \text{and} \quad \mathcal{M}_{1,b} \Rightarrow \mathcal{M}_{1,a}.$$

15. 6. Conjugate capacity of a capacity. — If φ is a capacity defined on a class $\mathcal{E}$, which is assumed to be absorbing, then the conjugate function φ' corresponding to φ is not in general a capacity because, firstly, $\mathcal{E}'$ is not in general absorbing, and secondly, φ' is not in general continuous on the right.

However, if $\mathcal{E}$ is an *absorbing class of closed subsets of* E then for every capacity φ defined on $\mathcal{E}$ another capacity $\bar{\varphi}$ may be associated with it which is also defined on $\mathcal{E}$. This is done by setting $\bar{\varphi}(X) = -\varphi(\complement X)$ for every $X \in \mathcal{E}$. The definition is meaningful since $\complement X$ is an open set and hence a set for which φ is defined.

The function $\bar{\varphi}$ is obvious increasing. It is also continuous on the right. This is due to the fact that by the definition of φ_*, every open set $\complement X$ contains closed sets belonging to $\mathcal{E}$, such that their capacity differs from that of $\complement X$ by an arbitrarily small value. Hence $\overline{\varphi}(X)$ is a capacity.

Let us further assume now $\mathcal{E}$ is the class of *all* closed subsets of E (E is additive, and therefore absorbing). For every open subset ω of E we have,

$$\overline{\varphi}(\omega) = \sup_{\substack{X\in\mathcal{E}\\ X\subset\omega}} \varphi(X) = -\inf_{\substack{G\supset\complement\omega\\ G \text{ open}}} \varphi(G).$$

Clearly, $\inf \varphi(G) = \varphi\left(\complement\omega\right) = -\varphi'(\omega)$, where φ' is the conjugate function corresponding to φ, φ' being defined on the class $\mathcal{E}'$ of all open subsets of E. It follows that $\overline{\varphi}(\omega) = \varphi'(\omega)$.

As a definition, the function $\overline{\varphi}$ will therefore be called the *conjugate capacity of* φ.

Moreover, for every $X \in \mathcal{E}$ we have

$$\overline{\overline{\varphi}}(X) = -\overline{\varphi}\left(\complement X\right) = -\varphi'\left(\complement X\right) = \varphi(X).$$

It can also be immediately verified that for every $A\subset E$,

$$\varphi_*(A) + \overline{\varphi}^*\left(\complement A\right) = 0 \qquad \text{and} \qquad \varphi^*(A) + \overline{\varphi}\left(\complement A\right) = 0.$$

Thus the operation $\complement$ (complementation) establishes a canonical correspondence between the φ-capacitable and the $\overline{\varphi}$-capacitable sets.

15. 7. If φ is of order $\mathfrak{M}_{1,a}$ ($\mathfrak{A}_{1,a}$), then $\overline{\varphi}$ is of order $\mathfrak{A}_{1,a}$ ($\mathfrak{M}_{1,a}$). This is an immediate consequence of the last two equalities.

If φ is of order $\mathfrak{M}_\alpha$ (for $\alpha = (1, b)$ or $\alpha = n \geqq 2$), then $\overline{\varphi}$ is of order $\mathfrak{A}_\alpha$. For the proof of this correspondence it is sufficient to show that the fundamental inequalities which define a class $\mathfrak{M}_\alpha$ still hold when the closed sets are replaced by open sets, a result obtained without difficulty from the following lemma.

15. 8. Lemma. — *Let* $\{\omega_i\}_{i\in I}$ *be a finite family of open subsets of* E *such that* $\varphi(\omega_i)$ *is finite for each* i. *To each* $\varepsilon > 0$ *there corresponds a family* $\{X_i\}_{i\in I}$ *of closed sets, with* $X_i \subset \omega_i$ *for every* i, *and such that* $\varphi\left(\bigcap_{i\in J}\omega_i\right) - \varphi\left(\bigcap_{i\in J} X^i\right) < \varepsilon$, *for every* $J\subset I$.

In fact, for arbitrary $J \subset I$, consider a closed set

$$X_J \subset \omega_J = \bigcap_{i \in J} \omega_i$$

such that $\varphi(\omega_J) - \varphi(X_J) < \varepsilon$.

If we set $X_i = \bigcup_{i \in J} X_J$, we obtain $X_i \subset \omega_i$. On the other hand, $\bigcap_{i \in J} X_i \supset X_J$ and hence the sets X_i satisfy the desired relation.

15. 9. If φ is of order $\mathcal{A}_\alpha$ ($\alpha = 1, b$ or $\alpha = n \geqq 2$), then $\bar{\varphi}$ is not necessarily of order $\mathcal{M}_\alpha$, except in the case when E is a *normal space*.

In this case it can be shown (see next Chapter 17. 9. and 17. 10.) that the inequalities defining a class $\mathcal{A}_\alpha$ are still valid if the closed sets are replaced by open sets. The inequalities which define the class $\mathcal{M}_\alpha$ are then obtained by complementation. Thus we see that a perfect duality does not exist between the alternating and monotone capacities. This is due to the fact that the definitions of φ_* and φ^* are not parallel; φ^* can be defined only after φ_* has been defined.

CHAPTER IV

EXTENSION AND RESTRICTION OF A CAPACITY

16. **Extension of a capacity.** — Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two classes of subsets of a topological space E such that $\mathcal{E}_1 \subset \mathcal{E}_2$, and let f_1 be a capacity on $\mathcal{E}_1$. We shall always suppose $\mathcal{E}_1$ to be such that each element of $\mathcal{E}_1$ is f_1-capacitable, which is the case, as we have seen, when $\mathcal{E}_1$ is absorbing (for example, additive).

16. 1. Definition. — *The function* f_2 on $\mathcal{E}_2$ *defined by* $f_2(X) = f_1^*(X)$ *for each* $X \in \mathcal{E}_2$ *is called* the extension of f_1 to $\mathcal{E}_2$. *It is indeed an extension in the ordinary sense for if* $X \in \mathcal{E}_1$, *we have* $f_2(X) = f_1^*(X) = f_1(X)$.

This function f_2 is a *capacity*. First, it is obviously increasing. On the other hand, for each $A \subset E$ such that, for example, f_1 is finite, and for each $\varepsilon > 0$, there exists an open set ω containing A and such that $f_1(\omega) - f_1^*(A) < \varepsilon$; hence, if $A \in \mathcal{E}_2$, we have the inequality $f_2(B) - f_2(A) < \varepsilon$ for each $B \in \mathcal{E}_2$ such that $A \subset B \subset \omega$. This fact shows that f_2 is continuous on the right.

Since $\mathcal{E}_1 \subset \mathcal{E}_2$, we have $f_{1_*}(X) \leqq f_{2_*}(X)$ for each set X. But this inequality obviously becomes an equality for open sets. It follows that $f_1^*(X) = f_2^*(X)$ for each X. In particular we have, for $A \in \mathcal{E}_2$, $f_1^*(A) = f_2(A) = f_2^*(A)$, and it follows that A is f_2-capacitable, although we have made on $\mathcal{E}_2$ no restrictive hypothesis such as « $\mathcal{E}_2$ is absorbing ».

It also follows from these relations that if an $X \subset E$ is f_1-capacitable, it is also f_2-capacitable and we have $f_1(X) = f_2(X)$. In short:

16. 2. Theorem. — *The extension* f_2 *of a capacity* f_1 *is a capacity and*

$$f_{1*} \leqq f_{2*}, \qquad f_1^* = f_2^*.$$

There are more f_2*-capacitable sets than* f_1*-capacitable sets.*

An example of extension. — If we take for $\mathcal{E}_2$ the class of all the subsets of E, each $X \subset E$ becomes f_2-capacitable. This example shows clearly that it is not of interest to make extensions to classes which are too large. Of course, extensions enrich the class of capacitable sets, but we lose preciseness in the process since now $f_{2*} \equiv f_2^*$.

16. 3. THEOREM. — *If* $X \subset E$ *is such that each element of* $\mathcal{E}_2$ *contained in* X *is* f_1*-capacitable, or is contained in an* f_1*-capacitable subset of* X, *we have* $f_{1*}(X) = f_{2*}(X)$; *thus, if this* X *is* f_2*-capacitable, it is also* f_1*-capacitable.*

In fact, if $A \subset B \subset X$ with $A \in \mathcal{E}_2$ and $f_{1*}(B) = f_1^*(B)$, we have $f_2(A) = f_1^*(A) \leqq f_1^*(B) = f_{1*}(B) \leqq f_{1*}(X)$. By comparing the extremes it follows that $f_{2*}(X) \leqq f_{1*}(X)$. Since we have already the inequality $f_{1*} \leqq f_{2*}$, we have, indeed, the equality $f_{1*}(X) = f_{2*}(X)$.

16. 4. Applications of theorem 16. 3.

16. 5. **First application.** — Suppose that each element of $\mathcal{E}_2$ is f_1-capacitable. The preceding theorem is applicable then to each $X \subset E$. Therefore we have the identities $f_{1*} \equiv f_{2*}$ and $f_1^* \equiv f_2^*$. In particular, the f_1-capacitability is identical to the f_2-capacitability.

EXAMPLE 1. — If $\mathcal{E}_2$ is the class of f_1-capacitable subsets of E, it is the largest extension of f_1 which does not change the capacitability. We shall say that it is the *canonical extension* of f_1.

EXAMPLE 2. — Suppose that there exists a closed set $N \subset E$ which contains each element of $\mathcal{E}_1$ and that, for each element $A \in \mathcal{E}_2$, the set $A \cap N$ is f_1-capacitable.

Then each element $A \in \mathcal{E}_2$ is f_1-capacitable.

In fact, we have $f_{1*}(X) = f_{1*}(X \cap N)$ for each $X \subset E$. Furthermore, for each open set ω, we have $f_1(\omega) = f_1\left(\omega \cup \complement N\right)$; this shows that $f_1^*(X) = f_1^*(X \cap N)$ for each $X \subset E$.

Thus each X such that $X \cap N$ is f_1-capacitable is also f_1-capacitable.

We have, for example, one such circumstance in taking for $\mathcal{E}_1$ the set of closed subsets of a closed N of E and for $\mathcal{E}_2$ the set of all closed subsets of E.

16. 6. **Second application.** — Suppose that each element of $\mathcal{E}_2$ contained in an element of $\mathcal{E}_1$ is an element of $\mathcal{E}_1$.

EXAMPLE. — Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be hereditary classes of closed subsets of E. Then theorem 16. 3. is applicable to each X which is a subset of an element of $\mathcal{E}_1$, that is, $f_{1*} = f_{2*}$ for this X.

Special case. — If there exists a subset $N \subset E$ which contains each element of $\mathcal{E}_1$, and if each element of $\mathcal{E}_2$ contained in N is an element of $\mathcal{E}_1$, the theorem is applicable to each $X \subset N$. We obtain an example of this situation by taking $\mathcal{E}_1$ as the set of all compacts contained in a set N in a Hausdorff space E and $\mathcal{E}_2$ as the set of all compacts in the same space E.

16. 7. **Transitivity of the extensions.** — If $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3$, if f_1 is a capacity on $\mathcal{E}_1$ and f_2, f_3 its extensions to $\mathcal{E}_2$ and $\mathcal{E}_3$ respectively, it is obvious that f_3 is identical to the extension of f_2 to $\mathcal{E}_3$. Indeed, the exterior capacity of a set remains invariant in each extension. This amounts to saying that the extension is a transitive operation.

17. Invariance of the classes $\mathfrak{A}_\alpha$ by extension.

17. 1. **Classes $\mathfrak{A}_{1,a}$.** — From the identity $f_1^* \equiv f_2^*$ it follows immediately that, if f_1 is in the class $\mathfrak{A}_{1,a}$, then each extension f_2 of f_1 is in the same class.

17. 2. **Classes $\mathfrak{A}_\alpha$ of order α greater than $(1, a)$.** — In order to study these classes, we will need a new definition.

17. 3. DEFINITION. — *A class $\mathcal{E}$ of subsets of* E *is* rich *if, for each couple of open sets ω_1, ω_2 of* E *and each element* A *of $\mathcal{E}$ such that $A \subset (\omega_1 \cup \omega_2)$, there exist two elements A_1 and A_2 of $\mathcal{E}$ such that $A_1 \subset \omega_1$, $A_2 \subset \omega_2$, $A \subset A_1 \cup A_2$.*

17. 4. EXAMPLE. — If E is a normal space, each hereditary set of closed subsets of E is rich. In fact, by duality, it is sufficient to prove the following: if F_1 and F_2 are two closed sets of E, and if G is an open set such that $G \supset F_1 \cap F_2$,

then there exist two open sets G_1 and G_2 which contain G and such that

$$G_1 \supset F_1, \qquad G_2 \supset F_2, \qquad \text{and} \qquad G = G_1 \cap G_2.$$

Now the sets $(F_1 - G)$ and $(F_2 - G)$ are closed and disjoint; the normality of E implies that they are respectively contained in two disjoint open sets g_1 and g_2. It is then sufficient to take $G_1 = G \cup g_1$ and $G_2 = G \cup g_2$; these open sets have the desired properties.

17. 5. Example. — More generally, if $\mathcal{E}$ is a hereditary class of closed sets of E such that each element of $\mathcal{E}$ is normal, then $\mathcal{E}$ is rich. This example is a generalization of the preceding one. The preceding proof is applicable provided that the passage to the complements is made with respect to the closed set A of $\mathcal{E}$ which is to be covered by A_1 and A_2.

17. 6. Example. — If $\mathcal{E}$ is a class of compacts of E such that, for each $K \in \mathcal{E}$, each compact $k \subset K$ and each neighborhood V of k, there exists an element X of $\mathcal{E}$ such that $k \subset X \subset V$, then $\mathcal{E}$ is rich.

In fact, when the open sets ω_1, ω_2 are given as well as the compact $K \in \mathcal{E}$ such that $K \subset (\omega_1 \cup \omega_2)$, we find immediately, by using the information in Example 17. 5., two subcompacts k_1 and k_2 of K such that $k_1 \subset \omega_1$, $k_2 \subset \omega_2$, and $K = k_1 \cup k_2$. By virtue of the hypothesis, there exist $K_1 \in \mathcal{E}$ and $K_2 \in \mathcal{E}$ such that $k_1 \subset K_1 \subset \omega_1$ and $k_2 \subset K_2 \subset \omega_2$. These compacts form the desired covering.

17. 7. Lemma. — *Let f be a capacity on a class $\mathcal{E}$ of subsets of* E, *and let $\{X_i\}_{i \in I}$ be a finite family of arbitrary subsets of* E *with $f^*\left(\bigcup_{i \in J} X_i\right)$ finite for each* $J \subset I$.

For each $\varepsilon > 0$, there exists a family $\{\omega_i\}$ of open sets of E *such that $X_i \overline{\subset} \omega_i$ for each i and*

$$f\left(\bigcup_{i \in J} \omega_i\right) - f^*\left(\bigcup_{i \in J} A_i\right) \leq \varepsilon \qquad \text{for each} \qquad J \subset I.$$

In fact, for each $J \subset I$, there is an open ω_J such that

$$\left(\bigcup_{i \in J} A_i\right) \subset \omega_J \qquad \text{and} \qquad f(\omega_J) - f^*\left(\bigcup_{i \in J} A_i\right) < \varepsilon.$$

If we set $\omega_i = \bigcap_{J \ni i} \omega_J$, the family $\{\omega_i\}_{i \in I}$ obviously has the desired properties.

17. 8. Lemma. — *Let f be a capacity on an additive and rich class $\mathcal{E}$ of subsets of a space* E, *and let $\{\omega_i\}_{i \in I}$ be a finite family of open sets of* E, *with $f\left(\bigcup_{i \in J} \omega_i\right)$ finite for each* $J \subset I$. *For each $\varepsilon > 0$ there exists a family $\{A_i\}_{i \in I}$ of elements of $\mathcal{E}$ such that $A_i \subset \omega_i$ for each $i \in I$, and such that for each $J \subset I$, we have*

$$f\left(\bigcup_{i \in J} \omega_i\right) - f\left(\bigcup_{i \in J} A_i\right) < \varepsilon.$$

(Note that the restriction that the $f(\omega_i)$ are finite is not essential; we would have an analogous statement if some of the $f(\omega_i)$ were $-\infty$ or $+\infty$).

For each $J \subset I$, let A_J be an element of $\mathcal{E}$ such that

$$A_J \subset \bigcup_{i \in J} \omega_i \qquad \text{and such that} \qquad f\left(\bigcup_{i \in J} \omega_i\right) - f(A_J) \leq \varepsilon.$$

By using the fact that $\mathcal{E}$ is rich we can, for each J, cover A_J by a family of elements $\{A_{i,J}\}_{i \in J}$ of $\mathcal{E}$ such that $A_{i,J} \subset \omega_i$ for all $i \in J$. The proof follows immediately if J contains only two indices; in the general case we apply the same process repeatedly (exactly $(\overline{\overline{J}} - 1)$ times). Then for each $i \in I$ let

$$A_i = \bigcup_{J \ni i} A_{i,J}.$$

It follows immediately that the family $\{A_i\}$ has the desired properties.

17. 9. Lemma. — *Let f be a capacity on an* additive and rich *class $\mathcal{E}$ of subsets of a space* E. *Let* I *be a finite set of indices and $\Phi(\{x_J\})$ a continuous real function of real variables $x_J (J \subset I)$. If for each family $\{A_i\}_{i \in I}$ of elements of $\mathcal{E}$ we have $\Phi(\{x_J\}) \geq 0$ where $x_J = f\left(\bigcup_{i \in J} A_i\right)$, we have the same inequality when we replace the sets A_i by arbitrary subsets of* E *and each x_J* by $f^*\left(\bigcup_{i \in J} A_i\right)$.

In order to simplify the proof we shall assume again that the capacities which occur are all finite. In order to include the case where they are not, it would be necessary to give a precise definition of the continuity of Φ at infinity. This definition is easy to formulate in the particular case (case of Φ linear) where we shall have to use it.

The inequality $\Phi(\{x_J\}) \geqq 0$ is satisfied when we take elements of $\mathcal{E}$ for the A_i, therefore also, by virtue of lemma 17. 7. and the continuity of Φ, when the sets A_i are open. Lemma 17. 9. then follows because of the continuity of Φ, from Lemma 17. 7. which asserts the possibility of approximating in a suitable way each of the A_i of a given family by an open set ω_i.

17. 10. Application. — Let E be a Hausdorff space containing a countable sub-set which is everywhere dense, and let f be a positive, sub-additive capacity defined on the class $\mathcal{E} = \mathcal{K}(E)$ of all compact sub-sets of E such that $f(X) = 0$ whenever X contains not more than one point.

Then there exists a sub-set $A \subset E$ *which is a* G_δ *everywhere dense in* E *(hence* A *is a residual of* E *when* E *is a complete metric space) and such that* $f_*(A) = f^*(A) = 0$.

For, let $D = \{a_1, a_2, \ldots, a_n, \ldots\}$ be a countable sub-set which is everywhere dense in E, and ε an arbitrary positive number.

There exists, for each n, an open set ω_n such that $f(\omega_n) < \varepsilon/2^n$, and $a_n \in \omega_n$.

If we set $\Omega_n = \bigcup_1^n \omega_i$ and $\Omega = \bigcup_1^\infty \omega_n$, then, from the above lemma

$$f(\Omega_n) < \sum_1^n f(\omega_i) < \varepsilon.$$

On the other hand, since the sequence Ω_n is increasing, and since each element of $\mathcal{E}$ is compact, it can be easily shown that $f(\Omega) = \lim f(\Omega_n)$ (see end of 28. 2., Chap IV).

It follows that $f(\Omega) \leqq \varepsilon$. Now, Ω is an open set which is everywhere dense in E. Hence, there exists a sequence of open sets G_n which are everywhere dense in E, and whose capacities tend to 0. Their intersection is the desired set A [12].

(12) Mazurkiewicz [1] has proved a weaker result, concerning only the interior capacity of A, whenever E is a compact sub-set of a Euclidean space, and f is the Newtonian capacity.

17. 11. THEOREM. — *If f is a capacity of arbitrary order $\mathcal{C}_\alpha$ on an* additive and rich *class $\mathcal{E}$ of subsets of a space* E, *each extension of f to an* additive *family is also of order $\mathcal{C}_\alpha$.*

Proof. For the class $\mathcal{C}_{1,a}$ we have already seen that this statement is satisfied, even without assuming that $\mathcal{E}$ is additive and rich.

For $\alpha \geqq (1, b)$ it is sufficient to remark that each class $\mathcal{C}_\alpha$ is defined by a system of inequalities of the form $\Phi \geqq 0$ ([13]) where Φ is a continuous function of capacities $f\left(\bigcup_{i \in J} A_i\right)$. These inequalities remain valid, according to Lemma 17. 9., for the exterior capacities $f^*\left(\bigcup_{i \in J} A_i\right)$, where the A_i are (for example) elements of the set $\mathcal{E}_2$ on which the extension f_2 of f is defined. Since $f_2\left(\bigcup_{i \in J} A_i\right) = f^*\left(\bigcup_{i \in J} A_i\right)$ by the definition of f_2, the inequalities $\Phi \geqq 0$ remain true for f_2.

17. 12. COROLLARY. — *If a capacity f on an additive and rich class $\mathcal{E}$ is of order $\mathcal{C}_n (n \geqq 2)$, each of the inequalities $\nabla_p \leqq 0$ $(p \leqq n)$ can be extended to the exterior capacities of arbitrary subsets of* E.

This corollary is actually an immediate consequence of Lemma 17. 9.

18. Invariance of the classes $\mathcal{M}_\alpha$ by extension.

18. 1. **The class $\mathcal{M}_{1,a}$.** If a capacity f_1 is of order $\mathcal{M}_{1,a}$ on $\mathcal{E}$, we have $f_{1*}(A_n) \to f_{1*}\left(\bigcap A_n\right)$ for each sequence A_n. However, since we know only that $f_{2*} \geqq f_{1*}$, we cannot show that $f_{2*}(A_n) \to f_{2*}\left(\bigcap A_n\right)$. Therefore, the order $\mathcal{M}_{1,a}$ is not conserved by extension.

18. 2. **Classes $\mathcal{M}_\alpha$ for $\alpha \geqq (1, b)$.**

18. 3. LEMMA. — *Let f be a capacity on an* additive and multiplicative *class $\mathcal{E}$ of subsets of* E, *and let $\{X_i\}_{i \in I}$ be a finite*

([13]) This statement is less obvious for the class $\mathcal{C}_{1,b}$. However, notice that the condition which defines $\mathcal{C}_{1,b}$ may be formulated as follows: for $a_1 \subset A_1$ and $a_2 \subset A_2$, we have $f(A_1 \cup A_2) - f(a_1 \cup a_2) \leqq \Psi\left[(f(A_1) - f(a_1)), (f(A_2) - f(a_2))\right]$ where $\Psi(u, v) \to 0$ with u and v.

family of subsets of E *such that* $f_*\left(\bigcap_{i \in J} X_i\right)$ *is finite for each* $J \subset I$. *For each* $\varepsilon > 0$, *there exists a family* $\{A_i\}_{i \in I}$ *of elements of* $\mathcal{E}$ *such that* $A_i \subset X_i$ *for each* i, *and such that, for each* $J \subset I$, *we have*

$$f_*\left(\bigcap_{i \in J} X_i\right) - f\left(\bigcap_{i \in J} A_i\right) \leqq \varepsilon.$$

Indeed, for each $J \subset I$ let A_J be an element of $\mathcal{E}$ such that $A_J \subset \bigcap_{i \in J} X_i$ and $f\left(\bigcap_{i \in J} X_i\right) - f(A_J) \leqq \varepsilon$. Then let $A_i = \bigcup_{J \ni i} A_J$ for each $i \in J$. This family obviously satisfies the condition stated.

18. 4. Lemma. — *Let f be a capacity on an* additive and multiplicative *class* $\mathcal{E}$. *With the same conventions as in Lemma* 17. 9., *if we have* $\Phi(\{x_J\}) \geqq 0$, *with* $x_J = f\left(\bigcap_{i \in J} A_i\right)$, *for each choice of the family* $\{A_i\}$ *of elements of* $\mathcal{E}$, *we have the same inequality when we replace the* A_i *by arbitrary subsets* X_i *of* E *and each* x_J *by* $f_*\left(\bigcap_{i \in J} X_i\right)$.

This lemma is an immediate consequence of the continuity of Φ and of Lemma 18. 3.

18. 5. Definition. — *A class* $\mathcal{F}$ *of subsets of a topological space* E *is called* G-separable *if for each couple* X_1 *and* X_2 *of disjoint subsets of* E *each of which is either an element of* $\mathcal{F}$ *or the intersection of one such element with a closed set of* E, *there exist two disjoint open sets* ω_1 *and* ω_2 *of* E *such that* $X_1 \subset \omega_1$ *and* $X_2 \subset \omega_2$.

The following are examples of G-separable sets $\mathcal{F}$:

18. 6. Any class $\mathcal{F}$ of compacts in a Hausdorff space E.

18. 7. Any class $\mathcal{F}$ of closed sets in a normal space E.

It is obvious that, if $\{X_i\}_{i \in I}$ is a finite family of mutually disjoint sets each of which is either an element of $\mathcal{F}$ or the intersection of such an element with a closed set of E, then there exists a family $\{\omega_i\}_{i \in I}$ of open sets of E such that $X_i \subset \omega_i$ for each i and $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$.

18. 8. LEMMA. — *Let f be a capacity on a set $\mathcal{E}$ of subsets of* E, *and let* $\{X_i\}_{i\in I}$ *be a finite family* of *subsets of* E *such that each* $f^*(X_J)$ *is finite, where* $X_J = \bigcap_{i\in J} X_i$ $(J \subset I)$.

When all the X_J *make up a* G-*separable set* $\mathcal{F}$, *there exists for each* $\varepsilon > 0$ *a family of open sets* $\{\omega_i\}_{i\in I}$ *of* E *such that* $X_i \subset \omega_i$ *for each* i *and*

$$f(\omega_J) - f^*(X_J) \leqq \varepsilon \quad \text{for each} \quad J \subset I, \quad \text{where} \quad \omega_J = \bigcap_{i\in J} \omega_i.$$

Proof. We can easily construct a family of open sets Ω_J such that

$$X_J \subset \Omega_J; \qquad f(\Omega_J) - f^*(X_J) \leqq \varepsilon$$

for each

$$J \subset I; \qquad \Omega_{J_1} \subset \Omega_{J_2} \quad \text{whenever} \quad J_2 \subset J_1.$$

This family plays a transitory role in the construction.

First let $\omega_I = \Omega_I$. Then we suppose the ω_J defined for all J of cardinal number $\overline{\overline{J}} > p$, and in such a way that for each such J we have

$$(1) \qquad X_J \subset \omega_J; \qquad \omega_J \subset \Omega_J; \qquad \omega_J = \bigcap_{J' \supset J} \omega_{J'}.$$

For each J such that $\overline{\overline{J}} = p$, we then define

$$Y_J = X_J \cap \Big(\bigcap_{J' \supset J} \complement \omega_{J'} \Big).$$

These Y_J thus defined are mutually disjoint; they are therefore separable by some open sets G_J which one can in addition restrict by the condition $G_J \subset \Omega_J$. We then define ω_J as follows:

$$\omega_J = G_J \cup \Big(\bigcup_{J' \supset J} \omega_{J'} \Big).$$

It is obvious that the family of ω_J thus increased $(\overline{\overline{J}} \geqq p)$ possess the three properties stated in (1) above. We continue the construction until we obtain the ω_J with $\overline{\overline{J}} = 1$; they are the desired ω_i.

18. 9. COROLLARY OF LEMMAS 18. 4. AND 18. 8. — *Let f be a capacity on an* additive and multiplicative class $\mathcal{E}$. *With the*

same conventions as in Lemma 18. 4, *we have the inequality* $\Phi(\{x_J\}) \geqq 0$, *where* $x_J = f^*(X_J)$ *and* $X_J = \bigcap_{i \in J} X_i$, *for each family* $\{X_i\}_{i \in I}$ *such that the set of* X_J *is G-separable.*

This corollary is an immediate consequence of lemmas 18. 4. and 18. 8. and of the continuity of Φ (we use Lemma 18. 4. in the particular case where the X_i are open sets).

18. 10. Theorem. — *If f is a capacity of order* $\mathfrak{M}_\alpha$ $(\alpha \geqq 1, b)$ *on an additive and multiplicative set* $\mathcal{E}_1$, *the extension of f to a multiplicative set* $\mathcal{E}_2$ *is also of order* $\mathfrak{M}_\alpha$ *when the set* $\mathcal{E}_2$ *is G-separable (for example, if each element of* $\mathcal{E}_2$ *is compact and* E *is a Hausdorff space, or if each element of* $\mathcal{E}_2$ *is closed and* E *is normal).*

This theorem is an immediate consequence of corollary 18. 9.

19. Extension of a class $\mathcal{E}_1$ by a limit procedure.

We are now going to study the extension of a capacity f in a case where the set $\mathcal{E}_2$ is deduced from $\mathcal{E}_1$ by a process independent of the given capacity f.

19. 1. Theorem. — *Let* $\mathcal{E}_1$ *be a multiplicative class of compacts of a space* E, *and let* $\mathcal{E}_2$ *be the set of arbitrary intersections of elements of* $\mathcal{E}_1$. *If* f_2 *is the extension to* $\mathcal{E}_2$ *of an arbitrary capacity* f_1 *on* $\mathcal{E}_1$, *then for each* $A_2 \in \mathcal{E}_2$,

$$f_2(A_2) = \inf f_1(X) \qquad (A_2 \subset X;\ X \in \mathcal{E}_1).$$

If f_1 *is of order* $\mathfrak{M}_\alpha$ $(\alpha \geqq 1, b)$, *then* f_2 *is of the same order.*

If $\mathcal{E}_1$ *is additive as well as multiplicative,* $\mathcal{E}_2$ *has the same property; if then* f_1 *is of order* $\mathfrak{A}_\alpha$, f_2 *is of the same order.*

Proof. We use the fact that, for each $A \in \mathcal{E}_2$ and for each open set ω containing X, there exists an element $B \in \mathcal{E}_1$ such that $A \subset B \subset \omega$. This statement is an immediate consequence of the fact that A is the intersection of a filtering decreasing family of compacts which are elements of $\mathcal{E}_1$.

It follows that for any finite family $\{A_i\}_{i \in I}$ of elements of $\mathcal{E}_2$, and for any two families of open sets $\{\omega_J\}$ and $\{\Omega_J\}$ such that

$$\bigcap_{i \in J} A_i \subset \omega_J \qquad \bigcup_{i \in J} A_i \subset \Omega_J \qquad \text{for each} \qquad J \subset I,$$

there exists a family $\{B_i\}_{i\in I}$ of elements of $\mathcal{E}_1$ such that, for any indices i and J

$$A_i \subset B_i \qquad \bigcap_{i\in J} B_i \subset \omega_J \qquad \bigcup_{i\in J} B_i \subset \Omega_J.$$

These relations show that we shall be able to approximate each finite family of elements of $\mathcal{E}_2$ from above by elements of $\mathcal{E}_1$ in such a way that this approximation is preserved by the operations of intersection and union. The formula

$$f_2(A_2) = \inf f_1(X) \qquad (A_2 \subset X;\ X \in \mathcal{E}_1)$$

follows from the fact that f_2 is continuous on the right, that $f_1(X) = f_2(X)$, and that we can approximate A_2 from above by some X.

Henceforth, for each inequality

$$\Phi(\{x_J\}) \geqq 0, \qquad \text{where} \qquad x_J = f\Big(\bigcap_{i\in J} A_i\Big),$$

which is valid for f_1, it is sufficient to carry out a passage to the limit in order to obtain the same inequality for f_2. This remark establishes the second assertion of the theorem.

When $\mathcal{E}_1$ is additive, the additivity of $\mathcal{E}_2$ follows immediately, and the process which we have just used for $f\Big(\bigcap_{i\in J} A_i\Big)$ is also valid for $f\Big(\bigcup_{i\in I} A_i\Big)$. This fact proves the last part of the theorem.

20. **Restriction of a capacity.** — Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two classes of subsets of a space E with $\mathcal{E}_1 \subset \mathcal{E}$, and let f_2 be a capacity on $\mathcal{E}_2$.

The restriction *of f_2 to $\mathcal{E}_1$ is the function f_1 defined on $\mathcal{E}_1$ by the relation $f_1(A) = f_2(A)$ for each $A \in \mathcal{E}_1$.*

It follows immediately that f_1 is a capacity. We suppose, as everywhere else, that the given data are such that for f_1 (and f_2) every element of $\mathcal{E}_1$ (respectively $\mathcal{E}_2$) is capacitable, for example, because $\mathcal{E}_1$ and $\mathcal{E}_2$ are additive or absorbing.

The following relations hold for each $X \subset E$:

$$f_{1*}(X) \leqq f_{2*}(X) \qquad f_1^*(X) \leqq f_2^*(X).$$

If f_2 is of order $\mathfrak{A}_{1,a}(\mathfrak{M}_{1,a})$, we cannot therefore conclude that f_1 is of the same order. But, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are additive (multiplicative), and if f_2 is of order $\mathfrak{A}_\alpha$ (with $\alpha \geqq (1, b)$ (or respectively of order $\mathfrak{M}_\alpha$), it follows immediately that f_1 is of the same order.

This operation is interesting in a special case.

20. 1. **Special case.** — *Let* N *be a subset of* E *such that, for each open set* $\omega \subset E$, *the set* $(N \cap \omega)$ *is* f_2*-capacitable (for example, if* N *be open).*

If we take for $\mathcal{E}_1$ *the set of elements of* $\mathcal{E}_2$ *included in* N, *we have for each* $X \subset N$ *the equalities*

$$f_{1*}(X) = f_{2*}(X) \qquad f_1^*(X) = f_2^*(X).$$

In particular, each subset X *of* N *is simultaneously* f_1 *and* f_2*-capacitable or non-capacitable.*

The first of these relations follows immediately. In order to show the second, we shall suppose, for example, that $f_1^*(X)$ is finite. For each $\varepsilon > 0$ there exists an open set ω such that $X \subset \omega$ and $f_1(\omega) - f_1^*(X) < \varepsilon$. Now we have the following sequence of inequalities:

$$f_1^*(X) \leqq f_2^*(X) \leqq f_2^*(N \cap \omega) = f_{2*}(N \cap \omega) = f_{1*}(N \cap \omega) \leqq f_{1*}(\omega) = f_1(\omega).$$

It follows that $f_2^*(X) - f_1^*(X) \leqq \varepsilon$ for each $\varepsilon > 0$, and the desired result follows.

20. 2. **Application of the preceding operations.** — We shall use these operations especially in the study of the capacitability of sets. In fact, it is often convenient in this study to suppose that the space E and set $\mathcal{E}$ possess a certain regularity. The operation of restriction will permit us to replace E by a subspace N; then the extension operation will permit enrichment of the new class $\mathcal{E}_1$ thus obtained, a step which often proves useful.

20. 3. Example. — Let E be a Hausdorff space, and let $\mathcal{E}$ be an *additive and hereditary* class of compacts of E. Let f be a capacity on $\mathcal{E}$.

Let X be a subset of E such that every compact contained in X is f-capacitable; and suppose that there exists a *completely regular* set N such that $X \subset N \subset E$, and such that each subset of N which is open relative to N is f-capacitable (if X possesses a completely regular neighborhood, we shall take for N the interior of this neighborhood; if the element A of $\mathcal{E}$ is metrizable, we shall take for N the set A if the capacity f is such that each K_σ of E is capacitable).

We wish to show how we can replace the study of interior and exterior capacities of X *by the same study in a simpler case.*

Let $\mathcal{E}_1$ be the additive and hereditary set of elements of $\mathcal{E}$ contained in N. If f_1 is the restriction of f to $\mathcal{E}_1$, we have

$$f_{1*}(X') = f_*(X') \quad \text{and} \quad f_1^*(X') = f^*(X') \quad \text{for every} \quad X' \subset N.$$

Observe, on the other hand, that we can consider f_1 as a capacity on the set $\mathcal{E}_1$ of subsets of *the space* N; we obtain for each $X' \subset N$ the same values for the interior and exterior f_1-capacities when we consider $\mathcal{E}_1$ as a class of subsets of E or of an arbitrary space in which N is imbedded. This remark will allow us to imbed N in a new normal space as follows:
Since N is completely regular, it can be imbedded in a compact space F. Designate by $\mathcal{E}_2$ the set of its compacts and by f_2 the extension of f_1 to $\mathcal{E}_2$.

According to theorems 16. 2. and 16. 3, as every compact included in X is f-capacitable, and then also f_1-capacitable, we have,

$$f_{1*}(X) = f_{2*}(X) \qquad f_1^*(X) = f_2^*(X).$$

It follows that the interior and exterior capacities of X are the same for f and for f_2.

Now f_2 has the advantage of being a capacity defined on the set of subcompacts of a compact space.

Let us show in addition that if f is of any order $\mathcal{A}_\alpha$ *or of order $\mathcal{M}_\alpha$ with $\alpha \geqq (1, b)$, the capacity f_2 is of the same order.*

It is obvious that f_1 is of the same order as f. Then Theorem 17. 11. shows that if f_1 is of order $\mathcal{A}_\alpha$, f_2 is also. And Theorem 18. 10. shows that if f_1 is of order $\mathcal{M}_\alpha$ with $\alpha \geqq (1, b)$, then f_2 is of the same order.

CHAPTER V

OPERATIONS ON CAPACITIES AND EXAMPLES OF CAPACITIES

In this chapter we shall study first some operations which transform capacities of a given class into capacities of the same class, and then several examples of capacities, some of which are important and will be used in the following chapters.

21. Operations on the range of capacities.

21. 1. If $\Phi(\{x_i\})$ is a continuous, increasing function of the real variables x_i $(i \in I)$, and if $(f_i)_{i \in I}$ denotes a finite family of capacities defined on a class $\mathcal{E}$ of subsets of a space E, then the function $f(X)$, defined by $f(X) = \Phi(\{f_i(X)\})$ is a capacity on $\mathcal{E}$, and we have

$$f_* = \Phi(\{f_{i*}\}) \qquad \text{and} \qquad f^* = \Phi(\{f_i^*\}).$$

If each of the f_i is of order $\mathfrak{A}_{1,a}(\mathfrak{M}_{1,a})$ then the same holds for f.

If Φ is a linear form with non-negative cœfficients, and if each f_i is of arbitrary order $\mathfrak{A}_\alpha(\mathfrak{M}_\alpha)$, then the same is true for f.

21. 2. If (f_n) is a sequence of capacities defined on the same $\mathcal{E}$, and if the f_n *converge uniformly* on $\mathcal{E}$ to a function f, then this function is a capacity. The f_{n*} converge uniformly to f_*, and the f_n^* converge uniformly to f^*. If each f_n is of order $\mathfrak{A}_\alpha(\mathfrak{M}_\alpha)$, then f is of the same order.

21. 3. If (f_n) is a *decreasing* sequence of capacities defined on the same $\mathcal{E}$, then the limit f of this sequence is a capacity.

We have $f_* \leqq \lim f_{n*}$, but not necessarily: $f_* = \lim f_{n*}$.

If each f_n is of class $\mathfrak{A}_\alpha(\mathfrak{M}_\alpha)$, with $\alpha \geqq (1, b)$, then the same holds for the limit f.

We shall not give the very easy proofs of these statements.

21. 4. *If $\Phi(u)$ is an increasing concave function of the real variable u, and if f is a capacity of order $\mathfrak{A}_2$ on an additive class $\mathcal{E}$, then the function $g = \Phi(f)$ is also a capacity of order $\mathfrak{A}_2$.*

Proof. The assumptions on Φ imply its continuity; hence g is a capacity. Let us show that $\nabla_2(X;\ A,\ B)_g \leqq 0$. We know that $\nabla_2(X;\ A,\ B)_f \leqq 0$ and that the ∇_1 and ∇_2 with respect to the function Φ are non-positive since Φ is increasing and concave. If we set

$$\nabla_2(X;\ A,\ B)_f = -\lambda_{AB}, \qquad \nabla_1(X \cup A;\ B)_f = -\lambda_B,$$
$$\nabla_1(X \cup B;\ A)g = -\lambda_A, \qquad -f(X) = -\lambda_0,$$

then the λ_A, λ_B, λ_{AB}, are non-negative and we have

$$\begin{aligned} &f(X) = \lambda_0, \\ &f(X \cup A) = \lambda_0 + \lambda_A + \lambda_{AB}, \\ &f(X \cup B) = \lambda_0 + \lambda_B + \lambda_{AB}, \\ &f(X \cup A \cup B) = \lambda_0 + \lambda_A + \lambda_B + \lambda_{AB}. \end{aligned}$$

If we add the two relations

$$\nabla_2(\lambda_0;\ \lambda_A,\ \lambda_B + \lambda_{AB})_\Phi \leqq 0 \quad \text{and} \quad \nabla_1(\lambda_0 + \lambda_A;\ \lambda_{AB})_\Phi \leqq 0$$

term by term, we obtain

$$\Phi(\lambda_0) - \Phi(\lambda_0 + \lambda_A + \lambda_{AB}) - \Phi(\lambda_0 + \lambda_B + \lambda_{AB}) + \Phi(\lambda_0 + \lambda_A + \lambda_B + \lambda_{AB}) \leqq 0,$$

which may be written also as follows:

$$\nabla_2(X;\ A,\ B)_g \leqq 0.$$

21. 5. **Generalization.** — An analogous result is obtained if Φ is replaced by a function of several real variables whose ∇_1 and ∇_2 are non-positive.

More generally, one could show that by composing two alternating functions of order n (in the sense of Chapter III*), the resulting function is alternating of the same order. The proof of this last result is not simple; we shall not give it here.*

21. 6. *If $\Phi(u)$ is an increasing, convex function of the real variable u, and if f is a capacity of order $\mathfrak{M}_2$ on a multiplicative class $\mathcal{E}$, then the function $g = \Phi(f)$ is also a capacity of order $\mathfrak{M}_2$.*

This statement is equivalent to the preceding one, for observe that, if we set $\Phi'(u) = -\Phi(-u)$ and $f' = -f$, then the function Φ' is increasing and concave, and f' is alternating of order 2 on the multiplicative semi-group $\mathcal{E}$.

An extention analogous to that of 21.4, can be obtained in this case also.

21. 7. *If f a capacity of order* $\mathfrak{A}_n (n \geq 3)$ *on an additive class* $\mathcal{E}$, *then* $\nabla_1(X, A_1)$ *is a capacity of order* $\mathfrak{A}_{n-1}$ *on* $\mathcal{E}$ *for every* $A_1 \in \mathcal{E}$.

The continuity on the right of $\nabla_1(X; A_1)$ is obvious. On the other hand, every difference of order $(n-1)$ of this difference ∇_1 is a difference ∇_n of f; it is therefore non-positive.

An analogous statement concerning the capacities of order $\mathfrak{M}_n (n \geq 3)$ on a multiplicative class $\mathcal{E}$ is obtained if ∇_1 is replaced by Λ_1.

22. Change of variable in a capacity.

22. 1. *Let* E *and* F *be two topological spaces and* $\mathcal{E}$ *and* $\mathcal{F}$ *two classes of subsets of* E *and* F *respectively. A mapping* $Y = \varphi(X)$ *from* $\mathcal{E}$ *into* $\mathcal{F}$ *will be called* increasing and continuous on the right *if*.

a) $(A_1 \subset A_2) \Longrightarrow (\varphi(A_1) \subset \varphi(A_2))$, *for any elements* A_1 *and* A_2 *of* $\mathcal{E}$;

b) *for every neighborhood* V_1 *of* $\varphi(A_1)$, *there exists a neighborhood* U_1 *of* A_1 *such that the relation* $\varphi(X) \subset V_1$ *holds for every* $X \in \mathcal{E}$ *such that* $A_1 \subset U_1$.

If f is a capacity on $\mathcal{F}$, *then the function* $e(X) = f(Y)$, *where* $Y = \varphi(X)$ *with* $X \in \mathcal{E}$, *is obviously a capacity on* $\mathcal{E}$. *We shall say that e* is derived from *f* by the change of variable $Y = \varphi(X)$.

22. 2. Example. — Let $y = \varphi(x)$ be a continuous mapping from E into F. For any class $\mathcal{E}$ of subsets of E, we shall still denote the extension of φ to $\mathcal{E}$ by φ, and let $\mathcal{F}$ be the image of $\mathcal{E}$ by φ. This mapping φ from $\mathcal{E}$ onto $\mathcal{F}$ is increasing and continuous on the right.

If f is a capacity on $\mathcal{F}$, then for every $B \subset F$ we have

$$e_*(\varphi^{-1}(B)) = f_*(B) \quad \text{and} \quad e^*(\varphi^{-1}(B)) \leq f^*(B);$$

and the last relation is an equality if the mapping φ from E into F is an open mapping, that is, if it maps open sets on open sets.

More generally, the following relations hold for every $A \subset E$:

$$e_*(A) \leq f_*(\varphi(A)) \quad \text{and} \quad e^*(A) \leq f^*(\varphi(A)).$$

An important special case is the following:

For E we take the product space of two Hausdorff spaces F and G; and let φ be the canonical projection from E on F. Let us suppose that every element of $\mathcal{E}$ is compact, that $\mathcal{F} = \varphi(\mathcal{E})$, and that the condition $\varphi(K) \in \mathcal{F}$ implies $K \in \mathcal{E}$ for every compact subset K of E. Using the notation employed in the preceding, we assert that the relation $e(\omega) = f(\varphi(\omega))$ holds for every open subset ω of E.

Indeed, for every compact subset $B \subset \varphi(\omega)$, there exists a compact subset $A \subset \omega$ such that $B = \varphi(A)$; this statement is easily deduced from the fact that B is compact. If we take, for the sets B, elements of $\mathcal{F}$ whose f-capacity approaches that of $\varphi(\omega)$, we obtain $e(\omega) \geq f(\varphi(\omega))$; but we know already that $e(\omega) \leq f(\varphi(\omega))$, hence the equality.

It follows that $e^*(X) = f^*(\varphi(X))$ for every subset $X \subset E$. Since we know already that $e_*(X) \leq f_*(\varphi(X))$, the e-capacitability of X, that is, the condition $e_*(X) = e^*(X)$, implies $f^*(\varphi(X)) \leq f_*(\varphi(X))$, whence $f^*(\varphi(X)) = f_*(\varphi(X))$.

22. 3. Theorem. — *The e-capacitability of* X *implies the f-capacitability of its projection* $\varphi(X)$, *and we have* $e(X) = f(\varphi(X))$.

23. Study of $\cup$-homomorphisms continuous on the right.

We shall now suppose that $\mathcal{E}$ is additive. We shall say that the mapping φ from $\mathcal{E}$ into $\mathcal{F}$ is a $\cup$-homomorphism continuous on the right if it is continuous on the right in the previously defined sense, and if $\varphi(A_1 \cup A_2) = \varphi(A_1) \cup \varphi(A_2)$ *whenever* A_1 *and* $A_2 \in \mathcal{E}$. *Such a mapping is clearly increasing.*

23. 1. General examples of $\cup$-homomorphisms continuous on the right.

(α) Let $E \equiv F$ and let $\mathcal{E}$ be an additive class of subsets of E.

(i) For every $A \subset E$, let $\mathcal{F}_A$ be the class of those subsets of E which are of the form $(X \cup A)$, where $X \in \mathcal{E}$. Then the mapping $X \to (X \cup A)$ from $\mathcal{E}$ onto $\mathcal{F}_A$ is a $\cup$-homomorphism continuous on the right.

(ii) For every *closed* subset A of E, let $\mathcal{F}_A$ be the class of those subsets of E which are of the form $(X \cap A)$, where $X \in \mathcal{E}$. Then the mapping $X \to (X \cap A)$ from $\mathcal{E}$ onto $\mathcal{F}_A$ has the desired property.

(β) Let $E \equiv F$. If $\mathcal{E}$ is the class of all subsets of E, and if $\mathcal{F}$ is the class of all closed subsets of E, then the mapping φ from $\mathcal{E}$ onto $\mathcal{F}$ which is defined by $\varphi(A) = \overline{A}$ has the desired property.

(γ) Let $x = \psi(y)$ be a continuous mapping from a compact space F into a Hausdorff space E. Then for every class $\mathcal{E}$ of subsets of E, the mapping $\varphi = \psi^{-1}$ from $\mathcal{E}$ into the class of all subsets of F has the desired property.

Indeed, for any $A \in \mathcal{E}$ let $B = \varphi(A) = \psi^{-1}(A)$. Let V be an open neighborhood of B. For every point $x \in A$ there exists an open neighborhood u_x of x such that $\psi^{-1}(u_x) \subset V$. If $U = \bigcup_{x \in A} u_x$, then U is an open neighborhood of A such that $\psi^{-1}(U) \subset V$, which proves the continuity on the right of φ.

(δ) More generally, let E be an arbitrary topological space, F a compact space, and A a closed subset of $(E \times F)$. For every $X \subset E$, let $Y = \varphi(X)$ be the set of those points y of F for which $(x, y) \in A$ for at least one $x \in X$. Then the mapping $A \to \varphi(A)$ is again a $\cup$-homomorphism which is continuous on the right.

To these results there corresponds a reciprocal proposition which shows, in an important special case, how every $\cup$-homomorphism which is continuous on the right can be obtained.

Let $\mathcal{E}$ *be an* additive, hereditary class of compact subsets *of a topological space* E, *and let* $Y = \varphi(X)$ *be a* $\cup$*-homomorphism, continuous on the right, from* $\mathcal{E}$ *onto a class* $\mathcal{F}$ *of* compact subsets *of a Hausdorff space* F. *Then there exists in* (E $\times$ F) *a closed subset* A *which satisfies the following relation*:

For every $X \in \mathcal{E}$, $\varphi(X)$ *is the set of all points* y *of* F *such that* $(x, y) \in A$ *for at least one* $x \in X$.

An equivalent statement is the following: if $pr_E(m)$ and $pr_F(m)$ denote the projections of a point m of A on E and F, respectively, then $\varphi(X) = pr_F(pr_E^{-1}(X))$.

We leave the verification of this proposition to the reader.

(ε) Let $y = \varphi(x)$ be a continuous mapping from E into F; then the extention of φ to an additive class $\mathcal{E}$ of subsets of E has the desired property. We have already used this example and stated an important special case of it in 22. 2.

23. 2. **Preservation of the class $\mathcal{A}_\alpha(\alpha \geqq (1, b))$ by the $\cup$-homomorphisms continuous on the right.** — Let E and F be two topogical spaces, $\mathcal{E}$ and $\mathcal{F}$ two additive classes of subsets of E and F, respectively, and let φ be a $\cup$-homomorphism, continuous on the right, from $\mathcal{E}$ into $\mathcal{F}$.

If f *is a capacity of order* $\mathcal{A}_\alpha(\alpha \geqq (1, b))$ *on* $\mathcal{F}$, *then the capacity* $e(X) = f(\varphi(X))$ on $\mathcal{E}$ is also of order $\mathcal{A}_\alpha$. This result is an immediate consequence of the fact that the definitions of the classes $\mathcal{A}_\alpha$ involve the operation $\cup$ only.

24. Study of $\cap$-homomorphisms continuous on the right.

Let us suppose that $\mathcal{E}$ *is multiplicative. We shall say that the mapping* φ *from* $\mathcal{E}$ *into* $\mathcal{F}$ *is a* $\cap$-*homomorphism continuous on the right if it is continuous on the right, and if* $\varphi(A_1 \cap A_2) = \varphi(A_1) \cap \varphi(A_2)$ *whenever* A_1 *and* A_2 *are elements of* $\mathcal{E}$. *Such a mapping is obviously increasing.*

24. 1 **General examples of $\cap$-homomorphisms continuous on the right.**

(α) Under the conditions specified in example 23. 1. (α), the mappings $X \to X \cup A$ and $X \to X \cap A$ are $\cap$-homomorphisms, continuous on the right, whenever $\mathcal{E}$ is multiplicative.

(β) Let $E \equiv F$. If $\mathcal{E}$ is the class of all subsets of E, and if $\mathcal{F}$ is the class of all open subsets of E, then the mapping φ from $\mathcal{E}$

onto $\mathcal{F}$ defined by $\varphi(A) = \mathring{A}$ (interior of A) has the desired property.

(γ) The mapping $\varphi = \psi^{-1}$ defined in example 23. 1. (γ), has the desired property. Thus, this mapping is both a $\cup$-and a $\cap$-homomorphism, continuous on the right, whenever $\mathcal{E}$ is both additive and multiplicative.

(δ) Let $y = \varphi(x)$ be a continuous, one-to-one mapping from E into F, or more generally, let φ be continuous and such that $\varphi(A_1 \cap A_2) = \varphi(A_1) \cap \varphi(A_2)$ whenever A_1 and A_2 are elements of $\mathcal{E}$. Then the extension of φ to $\mathcal{E}$ has the desired property.

(ε) Suppose that E is a Hausdorff space and that every element of $\mathcal{E}$ is compact.

(i) If F is the topological space of all compact subsets of E, and if, for every $A \in \mathcal{E}$, we define φ by $\varphi(A) = \mathcal{K}(A)$, where $\mathcal{K}(A)$ is the class of all compact subsets of A, then the mapping φ has the desired property. For, on the one hand,

$$\varphi(A_1 \cap A_2) = \varphi(A_1) \cap \varphi(A_2)$$

and, on the other, the continuity on the right follows from the definition of the classical topology of F.

(ii) Let I be any set of indices, and F the topological space E^I. If, for every $A \in \mathcal{E}$, we set $B = \varphi(A) = A^I$, then the mapping φ has the desired property.

24. 2. Problem. It would be interesting to find a simple method for the construction of every $\cap$-homomorphism, continuous on the right, from the class $\mathcal{E}$ of all compact subsets of a compact space E into the class $\mathcal{F}$ of all compact subsets of another compact space F.

24. 3. Preservation of the classes $\mathcal{M}_\alpha(\alpha \geq (1, b))$ by the $\cap$-homomorphisms continuous on the right. — Since there is a perfect analogy with the proposition concerning the preservation of the classes $\mathcal{A}_\alpha$ by the $\cup$-homomorphisms (see 23. 2.), the results will not be stated in detail.

24. 4. Study of other changes of variables. — There are other changes of variable, such as for instance those which transform a capacity of order $\mathcal{A}_\alpha$ into a capacity of order $\mathcal{M}_\alpha$, or conversely. They are of particular interest when the classes $\mathcal{E}$ and $\mathcal{F}$ to which they apply are classes of compact sets.

In this connection, it would be interesting to find a simple method for constructing every mapping φ of the following types: E and F are two compact spaces, $\mathcal{E}$ and $\mathcal{F}$ the classes of all compact subsets of E and F, respectively, and φ is a mapping from $\mathcal{E}$ into $\mathcal{F}$ which satisfies either

$$\varphi(A_1 \cup A_2) = \varphi(A_1) \cap \varphi(A_2)$$

or

$$\varphi(A_1 \cap A_2) = \varphi(A_1) \cup \varphi(A_2).$$

The first of these two functions may be called an exponential and the second a logarithm. Both are decreasing; it is, therefore, no longer possible to speak of their continuity on the right. In each particular case one should impose the type of continuity which is the most suitable.

Example of an exponential. — For a given E let T be an auxiliary compact space, F a compact topological space of continuous mappings from E into T, and A a compact subset of T. For every compact subset X of E, we denote by $Y = \varphi(X)$ the class of all continuous mappings from E into T which belong to F and which map X into A. Then Y is compact, and obviously satisfies the relation $\varphi(X_1 \cup X_2) = \varphi(X_1) \cap \varphi(X_2)$.

25. Construction of alternating capacities of order 2.

Although the most interesting capacities to study are those of order $\mathfrak{A}_\infty$ or $\mathfrak{M}_\infty$, the fact that the capacities of order $\mathfrak{A}_2$ and $\mathfrak{M}_2$ lead to a complete theory of capacitability induces us to investigate the operations which lead to such capacities. We shall study here an operation which leads to functions which are alternating of order 2.

25. 1. Study of the Greenian capacity by means of the Dirichlet integral. — Let D be a Greenian domain of R^n. Let $\mathfrak{D}$ be the set of absolutely continuous functions which are : non-negative on D, zero on the boundary of D, and possess a finite Dirichlet integral

$$f(\varphi) = \int (\text{grad } \varphi)^2 \, dx.$$

It can be shown that if φ_1 and $\varphi_2 \in \mathfrak{D}$, $\varphi_1 \smile \varphi_2$ and $\varphi_1 \frown \varphi_2$ are also in $\mathfrak{D}$, and that

$$f(\varphi_1 \smile \varphi_2) + f(\varphi_1 \frown \varphi_2) = f(\varphi_1) + f(\varphi_2).$$

Then let K be a compact subset of D. If we set

$$\text{cap}(K) = \inf f(\varphi) \qquad \text{for all} \qquad \varphi \geqq 1 \qquad \text{on } K,$$

it can be shown that, within a constant factor, this capacity is precisely the Greenian capacity of K that we have studied in Chapter II. Let us show that cap(K) is an alternating capacity of order $\mathfrak{a}_2$ (which we know already, but this new proof can be extended to new cases).

The fact that it is increasing and continuous on the right is immediate. Then let K_1 and K_2 be two compacts of D and let $\varepsilon > 0$. Let φ_1, φ_2, be two elements of $\mathfrak{D}$ such that:

$$f(\varphi_i) - \text{cap}(K_i) \leqq \varepsilon \qquad \text{and} \qquad \varphi_i(x) \geqq 1 \qquad \text{on } K_i \qquad (i = 1, 2).$$

We have therefore

$$f(\varphi_1 \smile \varphi_2) + f(\varphi_1 \frown \varphi_2) \leqq \text{cap } K_1 + \text{cap } K_2 + 2\varepsilon.$$

Now

$$(\varphi_1 \smile \varphi_2 \geqq 1 \qquad \text{on} \qquad (K_1 \cup K_2)$$

and

$$(\varphi_1 \frown \varphi_2) \geqq 1 \qquad \text{on} \qquad (K_1 \cap K_2).$$

It follows that

$$f(K_1 \cup K_2) + f(K_1 \cap K_2) \leqq f(K_1) + f(K_2).$$

Since this inequality is sufficient to obtain the most precise results of the theory of capacitability, it is interesting to try to apply the above reasoning to a more general case.

N. Aronszajn [1] in his study of functional completion and of exceptional sets associates a set function to each normed space $\mathfrak{L}$ of real functions on a given set E, in the following way: Let $||\varphi||$ be the norm on $\mathfrak{L}$. For each $X \subset E$ we set

$$F(X) = \inf ||\varphi|| \text{ for all } \varphi \text{ which are } \geqq 1 \text{ on } X.$$

If there exists no φ which is $\geqq 1$, on X, we set $F(X) = +\infty$.

In the case where $\mathfrak{L}$ is the linear space generated by the set $\mathfrak{D}$ introduced above, the Greenian capacity of a compact set X is in fact the square of the expression F(X) corresponding to the norm $||\varphi|| = \sqrt{f(\varphi)}$; but this difference is not trouble some for the theory of capacity.

The following contains a theorem which leads to some general cases where the above function F(X) is alternating of order 2.

25. 2. **Alternating functions associated with a subvaluation on a lattice.** — *Let* L *be a lattice,* L′ *a sublattice of* L *such that each* $a \in L$ *is majorated by an* $a' \in L'$, *and* f *a real function on* L′ *such that*

$$f(a \smile b) + f(a \frown b) \leqq f(a) + f(b).$$

We say that f *is a* sub-valuation *on* L′.

When L′ = L is a distributive lattice and when f is an *increasing* valuation on L, we shall see (26. 4.) that f is alternating of order ∞ on L, relative to the operation $\smile$.

If f is not an increasing valuation, this is no longer true in general. However, we shall see how we can still associate to each valuation and likewise to each sub-valuation on L′ an alternating function of order 2 on L, even if L and L′ are not distributive.

For each $x \in L$, we set

cap $(x) = \inf f(a)$ for all a such that $x \prec a$ and $a \in L'$.

Theorem. — *The function cap* (x) *is an alternating function of order* 2 *on* L, *relative to the operation* $\smile$.

Proof. That cap (x) is increasing is immediate; and the inequality

$$\text{cap}\,(a \smile b) + \text{cap}\,(a \frown b) \leqq \text{cap}\,(a) + \text{cap}\,(b)$$

is proved exactly as in the case where f is the Dirichlet integral. of φ.

Examples. — Usually, the sub-valuation f will be a valuation Here are some examples.

If D is a domain of R^n, we take $L = SS_+$ [14] and take for L′ the set of real positive functions φ which are: continuous on D, zero outside of a compact set, and Lipschitzian. If we set

[14] SS_+ denotes the cone of all positive and upper semi-continuous functions on D which vanish outside of some compact.

$\varphi_1 \prec \varphi_2$ when $\varphi_1(x) \leq \varphi_2(x)$ for every x, L is a lattice and L′ is a sublattice.

Let $\Phi\left(x, \varphi, \frac{\partial\varphi}{\partial x}\right)$ be a continuous function of x, φ, and of the partial derivatives of the first order of φ, such that $\int \Phi(x, 0, 0)\, d\mu$ (where μ is the Lebesgue measure or any other fixed absolutely continuous measure on D) has a sense. We set

$$f(\varphi) = \int_D \Phi\left(x, \varphi, \frac{\partial\varphi}{\partial x}\right) d\mu \quad \textit{for each} \quad \varphi \in L'.$$

It is immediate that if $\varphi_1 \prec \varphi_2$, we have

$$f(\varphi_1 \smile \varphi_2) + f(\varphi_1 \frown \varphi_2) = f(\varphi_1) + f(\varphi_2).$$

The fact that this relation holds when φ_1 and φ_2 are arbitrary is due to the following facts:

(*a*) the set of points of D where $\varphi_1 \neq \varphi_2$ is a denumerable union of partial domains in each of which we have either

$$\varphi_1 \prec \varphi_2 \quad \text{or} \quad \varphi_2 \prec \varphi_1;$$

(*b*) the set of points of D where $\varphi_1 = \varphi_2$ is the union of two borelian sets A and B such that on A the functions φ_1 and φ_2 have equal differentials, and B has Lebesgue measure zero.

It is often useful to notice that for each $\varepsilon > 0$, and for each neighborhood V of the support of any $\varphi \in L'$, there exists a function φ' indefinitely differentiable, zero outside of V, with

$$|\varphi - \varphi'| < \varepsilon \quad \text{and} \quad |f(\varphi) - f(\varphi')| < \varepsilon.$$

These conclusions would no longer hold if in the function Φ some partial derivatives of φ of order ≥ 2 occurred.

Special cases.

(*a*) $\Phi = \varphi^p$ leads to the norm of the spaces L^p.

(*b*) $\Phi = (\text{grad } \varphi)^2$ leads to the Dirichlet integral.

(*c*) $\Phi = (1 + \text{grad}^2\, \varphi)^{1/2}$ leads to the «area» of the graph of φ.

When Φ is homogeneous of degree α with respect to φ and $\frac{\partial\varphi}{\partial n}$, the function $[\text{cap}(\varphi)]^{1/\alpha}$ is homogeneous of degree 1 and, if $\alpha > 1$, the fact that $v = u^{1/\alpha}$ is then an increasing and concave function implies that $[\text{cap}(\varphi)]^{1/\alpha}$ is alternating of order 2 whenever $f(\varphi)$ is ≥ 0 and alternating of order 2 (see 21. 4.).

25. 3. **Equilibrium.** — The definition given above of the function cap (x) is more general, even in the setting of the classical Greenian capacity, than the ordinary definition, since it defines not only the capacity of the characteristic functions of compacts, but also the capacity of any element $\varphi \in SS_+$

We could associate to every element $\varphi \in SS_+$ a sub-harmonic function analogous to an equilibrium potential. We shall show, in the general scheme introduced above, how we can define such an equilibrium in very general cases.

Let us use the notations introduced in the above theorem. For each $a_0 \in L$, let $L(a_0)$ be the set of elements a of L such that $a_0 \prec a$ and $\text{cap}(a) = \text{cap}(a_0)$. $L(a_0)$ *is a sub-lattice of* L; in fact, if $a_1, a_2 \in L(a_0)$, we have

$$a_0 \prec a_1 \frown a_2 \prec a_1,\ a_2$$

so that since cap (x) is increasing, we have

$$\text{cap}\,(a_1 \frown a_2) = \text{cap}\,(a_0),$$

and hence

$$a_1 \frown a_2 \in L(a_0).$$

On the other hand,

$$\text{cap}\,(a_1 \smile a_2) + \text{cap}\,(a_1 \frown a_2) \leqq \text{cap}\,(a_1) + \text{cap}\,(a_2);$$

hence

$$\text{cap}\,(a_1 \smile a_2) \leqq \text{cap}\,(a_0);$$

and since cap (x) is increasing, it follows that

$$a_1 \smile a_2 \in L(a_0).$$

The lattice $L(a_0)$ possesses a smallest element, which is a_0; it can have only one largest element; when the latter exists, we shall denote it $\widehat{a_0}$; it is the *equilibrium element* associated with a_0.

A case where $\widehat{a_0}$ always exists whenever a_0 is such that $L(a_0)$ is bounded above is when

(*a*) $L = L'$;

(*b*) each subset of L bounded above possesses an upper bound;

(*c*) $f(x)$ is lower semi-continuous on the left, which means that for each subset (a_i) of L, filtering on the right, and having an upper bound a_ω, we have

$$f(a_\omega) \leqq \liminf f(a_i).$$

This semi-continuity occurs, for example, when $f(\varphi)$ is the integral on D of a function $\Phi\left(x, \varphi, \frac{\partial\varphi}{\partial x}\right)$ which is $\geqq 0$ and has, in a certain sense, a convex indicatrix when considered as a function of $\frac{\partial\varphi}{\partial x}$. (Example : $\Phi = \text{grad}^2\varphi$ or $\Phi = (1 + \text{grad}^2\varphi)^{1/2}$.)

It can happen that for some $a_0 \in$ L, L(a_0) is not bounded above, but that by introducing a convenient notion of exceptional set, L(a_0) possesses a quasi-upper bound. This happens for example in the classical potential theory.

26. Examples of alternating capacities of order $\mathfrak{A}_\infty$.

In all of the following examples, the capacities under consideration are always tacitly assumed to be defined on the class $\mathcal{E} = \mathcal{K}(E)$ of *all compact* subsets of the space E in question unless otherwise indicated. We shall give here only examples of capacities of order $\mathfrak{A}_\infty$. Let us notice here that many capacities which occur naturally in analysis are obtained from Radon measures by a small number of operations such as $\cap$, $\cup$, $\int$, max, min, and that in general, the capacities obtained in this way either fail to be of any order $\mathfrak{A}_\alpha$ or $\mathfrak{M}_\alpha$ or they are of order $\mathfrak{A}_\infty$ or $\mathfrak{M}_\infty$.

26. 1. Alternating family of elements of a commutative ordered group. — Let G be a commutative ordered group, and I a finite set. Every function, alternating of order ∞, which is defined on the class $\mathcal{E} = 2^{\text{I}}$ of all subsets of I and whose values are in G is called an *alternating family* $(x_{\text{J}})_{\text{J}\subset\text{I}}$ of elements of G. Thus, if $x_{\text{J}} = f(\text{J})$, all the ∇_f are supposed to be non-positive. Let us set, conforming to a notation already used before,

$$\nabla[(\text{I}-\text{J});\ \{i_p\}_{p\in\text{J}}] = -\lambda_{\text{J}} \qquad (\text{J} \subset \text{I, with J} \neq \emptyset)(\lambda_{\text{J}} \geqq 0).$$

By an already familiar computation we deduce from these relations the following:

$$x_J = x_\emptyset + \sum_{K \cap J \neq \emptyset} \lambda_K.$$

If $x_\emptyset$ is not defined, an arbitrary value such that λ_I is non-negative is assigned to $X_\emptyset$; this assignment is always possible. Conversely, it is easily verified that every family x_J which is defined by equalities of this form with numbers $\lambda_K \geqq 0$ is indeed an alternating family.

EXAMPLE. — If I contains two elements 1, 2, then every alternating family on I is of the following form:

$$x_1 = x_\emptyset + \lambda_1 + \lambda_{1,2}; \qquad x_2 = x_\emptyset + \lambda_2 + \lambda_{1,2}$$
$$x_{1,2} = x_\emptyset + \lambda_1 + \lambda_2 + \lambda_{1,2}.$$

26. 2. Operation « sup » in a commutative lattice group [15]. — Let G be a commutative lattice group and I any set. Also, let $i \to x_i$ be a mapping φ from I into G.

Set $f(X) = \sup_{i \in X} (x_i)$ for every finite $X \subset I$. The function f is thus defined on the additive class $\mathcal{E}$ of finite subsets of I.

We wish to prove that the ∇_f are non-positive and more precisely, that

$$\nabla(X; \{A_p\})_f = \inf[f(X), \{f(A_p)\}] - \inf[\{f(A_p)\}].$$

It is equivalent to prove that for arbitrary elements x, a_p of G ($p = 1, 2, \ldots$), we have

$$\nabla(x; \{a_p\})_{\sup} = \inf(x, a) - a \qquad \text{where} \qquad a = \inf\{a_p\}.$$

We recall the following identity: $\inf(u, v) + \sup(u, v) = u + v$. It follows that $\nabla_1(x; a_1) = x - \sup(x, a_1) = \inf(x, a_1) - a_1$. The general formula follows from this one by induction: the proof is entirely analogous to that of 14. 5. for functions which are both alternating and monotone of order 2. Thus, we can state that the operation « sup » in a commutative lattice group is an alternating function of order infinity.

(15) A commutative lattice group G is an ordered group such that any two elements X_1 and X_2 of G always have a least upper bound, $\sup(X_1, X_2)$, and a greatest lower bound, $\inf(X_1, X_2)$, sometimes denoted by $X_1 \smile X_2$ and $X_1 \frown X_2$, respectively.

For the operation inf there is a formula which is the dual of the preceding one; hence, the $\bigvee_{\inf}$ will be non-negative. Thus, if for every $X \subset I$ we set

$$\omega(X) = [\sup_{i \in X} (x_i) - \inf_{i \in X} (x_i)],$$

the oscillation $\omega(X)$ is an alternating function of order infinity.

If G is in addition a complete lattice ([16]), these results may be extended to additive classes $\mathcal{E}$ of subsets X of I such that every $\varphi(X)$ is bounded from above (and also bounded from below if $\omega(X)$ is being considered).

APPLICATION. — Let $\varphi(x)$ be a real-valued *continuous* function on a topological space E. For every $X \subset E$ we shall denote by $f(X)$ and $\omega(X)$ the least upper bound and the oscillation of φ on X. These two functions are alternating capacities of order $\mathfrak{A}_\infty$ on each additive class $\mathcal{E}$ of subsets of E (on which they are assumed to be finite, for simplification). When $\varphi(x)$ is only upper semicontinuous on E, $f(X)$ only is a capacity of order $\mathfrak{A}_\infty$.

EXAMPLE. — If $\delta(X)$ denotes the diameter of a compact subset X of the real line, then since $\delta(X)$ is the oscillation of the function x on X, this diameter is a capacity of order $\mathfrak{A}_\infty$ of X. (It should be remarked that if one wants to assign a value to $\delta(\emptyset)$, this value should be $-\infty$).

On the other hand, the diameter of a compact set X in an arbitrary metric space E is not of order $\mathfrak{A}_\infty$. This diameter is equal to the maximum of a function which is defined on E^2 and not on E. We therefore have only $\delta(X) = f(X^2)$ where f is a capacity of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E^2)$.

26. 3. **Generalization: valuation on a distributive lattice.** — *Let* L *be a distributive lattice and f a mapping from* L *into a commutative ordered group* G. *We shall say that f is a valuation, if*

$$f(a \smile b) + f(a \frown b) = f(a) + f(b).$$

26. 4. THEOREM. — *If f is an increasing valuation (that is, if* $(a \prec b) \Longrightarrow f(a) \leq f(b)$), *and if we set* $g(X) = f(\sup X)$ *for every*

([16]) A lattice G is said to be complete if and only if every subset of G which is bounded from above possesses a least upper bound (and likewise for the greatest lower bound).

finite subset X *of* A, *then the function* $g(X)$, *which is defined on the additive class* $\mathcal{E}$ *of all finite subsets of* L, *is alternating of order* ∞, *and*

$$\bigvee (X; \{A_i\})_g = \inf (g(X), \{g(A_i)\}) - \inf\{g(A_i)\}.$$

An analogous statement holds in the case where L is a complete distributive lattice and where $\mathcal{E}$ is the class of all bounded subsets of L.

Corollary. — *With the same notations, the function* $f(x)$ *is alternating of order* ∞ *on the ordered semi-group* L *with the operation sup.*

26. 5. Examples of such valuations.

(i) The dimension of a variety in projective geometry or in Von Neumann's continuous dimensional projective geometry.

(ii) For L we take the set of all positive integers, ordered by the relation $a \prec b$ if b is a multiple of a and we set :

$f(x) = \text{Log}\,(x)$
$g(X) = \text{Log}\,[\text{l.c.m.}(X)]$ for every $X \subset L$, with X finite.

26. 6. Non-negative Radon measures. — If E is a locally compact space, a function f defined on $\mathcal{K}(E)$ defines a non-negative Radon measure if and only if

(i) f is finite for every $K \in \mathcal{K}(E)$.
(ii) $f(\emptyset) = 0$.
(iii) f is increasing and continuous on the right.
(iv) $f(K_1 \cup K_2) + f(K_1 \cap K_2) = f(K_1) + f(K_2)$.

These conditions are equivalent to stating that f is a capacity on $\mathcal{K}(E)$ of orders $\mathcal{A}_\infty$ and $\mathcal{M}_\infty$ which is finite and such that $f(\emptyset) = 0$.

More generally, if E is any Hausdorff space, any function f which is defined on an additive and hereditary class $\mathcal{E}$ of compact subsets of E, and which satisfies the conditions (i), (ii), (iii), (iv), will be called a *generalized non-negative Radon measure on* $\mathcal{E}$. Here again, these conditions are equivalent to the statement that f is a capacity on $\mathcal{E}$ of orders $\mathcal{A}_\infty$ and $\mathcal{M}_\infty$ which is finite and such that $f(\emptyset) = 0$.

We further remark that, since the class $\mathcal{E}$ is *rich* (see

Chapter IV, 17. 3.) the extension of f to the class $\mathcal{K}(E)$ of all compact subsets of E is still of order $\mathfrak{A}_\infty$ by virtue of Theorem 17. 10. of Chapter IV. Since, on the other hand, $\mathcal{K}(E)$ is G-separable (see definition 18. 5.), this extension is also of order $\mathfrak{M}_\infty$ by Theorem 18. 11. of Chapter IV. Thus, this extension to $\mathcal{K}(E)$ is a capacity of order $\mathfrak{A}_\infty$ and $\mathfrak{M}_\infty$ such that $f(\emptyset) = 0$. But it may happen that for this extension $f(K) = +\infty$ for certain compact sets K.

Let us show that *if f is any function which is defined on an additive, hereditary class $\mathcal{E}$ of compact subsets of* E, *and which satisfies conditions* (i), (ii), *and* (iii), *the condition* (iv) *is equivalent to the following condition:*

(iv′) $f(K_1 \cup K_2) \leqq f(K_1) + f(K_2)$, *and this inequality becomes an equality whenever* $K_1 \cap K_2 = \emptyset$ (K_1 *and* K_2 *are elements of* $\mathcal{E}$).

Indeed, since $f \geqq 0$ and $f(\emptyset) = 0$, (iv) implies (iv′). Conversely, let us suppose that (iv′) is satisfied. We wish to show that, if K_1 and K_2 are elements of $\mathcal{E}$, then

$$f(K_1 \cup K_2) + f(K_1 \cap K_2) = f(K_1) + f(K_2).$$

If $K_1 \cap K_2 = \emptyset$, the desired relation obviously holds. If $K_1 \cap K_2 \neq \emptyset$, let ε be an arbitrary non-negative number, and let V be a compact neighborhood of $(K_1 \cap K_2)$ in $(K_1 \cup K_2)$ such that

$$f(V) - f(K_1 \cap K_2) \leqq \varepsilon.$$

Set $(K_i - \mathring{V}) = k_i$ ($i = 1, 2$). The compact sets k_i and $(K_1 \cap K_2)$ are disjoint and

$$((K_1 \cap K_2) \cup k_i) \subset K_i \subset (V \cup k_i) \quad (i = 1, 2).$$

Hence, by virtue of property (iv′),

$$f(k_i) + f(K_1 \cap K_2) \leqq f(K_i) \leqq f(k_i) + f(V),$$

and

$$f(k_1) + f(k_2) + f(K_1 \cap K_2) \leqq f(K_1 \cup K_2) \leqq f(k_1) + f(k_2) + f(V).$$

Therefore,

$$f(K_1) + f(K_2) = f(k_1) + f(k_2) + 2f(K_1 \cap K_2) + \eta,$$

where $$0 \leqq \eta \leqq 2\varepsilon,$$

and $$f(K_1 \cup K_2) = f(k_1) + f(k_2) + f(K_1 \cap K_2) + \eta',$$

where $$0 \leqq \eta' \leqq \varepsilon.$$

Thus $$f(K_1 \cup K_2) + f(K_1 \cap K_2) = f(K_1) + f(K_2) + \eta'',$$

where $$0 \leqq \eta'' \leqq 2\varepsilon.$$

Generalization. — There exist functions of compact sets which are closely analogous to the generalized Radon measures but which are not continuous on the right. For instance, the linear measure of Caratheodory, defined on the class of all compact subsets of the Euclidean plane, is such a function. In this connection, it is of interest to introduce the following definition.

We shall call any function f, defined on an additive hereditary class $\mathcal{E}$ of compact subsets of a Hausdorff space E *a* Caratheodory measure *if for every element* K *of* $\mathcal{E}$ *its restriction to the class of all compact subsets of* K *is a non-negative Radon measure on* K.

26. 7. **Newtonian or Greenian capacity.** — If E is a domain in the Euclidean space R^n, or more generally, if E is a conformal or locally Euclidean space which possesses a Green's function (see Brelot and Choquet [1]), then the capacity of a compact subset $K \subset E$ with respect to this Green's function is of order $\mathfrak{A}_\infty$. We have studied these capacities in detail in Chapter II.

26. 8. **Fundamental scheme of the capacities of order $\mathfrak{A}_\infty$.** — Let E and F be two sets (without topologies), A a subset of $(E \times F)$, and μ a non-negative additive function defined on a ring (17) $\mathcal{F}$ of subsets of F. For every subset X of E, let $\varphi(X)$ be the projection on F of the set of those points of A whose projection on E lies in X. In other words

$$\varphi(X) = \mathrm{pr}_F[A \cap (X \times F)].$$

The mapping $X \to \varphi(X)$ is a $\cup$-homomorphism.

Let $\mathcal{E}$ be an additive class of subsets of E such that $\varphi(\mathcal{E}) \subset \mathcal{F}$.

(17) A set which is closed under finite union and under difference, hence also under finite intersection.

The function μ is alternating of order ∞ on $\mathcal{F}$ (see Chapter III, 14. 5.). Hence if we set

$$f(\mathrm{X}) = \mu(\varphi(\mathrm{X})) \qquad \text{for every} \qquad \mathrm{X} \in \mathcal{E},$$

the function f is alternating of order ∞ on $\mathcal{E}$.

For instance, if E is a Hausdorff space, F a locally compact space, A a closed subset of $(\mathrm{E} \times \mathrm{F})$, μ a non-negative Radon measure on F, and $\mathcal{E}$ the class $\mathcal{K}(\mathrm{E})$ of all compact subsets of E, then it is easy to show that $\varphi(\mathrm{X})$ is closed for every $\mathrm{X} \in \mathcal{E}$, hence measurable with respect to the measure μ, and the preceding definition is applicable. If one can show in addition that $f(\mathrm{X})$ is continuous on the right, one can then state that $f(\mathrm{X})$ is a capacity of order $\mathcal{A}_\infty$. This case will be realized, for instance, if the set A is compact, or more generally, if $\varphi(\mathrm{X})$ is compact for every compact $\mathrm{X} \subset \mathrm{E}$.

We shall say that f is the function (or the capacity) obtained by the *fundamental scheme* (E, F, A, μ).

It is clear that in this scheme the additive function μ could be replaced by any alternating function of order ∞, but this generalization is not of great interest; on the other hand, we shall see that the importance of this scheme lies in the fact that it provides a canonical representation of every capacity of order ∞ on E, provided only that this capacity satisfies some conditions of regularity.

26. 9. **Game of « Heads or tails ».** — Let E be a finite set of throws in a game of « heads or tails ». For every $\mathrm{K} \subset \mathrm{E}$, let $f(\mathrm{K})$ be the probability of the event that tails occurs at least once on K. The function $f(\mathrm{K})$ may be obtained by the following scheme : let $\mathrm{F} = 2^{\mathrm{E}}$ be the class of all subsets of E (including $\emptyset$), and let $\mathrm{A} \subset (\mathrm{E} \times \mathrm{F})$ be the set of all points (x, X) such that $x \in \mathrm{X}$.

If μ is the measure on F defined by the condition that the measure of each of the 2^n points of F be $1/2^n$, then f is the function obtained by the scheme (E, F, A, μ). Thus f is alternating of order $\mathcal{A}_\infty$.

We remark that $f(\mathrm{K})$ depends only on the number of elements of K; if that number is n, then $f(\mathrm{K}) = \varphi(n)$.

Now if X, $\mathrm{A}_1, \ldots, \mathrm{A}_p$ are subsets of E which are mutually disjoint, with cardinal numbers n, $a_1, \ldots, a_p$, respectively, then we have obviously

$$\bigvee(X;\ \{A_i\})_f = \nabla(n;\ \{a_i\})_\varphi,$$

and this equality shows that φ is a function of n which is alternating of order infinity. This can be verified by using the following explicit expression of φ : $\varphi(n) = (1 - 2^{-n})$.

26. 10. **Geometrical probability.** — Let E be a plane, D a line in the plane, and μ a non-negative Radon measure on D; for every compact subset K of E, let $f(K)$ be the μ-measure of the orthogonal projection of K on D. Then $f(K)$ is obviously a capacity of order $\mathcal{A}_\infty$ on $\mathcal{K}(E)$. (As an analogous example, we can consider the « angle » $f(K)$ under which a compact set K, assumed to be contained in (E — 0), is seen from a fixed point 0 of the plane.)

From this remark we might deduce that the measure (here assumed to be the classical invariant measure) of the set of all lines of the plane which meet a compact set K is a capacity of order $\mathcal{A}_\infty$. But it is more convenient and more interesting to prove this by means of the fundamental scheme as follows.

Let F be the topological space (which is locally compact) of all lines D of the plane; let μ be the invariant classical measure on F, and A the closed subset of $(E \times F)$ which consists of the pairs (x, D) for which $x \in D$.

The function $f(x)$ which is obtained by means of the scheme (E, F, A, μ) is obviously the measure of the lines D which meet the compact set D. (If K is convex, then $f(K)$ is, moreover, equal to twice the length of the boundary curve of K.)

Now let us consider only those compact sets K which are contained in a fixed circle Γ. If we set $p(K) = \dfrac{f(K)}{f(\Gamma)}$, then the function $p(K)$ represents the *probability* of the event that a line which meets Γ also meets K. As in the preceding example we have here exhibited a probability which is a capacity of order $\mathcal{A}_\infty$. We shall return to this investigation in the last chapter.

26. 11. Let μ be a non-negative Radon measure defined on a compact metric space E, and let $h(u, m)$, $(u \geq 0,\ m \in E)$, be a

continuous function of the point (u, m), which is decreasing in u for every m.

For every compact subset X of E, set

$$f(X) = \int h(u_m, m)\, d\mu_{(m)}.$$

where u_m denotes the distance from m to X.

We shall show that f is a capacity of order $\mathcal{A}_\infty$ on $\mathcal{K}(E)$. Indeed, f is obtained by means of the fundamental scheme $[E, \mathcal{K}(E), A_E, \mu_h]$, where $\mathcal{K}(E)$ denotes the compact topological space of all closed subsets of E, A_E is the closed subset of all points (x, X) of $(E \times \mathcal{K}(E))$ such that $x \in X$, and μ_h is a non-negative Radon measure on that subset B of $\mathcal{K}(E)$ which consists of all closed solid spheres $B(m, u)$ of E, with μ_h defined by the elementary measure $dh \,.\, d\mu(m)$.

For every $X \in \mathcal{K}(E)$ the class of the compact sets which meet X has μ_h-measure zero with the exception of those which are closed spheres; and the spheres $B(m, u)$ which meet X are those for which $u \geq u_m$. Hence the result.

26. 12. **Harmonic measure.** — Let E be a Greenian domain, and for every $m \in E$ and every compact subset X of E, let $h(X, m)$ be the harmonic measure of X with respect to the point m for the domain $(E - X)$. (When $m \in X$, we shall set $h = 1$, by definition.) We have already used the fact that this function is quasi-everywhere equal to the equilibrium potential of X for the Green's function of E. (See 11. 2.) Moreover, we have shown (see 7. 5.) that the equilibrium potential of X considered as a function of X, has all its differences $(\nabla)_h$ non-positive. Thus $h(X, m)$ is an alternating function of X, of order ∞, for every m. It is continuous on the right. This fact is obvious if $m \notin X$, and, if $m \in X$, then $h(X, m) = 1$; hence, we have also $h(X', m) = 1$ for $X' \supset X$. Thus h is indeed a capacity of order $\mathcal{A}_\infty$.

More general capacities of order $\mathcal{A}_\infty$ may be derived from this one by setting $f(X) = \int h(X, m)\, d\mu(m)$, where μ is a non-negative Radon measure on E of finite total mass.

We have given this example immediately after example 26. 11. because of their great similarity.

26. 13. Construction of Cantor-Minkowski and regularization of a capacity. — Let E be a metric space such that every closed sphere in it is compact. For every compact subset K of E, and for every number $\rho \geq 0$, let $K_{(\rho)}$ be the set of all points of E whose distance from K is at most ρ.

The mapping $K \to K_{(\rho)}$ is a $\cup$-homomorphism, continuous on the right, from $\mathcal{K}(E)$ into $\mathcal{K}(E)$. Hence, if g is a capacity of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$, then the same is true for f_ρ, where f_ρ is defined by $f_\rho(K) = g(K_{(\rho)})$. Moreover, f_ρ decreases to g as a limit, as $\rho \to 0$.

For example, in $E = R^n$, f_ρ may denote the Euclidean measure of $K_{(\rho)}$.

This construction may be used to show that *every capacity g of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$ is the limit of a decreasing sequence of capacities of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$, each of which is a* continuous *function of its variable* X.

For simplification, let us suppose that $g \geq 0$. Let $v = \varphi(u)$ be a real-valued function of the real variable u, defined and continuous on $[0, 1]$, decreasing, vanishing at $x = 1$, and such that $\int_0^1 \varphi \, du = 1$. For every $\lambda > 0$, we set

$$g_\lambda(K) = \int_0^\infty f_u(K)\lambda\varphi(\lambda u)\, du.$$

We may also write

$$g_\lambda(K) = \int_0^\infty f_{t/\lambda}(K)\varphi(t)\, dt,$$

which shows that $g_\lambda(K)$ is a decreasing function of λ. The function $g_\lambda(K)$ is on the other hand, clearly an alternating function of order ∞ of K since this is the case for $f_u(K)$ for every u. And since for $0 \leq t \leq 1$, $f_{t/\lambda}(K)$ tends uniformly to $g(K)$ as $\lambda \to \infty$, it follows that $g_\lambda(K) \to g(K)$.

It remains to show that $g_\lambda(K)$ is, for every λ, a continuous function of K considered as an element of the classical topological space of the compact subsets of E. If we use the classical metric δ for this space, the distances of any point of E to K_1 and K_2 differ by at most ε whenever $\delta(K_1, K_2) \leq \varepsilon$, which implies that $K_1(\rho) \subset K_2(\rho + \varepsilon)$ and $K_2(\rho) \subset K_1(\rho + \varepsilon)$. Thus $f_\rho(K_1) \leq f_{\rho+\varepsilon}(K_2)$ and $f_\rho(K_2) \leq f_{\rho+\varepsilon}(K_1)$, so that

$$\begin{aligned} g_\lambda(K_1) &\leq \int_0^\infty f_{u+\varepsilon}(K_2)\lambda\varphi(\lambda u)\, du \\ &= g_\lambda(K_2) + \int_0^\infty [f_{u+\varepsilon}(K_2) - f_u(K_2)]\lambda\varphi(\lambda u)\, du. \end{aligned}$$

But

$$\int_0^{\infty} [f_{u+\varepsilon}(K_2) - f_u(K_2)]\lambda\varphi(\lambda u)\, du$$
$$= \int_{\varepsilon}^{\infty} f_u(K_2)\lambda[\varphi(\lambda(u-\varepsilon)) - \varphi(\lambda u)]\, du - \int_0^{\varepsilon} f_u(K_2)\lambda\varphi(u)\, du$$

so that

$$g_\lambda(K_1) - g_\lambda(K_2) \leqq \int_{\varepsilon}^{\infty} f_u(K_2)\lambda[\varphi(\lambda(u-\varepsilon) - \varphi(\lambda u)]\, du.$$

Let $M = f(K)$ for

$$K = \left[K_1\left(\frac{1}{\lambda} + \varepsilon_0\right)\right] \cup \left[K_2\left(\frac{1}{\lambda} + \varepsilon_0\right)\right]$$

where $\varepsilon_0 > 0$.

For every $\varepsilon < \varepsilon_0$, we have

$$g_\lambda(K_1) - g_\lambda(K_2) \leqq \int_{\varepsilon}^{\infty} M\lambda[\varphi(\lambda(u-\varepsilon)) - \varphi(\lambda u)]du = M\int_0^{\varepsilon\lambda}\varphi(t)\, dt,$$

and an analogous inequality by interchanging K_1 and K_2. Thus

$$|g_\lambda(K_1) - g_\lambda(K_2)| \leqq \int_0^{\varepsilon\lambda}\varphi(t)\, dt \qquad \text{for every} \quad \varepsilon < \varepsilon_0,$$

which shows that $g_\lambda(K)$ is continuous.

Note that any alternate capacity on $\mathcal{K}(E)$ is upper semi-continuous on the topological space $\mathcal{K}(E)$. We have just shown that it is a decreasing limit of continuous capacities of the same order.

26. 14. **Elementary capacities of order $\mathcal{A}_\infty$.** — Let E be a Hausdorff space and f a capacity on $\mathcal{K}(E)$ which is sub-additive and whose range contains at most the values 0 and 1.

Every element $A \in \mathcal{K}(E)$ such that $f(A) = 0$ has an open neighborhood ω such that, for every compact $X \subset \omega$ we have $f(X) = 0$. Let Ω be the union of the open sets ω.

Every compact $B \subset \Omega$ is covered by a finite family (ω_i) of these open sets ω; therefore that compact B is the union of a finite number of subcompacts each of which is contained in one of the ω_i (see, for instance, 17. 4. in Chapter IV).

Therefore $f(B) = 0$. In other words, for every $X \in \mathcal{K}(E)$, the necessary and sufficient condition that $f(X) = 0$ is that $X \subset \Omega$. Let $T = \complement\Omega$.

We have $$f(X) = \begin{cases} 0 & \text{if} \quad X \cap T = \emptyset. \\ 1 & \text{if} \quad X \cap T \neq \emptyset. \end{cases}$$

Consersely, if T is any non-empty closed subset of E, the function $f_T(X)$ defined by the preceding relations is obviously a subadditive capacity on $\mathfrak{K}(E)$.

We shall prove in Chapter VII as a special case of a general theorem that every $f_T(X)$ is a capacity of order $\mathfrak{A}_\infty$ and that these capacities are the extremal elements of the convex cone of positive capacities of order $\mathfrak{A}_\infty$ on $\mathfrak{K}(E)$.

The function $f_T(X)$ will be called the *elementary capacity* (with index T) of order $\mathfrak{A}_\infty$ on $\mathfrak{K}(E)$.

27. **Examples of capacities which are monotone of order $\mathfrak{M}_\infty$.** — We shall give here fewer examples than for capacities of order $\mathfrak{A}_\infty$, at first because monotone capacities do not occur as often as alternating capacities and also because they seem to be less useful.

27. 1. Every non-negative additive set function is monotone of order ∞. Thus each non-negative Radon measure on a locally compact space E is a capacity of order $\mathfrak{M}_\infty$ on $\mathfrak{K}(E)$.

27. 2. The fundamental scheme of alternating capacities is replaced here by a scheme that we shall indicate in a special case.

Let E be a locally compact space, F the topological space of its compact subsets, and μ a non-negative Radon measure on F. If, for every $K \subset E$, we set $f(K) = \mu(\mathfrak{K}(K))$ where $\mathfrak{K}(K)$ denotes the subset of F consisting of all the subcompacts of K, then $f(K)$ is a capacity of order $\mathfrak{M}_\infty$.

The interest of this scheme lies in the fact that it leads to a canonical representation of all positive capacities of order $\mathfrak{M}_\infty$ on $\mathfrak{K}(E)$, as we shall see in Chapter VII.

27. 3. Let μ be a non-negative Radon measure on a locally compact space E, and let $h(P, Q)$ be a non-negative continuous real-valued function of the couple (P, Q), or more generally a Baire function (with, if necessary, the restriction that $P \neq Q$).

The function $f(K) = \int_{K^2} h(P, Q)\, d\mu(P)\, d\mu(Q)$ is a capacity of order $\mathfrak{M}_\infty$ on $\mathfrak{K}(E)$, for the mapping $K \to K^2$ is a $\cap$-homomorphism continuous on the right, and $h(P, Q)\,d\mu(P)\,d\mu(Q)$ defines a Radon measure on E^2 (with possible value $+\infty$).

Let us remark that $f(K)$ can be interpreted as the energy of the restriction of μ to K for the kernel $h(P, Q)$.

There are analogous statements for a function h of n variables defined on E^n.

27. 4. On $E = R^n$, if we define $f(K)$ to be the Euclidean measure of the set of centers of circles of radius 1 contained in the compact K, f is of order $\mathfrak{M}_\infty$.

27. 5. On $E = R^n$, we set $f(K) = h(\rho(K))$, where $\rho(K)$ denotes the radius of the largest sphere with center 0 contained in K, and $h(u)$ a function of the real variable $u \geqq 0$ which is non-decreasing and continuous on the right.

The function f can be obtained by the scheme of 27. 2. above where μ is the Radon measure defined by $dh(u)$ on the set of spheres with center 0. Then f is a capacity of order $\mathfrak{M}_\infty$.

27. 6. Let E be a finite set of throws in a game of heads or tails. For every $K \subset E$, let $f(K)$ be the probability that tails occur nowhere except possibly on K.

This probability is within a constant the conjugate function of the probability that tails occurs at least once on K.

It is then of order $\mathfrak{M}_\infty$. If $\overline{\overline{K}} = n$ and $\overline{\overline{E}} = a$, then $f(K) = 2^n/2^a$; and it can be verified that $f(K)$ is a totally monotone function of n in the classical sense.

27. 7. **Elementary capacities of orders $\mathfrak{M}_\infty$.** — Let E be a Hausdorff space and f a capacity on $\mathcal{K}(E)$ which is of order $\mathfrak{M}_2$ and whose range contains at most the values 0 and 1. If $f(X_1) = f(X_2) = 1$, we have also $f(X_1 \cap X_2) = 1$ and unless $f \equiv 1$, we have $X_1 \cap X_2 \neq \emptyset$. Therefore the set of elements $X \in \mathcal{K}(E)$ for which $f(X) = 1$ does not contain $\emptyset$ and is multiplicative.

Let T be the non-empty intersection of those compacts; as T is also the limit of that decreasing filtering set of compacts and since f is continuous on the right, we have $f(T) = 1$.

In other words, in order that $f(X) = 1$, it is necessary and sufficient that $T \subset X$.

Conversely, for every *compact* $T \subset E$, let

$$f_T(X) = \begin{cases} 1 & \text{if} \quad T \subset X. \\ 0 & \text{if} \quad T \not\subset X, \end{cases}$$

It is obvious that $f_T(X)$ satisfies the identity:

$$f_T(X_1 \cap X_2) = f_T(X_1) . f_T(X_2).$$

It follows from this (and it will be a particular case of a theorem of Chapter VII) that $f_T(X)$ is of order $\mathfrak{M}_\infty$, and that these capacities are the extreme elements of the convex cone of positive capacities of order $\mathfrak{M}_\infty$ on $\mathcal{K}(E)$.

The function $f_T(X)$ will be called the *elementary capacity* (*with index T*) *of order* $\mathfrak{M}_\infty$ on $\mathcal{K}(E)$.

CHAPTER VI

CAPACITABILITY. FUNDAMENTAL THEOREMS.

28. Operations on capacitable sets for capacities of order $\mathcal{A}_\alpha$. — In this chapter we shall study the invariance of capacitability under the operations of denumerable union and intersection, as well as capacitability of analytic sets. We shall see that we can obtain subtantial results for capacities which satisfy sufficiently strict inequalities, for example, those which define the classes $\mathcal{A}_2$ or $\mathcal{M}_2$. In order to avoid some complications of terminology we shall suppose always that $\emptyset$ is an element of every class $\mathcal{E}$ of sets.

28. 1. THEOREM. — *Let $\mathcal{E}$ be an additive and rich set of subsets of a topological space* E, *and let f be a capacity of order $\mathcal{A}_\alpha$ $(\alpha \geqq (1, b))$ on $\mathcal{E}$.*

(i) *Each finite union of f-capacitable sets of capacity $\neq -\infty$ is also* f*-capacitable.*

(ii) *If f is such that for each increasing sequence $\{\omega_n\}$ of open sets of* E *we have $f(\omega_n) \to f\left(\bigcup \omega_n\right)$ (for example, if each element of $\mathcal{E}$ is compact), then*

(*a*) *f is of order $\mathcal{A}_{1,a}$; in other words, $f^*(A_n) \to f^*\left(\bigcup A_n\right)$ for each increasing sequence of sets $A_n \subset E$ such that $f^*(A_n) \neq -\infty$; and*

(*b*) *each denumerable union of capacitable sets of capacity $\neq -\infty$ is also* f*-capacitable.*

Proof. Notice that if f is of order $\mathcal{A}_n$ $(n \geqq 2)$, f is also of order $\mathcal{A}_2$. On the other hand, the inequality which defines the class $\mathcal{A}_{1,b}$ is highly analogous to the inequality

$$f(A_1 \cap A_2) - f(a_1 \cup a_2) \leqq [f(A_1) - f(a_1)] + [f(A_2) - f(a_2)],$$

which is satisfied for the class $\mathfrak{A}_2$. Thus, in order to simplify the notations, we shall give the proof only for the class $\mathfrak{A}_2$.

We recall first that, when f is of order $\mathfrak{A}_2$ and is additive and rich, by virtue of the inequality in 14. 3. and by Lemma 17. 9., we have

28. 2. $$f^*\left(\bigcup A_i\right) - f^*\left(\bigcup a_i\right) \leqq \sum \left(f^*(A_i) - f^*(a_i)\right),$$

where $a_i \subset A_i \subset E$ for each i.

Proof of (i). It is obviously sufficient to prove the theorem for the union of two sets A_1 and A_2. Moreover, if one of these sets, say A_1, has a capacity $f(A_1) = +\infty$, the set $A_1 \cup A_2$ has an interior capacity equal to $+\infty$; therefore, it is capacitable. We shall suppose therefore that $f(A_1)$ and $f(A_2)$ are finite.

For each $\varepsilon > 0$, there exists a set $X_i \in \mathcal{E}$ and an open set $\omega_i \subset E$ such that $X_i \subset A_i \subset \omega_i$ and $f(\omega_i) - f(X_i) < \varepsilon$ (for $i = 1, 2$).

We have therefore, by applying the above inequality 28. 2., $f(\omega_1 \cup \omega_2) - f(X_1 \cup X_2) \leqq [f(\omega_1) - f(X_1)] + [f(\omega_2) - f(X_2)] < 2\varepsilon$. Since $(X_1 \cup X_2) \subset (A_1 \cup A_2) \subset (\omega_1 \cup \omega_2)$ and $(X_1 \cup X_2) \in \mathcal{E}$, and since $(\omega_1 \cup \omega_2)$ is open, the set $(A_1 \cup A_2)$ is capacitable.

Proof of (ii). The proof of (a) will be given first. Let $\{A_n\}$ be an increasing sequence of subsets of E such that $f^*(A_n) \neq -\infty$ for every n.

If for $n = n_0$ we have $f^*(A_n) = +\infty$, it is obvious that $f^*(A_n) \to f^*\left(\bigcup A_n\right)$.

Otherwise for each $\varepsilon > 0$ and for each n there exists an open set ω_n such that $A_n \subset \omega_n$ and $f(\omega_n) - f^*(A_n) < \varepsilon/2^n$.

We have, therefore, by applying the inequality 28. 2. above, and by remarking that $\bigcup_1^n A_i = A_n$,

$$f(\Omega_n) - f^*(A_n) \leqq \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^n} \leqq \varepsilon \quad \text{where} \quad \Omega_n = \bigcup_1^n \omega_i.$$

Now if we set $\Omega = \bigcup \Omega_n = \bigcup \omega_n$, we know by hypothesis that $f(\Omega_n) \to f(\Omega)$. Therefore, $f(\Omega) \leqq \lim f^*(A_n) + \varepsilon$, and, since $\bigcup A_n \subset \Omega$, we have a fortiori

$$\lim f^*(A_n) \leqq f^*\left(\bigcup A_n\right) \leqq \lim f^*(A_n) + \varepsilon.$$

Since ε is arbitrary, we have $\lim f^*(A_n) = f^*\left(\bigcup A_n\right)$.

The proof of (*b*) will be given next. Let $\{A_n\}$ be an arbitrary sequence of capacitable sets such that $f(A_n) \neq -\infty$. Let $B_n = \bigcup_{1}^{n} A_i$. We have $\bigcup B_n = \bigcup A_n$, and moreover each B_n is capacitable according to the first part of the theorem. Since the sequence B_n is increasing we have

$$\lim f^*(B_n) = \lim f_*(B_n) = f^*\left(\bigcup B_n\right).$$

On the other hand we have

$$\lim f_*(B_n) \leqq f_*\left(\bigcup B_n\right)$$

Hence, $f^*\left(\bigcup B_n\right) \leqq f_*\left(\bigcup B_n\right)$; the capacitability of $\left(\bigcup B_n\right)$ follows from this inequality.

We shall now show that if each element of $\mathcal{E}$ is compact, the condition $\lim f(\omega_n) = f\left(\bigcup \omega_n\right)$ is satisfied.

We have at once that $\lim f(\omega_n) \leqq f(\Omega)$, where $\Omega = \bigcup \omega_n$. On the other hand, if $f(\Omega) < +\infty$, for each $\varepsilon < 0$ there exists a compact $K_\varepsilon \in \mathcal{E}$ such that $f(\omega) - f(K_\varepsilon) < \varepsilon$. Now $K_\varepsilon \subset \bigcup \omega_n$; therefore, since K_ε is compact and since the sequence ω_n is increasing, there exists an $n = n_\varepsilon$ such that $K_\varepsilon \subset \omega_{n_\varepsilon}$. It follows that $f(\omega) - f(\omega_{n_\varepsilon}) < \varepsilon$. Therefore, $f(\omega) \leqq \lim f(\omega_n)$; hence the equality.

In the case where $f(\Omega) = +\infty$, the proof is similar to that given.

We remark that this result about open sets is valid for any capacity on a class $\mathcal{E}$ of compacts.

28. 3. Corollary. — *Let $\mathcal{E}$ be an additive and hereditary set of compacts of E.*

If f is a capacity of order $\mathfrak{A}_\alpha$ ($\alpha \geqq (1, b)$) on $\mathcal{E}$ with $f > -\infty$, each denumerable union of capacitable sets is capacitable, and for each increasing sequence of sets $A_n \subset E$, we have

$$\lim f^*(A_n) = f^*\left(\bigcup A_n\right).$$

If f is of order $\mathfrak{A}_2$, then for arbitrary finite or infinite sequences of subsets (A_n) and (a_n) of E, with $a_n \subset A_n$ for each n, we have

$$f^*\left(\bigcup A_n\right) - f^*\left(\bigcup a_n\right) \leqq \sum (f^*(A_n) - f\ (a_n)).$$

29. **A capacitable class of sets.** — We shall introduce first a convenient terminology.

29. 1. Definition. — *Let* $\mathcal{E}$ *be a class of subsets of a set* E. *We shall let* $\mathcal{E}_\sigma$ *denote the class of sets* $A \subset E$ *where* A *is a denumerable, union of elements of* $\mathcal{E}$.

We shall let $\mathcal{E}_{\sigma\delta}$ *denote the class of sets* $A \subset E$ *where* A *is a denumerable intersection of elements of* $\mathcal{E}_\sigma$.

We want to show, that under certain hypotheses each element of $\mathcal{E}_{\sigma\delta}$ is capacitable. We cannot use for the proof the fact that each denumerable intersection of capacitable sets is capacitable, for this fact is already false for finite intersections as we shall show later. We will therefore have to use in a precise way the fact that $\mathcal{E}_{\sigma\delta}$ is constructed from elements of $\mathcal{E}$. the set $\mathcal{E}$ satisfying in addition certain restrictions.

29. 2. Theorem. — *If* $\mathcal{E}$, *additive and denumerably multiplicative, is such that, for each decreasing sequence* $\{A_n\}$ *of elements of* $\mathcal{E}$ *and every neighborhood* V *of* $A = \bigcap A_n$, *we have* $A_n \subset V$ *for n sufficiently large, and if f is of order* $\mathfrak{A}_{1,a}$, *each element of* $\mathcal{E}_{\sigma\delta}$ *is f-capacitable.*

Proof. Let $A \in \mathcal{E}_{\sigma\delta}$. Then $A = \bigcap A_n$, where $A_n \in \mathcal{E}_\sigma$; in other words, $A_n = \bigcup_{p=1}^{p=\infty} A_n^p$ where $A_n^p \in \mathcal{E}$.

We can always suppose, since $\mathcal{E}$ is additive, that A_n^p increases with p.

Set $f^*(A) = l$. If $l = -\infty$, we have $f_*(A) = -\infty$ also and A is capacitable. Otherwise it is finite or equal to $+\infty$. We shall give the proof in the case in which l is finite; the case in which $l = +\infty$ could be treated in an entirely analogous manner.

(Besides, the case in which $l = +\infty$ can always be reduced to the case where l is finite by replacing f by $g = -e^{-f}$. The function $(-e^{-u})$ is continuous and strictly increasing; hence if f is a capacity of order $\mathfrak{A}_{1,a}$, g is also. Furthermore, f-capacitability is equivalent to g-capacitability.)

Let ε be any positive number. The set $a_1^p = A \cap A_1^p$ is increasing with p, and we have $A = \bigcup_{p=1}^{p=\infty} a_n^p$. Therefore, since f is of order $\mathfrak{A}_{1,a}$, we have $f^*(A) = \lim f^*(a_1^p)$.

Therefore there exists an index p, say p_1, such that

$$f^*(A) - f^*(a_1^{p_1}) < \frac{\varepsilon}{2}.$$

Suppose that the sets $a_i^{p_i}$ have been defined for each $i \leqq n$ in such a way that for each i, $f^*(a_i^{p_i})$ is finite and that $a_i^{p_i} \subset A$. Set $a_{n+1}^p = a_n^{p_n} \cap A_{n+1}^p$. This set is increasing with p, and we have $a_n^{p_n} = \bigcup_{p=1}^{p=\infty} a_{n+1}^p$, from which it follows that

$$f^*(a_n^{p_n}) = \lim_{p \to \infty} f^*(a_{n+1}^p).$$

There exists therefore an index p, say p_{n+1}, such that

$$f^*(a_n^{p_n}) - f^*(a_{n+1}^{p_{n+1}}) < \frac{\varepsilon}{2^{n+1}}.$$

If we add the first n inequalities thus obtained, we get

29. 3. $$f^*(A) - f^*(a_n^{p_n}) < \varepsilon.$$

The $a_n^{p_n}$ constitute a decreasing sequence of sets, all contained in A. Set $a_\varepsilon = \bigcap a_n^{p_n}$. We can also write $a_\varepsilon = A \cap \left[\bigcap A_n^{p_n}\right]$. Now $A_n^{p_n} \subset A_n$, so that $\bigcap_n A_n^{p_n} \subset A$; hence, $a_\varepsilon = \bigcap_1^\infty A_n^{p_n}$.

If we set $B_n = \bigcap_1^n A_n^{p_n}$, the B_n constitute a decreasing sequence of elements of $\mathcal{E}$ and $a_\varepsilon = \bigcap B_n$. Now a_ε is again an element of $\mathcal{E}$; therefore, according to the continuity on the right of f and the given hypothesis on the mode of convergence of decreasing sequences of elements of $\mathcal{E}$, we have

$$f(a_\varepsilon) = \lim_{n \to \infty} f(B_n).$$

Since $a_\varepsilon \subset a_n^{p_n} \subset B_n$, we have also $f(a_\varepsilon) = \lim f^*(a_n^{p_n})$. The above inequality 29. 3. becomes $f^*(A) - f(a_\varepsilon) \leqq \varepsilon$. Since $a_\varepsilon \in \mathcal{E}$, we have therefore $f^*(A) \leqq f_*(A) + \varepsilon$ for each ε. Hence $f^*(A) = f_*(A)$.

29. 4. Corollary. — *If $\mathcal{E}$ is an additive and hereditary set of compacts of* E, *and if f is of arbitrary order $\mathfrak{A}_\alpha$ on $\mathcal{E}$, with $f > -\infty$, each element of $\mathcal{E}_{\sigma\delta}$ is f-capacitable.*

In fact, according to the Corollary 28. 3. of Theorem 28. 1., f is then of order $\mathfrak{A}_{1,a}$ and on the other hand, since each element of $\mathcal{E}$ is compact, each decreasing sequence of elements of $\mathcal{E}$ satisfies the exact conditions of Theorem 29. 2. Therefore, this theorem can be applied.

Notice that in this case each element of $\mathcal{E}_{\sigma\delta}$ is a $K_{\sigma\delta}$. But it is not true that each $K_{\sigma\delta}$ of E is always f-capacitable. We can indeed construct examples where there are some compacts of E which are not f-capacitable, even if f is of order $\mathcal{A}_\infty$.

The following is rather instructive as an example. Let E be the Euclidean plane R^2, $\mathcal{E}$ the set of compacts K of the plane such that K is contained in a finite union of straight lines parallel to a given fixed line Δ. For each $K \in \mathcal{E}$ we set

$$f(K) = \text{linear measure of the projection of K on } \Delta.$$

It is immediate that f is continuous on the right on $\mathcal{E}$ and alternating of order $\mathcal{A}_\infty$; on the other hand, $\mathcal{E}$ is additive and hereditary.

Now for each compact $K \subset E$ we have $f^*(K) =$ linear measure of the projection of K on Δ; and if K is such that each intersection of K with a line parallel to Δ has a zero linear measure, we have $f_*(K) = 0$.

For example each arc of a circle is non-capacitable for f. Here the elements of $\mathcal{E}_{\sigma\delta}$ are the denumerable unions of sets each of which is located on a line parallel to Δ and is any $K_{\sigma\delta}$ on such a line.

30. **Capacitability of K-borelian and K-analytic sets.** — We shall extend Corollary 29. 4. to the K-borelian and K-analytic sets.

30. 1. Theorem. — *If $\mathcal{E}$ is an additive and hereditary class of compacts of a Hausdorff space* E, *and if f is of arbitrary order $\mathcal{A}_\alpha$ on $\mathcal{E}$ and $f > -\infty$, any* K*-analytic set* A *of* E *is f-capacitable in each of the following two cases.*

(i) $A \subset B$ *where* $B \in \mathcal{E}_\sigma$ (*example* : A *is an element of the borelian field generated by* $\mathcal{E}$).

(ii) $A \subset \omega$ *where* ω *is a completely regular open set*; *and in addition* $\mathcal{K}(A) \subset \mathcal{E}_\sigma$, *that is, each compact* $K \subset A$ *is an element of* $\mathcal{E}_\sigma$.

Proof. In each of the two cases considered, A is such that each compact K contained in A is an element of $\mathcal{E}_\sigma$, and hence is f-capacitable.

Therefore, according to Theorem 16. 3. it is sufficient, in order to show the f-capacitability of A, to prove that A is capacitable for the extention f_2 of f to the set $\mathcal{K}(E)$ of all compacts of E.

Now $\mathcal{E}$ being additive and rich and $\mathcal{K}(E)$ being additive (E is Hausdorff), this extension f_2 is, according to Theorem 17. 10. of class $\mathcal{A}_\alpha$.

Thus Theorem 30. 1. will be established if it is proved in the simpler case where $\mathcal{E} = \mathcal{K}(E)$.

We shall now simplify case (ii). It is sufficient to remark that, since A is contained in a completely regular open set, we can apply the method explained in Example 20. 3. to reduce the problem immediately to the case where the space E is compact.

In short, the two cases (i) and (ii) are both reduced to the following simpler case : A is contained in a K_σ of E and $\mathcal{E} = \mathcal{K}(E)$.

Now according to Theorem 5. 1. there exists a compact space F and a set $\Gamma \subset E \times F$ such that Γ is a $K_{\sigma\delta}$, and such that its projection on E is identical with A.

Let us designate then by g the capacity defined on $\mathcal{K}(E \times F)$ by the equality $g(X) = f(\mathrm{pr}_E X)$, where $(\mathrm{pr}_E X)$ means the projection of the compact X on E.

According to 22.2 and 23. 2. in Chapter v, the capacity g is of order $\mathcal{A}_\alpha$; since in addition $g > -\infty$, according to Corollary 29. 4., Γ is g-capacitable. Therefore according to Theorem 22. 3. in Chapter v, its projection A on E is f-capacitable.

30. 2. COROLLARY. — *If* E *is a space which is homeomorphic to a borelian or analytic subset (in the classical sense) of a separable complete metric space, and if f is a capacity $> -\infty$, defined on the set $\mathcal{K}(E)$ of the compacts of* E *and of arbitrary order $\mathcal{A}_\alpha$, each borelian or analytic (in the classical sense) subset* A *of* E *is f-capacitable.*

In fact, Theorem 30. 1. is applicable to A since A is contained in the open set E which is completely regular, and since A is K-analytic (according to the classical theory A is the continuous image of the set of irrational numbers of [0, 1], which is a $K_{\sigma\delta}$).

31. Capacitability for the capacities which are only subadditive. — *We shall now construct an example of a capacity $f \geqq 0$, sub-additive, defined on the set of all compacts of the plane*

$E = R^2$, *and for which there exists a closed subset* A *in* E (*hence* A *is at the same time a* K_σ *and a* G_δ) *which is not capacitable.*

For each compact $K \subset R^2$, denote by $\Delta_K(y)$ and $\delta_K(y)$ the respective diameters of the sets $K \cap Dy$ and $K \cap dy$, where Dy and dy designate respectively the half-lines $(x \geqq 0, y)$ and $(x \leqq 0, y)$.

Let $\varphi(u)$ be a continuous and increasing real function, defined for $u \geqq 0$, and such that $\varphi(0) = 1$ and $\varphi(+\infty) = 2$. (For example, $\varphi(u) = 2 - e^{-u}$).

Set $\psi_K(y) = \varphi(\Delta_K(y) . \delta_K(y))$ and $f(K) = \int_{P(K)} \psi_K(y)\, dy$,

the integral being taken on the projection P(K) of K on the y-axis. This integral has a sense, for $\psi_K(y)$ is upper semi-continuous. Since $1 \leqq \psi_K \leqq 2$, $f(K)$ is clearly sub-additive; it is on the other hand increasing and continuous on the right; and we can add that $f(K) = 0$ for each compact K whose projection on $0y$ is of linear measure zero.

Now let A be the closed set $(x \geqq 0;\ 0 \leqq y \leqq 1)$.

$$\text{We have} \qquad f_*(A) = 1 \qquad \text{and} \qquad f^*(A) = 2.$$

In fact, $\psi_K(y) \equiv 1$ for each $K \subset A$, from which follows $f_*(A) = 1$ and on the other hand we have $f(\omega) = 2$ for each open set ω containing A, for there exist compacts $K \subset \omega$ such that $\psi_K(y) > 2 - \varepsilon$ for each arbitrarily given $\varepsilon > 0$.

32. **Capacitability of sets which are not K-borelian.** — In this section we shall give two examples.

32. 1. Example. — *The following is an example of a capacity* $f \geqq 0$, *alternating of order* $\mathfrak{A}_\infty$, *defined on the set* $\mathcal{K}(E)$ *of all sub-compacts of a* compact *space* E, *for which there exists a non-capacitable set* $A \subset E$ *which is at the same time a* $K \cap G$ *and a* G_δ.

Let X be the compact space obtained by adding the point of Alexandroff ω to a discrete space of cardinal number $2^{\aleph_0}$. Let Y be the segment [0, 1] and let $E = X \times Y$. For each compact $K \subset E$, let

$f(K)$ = the linear measure of the projection of K on Y.

Then f is indeed a capacity of order $\mathfrak{A}_\infty$.

Now by hypothesis there exists a 1-1 correspondence given by $y = \varphi(x)$ from $(X - \omega)$ onto Y. Designate by A the graph of φ (that is, the set of points $(x, \varphi(x))$ where $x \in (X - \omega)$). This set is of the form $K \cap G$; on the other hand, for each $\varepsilon > 0$, the set A_ε of points (x, y) such that $|y - \varphi(x)| < \varepsilon$ and $x \in (X - \omega)$ is open and so $A = \bigcap A_\varepsilon$ is also a G_δ.

Now each sub-compact of A is discrete, and hence finite, from which it follows that $f_*(A) = 0$. Each open set containing A projects onto Y; it follows that $f^*(A) = 1$.

Hence, A is not f-capacitable.

32. 2. Example. — *We shall now present an example of a capacity $f \geq 0$, alternating of order $\mathfrak{A}_\infty$ defined on the set $\mathcal{K}(E)$ of compacts of a* locally compact *space* E *and for which there exists a* closed *set* $A \subset E$ *which is not f-capacitable.*

It suffices to modify the preceding example by designating by X a discrete space of cardinal number $2^{\aleph_0}$. The space $E = X \times Y$ is locally compact and the graph A of φ is the required closed set.

32. 3. Remark. — These two examples show that the statements of the preceding theorems cannot be extended, without some restrictive hypothesis on the space E, to every element of the borelian field generated by the open and closed sets of E even when we impose on f the greatest regularity possible; examples of restrictive hypotheses on E which would be sufficient are the following; E is a complete, separable metric space; or E is compact and such that each open set G of E is a K_σ. Examples 32. 1. and 32. 2. justify the use of the K-borelian and K-analytic sets.

33. **Capacitability of sets CA.** — It is well known that, for each Radon measure μ, which is defined, for example, on the plane R^2, each set CA (that is to say the complement of an analytic set) is μ-measurable. We cannot state the same result for capacities however regular they may be. More precisely, we have the following theorem.

33. 1. Theorem. — If $E = R^2$ *and* $\mathcal{E} = \mathcal{K}(E)$, *the statement « there exists a capacity $f \geq 0$ of order $\mathfrak{A}_\infty$ on $\mathcal{E}$, and a* $CA \subset E$ *which is not f-capacitable » is not in contradiction with the ordinary axioms of set theory.*

Proof. According to a result of Novikov [1] which appears to have been previously stated without proof by Goedel, the statement « there exists on the real line R a projective set of class P_2 which is not measurable in the sense of Lebesgue » is not in contradiction with the ordinary axioms of the theory of sets (being admitted that these axioms are consistent).

Now let Δ be a straight line of $E = R^2$ and B a subset of Δ which is projective of class P_2 and is not measurable in the sense of Lebesgue. For each compact $K \subset E$, set $f(K)$ equal to the linear measure of the projection of K on Δ. It is a capacity which is ≥ 0, and it is of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$.

There exists [18] a subset $A \subset E$ whose projection on Δ is identical to B, and which is of class C_1, that is, the complement of an analytic set.

This set A cannot be f-capacitable, otherwise the set B would be measurable in the sense of Lebesgue, according to Theorem 22. 3. of Chapter v.

In what follows we shall make use of the fact that there even exists [19] in R^2 a set CA of interior f-capacity zero and whose orthogonal projection on Δ is identical to Δ :

Indeed, the projective set of Novikov is of class B_2; that is, the projective set and its complement are of class P_2. It follows easily that there exists a partition of Δ into two sets of class P_2 each of which has its interior measure zero and its exterior measure infinite.

Each of these two sets is the projection of a set, say A_i $(i = 1, 2)$, of R^2 which is of class CA, and we can always make them such that A_1 and A_2 are contained in two disjoint open sets. As a result of this precaution and since $f_*(A_1) = f_*(A_2) = 0$, we also have $f_*(A_1 \cup A_2) = 0$. The set $(A_1 \cup A_2)$, which is still of class CA, possesses the required property.

33. 2. **Consequence.** — It follows immediately that if, in $E = R^2$ for example, a set is measurable for each positive Radon measure, it is not necessarily capacitable for each capacity which is ≥ 0 and is of order $\mathfrak{A}_\infty$.

In the same line, we can set the following problem.

(18, 19) The words « there exists » are a convenient abbreviation for « there is no contradiction in supposing that there exists ».

33. 3. **Problem.** — If A is a subset of the plane $E = R^2$ (for example) which is of measure zero for each Radon measure without point masses, is A capacitable for each capacity $f \geq 0$ and of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$?

34. **Construction of non-capacitable sets for each sub-additive capacity.** — Let $\mathcal{E}$ be an additive and hereditary set of compacts of a Hausdorff space E, and let f be a sub-additive capacity (hence ≥ 0) on $\mathcal{E}$. (For example, f is ≥ 0 and of order $\mathfrak{A}_n$ with $n \geq 2$.

According to Lemma 17. 9., we have, for any A, $B \subset E$:

34. 1. $$f^* A \cup B) \leq f^*(A) + f^*(B).$$

Furthermore, let K be such that $K \subset (A \cup B)$ with $K \in \mathcal{E}$. For each open set ω such that $B \subset \omega$, we have

$$K = (K - \omega) \cup (K \cap \omega), \qquad \text{with} \qquad (K - \omega) \in \mathcal{E}.$$

Therefore

$$f(K) \leq f^*(K - \omega) + f^*(K \cap \omega)$$
$$= f(K - \omega) + f^*(K \cap \omega) \leq f_*(A) + f^*(\omega).$$

We can find a sequence (K_n, ω_n) such that $f(K_n) \rightarrow f_*(A \cup B)$ and $f(\omega_n) \rightarrow f^*(B)$. Passing to the limit, it follows that

34. 2. $$f_*(A \cup B) \leq f_*(A) + f^*(B).$$

We have, of course, an analogous formula by interchanging A and B.

Then let C be an f-capacitable set with $f(C) > 0$. If there exists a partition of C into two sets A, B such that $f_*(A) = f_*(B) = 0$, the inequality 34. 2. gives $f(C) \leq f^*(B)$; and since $B \subset C$, we have $f^*_*(B) = f(C) > 0$.

Similarly $$f^*(A) = f(C) > 0.$$

The sets A and B are therefore not f-capacitable.

Suppose now that C is a metrizable compact having the cardinal $2^{\aleph_0}$. By using the *axiom of choice*, we can easily partition C into two sets A and B such that each subcompact of C having the cardinal $2^{\aleph_0}$ intersects A and B. In other words, each subcompact of A or B will be at most denumerable.

Now if f is such that $f(K) = 0$ for each compact containing only one point, the sub-additivity of f implies $f(X) = 0$ for each X which is at most denumerable. Then if $f(C) > 0$, we have the following for the sets A and B :

$$f_*(A) = f_*(B) = 0 \qquad \text{and} \qquad f^*(A) = f^*(B) = f(C) > 0.$$

They are therefore not capacitable.

35. **Intersection of capacitable sets.** — We have stated previously, that for the capacities f of order $\mathfrak{A}_\alpha$, the intersection of two f-capacitable sets need not be f-capacitable. The reason for this is as follows. Let A be a set which is not f-capacitable and let B_1, B_2 be two disjoint sets such that $(A \cup B_1)$ and $(A \cup B_2)$ are f-capacitable; their intersection is identical to A, which is not f-capacitable.

Here are two examples where this construction is applicable.

35. 1. Example. — Let f denote the Newtonian capacity in the space $E = R^3$. We shall designate by A a bounded non-capacitable set (there exists such according to section 34) and by B_1, B_2 two disjoint concentric spheres each of which contains A. We have $f_*(A \cup B_i) = f^*(A \cup B_i) = f(B_i)$ according to the classical theory of potential; hence, $(A \cup B_1)$ and $(A \cup B_2)$ furnish the required example.

35. 2. Example. — Let $f(K)$ be defined on the set of compacts of the plane $E = R^2$ as follows: $f(K) =$ linear measure of the orthogonal projection of K on a straight line Δ of R^2. Let A again denote a bounded non-capacitable set (construct A by the method of section 34 or by using Theorem 33. 1.). This time B_1 and B_2 are two disjoint concentric circumferences containing A. It is immediate here that

$$f_*(A \cup B_i) = f^*(A \cup B_i) = f(B_i) \quad (i = 1,\ 2).$$

36. **Decreasing sequences of capacitable sets.** — In spite of the fact that for the capacities of order $\mathfrak{A}_\infty$, the intersection of two capacitable sets is not always capacitable, we could hope that the intersection A of a decreasing sequence of capacitable sets A_n is capacitable and that $\lim f(A_n) = f(A)$. Let us show that neither of these two results is correct.

Recall, for instance, Example 35. 2. The set A will still denote a bounded non-capacitable set. Let B_0 be the circumference of a circle of radius ρ containing A and let B_x be the circumference of a circle concentric to B_0 and of radius $(\rho + x)$ where $x > 0$. Denote by C_x the open annulus bounded by B_0 and B_x.

If we set $A_n = (A \cup C_{1/n})$ we have $A = \bigcap A_n$ since $\bigcap C_{1/n} = \emptyset$.

Now each of the sets A_n is f-capacitable, the sequence of A_n is decreasing, but their intersection is not f-capacitable.

On the other hand, we have $f(C_{1/n}) = 2\left(\rho + \frac{1}{n}\right)$ and $f(\emptyset) = 0$.

Hence it is not true that $\lim f(C_{1/n}) = f\left(\bigcap C_{1/n}\right)$ although the $C_{1/n}$ constitute a decreasing sequence of plane open sets, that is, sets of very regular topological structure.

We could easily construct an analogous example for the Newtonian capacity f in the space R^3.

37. Application of the theory of capacitability to the study of measure. — We shall give three examples of the application of the theory of capacitability to the study of measure.

37. 1. Example. — Let A be a borelian or analytic set in the plane $E = R^2$, and let Δ be a straight line in the plane. Let us suppose that the projection $(pr_\Delta A)$ of A on Δ has a non-zero linear measure. Since A is analytic, it is f-capacitable for the capacity f defined in Example 35. 2.

Therefore, for each $\varepsilon > 0$, A *contains a compact* K *such that*

$$mes^* pr_\Delta A - mes\ pr_\Delta K < \varepsilon.$$

This result can easily be improved in the sense that we can choose the compact K such that it contains at most one point on each straight line perpendicular to Δ; the projection then defines a homeomorphism between K and $(pr_\Delta K)$.

Notice that the same property cannot be demonstrated if we replace A by a set which is the complement of an analytic set, even if its projection on Δ is identical to Δ. This follows from the second example studied in section 33.

37. 2. Example. — More generally let A be a K-analytic and completely regular space, and let φ be a continuous map

of A into a locally compact space F on which there is defined a positive Radon measure μ.

There exist some compacts $K \subset A$ *such that* $\mu(\varphi(K))$ *approximates* $\mu(\varphi(A))$ *arbitrarily closely.*

The proof is entirely analogous to that of the preceding example.

37. 3. Example. — Let A be a K-analytic subset of a compact space E, and let $\mathcal{K}$ be a set of subcompacts of E each of which intersects A. Let us suppose that $\mathcal{K}$, in the topological space F of subcompacts of E, is μ-measurable for a certain Radon measure $\mu \geq 0$ on this space F, and that $\mu(\mathcal{K}) > 0$. Then for each $\varepsilon > 0$ there exists a subcompact $K \subset A$ such that, if $\mathcal{K}'$ denotes the set of elements of $\mathcal{K}$ which intersect K, we have $\mu(\mathcal{K}) - \mu(\mathcal{K}') < \varepsilon$.

The very simple proof uses the fundamental scheme of capacities of order $\mathcal{A}_\infty$.

38. The study of monotone capacities of order $\mathcal{M}_\alpha$. — We shall not make a direct study of capacities of order $\mathcal{M}_\alpha$, but we shall use the properties already established for the capacities of order $\mathcal{A}_\alpha$. Thanks to the notion of conjugate capacities which we introduced at the end of Chapter III (see 15. 6.), to each of the properties of capacities of order $\mathcal{A}_\alpha$ there corresponds a dual property for capacities of order $\mathcal{M}_\alpha$. This duality gives some substantial results only for capacities defined on a set of *closed* subsets of the space E, but this particular case appears to be sufficient for the study of capacities of order $\mathcal{M}_\alpha$.

38. 1. Theorem. — *Let* E *be a completely regular Hausdorff space, let* $\mathcal{E}$ *be an additive and hereditary class of sub-compacts of* E, *and let f be a capacity of order* $\mathcal{M}_\alpha$ ($\alpha \geq 1$, *b*) *defined on* $\mathcal{E}$ *with* (*sup f*) $< +\infty$.

(i) *Each* $A \subset E$ *such that* $(\widehat{E} - A)$ *is* K-*analytic for one of the compact extensions* [20] $\widehat{E}$ *of* E, *and such that* $\mathcal{K}(A) \subset \mathcal{E}$, *is f-capacitable.*

[20] It would be interesting to find general cases where this property (that $(\widehat{E} - A)$ is K-analytic) would be independent of the considered compact extension $\widehat{E}$. We find in Šneĭder [1], [2], some theorems in this sense, when the considered compact extensions have a certain character of denumerability.

(ii) *If in addition* $\mathcal{E}$ *is identical to the set* $\mathcal{K}(E)$ *of compacts of* E, *we have* $f_*\left(\bigcap A_n\right) = \lim f_*(A_n)$ *for each decreasing sequence of subsets* A_n *of* E (*property* $\mathcal{A}_{1,a}$); *and each denumerable intersection of f-capacitable sets is f-capacitable.*

Proof of (i). Let us designate by f_2 the extension of f to the set $\mathcal{K}(E)$ of compacts of E. According to Theorem 16. 3., for each $A \subset E$ such that $\mathcal{K}(A) \subset \mathcal{E}$, the f-capacitability of A is equivalent to its f_2-capacitability. In order to study the f-capacitability of such sets A, we can therefore suppose henceforth that $\mathcal{E} = \mathcal{K}(E)$.

The hypothesis on the class of f remains the same because according to Theorem 18. 11., since $\mathcal{K}(E)$ is G-separable and multiplicative, the extension of f is still of class $\mathcal{M}_\alpha$ and we still have $(\sup f) < \infty$.

Let $\widehat{E}$ be a compact extension of E. According to the remarks at the end of section 20, the extension of f to the set $\mathcal{K}(\widehat{E})$ is still of order $\mathcal{M}_\alpha$ since $\alpha \geqq (1, b)$, and the character of capacitability of the sets A which were considered remains unchanged in this new extension; their interior and exterior capacities also remain unchanged.

We are therefore brought back to the study of the much simpler case where the space E is compact and where $\mathcal{E} = \mathcal{K}(E)$. (We now use the notation E in place of $\widehat{E}$).

Then let $\bar{f}$ be the conjugate capacity of f which was defined at the end of Chapter III (see 15. 6.). This capacity is of order $\mathcal{A}_\alpha$ and is $> -\infty$. Therefore, according to Theorem 30. 1. above, if $(E - A)$ is a K-analytic set, $(E - A)$ is $\bar{f}$-capacitable, and thus A is f-capacitable.

Proof of (ii). If $\mathcal{E} = \mathcal{K}(E)$, we need only the second extension used above in order to reduce the proof to the case where E is compact. Now if A_n is an arbitrary sequence of subsets of E, their interior and exterior capacities and those of $A = \bigcap A_n$ remain unchanged in this extension.

We can therefore suppose that E is compact, and the conjugate capacity $\bar{f}$ allows us to interpret Corollary 28. 3. and to obtain the second part of the theorem.

38. 2. Corollary. — *If* E *is homeomorphic to a borelian* (*in the classical sense*) *subset of a complete, separable metric space,*

and if f is a capacity of order $\mathcal{M}_\alpha$ $(\alpha \geq (1, \mathrm{b}))$ *which is defined on the set* $\mathcal{K}(\mathrm{E})$ *of compacts of* E, *with* $(\sup f) < \infty$, *then each borelian subset of* E *or each set whose complement is* K-*analytic is f-capacitable.*

Indeed, there exists in this case an extension $\hat{\mathrm{E}}$ of E such that $\hat{\mathrm{E}}$ is compact and metrizable. If A is borelian in E, or has a complement $\complement \mathrm{A}$ which is analytic, the same holds in $\hat{\mathrm{E}}$. Hence, we can apply Theorem 38. 1.

38. 3. Remark. — Since the CA sets, whose topological nature is not well known, are those which are capacitable of order $\mathcal{M}_\alpha$, it follows that the capacities of order $\mathcal{M}_\alpha$ are in a certain sense less « natural » than capacities of order $\mathcal{A}_\alpha$.

Starting with these two classes, one can construct capacities with curious properties. For instance, if f is the sum of a capacity of order $\mathcal{A}_2$ and a capacity of order $\mathcal{M}_2$ on the set of subcompacts of $\mathrm{E} = \mathrm{R}^2$, for example, every borelian set $\mathrm{A} \subset \mathrm{E}$ is f-capacitable, but it is possible to construct f in such a way that « there exist » analytic sets and sets CA which are not f-capacitable.

38. 4. Remark. — The following will show that *the restriction* $(\sup. f) < \infty$ *is essential in the preceding theorem.*

Let $\mathrm{E} = \mathrm{R}^2$, and let $x'x$, $y'y$ be two perpendicular axes in E. For each compact $\mathrm{K} \subset \mathrm{E}$, let $\mu_x(\mathrm{K})$ and $\mu_y(\mathrm{K})$ be the linear measure of the intersection of K with $x'x$ and $y'y$ respectively.

$$\text{Set} \quad f(\mathrm{K}) = \mu_x(\mathrm{K}) . \mu_y(\mathrm{K}).$$

Then f is a capacity of order $\mathcal{M}_\infty$. The continuity on the right is obvious. Let us now set $\mathrm{K}_x = \mathrm{K} \cap (x'x)$ and $\mathrm{K}_y = \mathrm{K} \cap (y'y)$. The applications $\mathrm{K} \to \mathrm{K}_x$ and $\mathrm{K} \to \mathrm{K}_y$ are $\cap$-homomorphisms and so is the application $\mathrm{K} \to \mathrm{K}_x \times \mathrm{K}_y$.

Now if ν denotes the Lebesgue measure in R^2, we have

$$f(\mathrm{K}) = \mu_x(\mathrm{K}) . \mu_y(\mathrm{K}) = \nu(\mathrm{K}_x \times \mathrm{K}_y).$$

Since ν is of order $\mathcal{M}_\infty$, then f is also.

Now if A is the straight line $x'x$, we have $f_*(\mathrm{A}) = 0$. However, for each open set ω containing A, we have $f(\omega) = +\infty$, and hence $f^*(A) = +\infty$. Then A is not capacitable, although A and its complement are very simple borelian sets.

Observe that if E is a compact space, if $\mathcal{E} = \mathcal{K}(E)$, and if $f(K) \neq +\infty$ for each $K \in \mathcal{E}$, we have $(\sup. f) \leq f(E) < +\infty$.

38. 5. Remark. — For capacities of order $\mathcal{A}_\alpha$, we do not have $f_*\left(\bigcap A_n\right) = \lim f_*(A_n)$ for each decreasing sequence of sets A_n. Similarly, we do not have $f^*\left(\bigcup A_n\right) = f^*(A_n)$ for every increasing sequence of A_n when f is of order $\mathcal{M}_\alpha$.

The following is an example in which the capacity f and the class $\mathcal{E}$ are, however, exceptionally regular. The set E is the segment [0, 1] and $\mathcal{E} = \mathcal{K}(E)$. We set $f(K) = 0$ except when $K = E$ ($f(E) = 1$). This capacity is of order $\mathcal{M}_\infty$ and every subset of E is capacitable. However, if A_n is a strictly increasing sequence of compacts of E such that $\bigcup A_n = E$, we have

$$\lim f(A_n) = 0 \qquad \text{and} \qquad f\left(\bigcup A_n\right) = 1.$$

CHAPTER VII

EXTREMAL ELEMENTS OF CONVEX CONES AND INTEGRAL REPRESENTATIONS. APPLICATIONS

39. **Introduction.** — We propose to study some convex cones whose elements are real-valued or vector-valued functions, to find their extremal elements, and to use these for integral representation of the elements of these cones.

These representations will furnish in certain cases a simple geometric interpretation of the elements being studied, and they will enable us to show their relations with other problems.

Throughout this chapter the vector spaces under consideration are assumed to be spaces over the real field R, and this fact will not be mentioned again. The same assumption is made for all cones.

Let us first recall a few classical definitions and results (see also Bourbaki [4]).

39. 1. **Extreme points and extremal elements.** — Let $\mathcal{L}$ be a vector space and $\mathcal{C}$ a convex subset of $\mathcal{L}$. We shall say that $a \in \mathcal{C}$ is an *extreme point* of $\mathcal{C}$ if no open segment of $\mathcal{C}$ contains a.

Now let $\mathcal{C}$ be a convex *cone* in $\mathcal{L}$ which contains no straight line passing through the origin. If $\mathcal{H}$ is an affine subspace of $\mathcal{L}$, which does not contain 0 and which meets every *ray* of $\mathcal{C}$, then $a \in \mathcal{C} \cap \mathcal{H}$ is an extreme point of $\mathcal{C} \cap \mathcal{H}$ if and only if the equation

$$a = a_1 + a_2 \qquad \text{with } a_1 \text{ and } a_2 \in \mathcal{C},$$

implies

$$a_1 = \lambda_1 a \qquad \text{and} \qquad a_2 = \lambda_2 a,$$

where λ_1 and λ_2 are non-negative.

Such an element $a \in \mathcal{C}$ is called an *extremal element* of the cone $\mathcal{C}$; obviously every λa $(\lambda > 0)$ is then also an extremal element of $\mathcal{C}$.

39. 2. **Theorem of Krein and Milmam.** — *If the vector space $\mathcal{L}$ is a locally convex Hausdorff space, and $\mathcal{C}$ a convex, compact subset of $\mathcal{L}$, then the set $e(\mathcal{C})$ of all extreme points of $\mathcal{C}$ has a convex hull whose closure is $\mathcal{C}$.*

In other words, if $x \in \mathcal{C}$, then there exists for every neighborhood V of x a finite number of positive point masses located at extreme points of $\mathcal{C}$ and having their center of gravity in V.

The set $e(\mathcal{C})$ is not necessarily compact, If it is compact, then the preceding theorem can be sharpened as we shall see.

39. 3. **Center of gravity.** — Let $\mathcal{C}$ be a convex compact subset of a space $\mathcal{L}$, and μ a positive Radon measure on $\mathcal{C}$.

It is possible to find an ultra-filter, weakly converging to μ, on the set of all elementary positive Radon measures μ_i defined on $\mathcal{C}$, each of which consists of a finite number of point masses. The centers of gravity $G(\mu_i)$ of these measures are in the compact set $\mathcal{C}$; hence, they converge with respect to the given ultra-filter to a point G of $\mathcal{C}$.

Let us show that G is unique. We have, for every continuous linear functional $l(x)$ on $\mathcal{L}$,

$$l(G(\mu_i)) \int d\mu_i = \int l(x)\, d\mu_i.$$

Since $l(x)$ is continuous, we obtain

$$l(G) \int d\mu = \int l(x)\, d\mu. \tag{1}$$

Now, if $l(G') = l(G)$ for every l, then $G' = G$. Thus G is well-defined by (1), which is sometimes written as

$$G \int d\mu = \int x d\mu.$$

In particular, let us suppose that $\mathcal{L}$ is the space of all real-valued functions $x = x(t)$ defined on a set E. We shall topologize $\mathcal{L}$ by means of the topology of simple convergence; that is, the point $x = 0$ is assumed to possess a neighborhood basis of the form $V(\varepsilon, t_1, \ldots, t_n)$ consisting of all points x for which $|x(t_i)| < \varepsilon$ $(i = 1, 2, \ldots, n)$. This space $\mathcal{L}$ is a locally convex Hausdorff space.

For every $t \in E$, the function $l(x) = x(t)$ is a linear continuous function on $\mathfrak{L}$. Hence, with the preceding notations, and designating by $x_0(t)$ the center of gravity of a measure μ on $\mathcal{C}$, we have

$$x_0(t) \int d\mu = \int x(t)\, d\mu \qquad \text{for every} \quad t \in E.$$

39. 4 THEOREM. — *If the vector space $\mathfrak{L}$ is a locally convex Hausdorff space, and if $\mathcal{C}$ is a convex compact subset of $\mathfrak{L}$, then for every $x_0 \in \mathcal{C}$ there exists a measure $\mu_0 \geq 0$ on $\overline{e(\mathcal{C})}$ whose center of gravity is x_0.*

Proof. For every neighborhood V of x, there is a measure μ_i on $e(\mathcal{C})$ of total mass 1, which consists of a finite number of point masses, and which has its center of gravity $G(\mu_i)$ in V. Hence, there exists an ultra-filter on the set of these μ_i such that the associated $G(\mu_i)$ converge to x_0. But, on the other hand, the measures μ_i converge weakly to a measure μ_0 whose support is $\overline{e(\mathcal{C})}$. The total mass of μ_0 is 1, and its center of gravity is indeed x_0.

If $e(\mathcal{C})$ is closed, $e(\mathcal{C})$ is obviously the support of μ.

39. 5. REMARK. — It would be interesting to know whether it is always possible to impose on the measure μ_0 the condition that its support be $e(\mathcal{C})$, in other words, that $[\mathcal{C} - e(\mathcal{C})]$ have μ_0-measure zero.

It should be observed that if $\mathfrak{L}$ is a normed vector space, then $e(\mathcal{C})$ is a G_δ. In the general case, little is known concerning the topological character of $e(\mathcal{C})$.

39. 6. APPLICATION. — Suppose that $\mathfrak{L}$ is the vector space of all real-valued functions defined on a space E, with the topology of simple convergence.

Let $\mathcal{C}$ be a convex cone of $\mathfrak{L}$, and assume that there exists a point $t_0 \in E$ such that $x(t_0) > 0$ for every $x \in \mathcal{C}$.

We designate by $\mathcal{C}_1$ the set of all normalized elements of $\mathcal{C}$, that is, the set of all $x \in \mathcal{C}$ for which $x(t_0) = 1$. We further designate by $e(\mathcal{C}_1)$ the set of all extreme points of $\mathcal{C}_1$.

If $\mathcal{C}_1$ is compact, then the above theorem shows that for every $x \in \mathcal{C}$ there exists a measure $\mu \geq 0$ on $\overline{e(\mathcal{C}_1)}$ such that

$$x(t) = \int x_e(t)\, d\mu\,(e) \qquad \text{for every} \quad t \in E.$$

In almost all the cases which we shall study, $e(\mathcal{C}_1)$ will be a compact set.

39. 7. **Uniqueness of the measure μ associated with an element $x \in \mathcal{C}$.** — Suppose again that $\mathcal{L}$ is locally convex, $\mathcal{C}$ a convex cone in $\mathcal{L}$ which contains no straight line passing through the origin, and $\mathcal{H}$ a closed linear variety in $\mathcal{L}$ which does not contain 0 and meets every ray of $\mathcal{C}$.

Let us assume that $\mathcal{C}_1 = \mathcal{C} \cap \mathcal{H}$ is compact and has the property (even in the case where $e(\mathcal{C}_1)$ is not compact) that there exists, for every $x \in \mathcal{C}_1$, *one and only one* measure of total mass 1 whose support is $e(\mathcal{C}_1)$ and whose center is x.

Then there exists, for every $x \in \mathcal{C}$, one and only one measure μ on $e(\mathcal{C}_1)$ for which $x = \int x_e \, d\mu(e)$. We shall denote this integral by $x(\mu)$.

This correspondence between the measures $\mu \geqq 0$ on $e(\mathcal{C})$ and the points of $\mathcal{C}$ is one-to-one, and since, moreover, $x(\mu_1 + \mu_2) = x(\mu_1) + x(\mu_2)$, and $x(\lambda\mu_1) = \lambda x(\mu_1)$ for $\lambda \geqq 0$, this correspondence is an isomorphism between the order structure of the set of the $\mu \geqq 0$ defined on $e(\mathcal{C}_1)$ and the order structure of $\mathcal{C}$ associated with

$$\mathcal{C}(a \prec b \text{ if } b = a + c).$$

Since the ordered set of the $\mu \geqq 0$ is a lattice, the ordered cone $\mathcal{C}$ is also a lattice. We can therefore state the following result :

39. 8. Theorem. — *If there exists a unique integral representation of the points of $\mathcal{C}$ by means of a measure on $e(\mathcal{C}_1)$, then the ordered cone $\mathcal{C}$ is a lattice.*

The fact that $\mathcal{C}$ is a lattice may be interpreted geometrically as follows : if $\mathcal{C}_1'$ and $\mathcal{C}_1''$ are the sets obtained from $\mathcal{C}_1$ by means of two positive homotheties, (with arbitrary centers) then the set $\mathcal{C}_1' \cap \mathcal{C}_1''$ is either empty or homothetic to $\mathcal{C}_1$ under a positive homothety.

The necessary condition for uniqueness given in the preceding theorem makes it often possible to determine a priori cases where uniqueness is lacking. It would be very interesting to know if the above condition is both necessary and sufficient for the existence and uniqueness of the integral representation.

39. 9. **Examples of cones which are lattices.** — (1) Let E be an ordered set, and $\mathfrak{I}$ the cone of all non-negative increasing real-valued functions defined on E. For any two elements f_1 and f_2 of $\mathfrak{I}$, the set of all $f \in \mathfrak{I}$ such that $f_1 \prec f$ and $f_2 \prec f$ has a smallest element f_0 in the sense that $f_0(x) \leqslant f(x)$ for every $x \in$ E, but in general it is not true that $f_0 \prec f$, so that $\mathfrak{I}$ is not a lattice.

But when E is totally ordered, $\mathfrak{I}$ is a lattice.

(2) E is a Greenian domain of R^n and $\mathcal{C}$ is the cone of real positive functions which are super-harmonic in E. The cone $\mathcal{C}$ is a lattice. It follows immediately that the sub-cone of $\mathcal{C}$ which consists of the positive and harmonic functions is also a lattice. The extremal elements of $\mathcal{C}$ are the multiples $\lambda G(P_0, Q)$ of the Green's function with pôle P_0 and certain limits of $\lambda G(P_0, Q)$ obtained by letting P_0 tend toward the frontier of E.

The set of normalized extremal elements is not compact in general; nevertheless, the integral representation by means of extremal elements exists and is unique (see Martin [1]).

(3) When $n = 1$ in example (2), $\mathcal{C}$ is identical to the set of positive and concave functions on an interval (a, b) of R. We might believe, more generally, that if E is a convex set of R^n and $\mathcal{C}$ is the set of all positive and concave functions on E, then $\mathcal{C}$ is a lattice. This is not true.

For example, let E be the circle $x^2 + y^2 \leqq 1$ of R^2, and let $f_1 \equiv 1 - x$, $f_2 \equiv 1 + x$. If $f_1 \smile f_2$ did exist, we should have $f_1 \smile f_2 \leqq f =$ inf. (linear functions greater than f_1 and f_2 on E). Now we have also $f =$ inf. (elements of $\mathcal{C}$ greater than f_1 and f_2 on E.). We would therefore have $f_1 \smile f_2 = f$. But then for each linear function $l \in \mathcal{C}$ such that $f_1, f_2 \leqq l$, since this implies $f_1, f_2 \prec l$, we would have $l = f + g$ where $g \in \mathcal{C}$. Since f is not linear, this equality is impossible.

Then the integral representation for $\mathcal{C}$ is not unique, as will be verified in the following particular case : It is immediate that the functions $(1-x)$, $(1+x)$, $(1-y)$, $(1+y)$ are extremal elements. Now $2 \equiv (1-x) + (1+x)$ and $2 \equiv (1-y) + (1+y)$. This proves the non-uniqueness of the representation of the function $f \equiv 2$.

40. Extremal elements of the cone of positive increasing functions. — If E and F are two ordered sets, each homomorphism of E into F is called an increasing application of E into F. If we wish to define the sum of two such applications and the product of one of these applications by a real constant, we are led to suppose that F is a vector space on the field R.

In order to obtain substantial results, we shall suppose moreover that F is a vector lattice on the field of reals, and that E is filtering on the right. We then have the following statement.

40. 1. THEOREM. — *Let $\mathfrak{J}$ be the convex cone of the positive and increasing applications f of a set* E, *which is ordered and filtering on the right, into a vector lattice* F. *The set of extremal elements of $\mathfrak{J}$ is identical with the set of elements f of $\mathfrak{J}$ which, besides the value* 0, *take at most only one value which is $\neq$* 0 *and is extremal on* F_+; *any such function f is of the form $f_{A,b}(x)$ where* A *is a subset of* E *which is hereditary to the left* ([21]), *and b is an extremal element on the cone* F_+ *of elements* $\geqq 0$ *of* F, *with*

$$f_{A,b}(x) = \begin{cases} 0 & \text{for} \quad x \in A, \\ b & \text{for} \quad x \in \complement A. \end{cases}$$

Proof. It is immediate that the set of positive and increasing applications f of E into F is a convex cone. It is likewise immediate that there is identity between the elements $f \in \mathfrak{J}$ such that $f(E)$ contains besides 0 only one extremal element of F_+ and the set of $f_{A,b}$.

Then let f be a function of the form $f_{A,b}(x)$.

Suppose that $f = f_1 + f_2$, where f_1 and $f_2 \in \mathfrak{J}$.

For each $x \in A$, we have $0 = f_1(x) + f_2(x)$ and thus

$$f_1(x) = f_2(x) = 0.$$

Let u and $v \in \complement(A)$; for each $w \geqq u$ we have

$$f_1(u) + f_2(u) = b = f_1(w) + f_2(w),$$

and thus

$$[f_1(w) - f_1(u)] + [f_2(w) - f_2(u)] = 0.$$

([21]) That is, such that $(x' \leqq x$ and $x \in A) \Longrightarrow (x' \in A)$.

It follows that

$$f_1(u) = f_1(w) \quad \text{and} \quad f_2(u) = f_2(w).$$

Now since E is filtering on the right, there exists a w greater than u and v. It follows therefore that

$$f_1(u) = f_1(v) \quad \text{and} \quad f_2(u) = f_2(v).$$

In other words, f_1 and f_2 on $\complement A$ take constant values b_1 and b_2 with $b = b_1 + b_2$. Moreover, since b is an extremal element of F_+, we have $b_1 = \lambda_1 b$ and $b_2 = \lambda_2 b$ where λ_1 and λ_2 are two real numbers ≥ 0. We have therefore $f_1 \equiv \lambda_1 f$ and $f_2 \equiv \lambda_2 f$ which shows that f is extremal.

Conversely, suppose now f is an extremal element of $\mathfrak{J}$. If $f(E)$ contains besides 0 only a single element $b \neq 0$, it is clear that f cannot be extremal if b is not extremal on F_+. Let us show that if $f(E)$ contains at least two elements b and c different from zero, f is not extremal.

We can always suppose $b < c$; for if $f(u)$ and $f(v)$ are two distinct elements of $f(E)$, there exists $w \in E$ such that u and $v \leq w$ and hence $f(u)$ and $f(v) \leq f(w)$. In other words, $f(E)$ contains two distinct comparable elements, $(f(u), f(w))$ or $(f(v), f(w))$.

Then let $f_1 \equiv \inf(f, b)$; $f_2 \equiv \sup(f, b) - b$.

We have $f \equiv f_1 + f_2$ by virtue of the identity

$$\inf(y, b) + \sup(y, b) = y + b.$$

Now $\inf(y, b)$ and $\sup(y, b)$ are two increasing functions of y; therefore f_1 and f_2 belong to $\mathfrak{J}$. Since $\sup(c, b) - b \neq 0$, f_2 is not identically zero. But we cannot have $f_2 \equiv \lambda_2 f$ since we have $f_2 = 0$ when $f = b$. Hence the decomposition $f = f_1 + f_2$ shows that f is not extremal.

40. 2. Remark. — It is sufficient to reverse the order in E in order to obtain a characterization of the extremal elements of the set of positive and *decreasing* applications into F of a set E which is ordered and filtering on the left.

40. 3. Remark. — If there exists on E a topology compatible with its structure of ordered set filtering on the right,

and on F a topology compatible with its structure of vector lattice, we can, instead of studying the cone $\mathfrak{J}$, study the subcone $\mathfrak{J}'$ of applications f which are also continuous on the right on E (in an obvious sense). A reasoning formally identical to the preceding furnishes as extremal elements of $\mathfrak{J}'$ the elements $f_{A,b}$ which are continuous on the right, that is, those for which A is a subset of E which is open on the right and hereditary on the left.

40. 4. **Integral representation of real-valued positive and increasing functions.** — We shall now suppose that E possesses a largest element ω, and that F is the real line R. Let $\mathfrak{J}_1$ be the set of elements f of $\mathfrak{J}$ such that $f(\omega) = 1$. It is seen immediately that $\mathfrak{J}_1$ is, in the vector space of real valued functions on E (with the topology of simple convergence) a convex and compact subset. On the other hand, the subset $e(\mathfrak{J}_1)$ of extreme points of $\mathfrak{J}_1$ is identical with the set of extremal elements $f = f_{A,1}$ of $\mathfrak{J}$ such that $f(\omega) = 1$; hence $e(\mathfrak{J}_1)$ is compact. According to the Application in 39. 6., it follows that *for each* $f \in \mathfrak{J}$, *there exists a positive Radon measure* μ *on* $e(\mathfrak{J}_1)$ *such that*

$$f(x) \equiv \int f_{A,1}(x)\, d\mu\, (A).$$

We can easily extend this result to the case where E does not possess a largest element, when we limit the study to the elements f of $\mathfrak{J}$ which are bounded on E. It is sufficient to extend these functions f to the set $\widehat{E}$ which results from E by adding to it a largest element ω.

40. 5. Example. — If E is the interval $[0, 1]$ of the real line, with the usual order, the extremal elements of $\mathfrak{J}_1$ are of the form $f_{A,1}$ with A = set of $x < a$ or set of $x \leqq a$, $(0 \leqq a \leqq 1)$. It is easy to see that $e(\mathfrak{J}_1)$ is homeomorphic to the set of all A with the topology of order (the order here being defined by the relation of inclusion, $A_1 \subset A_2$).

40. 6. **Interpretation of the formula** $f(x) = \int f_{A,1}(x)\, d\mu$. — The class of the $f_{A,1}$ can be identified with the class of subsets A of E which are hereditary on the right or with the class of

their complements A′ which are indeed the hereditary on the right subsets of E. Now

$$f_{A,1} = \begin{cases} 1 & \text{when} \quad x \in A', \\ 0 & \text{when} \quad x \in A. \end{cases}$$

Therefore, $f(x)$ is just the μ-measure of the compact set of those A′ which contain x. If the set of A′ is ordered by inclusion, and if we denote by $A'(x)$ the set A′ of points of E greater than x, we can still say that $f(x)$ is the μ-measure of the set of all $A' \succ A'(x)$.

40. 7. **Uniqueness of the measure μ.** — When E is the interval [0, 1] the cone $\mathfrak{J}$ is a lattice and it is well known that there is a unique measure μ determined by an increasing fonction f on E.

When E is the ordered set R^2_+ it is no longer true. For instance let $f_1(\xi, \eta) = 0$ if $\xi < 1$ and $f_1 = 1$ elsewhere; let $f_2(\xi, \eta) = 0$ if $\eta < 1$ and $f_2 = 1$ elsewhere. Those functions are extremal elements of $\mathfrak{J}$; however the function $f = f_1 + f_2$ has another representation in terms of extremal elements: $f = f_3 + f_4$ where $f_3 = 0$ if $\xi < 1$ and $\eta < 1$ and $f_3 = 1$ elsewhere; $f_4 = 0$ if $\xi < 1$ or $\eta < 1$ and $f_4 = 1$ elsewhere.

41. **Extremal elements of the cone of positive and increasing valuations on a distributive lattice.** — Let E be a distributive lattice and F an ordered vector space. Recall that a valuation f of E into F is an application of E into F such that

$$f(a \smile b) + f(a \frown b) = f(a) + f(b).$$

It is clear that these valuations constitute a vector space. We shall designate by $\mathfrak{V}$ the convex cone of valuations of E into F which are *positive* and *increasing*.

41. 1. Theorem. — *The set of extremal elements of $\mathfrak{V}$ is identical with the set of functions of the form $f_{P,\lambda}$, where* P *denotes a partition of* E *into two sub-lattices* $E_1(P)$ and $E_2(P)$ *with* $E_1(P)$ *hereditary on the left,* $E_2(P)$ *hereditary on the right, and where* λ *is an extremal element of the cone* F_+ *of positive elements of* F, *with*

$$f_{P,\lambda}(x) = \begin{cases} 0 & \text{if} \quad x \in L_1 \\ \lambda & \text{if} \quad x \in L_2. \end{cases}$$

Proof. (*a*) We easily demonstrate at once the identity between the functions $f \in \mathcal{V}$ which take only the values 0 and λ, and the functions of the form $f_{P,\lambda}$. Then, exactly in the same manner as in the preceding theorem, we show that each element $f_{P,\lambda}$ is extremal for $\mathcal{V}$.

(*b*) Conversely, suppose that f is an extremal element of $\mathcal{V}$. If $f(E)$ contains, other than zero, only one element $\lambda \neq 0$, it is clear that f cannot be extremal if f is not extremal on F_+. Let us show that if $f(E)$ contains at least two elements λ and μ which are different from zero, then f is not extremal. As in the proof of Theorem 40. 1. we can always suppose $\lambda < \mu$. Let a and b be two points of E such that $f(a) = \lambda$ and $f(b) = \mu$. Let

$$f_1(x) = f(x \smile a) - f(a); \qquad f_2(x) = f(x \frown a).$$

We clearly have f_1 and $f_2 \geq 0$ and f_1, f_2 increasing. On the other hand, since E is distributive, we verify easily that f_1 and f_2 are two valuations. We have therefore f_1 and $f_2 \in \mathcal{V}$ and $f = f_1 + f_2$.

Now, $f_1(a) = 0$ and $f_1(b) \neq 0$. Since $f(a) \neq 0$, we cannot have $f_1 = \lambda_1 f$ where λ_1 is a constant. This fact shows that f is not extremal.

41. 2. Remark. — We can remark, as for the theorem of 40. 1., that if E and F possess topologies compatible with their structures, the extremal elements of the sub-cone $\mathcal{C}' \subset \mathcal{C}$ made up of the continuous on the right elements of $\mathcal{C}$ are those of the functions $f_{P,\lambda}$ which are continuous on the right. In fact, in the preceding proof f_1 and f_2 are continuous on the right if f is continuous on the right.

41. 3. **Integral representation and interpretation.** — When F is the real line R, we obtain for the integral representation of the elements of $\mathcal{V}$ results quite analogous to those relative to the cone $\mathcal{J}$, either when E possesses a greatest element or more generally when the function f that we wish to represent is bounded on E. This integral representation results from the fact that the set of normalized extremal elements $f_{P,1}$ of $\mathcal{V}$ is compact for the topology of simple convergence. The formula

$$f(x) = \int f_{P,1}(x)\, d\mu(P),$$

valid for each x, shows that $f(x)$ is equal to the μ-measure of the set of partitions P such that $x \in L_2(P)$.

It would be interesting to know whether $\mathcal{V}$ is a lattice and whether the integral representation of the elements of $\mathcal{V}$ is unique.

42. **Application to the integral representation of simply additive measures.** — Let E be an algebra of subsets of a given set A; that is, if $X_1 \in E$ and $X_2 \in E$, we have $X_1 \cup X_2 \in E$ and $\complement(X_1) \in E$. The set E is a distributive lattice when E is ordered by the inclusion relation on A.

Let $\mathcal{M}_a$ be the set of positive and simply additive measures on E. Each of these measures is clearly a positive and increasing valuation on E. Conversely, each positive valuation f on E is of the form $f(\emptyset) +$ (a measure); in fact, $[f - f(\emptyset)] = g$ is a positive valuation on E with $f(\emptyset) = 0$. Now

$$g(X \cup Y) + g(\emptyset) = g(X) + g(Y) \quad \text{when} \quad X \cap Y = \emptyset;$$

it follows that.

$$g(X \cup Y) = g(X) + g(Y).$$

The extremal elements of $\mathcal{M}_a$ are, therefore, within a positive factor, the measures f on E which take only the values 0 and 1.

The set $\mathcal{M}_{a_1}$ of measures f on E such that $f(A) = 1$ and its subset $e(\mathcal{M}_{a_1})$ of measures with values 0 or 1 are compact for the topology of simple convergence.

Therefore, to each $f \in \mathcal{M}_{a_1}$ is associated a Radon measure $\mu \geqq 0$ on $e(\mathcal{M}_{a_1})$ such that

$$f(X) = \int f_e(X)\, d\mu(e) \qquad \text{for each} \qquad X \in E.$$

Let us study the extremal elements of $\mathcal{M}_{a_1}$. For each $f \in e(\mathcal{M}_{a_1})$ let $B(f)$ be the set of elements of E such that $f(X) = 1$. This set constitutes a base of a filter on A such that if $X \in B(f)$, then whenever two elements X_1, X_2 of E form a partition of X, one of them belongs to $B(f)$.

Conversely, if B is a base of a filter on A, made up of elements of E and possessing this last property, we say that B is *saturated* relative to E. To each B we can associate a function f_B on E by placing

$$f(X) = \begin{cases} 1 & \text{when X belongs to the filter of base B,} \\ 0 & \text{otherwise.} \end{cases}$$

It is seen immediately that $f_B \in e(\mathcal{M}_{a_1})$, and that $B(f_B) = B$. *We have then established a canonical and one-to-one correspondence between the normalized extremal measures on* E *and the bases of filters on* A *which are saturated relative to* E.

42. 1. Remark. — The property which defines the saturation of B resembles the property which characterizes the ultra-filters, and it is actually identical to it when $E \neq 2^A$. However, if $E \neq 2^A$, there exists some bases of filter B saturated relative to E and which are not bases of ultra-filters.

We are now going to make a more detailed study of $\mathcal{M}_a$ when $E = 2^A$.

42. 2. Extremal elements of the cone of positive measures on $E = 2^A$. — By using a method of Stone [1], [2], [3] with a slightly different language, we are going to show how one can interpret the extremal elements of $\mathcal{M}_{a_1}$ and represent each element f of $\mathcal{M}_a$.

The following could be extended to the case where E is an arbitrary algebra of subsets of A, but with a more complicated formulation.

(1) Extremal elements of $\mathcal{M}_{a_1}$. The bases of filters on A satured relative to E are identical with the ultra-filters on A. Therefore, the extremal elements of $\mathcal{M}_{a_1}$ are the functions $f_u(X)$ where u is an ultra-filter on A, with

$$f_u(X) = \begin{cases} 1 & \text{when } X \in u, \\ 0 & \text{when } X \notin u. \end{cases}$$

For example, for each $x_0 \in E$, the ultra-filter u_{x_0} of the sets containing x_0 corresponds to the point measure $f_{u_{x_0}} = \varepsilon_{x_0}$.

(2) Topology on the space U of ultra-filters. By definition this topology would be the topology of simple convergence on the set of associated measures f_u, The space U is therefore compact. For each $X \subset A$, let $\omega(X)$ be the set of ultra-filters on X. It is immediately seen that for each $u_0 \in U$

the set of $\omega(X)$, where $X \in u_0$, constitutes a base of neighborhoods of u_0.

In particular, for each $x_0 \in A$ the ultra-filter u_{x_0} possesses a base of neighborhoods formed by u_{x_0} itself. Hence u_{x_0} is isolated in U; conversely, each isolated element of U is of this form.

For each $X \subset A$, the set of ultra-filters on X is compact; hence, $\omega(X)$ is compact and so is $\omega(A - X)$. Now each ultra-filter on A is supported by either X or $(A - X)$; therefore $\omega(X)$ and $\omega(A - X)$ constitute a partition of U. Hence, the sets $\omega(X)$ are both open and closed.

In particular, each point of U possesses a base of neighborhoods of the form $\omega(X)$, and hence both open and closed. It follows conversely that each subset of U which is both open and closed can be written uniquely in the form $\omega(X)$.

For each open set $\Omega \subset U$, let I be the set of isolated points of Ω. Since each point of U is the limit of isolated points, we have $\bar{I} = \bar{\Omega}$. Now let X_I be the canonical image of I in A. We have $\bar{\Omega} = \bar{I} = \omega(X_I)$. Therefore, the closure of each open set is an open set.

The points of U represent the ultra-filters on A. Let us see how the filters on A are represented in U. Let $\mathcal{F}$ be a filter on E; there corresponds to it the filtering decreasing set of open and closed sets $\omega(X)$ where $X \in \mathcal{F}$. Set $\varphi = \bigcap_{X \in \mathcal{F}} \omega(X)$.
We therefore have associated to $\mathcal{F}$ the closed, non-empty $\varphi_{(\mathcal{F})}$ of U. Conversely let φ be any closed, non-empty set of U. The set of its open and closed neighborhoods has a canonical image in E which is clearly a filter which we denote by $\mathcal{F}(\varphi)$. It is seen immediately that $\mathcal{F}(\varphi(\mathcal{F}_0)) = \mathcal{F}_0$ for each filter $\mathcal{F}_0$ on A. *We have therefore established a canonical and one-to-one correspondence between the filters on* E *and the closed sets of* U.

In this correspondence, the intersection of a family of filters on A corresponds to the closure of the union of the corresponding closed sets in U; the upper bound of a family of filters on A corresponds to the intersection (assumed to be non-empty) of the corresponding closed sets in U.

We shall see later, in the study of capacities of order $\mathfrak{A}_\infty$, which topology it is natural to define on the set of filters on E.

42. 3. **Integral representation of a measure on $E = 2^A$.** — According to the general theorem 39. 4., there exists for each measure $f \in \mathcal{M}_a$ a positive Radon measure μ on U such that

$$f(X) = \int f_u(X)\, d\mu(u) \qquad \text{for each} \qquad X \subset E.$$

In other words, for each X we have

$f(X) = \mu$-measure of the set of u supported by X; that is, $f(X)$ is the measure $\mu[\omega(X)]$ of the image in U of the filter of the supersets of X.

This Radon measure on U is well defined for each open and closed set; since these sets constitute a base of open sets in U, this measure μ is *unique*.

Conversely, to each Radon measure μ on U is associated a measure f, which is simply additive on E, by the relation $f(X) = \mu[\omega(X)]$.

Each measure f on E can be extended to the set of filters on A by setting

$$f(\mathcal{F}) = \inf_{X \in \mathcal{F}} f(X) = \inf_{X \in \mathcal{F}} \mu[\omega(X)] = \mu[\varphi(\mathcal{F})]$$

for each filter $\mathcal{F}$ on A. Then $f(\mathcal{F})$ is just the μ-measure of its image $\varphi(\mathcal{F})$ in U.

In this interpretation the fact that an additive measure f on E is not completely additive follows from the fact that when $\omega_1, \ldots, \omega_n, \ldots$, is a sequence of not empty open and closed subsets of U which are mutually disjoint, we always have $\overline{\bigcup \omega_n} \neq \bigcup \omega_n$ and hence, in general, different μ-measures for these two sets.

We have defined on the space of measures f on E the topology of simple convergence on $E = 2^A$. Now these measures are in one-to-one correspondance with the positive Radon measures on U; hence, there exists a topology on the set of these measures μ. As the images $\omega(X)$ of the elements of E constitute a base of open and closed sets of U, this topology on the space of Radon measures on U is identical with the classical topology of vague convergence (see Bourbaki [3]).

43. Extremal elements of the cone of positive functions alternating of order ∞ on an ordered semi-group. — We have already emphasized the analogy between the capacities of order $\mathcal{A}_\infty$ and the functions of a real variable which are completely monotone. We are going to see that this analogy is not only formal, but also that these two types of functions belong to the same very general class of functions in which exponentials and additive functions play an essential role.

43. 1. Definitions. — Let E be an ordered commutative semi-group *with a zero*, all of whose other elements are *greater than zero.* Let F be an ordered vector space, and let $\mathcal{A}$ be the convex cone of the functions which are defined on E and take values in F_+, and which are alternating of order ∞ (see 13. 1., Chapter III).

We shall suppose F such that each $X \subset F_+$ which is bounded from above and which is filtering on the right ([22]) has an upper bound.

43. 2. Definition. — *Any application f of* E *into* F *such that $f(a \top b) = f(a) + f(b)$ will be called* linear.

It is obvious that any linear and positive f belongs to $\mathcal{A}$; the set of those functions is a sub-convex cone of $\mathcal{A}$, which we will denote by $\mathcal{L}$.

43. 3. Definition. — *We say that a function ψ on* E *is an* exponential *when ψ is a real-valued function such that*

$$0 \leq \psi \leq 1 \qquad \text{and} \qquad \psi(a \top b) \equiv \psi(a) \cdot \psi(b).$$

To each real, linear, and positive f on E corresponds the exponential $\psi = e^{-f}$ and, conversely, to each exponential ψ which does not assume zero values on E corresponds the positive linear function $f = \text{Log } 1/\psi$.

43. 4. Theorem. — *In order that an element f of the cone $\mathcal{A}$ be extremal, it is necessary and sufficient that it be of one of the two following forms.*

(1) *f is an extremal element of the cone $\mathcal{L}$ of linear elements of $\mathcal{A}$.*

([22]) We say that X′ is filtering on the right if for every $a, b \in X$, there is an element $c \in X$ such that $a, b \leq c$.

(2) $f = (1 - \psi)V$, *where* ψ *is an exponential on* E *and* V *is an extremal element of the convex cone* F_+.

Proof. Let f be an extremal element of $\mathfrak{a}$.

(1) If $f(E)$ contains at most two elements 0 and $V \neq 0$, we can set $f = (1 - \psi)V$, where ψ is a function which takes only the values 0 and 1 and is such that $(\nabla_n)_\psi \geqq 0$ for each n.

If $\psi(a) = 0$, it follows from $\nabla_1(a; b) \geqq 0$ that $\psi(a \top b) = 0$ for each b.

If $\psi(a) = \psi(b) = 1$, it follows from $\nabla_2(0; a, b) \geqq 0$ that $\psi(a \top b) = 1$. We have therefore

$$\psi(a \top b) \equiv \psi(a) . \psi(b),$$

and hence ψ is an exponential.

In order that such an f be extremal, it is clearly necessary that V be extremal on F_+.

(2) If $f(E)$ contains at least two distinct elements λ, μ such that $\lambda, \mu \neq 0$, we can clearly suppose that they are comparable in F. If $f(a) = \lambda$ and $f(b) = \mu$, with $\lambda < \mu$, set

$$f_1(x) = f(x \top a) - f(a),$$

and

$$f_2(x) = f(a) + \nabla_1(x; a)_f = f(a) + f(x) - f(x \top a).$$

Then, $f = f_1 + f_2$.

The function f_1 belongs to $\mathfrak{a}$ since the operation $x \to (x \top a)$ is a homomorphism of E into itself and since $f(x \top a) - f(a) \geqq 0$. Also f_2 belongs to $\mathfrak{a}$ since on the one hand $(\nabla_n)_{f_2} = (\nabla_{n+1})_f \leqq 0$ and on the other hand $f_2 \geqq 0$ since the inequalities $(\nabla_2)_f \leqq 0$ and $f \geqq 0$ imply $f(a) + f(x) - f(x \top a) \geqq 0$. Now

$$f_1(b) = f(a \top b) - f(a) \geqq \lambda - \mu > 0;$$

hence f_1 is not identically zero.

If f is extremal, we have $f_1 \equiv \lambda_a f$ where λ_a is a real number such that $0 < \lambda_a \leqq 1$; (this fact implies in particular that $f(0) = 0$ since $f_1(0) = 0$). We can then write

$$f(x \top a) - f(a) \equiv \lambda_a f(a).$$

Two cases are then possible.

(a) First assume that f is not bounded above on S, that is, assume that there exists no $\nu \in G$ such that $f(x) \leqq \nu$ for each x.

Now $f(x) \leqq f(x \top a)$, and hence $f(x) - f(a) \leqq \lambda_a f(x)$. We cannot have $\lambda_a \neq 1$, otherwise

$$f(x) \leqq \frac{f(a)}{1 - \lambda_a},$$

and f would be bounded. Therefore $\lambda_a = 1$, and hence

$$f(x \top a) = f(x) + f(a).$$

This equality is true for any x. It is true also for any a; indeed, the proof above shows that it is true for each a such that $f(a) > 0$ since, f being not bounded, f takes values $\mu > \lambda$; on the other hand if a is such that $f(a) = 0$ the relations

$$\nabla_1(x;\, a) \leqq 0, \qquad \nabla_2(0;\, a,\, x) \leqq 0, \qquad f(0) = 0$$

give

$$f(x) \leqq f(x \top a) \qquad \text{and} \qquad f(x) \geqq f(x \top a)$$

and so

$$f(x \top a) = f(x) = f(x) + f(a).$$

Then f is *linear*. If f is extremal on $\mathfrak{A}$, f is a fortiori extremal on $\mathfrak{L}$.

(*b*) Now suppose that f is bounded above on E. Let V be its upper bound ($\neq 0$), which exists since the set $f(\mathrm{E})$ is filtering on the right in G. Set $g = \mathrm{V} - f$. We have the identity

$$g(a) - g(x \top a) \equiv \lambda_a(\mathrm{V} - g(x))$$

or

$$(1) \qquad g(a) + \lambda_a g(x) = g(x \top a) + \lambda_a \mathrm{V}.$$

By the definition of V, we have

$$\inf g(x) = 0 \qquad \text{and} \qquad \inf g(x \top a) = 0.$$

By taking the lower bound of the two sides of (1) we obtain

$$(2) \qquad g(a) = \lambda_a \mathrm{V}.$$

This relation is valid for each a such that $0 < f(a) < \mathrm{V}$. If $f(a) = 0$, we have $g(a) = \mathrm{V}$; if $f(a) = \mathrm{V}$, we have $g(a) = 0$. We can therefore set $f(x) \equiv \varphi(x) \,.\, \mathrm{V}$, where φ is a real function such that $0 \leqq \varphi \leqq 1$. Also let us set $\psi = 1 - \varphi$, and so $g(x) = \psi \,.\, \mathrm{V}$.

The relations (1) and (2) can be written now as

$$\lambda_a \psi(x) = \psi(x \top a) \qquad \text{and} \qquad \psi(a) = \lambda_a;$$

hence

$$\psi(x \top a) = \psi(x) . \psi(a). \tag{3}$$

This relation is valid for each a such that $0 < f(a) < V$. If $f(a) = 0$, we have as in the preceding case $f(x \top a) = f(x)$; thus, $\psi(x \top a) = \psi(x)$ and, since $\psi(a) = 1$, the identity (3) is again satisfied.

If $f(a) = V$, we also have $f(a \top x) = V$. Therefore

$$\psi(x \top a) = \psi(a) = 0,$$

and the identity (3) is again satisfied. In other words, $f = (1 - \psi)V$, where ψ is an exponential.

Since f is supposed to be extremal, it is clear that V must be also an extremal element of the cone G_+.

CONVERSE. — We now have to show that the functions f on E that we have just studied are indeed the extremal elements of $\mathcal{A}$.

(1) If f is an extremal element of $\mathcal{L}$, and if $f = f_1 + f_2$ with $f_1, f_2 \in \mathcal{A}$, the f_1 and f_2 necessarily belong to $\mathcal{L}$.

In fact, $f(0) = 0$ and $(\nabla_2)_f \equiv 0$, which implies that $f_i(0) = 0$ and $(\nabla_2) f_i \equiv 0$ $(i = 1, 2)$.

Thus the relation $\nabla_2(0; a, b)_{f_i} = 0$ is

$$f_i(a + b) = f_i(a) + f_i(b) \quad (i = 1, 2).$$

Therefore, since f is extremal on $\mathcal{L}$, f_1 and f_2 are proportional to f; thus f is extremal on $\mathcal{A}$.

(2) Let $f = (1 - \psi)V$ where ψ is an exponential on S and where $V \in G_+$. We first have to show that we have $f \in \mathcal{A}$. Now $f \geqq 0$ since $0 \leqq \psi \leqq 1$; in order to show that $(\nabla_n)_f \leqq 0$, it is sufficient to establish the equivalent relation $(\nabla_n)_\psi \geqq 0$.

Now $\nabla_1(x; a)_\psi = \psi(x) - \psi(x \top a) = \psi(x)(1 - \psi(a))$ and more generally,

$$\nabla_n(x; \{a_i\}) = \psi(x) \prod (1 - \psi(a_i)) \geqq 0.$$

Finally, it remains to show that f is also an extremal element of $\mathcal{A}$. We shall be able to do so only after having introduced a suitable topology on $\mathcal{A}$ (see section 44 below).

43. 5. EXAMPLE. — $E = R^n_+$, that is, the set of elements of the group R^n with positive coordinates. Each positive linear

function f on E is of the form $\Sigma a_i x_i$ $(a_i \geqq 0)$. The extremal elements of $\mathcal{L}$ are then the positive multiples of the n functions $f \equiv x_i$.

Every exponential f which does not vanish on E is, according to a previous remark, of the form $e^{-a.x}$ where $a.x$ denotes the scalar product of the elements a and x of R^n_+.

More generally, each exponential $\psi((x_i))$can be written $\psi((x_i)) = \Pi \psi_i(x_i)$ where ψ_i denotes the restriction of ψ to the x_i-axis.

The restriction ψ_i is again an exponential function of a real variable. Now if $\psi_i(a) \neq 0$ for $a > 0$, then $\psi_i(na) \neq 0$ for every n; thus ψ_i cannot take the value zero. If $\psi_i(a) = 0$ for all $a > 0$, then $\psi(0) = 1$ or 0; it is easily seen that conversely, each of these functions is an exponential.

Thus each exponential $\psi(x)$ on S can be written as $\psi(x) = \Pi \psi_i(x_i)$ where

$$\psi_i(x_i) = e^{-a_i x_i}(a_i \geqq 0) \quad \text{or} \quad \psi_i(x_i) = e^{-\infty x_i} \quad \text{or} \quad \psi_i \equiv 0,$$

where

$$e^{-\infty x_i} = \begin{cases} 1 & \text{if } x_i = 0 \\ 0 & \text{if } x_i > 0. \end{cases}$$

43. 6. Example. — E is idempotent, that is, $x \top x = x$ for every $x \in E$.

Each linear function on E is identically zero since

$$f(x) = f(x \top x) = f(x) + f(x) \quad \text{implies} \quad f(x) = 0.$$

If ψ is an exponential on E, then

$$\psi(x) = \psi(x \top x) = \psi(x) . \psi(x);$$

hence $\psi(x) = 0$ or 1. The set of elements x of E, for which $\psi(x) = 1$ is a sub-semigroup σ of E, hereditary on the left (that is, $x' < x$ and $x \in \sigma$ implies $x' \in \sigma$). For if $\psi(a) = \psi(b) = 1$, then $\psi(a \top b) = 1$. If $\psi(b) = 1$ and $a < b$ then, since $a \top b = b$, $\psi(b) = \psi(a) . \psi(b)$ and hence $\psi(a) = 1$.

Conversely, for every sub-semigroup σ of E which is hereditary on the left, let $\psi(x) = 1$ if $x \in \sigma$ and $\psi(x) = 0$ if $x \notin \sigma$. Then if $\psi(a) = \psi(b) = 1$, is follows immediately that $\psi(a \top b) = \psi(a) . \psi(b)$.

If $\psi(a) = 0$ and $\psi(b) = 1$, then $a > b$ and therefore

$$a \top b = a; \qquad \text{thus} \qquad \psi(a \top b) = \psi(a) = \psi(a) . \psi(b).$$

If $\psi(a) = \psi(b) = 0$, then $\psi(a \top b) = 0$, since a and $b = a \top b$; hence again, $\psi(a \top b) = \psi(a)\,\psi(b)$.

There is thus a one-to-one canonical correspondence between the exponentials on E and the sub-semigroups of E which are hereditary on the left.

The extremal elements of $\mathfrak{A}$ are the functions $f_{\sigma, V}(x)$ on E defined by

$$f_{\sigma, V}(x) = \begin{cases} 0 & \text{if} \quad x \in \sigma \\ V & \text{if} \quad x \notin \sigma, \end{cases}$$

where σ is a sub-semigroup of E which is hereditary on the left and V is an extremal element of F_+.

43. 7. Example. — E is an additive class of sets. Let E be an additive class of subsets of a set A, the operation $\top$ being union and the order on E being inclusion.

To every exponential ψ on E there is a canonically associated sub-semigroup σ which is hereditary on the left. Let σ^* be the set of complements $\complement X$ of elements X of σ; except for the case where $A \in E$ and where $\psi \equiv 1$, σ^* is a base of a filter.

Conversely, to each filter $\mathcal{F}$ on A having a base consisting of elements of E^*, there is associated the exponential on E defined by $f(X) = 1$ if $\complement X \in \mathcal{F}$ and $f(X) = 0$ if $\complement X \notin \mathcal{F}$.

Thus exponentials, filters *and* extremal elements of the cone $\mathfrak{A}$ *are in this example three aspects of the same mathematical object.*

The preceding interpretation of exponentials in terms of filters now permits a better study of the normalized extremal elements f of $\mathfrak{A}$ whenever F is the additive group R, and an extension of the definition of f to the set of filters on A.

For such an element $f = (1 - \psi)$, let T be the filter on A associated with ψ. Then

$$f(x) = \begin{cases} 0 & \text{if for some} \quad Y \in T, \quad X \cap Y = \emptyset, \\ 1 & \text{if for every} \quad Y \in T, \quad X \cap Y \neq \emptyset. \end{cases}$$

More generally, let T_1 and T_2 be two filters on A, and let

$$f(T_1, T_2) = \begin{cases} 0 & \text{if} \quad T_1 \smile T_2 \quad \text{does not exist,} \\ 1 & \text{if} \quad T_1 \smile T_2 \quad \text{exists;} \end{cases}$$

that is, $f(T_1, T_2)$ equals

$$\begin{cases} 0 & \text{if there exists two elements of } T_1 \text{ and } T_2 \text{ which are disjoint,} \\ 1 & \text{otherwise.} \end{cases}$$

For each fixed T_1 it is easily seen that the function $f_{T_1}(T) = f(T_1, T)$ is an alternating function of order ∞ on the semigroup of the filters T on A, with the operation $\top$ denoting the intersection, that is, $T \top T'$ denoting the filter each of whose elements is the union of an element of T and an element of T'.

When T denotes the filter of supersets of a set $X \subset A$, the function $f_{T_1}(T)$ is identical with the function $f_{T_1}(X)$ considered earlier. The function $f_{T_1}(T) = f(T_1, T)$ is called the *elementary function alternating of order* ∞ *and of index* T_1.

43. 8. **Special case.** — If E is the set of compact subsets of a Hausdorff space A, and if f is continuous on the right, the filter T associated with f is just the filter of neighborhoods of a closed subset of A. This was shown above (see section 26. 14., Chap. v).

44. **Topology of simple convergence on $\mathfrak{A}$. Application.** — Let us come back to the general case assuming simply that F is identical with R, and introduce on $\mathfrak{A}$ the topology of simple convergence on E.

The set of exponentials on E is clearly compact in the topology of simple convergence; the same is true of the set of elements of $\mathfrak{A}$ of the form $(1 - \psi)$, where ψ is an exponential.

We shall now show, by using this compactness and the rate of the decrease of the exponentials ψ on E, that each element $(1 - \psi)$ is extremal on $\mathfrak{A}$.

We use the fact, which is easy to show, that if $\mathcal{C}$ denotes a convex and compact subset of a locally convex Hausdorff linear space, for each non-extreme point $m \in \mathcal{C}$, there is a measure $\mu \geq 0$ of total mass 1 which is supported by $\left[\overline{e(\mathcal{C})} - \{m\}\right]$

and whose center of gravity is m [where $e(\mathcal{C})$ denotes the set of extreme points of $\mathcal{C}$].

(1) We suppose first that E has a largest element ω. Each $f \in \mathfrak{A}$ is then bounded and the set $\mathfrak{A}_1$ of the elements of $\mathfrak{A}$ such that $f(\omega) = 1$ is compact. The set of extreme points of $\mathfrak{A}_1$ is identical with the set of extremal elements f of $\mathfrak{A}$ such that $f(\omega) = 1$. Now if ψ is an exponential not $\equiv 1$ on E, then $\inf \psi = 0$ (since $\psi(na) = (\psi(a)^n)$; hence, $\sup (1 - \psi) = 1$. Then the set $e(\mathfrak{A}_1)$ of extreme points of $\mathfrak{A}_1$ is contained in the compact set $\mathcal{E}_1$ of the elements $(1 - \psi)$ where $\psi \not\equiv 1$. We shall show that $e(\mathfrak{A}_1) = \mathcal{E}_1$.

Otherwise, suppose $f = (1 - \psi)$ is an element of $[\mathcal{E}_1 - e(\mathfrak{A}_1)]$. There is a measure μ on the compact set $(\mathcal{E}_1 - \{f\})$ such that

$$\int d\mu = 1 \qquad \text{and} \qquad (1-\psi) = \int (1-\psi_t)\, d\mu(t);$$

$$\text{hence} \qquad \psi = \int \psi_t \, d\mu(t).$$

For every $a \in$ E and for every $\varepsilon > 0$, the closed set of t for which $\psi_t(a) \geqq \psi(a) + \varepsilon$ is of μ-measure zero. For let $\mu(\varepsilon)$ be its measure. Then

$$(\psi(a))^n = \psi(na) = \int \psi_t(na)\, d\mu(t)$$
$$= \int [\psi_t(a)]^n \, d\mu(t) \geqq \mu(\varepsilon)(\psi(a) + \varepsilon)^n,$$

hence $\mu(\varepsilon) \leqq \left(\frac{\psi(a)}{\psi(a) + \varepsilon}\right)^n$, a quantity which tends to 0 as $n \to \infty$.

Then $\psi_t(a) \leqq \psi(a)$ for almost all $t \in \mathcal{E}_1$.

From the relation

$$\int (\psi(a) - \psi_t(a))\, d\mu(t) = 0, \qquad \text{since} \qquad (\psi(a) - \psi_t(a)) \geqq 0$$

almost everywhere, it follows that $\psi(a) = \psi_t(a)$ for almost every t.

By passing to the limit, this equality holds at each point of the compact support of μ. In other words, ψ is identical with each ψ_t for which t belongs to the support of μ. Then μ is a point mass supported by the representative point of $(1 - \psi)$, contrary to hypothesis.

(2) If E does not possess a largest element, denote by $\widehat{\mathrm{E}}$ the semi-group obtained by adjoining to E an element ω, by

definition greater than each element of E and such that $a \top \omega = \omega$ for every $a \in E$.

Let $\widehat{\mathcal{A}}$ be the set of applications which are alternating of order ∞ of $\widehat{\mathcal{A}}$ in F_+; then obviously, in order that $f \in \widehat{\mathcal{A}}$, it is necessary and sufficient that the restriction of f to E be an element of $\mathcal{A}$ such that $\sup_{x \in E} f(x) \leq f(\omega)$.

According to the preceding, the extremal elements of $\widehat{\mathcal{A}}$ are, within a factor, just the functions $(1 - \psi)$, where ψ is an exponential on E. Now, if ψ is an exponential on E such that $\psi \not\equiv 1$, then $\inf_{x \in E} \psi(x) = 0$. Also, if $\widehat{\psi}$ is the extension of ψ to $\widehat{E}$ obtained by setting $\psi(\omega) = 0$, then $\widehat{\psi}$ is an exponential on $\widehat{E}$. Then $(1 - \widehat{\psi})$ is extremal on $\widehat{\mathcal{A}}$. This implies that $(1 - \psi)$ is extremal on $\mathcal{A}$; otherwise $(1 - \psi) = f_1 + f_2$ with f_1 and $f_2 \in \mathcal{A}$ and f_1, f_2 not proportional to $(1 - \psi)$.

We have $\sup (1 - \psi) = 1 = \sup f_1 + \sup f_2$ (on E). Then if $\widehat{f_i}$ denotes the extension of f_i to $\widehat{E}$ obtained by setting $f_i(\omega) = \sup f_i$ on S, we have $(1 - \widehat{\psi}) = \widehat{f_1} + \widehat{f_2}$ with $\widehat{f_1}$ and $\widehat{f_2} \in \widehat{\mathcal{A}}$ and f_1, f_2 not proportional to $(1 - \widehat{\psi})$.

Thus the theorem is proved when $F = R$.

We now suppose F to be arbitrary. Let us prove that each $f = (1 - \psi)V$ (φ exponential, V extremal on F_+) is an extremal element of $\mathcal{A}$.

Assume that $f = f_1 + f_2$ $(f_1, f_2 \in \mathcal{A})$.

For any $x \in E$, if $f(x) = 0$, then $f_1(x) + f_2(x) = 0$; hence $f_1(x) = f_2(x) = 0$.

If $f(x) \neq 0$, $(1 - \psi)V$ is extremal on F_+, so that $f_1(x)$ and $f_2(x)$ are colinear to V. In other words, we may set $f_1 = \varphi_1 V$ and $f_2 = \varphi_2 V$, where φ_1 and φ_2 are two real, positive functions; it follows immediately that φ_1 and φ_2 are alternating of order infinity. But we know that the relation $(1 - \psi) = \varphi_1 + \varphi_2$ implies that φ_1 and φ_2 are proportional to $(1 - \psi)$; hence f is indeed extremal on $\mathcal{A}$.

44.1. Remark. — When E is idempotent, each exponential ψ takes only the values 0 and 1, so that the elements $(1 - \psi)$ of $\mathcal{A}$ are increasing functions on E (ordered by the convention that $a < b$ if $b = a + c$) which take only the values 0 and 1.

Since E thus ordered is filtering on the right, these functions are extremal on the cone of real, positive, increasing functions on E; they are, a fortiori, extremal on the cone $\mathcal{A}$.

Thus the proof that the functions $(1-\psi)$ are extremal on $\mathcal{A}$ is very simple in this particular case.

45. **Integral representation of the elements of $\mathcal{A}$.** — Let us suppose at first that E has a greatest element ω. We suppose here that $F = R$. With the notations used above, for each $f \in \mathcal{A}$ there exists a measure $\mu \geq 0$ on the compact set of extremal elements $(1-\psi)$ of $\mathcal{A}$ (with $\psi \not\equiv 1$) such that

$$f(x) = \int (1-\psi_t(x))\, d\mu(t) \qquad \text{for every } x \in E.$$

When E does not possess a greatest element, there still is such a representation whenever the given function f is bounded on E, f being considered as the restriction to E of a function defined on $\widehat{E} = (E \cap \omega)$.

We shall not consider in the general case, the question of uniqueness of the measure μ associated with the given f.

45. 1. **The case $E = 2^A$.** — We shall assume that E is the additive set of subsets of a set A, the order on E being inclusion; assume also that $F = R$.

The normalized extremal elements of $\mathcal{A}$ are the elementary alternating functions $f_T(X)$ associated with some filter T on A. We shall use the space U of the ultra-filters on A which has already been introduced.

With each filter T on E there is associated in U a closed set that will be denoted by $\omega(T)$ or simply by T. Thus

$$f_T(X) = \begin{cases} 0 & \text{if} \quad \omega(T) \cap \omega(X) = \emptyset \\ 1 & \text{if} \quad \omega(T) \cap \omega(X) \neq \emptyset. \end{cases}$$

With each element f of $\mathcal{A}$ there is associated the capacity φ of order $\mathcal{A}_\infty$ defined on the set of open and closed subsets of U by the relation $\varphi(\omega(X)) = f(X)$. This capacity φ can be extended to the set of all the closed sets of U by setting

$$\varphi(W) = \inf_{W \subset \omega(X)} (\omega(X))$$

for each such closed W.

This extension is equivalent to extending the function f to the ordered semi-group of all filters on E.

Conversely, each capacity $\varphi \geq 0$ of order $\mathcal{A}_\infty$ on $\mathcal{K}(U)$ is characterized by its restriction to the set of open and closed subsets $\omega(X)$; in other words, there corresponds to φ an element f of $\mathcal{A}$.

Summarizing, we have established a canonical one-to-one correspondence between the capacities ≥ 0 and of order $\mathcal{A}_\infty$ on $\mathcal{K}(U)$ and the functions $f \geq 0$ and alternating of order ∞ on 2^A.

The topology of simple convergence on the set of elementary functions $f_T(X)$ is identical with the classical topology on the space of closed sets T of U. This follows simply from the fact that such a closed set has a base of neighborhoods consisting of the sets $\omega(X)$.

To each element f of $\mathcal{A}$ there corresponds a Radon measure $\mu \geq 0$ on the space $\mathcal{K}(U)$ such that

$$f(X) = \int f_T(X)\, d\mu(T) \quad \text{for every} \quad X \subset A,$$

and, more generally, for each filter on A.

The uniqueness of μ will be proved later on when we study capacities of order $\mathcal{A}_\infty$ on an arbitrary locally compact space.

Let us add that the topology of simple convergence on $\mathcal{A}$ is identical with the vague topology (which we shall define also) on the set of capacities φ associated with the elements f of $\mathcal{A}$.

45. 2. **The case** $E = R^n_+$. Let $\mathcal{A}$ be the cone of real functions ≥ 0 and alternating of order ∞ on R^n_+. For a given $f \in \mathcal{A}$, if $f((1)) = 0$ (where (1) denotes the point each of whose coordinates is 1) then $f \equiv 0$ since each f is decreasing and concave on R^n_+.

Thus for each $f \not\equiv 0$, there is a $\lambda > 0$ such that $\lambda f((1)) = 1$. In other words, the closed hyperplane $f((1)) = 1$ of the vector space of real functions on R^n_+ intersects each ray of $\mathcal{A}$ at one and only one point. Let $\mathcal{A}_1$ be the set of elements f of $\mathcal{A}$ such that $f((1)) = 1$.

Since F is increasing and concave on R^n_+, each $f \in \mathcal{A}_1$ has the property that $f(x) < \sup [1, x_i]$. Then $\mathcal{A}_1$ is compact in the topology of simple convergence.

Now the extremal elements of $\mathcal{A}_1$ are the n functions $f \equiv x_i$ and the normalized functions

$$\frac{1 - e^{-\Sigma t_i x_i}}{1 - e^{-\Sigma t_i}},$$

where the t_i are $\geqq 0$ or $+\infty$, and $\Sigma t_i \neq 0$.

It follows that each $f \in \mathcal{A}$ *which is continuous* on R^n_+ has an integral representation of the form

$$f(x) = a.x + \lambda + \int \frac{1 - e^{-t.x}}{1 - e^{t.1}} d\mu(t),$$

where μ is a positive measure of finite total mass, on the set of non-zero vectors t of R^n_+.

The functions $f \in \mathcal{A}$ which are not continuous have formally the same representation but with suitable definitions to take care of vectors t with infinite coordinates.

When $n = 1$, this expression can be simplified and can be written, for every $f \in \mathcal{A}$, as

$$f(x) = \lambda + \int \frac{1 - e^{-tx}}{1 - e^{-t}} d\mu(t),$$

where μ is supported by the compact $[0, +\infty]$, with the convention that

$$\frac{1 - e^{-tx}}{1 - e^{-t}} \equiv x \quad \text{when} \quad t = 0.$$

46. Extremal elements of the cone of monotone functions of order ∞ on an ordered semi-group. — Let E and F once more be a semi-group and an ordered vector space respectively, which have the same properties as in section 43. Denote by $\mathcal{M}$ the convex cone of functions from E to F_+ which are monotone of order ∞, (that is, the ∇_n are $\geqq 0$).

For every $f \in \mathcal{M}$, we have $\nabla_1(0; a) \geqq 0$; hence, $(f(0) - f(a)) \geqq 0$. Then the function $g(x) = f(0) - f(x)$ is $\geqq 0$, and $(\nabla_n)_g \leqq 0$. Thus, g is a bounded element of $\mathcal{A}$.

Conversely, for each bounded element g of $\mathcal{A}$, if $g(\infty)$ denotes its upper bound, the function $f = g(\infty) - g(x)$ is an element of $\mathcal{M}$.

It follows easily that the extremal elements of $\mathcal{M}$ are the functions $f = \psi.V$, where ψ is an exponential on E and V an extremal element of F_+.

46. 1. EXAMPLE. — Let $E = 2^A$ be the set of subsets of a set A, the order on E being the inverse of that defined by inclusion, and the operation in E being intersection. Let F be identical with R.

The extremal elements of $\mathcal{M}$ are identical with the functions $f_T(X)$ associated with some filter T on E, where

$$f_T(X) = \begin{cases} 0 & \text{if} \quad X \notin T \\ 1 & \text{if} \quad X \in T. \end{cases}$$

46. 2. EXAMPLE. — Let E be the semi-group R_+ and $F = R$. The non-zero extremal elements are the exponentials e^{-tx}, where $0 \leq t \leq \infty$.

46. 3. APPLICATION. — The introduction of the topology of simple convergence on $\mathcal{M}$ leads to applications analogous to those obtained by considering the cone $\mathcal{A}$. For example every *continuous* function $f(x)$ of the real variable $x \geq 0$ such that $(-1)^n f^{(n)}(x) \geq 0$, for all $x > 0$, that is, every completely monotone function of x, has a representation of the form

$$f(x) = \int e^{-tx}\, d\mu(t),$$

where μ is defined on $[0, \infty[$ and has a finite total mass. This result is the classical Bernstein theorem.

There exists obviously an analogous representation for continuous completely monotone functions on R^n_+ :

$$f(x) = \int e^{-t.x}\, d\mu(t)$$

where μ a is positive measure, with a finite total mass, on R^n_+.

Similarly, we could state integral representations for positive completely monotone functions defined on the *open* positive half-line $x > 0$, or, more generally, on the interior of R^n_+. But these generalizations are merely special cases of a more general result concerning functions defined on an arbitrary semi-group, which we shall now study briefly.

47. Alternating or monotone functions of order ∞ on an arbitrary commutative semi-group. — Let E be any commutative semi-group, and let F be a vector space satisfying the same conditions as above. A function f from E to F_+ is alternating (respectively, monotone) of order ∞ if all its differences $\nabla_n(x; \{a_i\})$ are $\leqq 0$ (respectively, $\geqq 0$) for any x and a_i in E.

The convex cone of these functions is again denoted by $\mathfrak{A}$ (respectively $\mathfrak{M}$).

If a and $b \in$ E, we shall write $a \prec b$ if $a = b$ or if $b = a \top c$. This relation is reflexive and transitive.

If $a \prec b$ and $b \prec a$, we shall write $a \sim b$; this relation is an equivalence relation ρ compatible with the relation $\prec$.

Moreover, if $a \sim b$ and $a' \sim b'$ then $(a \top a') \sim (a' \top b')$. Then the quotient set E/ρ is an ordered semi-group in which the relation $x \prec y$ is equivalent to $x = y$ or $y = x + z$; that is, it is an ordered semi-group which we shall call *regular.*

In E, if $a \prec b$ and if $f \in \mathfrak{A}$, then $f(a) \leqq f(b)$; thus if $a \sim b$, $f(a) = f(b)$.

Then with the function f on E there is canonically associated on E/ρ a function alternating of order ∞. We obtain an analogous reduction when $f \in \mathfrak{M}$. Then in studying $\mathfrak{A}$ and $\mathfrak{M}$ it may always be supposed that E is a regular ordered semi-group; this assumption will be made henceforth.

If E possesses a neutral element 0, we have $0 \prec 0 + x$ or $0 \prec x$ for every x. Then this case has been studied in the preceding.

If not, we may embed E in the semi-group $\widehat{E}$ obtained by the addition of a neutral element 0 to E such that $0 \prec x$ for all x; the study of the elements f of $\mathfrak{A}$ and $\mathfrak{M}$ associated with E is then equivalent to the study of functions defined on the set of non-zero elements of a regular ordered semi-group with a zero. This remark simplifies sometimes the study of $\mathfrak{A}$ and $\mathfrak{M}$.

47. 1. Definitions. — (1) *An element a of a regular semi-group* E *without* 0 *is called* extremal *if the equality $a = b + c$ is impossible.*

(2) *For every a which is extremal, the function φ_a defined by $\varphi_a(x) = 0$ if $x \neq a$ and $\varphi_a(a) = 1$ is called the* singular *function with the pole a.*

(3) *An exponential on* E *is again a function $\psi(x)$ such that $0 \leqq \psi \leqq 1$ and $\psi(a \top b) = \psi(a).\psi(b)$.*

47. 2. **Theorem.** — *Let* E *be a regular ordered semi-group. In order that an element* f *of* $\mathcal{M}$ *be extremal, it is necessary and sufficient that it be of one of the following forms:*

$$f = \varphi V \qquad \textit{or} \qquad f = \psi V,$$

where V *is an extremal element of* F_+, φ *a singular function, and* ψ *an exponential.*

Proof. When E has a zero, no element of E is extremal and the theorem is a consequence of section 46. We assume, therefore, that E has no zero and suppose that f is extremal. For every $a \in E$,

$$f = f_1 + f_2, \quad \text{where} \quad f_1(x) = f(x \top a),$$

and

$$f_2(x) = f(x) - f(x \top a) = \nabla_1(x; a)_f.$$

The functions f_1 and f_2 belong to $\mathcal{M}$ so that

$$(1) \qquad f_1(x) = f(x \top a) \equiv \lambda_a f(x), \quad \text{where} \quad 0 \leqq \lambda_a \leqq 1.$$

Case 1. If there is an $a \in E$ such that $f(x \top a) \not\equiv 0$, then $\lambda_a \neq 0$.

Now $\lambda_a f(x) = \lambda_x f(a)$; hence, $f(x)/\lambda_x = f(a)/\lambda_a =$ some $V \neq 0$.

Then (1) can be written as $V.\lambda_{a \top x} = \lambda_a . V . \lambda_x$; hence, $\lambda_{a \top x} = \lambda_a \lambda_x$ for every a such that $f(a \top x) \not\equiv 0$.

But if a such that $f(a \top x) \equiv 0$, the identity (1) shows that $\lambda_a = 0$ and also $\lambda_{a \top x} = 0$. Then again $\lambda_{a \top x} = \lambda_a \lambda_x$.

Thus $f = \psi V$, where ψ is an exponential. If ψV is extremal, then V is obviously extremal on F_+.

Conversely, if ψ is an exponential on E, it is easily verified that $f = \psi V$ belongs to $\mathcal{M}$ for every $V \in F_+$.

When, moreover, V is extremal on F_+, it can be shown as before, by introducing the topology of simple convergence, that each $f = \psi V$ is an extremal element of $\mathcal{M}$.

Case 2. If $f(x \top a) \equiv 0$ for every a, $f(x)$ is zero at every non-extremal point of E.

Now every function f from E to F_+ which is 0 at every non extremal point of E is an element of $\mathcal{M}$. For in $\nabla(x; \{a_i\})_f$ all the terms are zero except possibly the first, $f(x)$, which is $\geqq 0$. Then in order for such an f to be extremal, it is necessary and sufficient that the set of points x where $f(x) \neq 0$ cannot be partitioned; in other words, that this set consists of a single extremal point of E and that the value of f at this point be an extremal element of F_+. Then $f = \varphi V$ where φ is a singular function and V an extremal element of F_+.

47. 3. Remark. — There is an analogous theorem concerning the extremal elements of $\mathcal{A}$.

47. 4. Example. — Let E be the additive semi-group of real numbers $x \geqq a > 0$.

The extremal points of E are the points x of the interval $[a, 2a[$. It is immediate that an exponential ψ on E other than $\psi \equiv 0$ is not zero at any point of E; then $\operatorname{Log} 1/\psi$ is a positive linear function on E. Now it is elementary to prove that such a function has the form tx. Thus, each exponential not $\equiv 0$ on E is of the form e^{-tx}.

If we remark, on the other hand, that $f(x + a) \leqq f(a)$, then we deduce that f is bounded on $[2a, \infty[$.

We can then prove easily that for every $f \in \mathcal{A}$ there exists a measure μ on $[0, \infty[$ such that $\int e^{-at} d\mu(t) < \infty$, and a function $s(x) \geqq 0$ defined for $x \geqq a$, with $s(x) = 0$ for $x \geqq 2a$ such that

$$f(x) = \int e^{-tx} d\mu(t) + s(x) \qquad \text{for every} \qquad x \geqq a.$$

This result is rather remarkable since it implies that f is analytic on $[2a, +\infty[$ although the conditions $\nabla_n \geqq 0$ imposed on f have no local interpretation (since the parameters a_i appearing in ∇_n are all $\geqq a$).

An analogous study of the semi-group E of real $x > 0$ (which contains no extremal point) would lead to the classical representation of positive and completely monotone functions on $]0, \infty[$.

48. **Vague topology on the cone of increasing functions.** — Let E be a *locally compact space* and $\mathcal{J}$ the convex cone of real, non-negative, and increasing functions f defined on the class $\mathcal{K}(E)$ of compact subsets of E. We have already introduced on $\mathcal{J}$ the topology of simple convergence. However, this topology is not satisfactory for investigation of the subcone

of $\mathfrak{J}$ consisting of positive capacities f (that is, the elements f of $\mathfrak{J}$ which are continuous on the right).

We shall therefore introduce a weaker topology by associating with each $f \in \mathfrak{J}$ a suitable functional defined on the convex cone Q_+ of functions $\varphi(x)$ defined on E, real and $\geqq 0$, and 0 outside of a compact set.

48. 1. **Functional on Q_+ associated with an element $f \in \mathfrak{J}$.** — Let $f \in \mathfrak{J}$, and let $\varphi \in Q_+$. For every number $\lambda > 0$ let E_λ be the set of points x of E such that $\varphi(x) \geqq \lambda$.

The set E_λ is a compact set which decreases when λ increases, with $E_\lambda \subset$ support of φ. Then $f(E_\lambda)$ is a positive, bounded decreasing function of λ. Set

$$\widehat{f}(\varphi) = \int_{I(\varphi)} f(E_\lambda)\, d\lambda \quad \text{where } I(\varphi) \text{ is the interval }]0, \max\varphi].$$

When $f(\emptyset) = 0$, this integral can be written as

$$\widehat{f}(\varphi) = \int_{0+}^{\infty} f(E_\lambda)\, d\lambda = -\int_0^{\infty} \lambda\, df.$$

In particular, when f is a Radon measure, $f(\varphi)$ is simply the integral $\int \varphi\, d\mu$.

48. 2. **Immediate properties of the functional $\widehat{f}(\varphi)$.**

(1) Clearly $\widehat{f}(\varphi) \geqq 0$.

(2) $\widehat{f}(\varphi_1) \leqq \widehat{f}(\varphi_2)$ if $\varphi_1 \leqq \varphi_2$; in other words, $\widehat{f}(\varphi)$ is increasing.

(3) For every $a \geqq 0$, $\widehat{f}(a\varphi) = a\widehat{f}(\varphi)$

But conversely, each functional defined on Q_+ and possessing these three properties is not necessarily the functional associated with some element $f \in \mathfrak{J}$. We shall see later interesting examples of this fact.

48. 3. **Regularization of elements f of $\mathfrak{J}$.** — Denote by $\widehat{\mathfrak{J}}$ the convex cone of functions $\widehat{f}$ associated with elements $f \in \mathfrak{J}$. The mapping $f \to \widehat{f}$ of $\mathfrak{J}$ into $\widehat{\mathfrak{J}}$ is linear. We shall investigate the inverse image of an element $\widehat{f}$ in this mapping.

For every $f \in \mathfrak{J}$, the *regularized function* associated with f is the function f_r on $\mathfrak{K}(E)$ defined by

$$f_r(K) = \inf_{K \subset \mathring{X}} f(X) \qquad (K \text{ and } X \in \mathfrak{K}(E)).$$

If W denotes any « proximity » of the uniform structure of E (associated with any of its compactifications), and if K_W denotes the neighborhood of order W of any compact subset of E, then the above may be written

$$f_r(K) = \lim_{W \to 0} f(K_W).$$

This form enables us to show that many properties of f are preserved by regularization. For example, if f is sub-additive, or alternating or monotone of order n or ∞, the same is true of f_r.

It is immediate that f_r is $\geqq 0$, increasing, continuous on the right, and is the smallest of the functions larger than f and possessing these properties. In particular, for every $f \in \mathfrak{J}$ the condition $f \equiv f_r$ is equivalent to the condition that f be continuous on the right.

An essential property of regularization is the equality $\widehat{f} = \widehat{f_r}$ *for every* $f \in \mathfrak{J}$.

Indeed, for any $\lambda > 0$, we have $f(E_\lambda) \leqq f_r(E_\lambda)$; and

$$f_r(E_{\lambda_2}) \leqq f(E_{\lambda_1}) \qquad \text{for} \qquad \lambda_1 < \lambda_2 \qquad \text{since} \qquad E_{\lambda_2} \subset \mathring{E}_{\lambda_1}.$$

Then if we set $u_\lambda = f(E_\lambda)$ and $u_r(\lambda) = f_r(E_\lambda)$, it follows that $u_r(\lambda) = \limsup_{\lambda' \to \lambda} u(\lambda')$ or $u_r(\lambda) =$ the smallest decreasing function greater than $u(\lambda)$ and continuous on the left.

Thus $\int f(E_\lambda)\, d\lambda = \int f_r(E_\lambda)\, d\lambda$.

The set of elements f of $\mathfrak{J}$ which are continuous on the right is clearly a convex cone, which will be denoted by $\mathfrak{J}_r$. The preceding shows that the canonical mapping of $\mathfrak{J}_r$ into $\widehat{\mathfrak{J}}$ is a mapping *onto* $\widehat{\mathfrak{J}}$.

Let us show that the canonical mapping $f \to \widehat{f}$ *of* $\mathfrak{J}_r$ *onto* $\widehat{\mathfrak{J}}$ *is one-to-one.*

It is sufficient to show that for every $f \in \mathfrak{J}_r$ and for every $K \in \mathfrak{K}(E)$, $f(K)$ may be determined when $\widehat{f}$ is known.

Now let $\psi(K)$ be the characteristic function of K; then

$$f(K) = \inf_{\psi(K) \leqq \varphi} \widehat{f}(\varphi).$$

Indeed, $f(K) \leqq \widehat{f}(\varphi)$ for every $\varphi \geqq \psi(K)$, and since f is continuous on the right, for every $\varepsilon > 0$ there is a compact neighborhood V of K such that $(f(V) - f(K)) < \varepsilon$. Now, since E is locally compact, there exists a function $\varphi \in Q_+$ such that $0 \leqq \varphi \leqq 1$, $\varphi = 0$ outside of V, and $\varphi = 1$ on K.

Then $\widehat{f}(\varphi) \leqq f(K) + \varepsilon$, which proves the equality.

48. 4. **Vague topology on $\mathfrak{J}$.** — The set of elements f of $\mathfrak{J}$ which have the same image $\widehat{f}$ in $\widehat{\mathfrak{J}}$ is identical with the set of elements f whose regularized function is the element f_r of $\mathfrak{J}_r$ which corresponds canonically to $\widehat{f}$. In other words, if H_r is the canonical map of $\mathfrak{J}$ onto $\mathfrak{J}_r$ and $\widehat{H}$ that of $\mathfrak{J}$ onto $\widehat{\mathfrak{J}}$, then $\widehat{H} = \widehat{H} \circ H_r$.

Let $\mathfrak{T}_S$ be the topology of simple convergence on $\mathfrak{J}$ and $\widehat{\mathfrak{T}}_S$ the topology of simple convergence on $\widehat{\mathfrak{J}}$. *The inverse image $\mathfrak{T}_V$ under $\widehat{H}$ of the simple topology $\widehat{\mathfrak{T}}_S$ on $\widehat{\mathfrak{J}}$ is called the* vague topology on $\mathfrak{J}$.

In other words, a filter on $\mathfrak{J}$ converges vaguely to an element f_0 of $\mathfrak{J}$ whenever, for every $\varphi \in \Omega_+$, the values $f(\varphi)$ converge to $f_0(\varphi)$ relative to this filter.

Or again, for every $f_0 \in I$, a base of vague neighborhoods of f_0 consists of the $V(\varepsilon, (\varphi_i))$ where this symbol denotes the set of $f \in \mathfrak{J}$ such that

$$|f(\varphi_i) - f_0(\varphi_i)| \leqq \varepsilon \quad \left(\varepsilon \geqq 0;\ \varphi_i \in Q_+ \quad \text{with} \quad i \in I \quad \text{and } \bar{\bar{I}} \text{ finite}\right).$$

The map $\widehat{H}$ of $\mathfrak{T}_V$ into $\widehat{\mathfrak{T}}_S$ is continuous by construction. Let us show that the map $\widehat{H}$ of $\mathfrak{T}_S$ into $\widehat{\mathfrak{T}}_S$ is also continuous.

This result follows immediately from the fact that, for every filter on $\mathfrak{J}$ which converges simply to an element f_0 of $\mathfrak{J}$, for every φ and every λ, the $f(E_\lambda)$ converge to $f_0(E_\lambda)$; then if $u(\lambda) = f(E_\lambda)$, the $u(\lambda)$ converge to $u_0(\lambda)$. Now $u(\lambda)$ is decreasing and thus

$$\int_{I(\varphi)} u(\lambda) \to \int_{I(\varphi)} u_0(\lambda).$$

This statement is equivalent to saying that the $f(\varphi)$ converge to $f_0(\varphi)$.

The restriction of $\widehat{H}$ to $\mathfrak{J}_r$ being a one-to-one map of $\mathfrak{J}_r$ onto $\widehat{\mathfrak{J}}$, $\widehat{H}$ defines a homeomorphism between $\mathfrak{J}_r$ with the vague topology and $\widehat{\mathfrak{J}}$ with the simple topology. But it must be noticed that the restrictions of $\mathfrak{T}_S$ and of $\mathfrak{T}_V$ to $\mathfrak{J}_r$ are not identical (except for very special cases). To prove this statement it is sufficient to take a sequence f_n of Radon measures on $E = [0, 1]$ each consisting of a point mass $+1$ at the point $x = 1/n$; this sequence converges vaguely to $f_0 =$ the point mass $+1$ at $x = 0$, but it does not converge simply to any function f_0.

The space $\mathfrak{J}$ with the topology $\mathfrak{T}_V$ is not a Hausdorff space; the associated Hausdorff quotient space is homeomorphic with $\widehat{\mathfrak{J}}$ or with $\mathfrak{J}_r$ with the vague topology.

It can be proved, but we shall not do so here, that with the topology $\mathfrak{T}_S$, $\mathfrak{J}_r$ is everywhere dense in $\mathfrak{J}$; and that, similary, if $\mathfrak{A}$ (respectively $\mathfrak{M}$) denotes the convex closed subcone of $\mathfrak{J}$ consisting of the alternating (respectively monotone) functions of order ∞, then $\overline{(\mathfrak{J}_r \cap \mathfrak{A})} = \mathfrak{A}$ (respectively $\overline{(\mathfrak{J}_r \cap \mathfrak{M})} = \mathfrak{M}$).

48. 5. **Study of the case where E is compact.** — For every $k > 0$, the set $\mathfrak{J}(k)$ of all those $f \in \mathfrak{J}$ for which $f(E) \leqq k$ is obviously compact with the topology of simple convergence. Thus since for every $f_0 \in \mathfrak{J}$, the set of those f for which $f(E) \leqq 2f_0(E)$ is obviously a neighborhood of f_0, $\mathfrak{J}$ is locally compact with the topology of simple convergence.

Now since the mapping $\widehat{H}$ from $\mathfrak{T}_S$ into $\widehat{\mathfrak{T}}_S$ is continuous, the image $\widehat{H}(\mathfrak{J}(k))$ is compact; but, since for every $f \in \mathfrak{J}$, $f(E) = \widehat{f}(1)$, the set $\widehat{H}(\mathfrak{J}(k))$ is identical with the set of all those $\widehat{f} \in \widehat{\mathfrak{J}}$ for which $\widehat{f}(1) \leqq \lambda$; moreover, every $f_0 (\not\equiv 0)$ has as neighborhood in the topology $\widehat{\mathfrak{T}}_S$ the set of all those $\widehat{f}$ for which $\widehat{f}(1) \leqq 2\widehat{f}_0(1)$.

Hence, $\widehat{\mathfrak{J}}$ *is locally compact.*

It follows that $\mathfrak{J}_r$ is locally compact in the vague topology. The same holds for the sub-cones $\mathfrak{A}_r$ (and $\mathfrak{M}_r$) of $\mathfrak{J}_r$ consisting of all positive alternating (monotone) capacities of order $\mathfrak{A}_\infty$ ($\mathfrak{M}_\infty$) on $\mathfrak{K}(E)$.

In $\mathfrak{J}$, $\mathfrak{A}$, and $\mathfrak{M}$, the subsets consisting of all functions $f (f \not\equiv 0)$ which take no values other than 0 and 1 are obviously compact (since $f(E) = 1$); the same is true for the canonical image of these sets into $\mathfrak{J}_r$, $\mathfrak{A}_r$, and $\mathfrak{M}_r$. Now, if $f \in \mathfrak{J}$, and if f takes no values other than 0 or 1, the same holds for the regularized f_r.

Thus, since these functions are the same, within a constant factor, as the normalized extremal elements of $\mathfrak{J}_r$, $\mathfrak{A}_r$ and $\mathfrak{M}_r$, those sets $e(\mathfrak{I}_r)$, $e(\mathfrak{A}_r)$, $e(\mathfrak{M}_r)$ are compact in the topology of vague convergence.

48. 6. **Study of the case where E is locally compact.** — The topology $\mathfrak{T}_S$ on $\mathfrak{I}$ is not locally compact in this case, but it is easily shown that $\mathfrak{I}$ is complete under the uniform structure associated with the topology of simple convergence. The same is true for the closed sub-cones $\mathfrak{A}$ and $\mathfrak{M}$.

Likewise, $\mathfrak{I}_r$, $\mathfrak{A}_r$ and $\mathfrak{M}_r$ are complete under the uniform structure associated with the topology of vague convergence.

It may be useful to remark that, for every $f_0 \in \mathfrak{I}$, the set of all $f \leqq f_0$ is compact under the topology of simple convergence. (The same holds for $\mathfrak{A}$ and $\mathfrak{M}$). This would still be true if f_0 were replaced by an arbitrary non-negative function defined on $\mathfrak{K}(E)$.

The same is true on $\mathfrak{I}_r$ (also on $\mathfrak{A}_r$ and $\mathfrak{M}_r$) with the vague topology.

The following is another restriction which leads to compact sets.

Let us set $f(\Phi) = \sup_{\varphi \geqq \Phi} f(\varphi)$ $(\varphi \in Q_+)$ for every real-valued non-negative function Φ which is continuous on E. Then the set of all $f \in \mathfrak{I}$ for which $f(\Phi) \leqq k$ is compact in the topology of simple convergence, for every constant $k > 0$. (The same holds for $\mathfrak{A}$ and $\mathfrak{M}$).

The above proposition holds also on $\mathfrak{I}_r$ (and $\mathfrak{A}_r$, $\mathfrak{M}_r$) with the vague topology.

48. 7. **Extension of $\widehat{f}(\varphi)$ to non-negative, upper semi-continuous functions which vanish on the complement of a compact set.** — We have associated $\widehat{f}(\varphi)$, defined on Q_+, with every $f \in \mathfrak{A}$.

Let us designate by SS_+ the set of all positive upper semi-continuous real-valued functions $\varphi(x)$ defined on E which vanish outside of some compact set.

Furthermore, for each $\varphi_0 \in SS_+$, set

$$\widehat{f}(\varphi_0) = \inf_{f_0 \leqq \varphi} f(\varphi) \quad (\varphi \in Q_+).$$

The notation $f(\varphi_0)$ is consistent since, if $\varphi_0 \in Q_+$, the extended function f takes the same value as the function which was originally defined on Q_+. Thus, we have indeed obtained an extension of $\widehat{f}$.

We shall henceforth assume that f is continuous on the right, that is, $f \in \mathfrak{I}_r$; in other words we shall assume that f is a positive capacity on $\mathcal{K}(E)$. Then we have $f(\varphi) = \widehat{f}(\varphi)$ whenever φ is the characteristic function of a compact set.

More generally, it is easily verified that, for every $\varphi \in SS_+$,

$$\widehat{f}(\varphi) = \int_{I(\varphi)} f(E_\lambda)\, d\lambda,$$

where E_λ again denotes the compact set of all points x for which $\varphi(x) \geqq \lambda$.

The lower integral of every positive function defined on E (relative to f) can then be defined by the classical procedure.

In particular, this lower integral is defined for every positive *lower* semi-continuous function defined on E. The upper integral of every positive function φ on E can then be defined as the infimum of the upper integrals of all lower semi-continuous functions greater than φ on E.

Hence we have a concept of a capacitable function. It would not be very difficult to extend this concept to functions of arbitrary sign on E.

In order to obtain significant theorems, it would be necessary to place certain restrictions on the function f, such as, for instance, that f be alternating of order 2.

49. **Integral representation of the non-negative capacities of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$.** — We make the initial assumption that E is a *compact space*. The cone $\mathfrak{A}_r$ of all positive capacities of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$ is therefore locally compact in the vague topology, and the set of its normalized extremal elements is compact. Let us recall that these normalized extremal elements are the functions $f_T(X)$ defined by

$$f_T(X) = \begin{cases} 0 & \text{for } X \cap T = \emptyset, \\ 1 & \text{for } X \cap T \neq \emptyset \end{cases} \qquad \text{where T is an arbitrary compact subset of E.}$$

Let $\mathcal{E}_r$ be the set of these elements $f_T(X)$ ($\mathcal{E}_r \subset \mathfrak{A}_r$).

The vague topology on $\mathcal{E}_r$ (distinct from the topology $\mathcal{T}_s$ even on this subset of $\mathfrak{A}_r$), which may be considered as a topology on the set of all elements T of $\mathcal{K}(E)$, is identical with the classical topology of $\mathcal{K}(E)$.

For we have for every $T \in \mathcal{K}(E)$, and each $\varphi \in Q_+$,

$$\widehat{f}_T(\varphi) = \max(\varphi) \text{ on } T.$$

It follows immediately that for every filter on $\mathcal{K}(E)$ which converges to T_0 in the classical topology, $\widehat{f}_T(\varphi)$ converges for every φ to $\widehat{f}_{T_0}(\varphi)$.

Conversely, assume that for some filter on $\mathcal{K}(E)$ the $\widehat{f}_T(\varphi)$ converge to $\widehat{f}_T(\varphi)$, and that (ω_i) is a finite covering of T_0 by open sets each of which meets T_0.

There exists $\varphi_0 \in Q_+$ with $0 \leqq \varphi_0 \leqq 1$, such that $\varphi_0 = 1$ on $\complement\left(\bigcup(\omega_i)\right)$ and $\varphi_0 = 0$ on T_0. For every i, there exists $\varphi_i \in Q_+$, with $0 \leqq \varphi_i \leqq 1$, such that $\varphi_i = 0$ on $\complement(\omega_i)$ and max (φ_i) on T_0 is 1.

Hence there exists a set belonging to the given filter such that every element T of this set is contained in $\bigcup(\omega_i)$ and meets each ω_i. Thus this filter converges to T_0 in the classical sense.

Hence, in view of the general theorem (see 39. 4.) there exists for every $f \in \mathfrak{A}_r$ a non-negative Radon measure μ on $\mathcal{K}(E)$ such that

$$\widehat{f}(\varphi) = \int \widehat{f}_T(\varphi)\, d\mu(T) \qquad \text{for every} \qquad \varphi \in Q_+.$$

This formula may be extended to every $\varphi_0 \in SS_+$. For, such a φ_0 is the limit of a decreasing filtering set of functions $\varphi \in Q_+$. Hence, $\widehat{f}(\varphi_0)$ is the limit of the $\widehat{f}(\varphi)$ with respect to this filtering set. On the other hand, $\widehat{f}_T(\varphi_0) = (\max(\varphi_0) \text{ on } T)$ is the limit of $(\max(\varphi) \text{ on } T)$ with respect to this filtering set. This function $\widehat{f}_T(\varphi_0)$ is upper semi-continuous on $\mathcal{K}(E)$, and its integral $\int \widehat{f}_T(\varphi_0)\, d\mu(T)$ is indeed the limit of $\int \widehat{f}_T(\varphi)\, d\mu(T)$.

In particular, if for φ we choose the characteristic function of a compact set $X \subset E$, then

$$f(X) = \int f_T(X)\, d\mu(T) \qquad \text{for every compact set} \qquad X \subset E.$$

In other words, $f(X)$ is the μ-measure of the set of all compact sets T which meet X. Thus, the capacity $f(X)$ may be obtained from the fundamental scheme (E, F, A, μ), where E is the given space, $F = \mathcal{K}(E)$ with the classical topology, A is the set of all points (x, X) of $(E \times F)$ for which $x \in X$, and μ is the Radon measure on $\mathcal{K}(E)$ which we have introduced above.

We had previously established that the set functions obtained from a Radon measure by means of a finite number of $\cup$-homomorphisms are capacities of order $\mathfrak{A}_\infty$. We have now proved the converse.

More precisely, we can state the following theorem.

49. 1. Theorem. — *Suppose that* $\mathcal{A}$ *is the smallest class of all real-valued functions f each of which is defined on the set* $\mathcal{K}(E)$ *of all compact subsets of some compact space* E, *such that the following conditions are satisfied*:

(1) $\mathcal{A}$ *contains every non-negative Radon measure defined on any compact space* E.

(2) *If* E *and* F *are two compact spaces, if* $Y = \varphi(X)$ *is a* $\cup$*-homomorphism which is continuous on the right from* $\mathcal{K}(E)$ *into* $\mathcal{K}(F)$, *and if* $f \in \mathcal{A}$ *is defined on* $\mathcal{K}(F)$, *then we have* $e(X) \in \mathcal{A}$ *where* $e(X)$ *is the function defined on* $\mathcal{K}(X)$ by $e(X) = f(\varphi(X))$.

This class $\mathcal{A}$ *is* identical *with the class of all positive capacities of order* $\mathfrak{A}_\infty$ *defined on the sets* $\mathcal{K}(E)$ *relative to any compact space* E.

49. 2. **Probabilistic interpretation of this result.** — We have already, in particular cases, interpreted the scheme (E, F, A, μ) as a probabilistic scheme.

More generally, let such a scheme be given, in which E and F are two abstract sets, μ a simply additive positive measure defined on an algebra $\mathfrak{R}$ of subsets of F with $\mu(F) = 1$; and let us denote by $\mathcal{E}$ an additive class of subsets of E such that for each $X \in \mathcal{E}$ the set $Y = \psi(X)$ obtained from X by means of the construction of 26. 8, Chapter v, belongs to $\mathfrak{R}$.

We know that the function $f(X) = \mu(Y)$ is alternating of order ∞ on $\mathfrak{R}$.

Now let us consider μ as an elementary probability on the set F of events. Let us consider E as another set of events, and A as the set of all *favorable encounters* (x, y) with $x \in E$ and

$y \in F$. Then $f(X)$ is obviously the probability that such a favorable encounter occurs at least once on the subset $X \subset E$.

Conversely, the preceding theorem shows that, if sufficient conditions of regularity are imposed on E, $\mathcal{E}$, f (compactness, continuity on the right), then every positive function of X which is alternating of order infinity expresses the probability that a favorable event occurs at least once on X.

Actually, the regularity need not be of such strong form. And the fact that the set of all non-negative functions which are alternating of order infinity on an additive class of sets has, as extremal elements, functions whose values are 0 and 1, shows that one could certainly always interpret such a function as a probability; but it would undoubtedly be necessary in this case to generalize the notion of additive measure μ on F.

Whenever one can prove that any function defined on the set $\mathcal{K}(E)$ of compact subsets of a compact space is a positive capacity of order $\mathcal{A}_\infty$, one is sure that it could be interpreted in terms of probabilities.

In the most interesting cases (such as the theory of potential), the space E is not in general compact, but only locally compact, and the function f is not bounded from above; hence it is not possible to give a direct probabilistic interpretation of f.

However, the brief study of the case where E is locally compact which follows in section 49. 5. will show that the fundamental scheme still exists in this case, and that it is therefore possible to give « locally » an interpretation of f in terms of probability theory. If in particular f is bounded on $\mathcal{K}(E)$, then it is sufficient to divide f by sup f in order to obtain the desired probabilistic interpretation.

49. 3. Example. — If E is a Greenian domain in the space R^n and P a fixed point of D, we denote by $f(X)$ the harmonic measure, for the domain $(D - X)$, of the compact subset X of D with respect to the point P. ($f(X) = 1$ if $P \in X$).

We know (26. 12, Chapter v) that $f(X)$ is a capacity of order $\mathcal{A}_\infty$ on $\mathcal{K}(E)$, and that $0 \leq f \leq 1$. Hence f must admit an interpretation in terms of probability. That interpretation is known (see Kac [1 and 2]); $f(X)$ is the probability that a particle issuing from P and undergoing a Brownian motion will meet X at least once before it meets the boundary of D.

The support of the measure μ is in this case the set of all supports of Brownian trajectories issuing from P and contained in D.

In this particular case $f(X)$ can be extended to the set of all compact subsets of the boundary of D; its restriction to the set of these compact sets is then a Radon measure, which is identical with the ordinary harmonic measure. Obviously, the boundary can be topologized by various topologies which lead to diverse harmonic measures used in modern potential theory: ramified, geodesic, and Greenian measures (see Brelot and Choquet [1]).

49. 4. Example. — The Newtonian or Greenian capacity F(X) of a compact subset X of a domain E admits a less simple interpretation; this situation is due to the fact that $f(X)$ is not bounded on $\mathcal{K}(E)$ (see Kac [2]).

49. 5. **Integral representation in the case where E is locally compact.** — Suppose that the space E is locally compact, but not compact, and that $\widehat{E}$ is the compact space obtained from E by adjoining the point ω. The locally compact topological space $[\mathcal{K}(\widehat{E}) - \{\omega\}]$, where $\{\omega\}$ is the element of $\mathcal{K}(E)$ consisting of the simple point ω, is isomorphic with the set $\mathcal{F}(E)$ of all non-empty subsets of E with a suitable topology. When we shall talk of $\mathcal{F}(E)$, it will be understood that that topology has been placed on $\mathcal{F}(E)$.

We have already shown that the extremal elements of the cone $\mathcal{A}_r$ of the capacities of order $\mathcal{A}_\infty$ on $\mathcal{K}(E)$ are the functions $f_T(X)$, where T is a non-empty closed subset of E, with

$$f_T(X) = \begin{cases} 0 & \text{if } (T \cap X) = \emptyset \\ 1 & \text{if } (T \cap X) \neq \emptyset. \end{cases}$$

For every $f \in \mathcal{A}_r$ and each compact set $K \subset E$, let us denote by f_K the capacity defined on $\mathcal{K}(E)$ by $f_K(X) = f(X \cap K)$.

There exists a measure μ_K defined on the compact subset $\mathcal{K}(K)$ of $\mathcal{F}(E)$ and corresponding to f_K, such that

$$f_K(X) = \int f_T(X \cap K)\, d\mu_K(T) = \int f_T(X)\, d\mu_K(T).$$

Using the facts that $f_K(X) \leqq f(X)$ and that $f(X) = \lim f_K(X)$ with respect to the set, filtering on the right, of all compacts K, one can show that the measures μ_K converge vaguely, with respect to this same filtering set, to a measure μ on $\mathcal{F}(E)$, and that

$$f(X) = \int f_T(X)\, d\mu(T), \quad \text{for every} \quad X \in \mathcal{K}(E).$$

We have again in this case, for every $\varphi \in SS_+$,

$$f(\varphi) = \int f_T(\varphi)\, d\mu(T).$$

The relation $\int d\mu \leqq \infty$ holds if and only if f is bounded; in this case the preceding formula is valid for every upper semi-continuous non-negative φ on E.

50. **Integral representation of the non-negative capacities of order $\mathfrak{M}_\infty$ on $\mathcal{K}(E)$. Uniqueness of this representation.** — The reasoning and the results in this case are closely analogous to those pertaining to the capacities of order $\mathfrak{A}_\infty$; when E is locally compact, the proof and results are even simpler than in the case of the capacities of order $\mathfrak{A}_\infty$.

Let us first suppose that E is compact. The extremal elements of M_r are the functions $f_T(X)$ (where T is a compact subset of E) defined by

$$f_T(X) = \begin{cases} 0 & \text{if} \quad T \not\subset X, \\ 1 & \text{if} \quad T \subset X. \end{cases}$$

For every $f \in \mathfrak{M}_r$, there exists a measure $\mu \geqq 0$ on $\mathcal{K}(E)$ such that

$$f(\varphi) = \int f_T(\varphi)\, d\mu(T) \quad \text{for every} \quad \varphi \in SS_+.$$

In particular, if we take for φ the characteristic function of a compact subset $X \subset E$, we see that $f(X)$ is the μ-measure of the set of all T for which $T \subset X$.

Hence the following geometrical interpretation: $\mathcal{K}(E)$ is a compact space, ordered by inclusion. For each $X \in \mathcal{K}(E)$, the set of all $T \subset X$ is a compact subset of $\mathcal{K}(E)$; we shall call this set the negative cone with vertex X in $\mathcal{K}(E)$.

For every $f \in \mathfrak{M}_r$ there exists a measure $\mu \geqq 0$ on $\mathcal{K}(E)$ such that we have, for every $X \in \mathcal{K}(E)$.

$f(X)$ = the μ-measure of the negative cone of vertex X.

We have thus in a particular case a new proof of a general theorem obtained by A. Revuz [3], which furnishes a simple integral representation of all « totally monotone » functions defined on a partially ordered set S (here $S = \mathcal{K}(E)$), when certain conditions of regularity are satisfied.

The functions studied by A. Revuz are identical with the functions monotone of order ∞ ($\nabla_n \geq 0$), defined on a semi-group consisting of an ordered set S on which the semi-group operation is the operation $(a \frown b)$ which is assumed to be always possible.

The general theorem of section 47. 2. shows that the cone of these functions admits as its only extremal elements exponentials (which take no values other than 0 and 1, since the semi-group is idempotent), because there are no extremal elements in S (we have $x = x \frown x$ for every x).

Now the set of all points x in S where a given exponential $\psi(x)$ takes the value 1 is invariant under the operation $\frown$, and it is hereditary on the left; conversely, one may associate with each subset of S having these properties an exponential whose value is 1 on that subset and 0 elsewhere.

It can thus be foreseen that, if every negative cone of S is compact, then it is possible to associate with each exponential ψ a point $P(\psi)$ of S such that $\psi(x)$ equals 1 or 0 according as $x \geq P(\psi)$ or not.

It follows that in cases of sufficient regularity there exists a representation of totally monotone functions f on S by means of measures $\mu \geq 0$ defined on S and such that:

$f(x) =$ the μ-measure of the negative cone with vertex x.

The very subtle analysis undertaken by A. Revuz enables him to show the uniqueness of that measure μ in general cases.

In particular, this measure is unique when S is the ordered set $\mathcal{K}(E)$ associated with the compact space E, in other words, if we are dealing with capacities of order $\mathfrak{M}_\infty$ on $\mathcal{K}(E)$.

This uniqueness makes it possible to extend these results immediately to the case where E is an arbitrary Hausdorff space.

More precisely, we have the following theorem.

50. 1. Theorem. — *If* E *is an arbitrary Hausdorff space, and f a non-negative capacity of order $\mathfrak{M}_\infty$ on $\mathcal{K}(E)$, then there*

exists one, and only one, generalized, non-negative Radon measure μ *(see* 26. 6, *Chapter* v) *defined on* $\mathcal{K}(E)$ *with the classical topology such that, for every compact set* $X \subset E$, $f(X)$ *is the* μ*-measure of the compact negative cone of vertex* X *in* $\mathcal{K}(E)$.

To prove this extension, it is sufficient to observe that, for every compact set $X \subset E$, the restriction of f to $\mathcal{K}(X)$ is associated with a Radon measure whose support is $\mathcal{K}(X)$, and that, if $X_1 \subset X_2$, then the measures thus associated with X_1 and X_2 are compatible on $\mathcal{K}(X_1)$ because of the uniqueness of these measures.

50. 2. **Probabilistic interpretation of the elements f of $\mathcal{M}_r$.** — We have already remarked that the probability f that a favorable event occurs at least once on a set $X \subset E$ is a function of X which is alternating of order ∞; thus, the function $g(X) = 1 - f\left(\complement(X)\right)$, which expresses the probability that this favorable event occurs never on the complement of X, is a function which is monotone of order ∞.

Conversely, the above result shows that, under the conditions of regularity which we have indicated, and if, moreover, E is compact and $f(E) = 1$, then each function $f(X)$ which is monotone of order ∞ expresses the probability that some favorable event never occurs on the complement of X.

51. **Uniqueness of the representation of a non-negative capacity of order $\mathcal{A}_\infty$ on $\mathcal{K}(E)$.** — Suppose that E is compact, and that f is a non-negative capacity of order $\mathcal{A}_\infty$ on $\mathcal{K}(E)$, and let μ be one of the Radon measures on $\mathcal{K}(E)$ associated with f.

Let $\bar{f}$, of order $\mathcal{M}_\infty$, be the conjugate capacity of f (see 15. 6. Chapter III).

If we set $g = f(E) + \bar{f}$, then the capacity g is non-negative and of order $\mathcal{M}_\infty$; hence, a uniquely determined Radon measure $\nu \geqq 0$ on $\mathcal{K}(E)$ is associated with g.

Now for every compact set $X \subset E$ we have

$f(X) =$ the μ-measure of the set of all T such that $X \cap T \neq 0$;

hence : $\quad g(X) = f(E) + \bar{f}(X) = f(E) - f(E - X)$

is the μ-measure of the set of all T which do not meet $(E - X)$ and which are therefore contained in X.

Since ν is unique, we have $\mu \equiv \nu$, hence the following theorem.

51. 1. Theorem. — *For every non-negative capacity f of order $\mathfrak{A}_\infty$ on $\mathcal{K}(E)$, where E is compact, the measure μ on $\mathcal{K}(E)$ associated with f is* unique; *furthermore, if ν denotes the measure on $\mathcal{K}(E)$ associated with the non-negative capacity g of order $\mathfrak{M}_\infty$, defined by $g = \bar{f}(E) + f$, then we have $\mu \equiv \nu$.*

We remark, without giving the proof, that this result can be extended to the case where E is *locally compact,* in the following form:

(1) μ *is unique;*

(2) $\mu \equiv \nu$ *whenever* $g = (f(E) + f)$ *is defined, that is, whenever f is bounded.*

When f is not bounded it is still possible to define a function g associated with f by using the following *definition*:

$$g(X) = \text{the } \mu\text{-measure of the set of all } T \subset X.$$

It can be shown that $g(X)$ is the limit, as K tends to D, of the functions

$$g_K(X) = f(K) + \bar{f}_K(X) = f(X) - f(K - X).$$

For example, suppose that f is the Greenian capacity relative to a domain D; it can be easily shown that, for every X, we have

$$[f(K) - f(K - X)] \to 0 \quad \text{as} \quad K \to D.$$

It follows that $g(X) \equiv 0$; this fact implies that the measure μ on $\mathcal{F}(E)$ has as its support the set of those closed subsets of D which are not compact.

52. **Functional study of the elements of $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{M}}$.** — We have defined, for every $f \in \mathfrak{I}$ and for every $\varphi \in Q_+$,

$$\hat{f}(\varphi) = \int_{I(\varphi)} f(E_\lambda)\, d\lambda.$$

As we know, it follows that f is positive, increasing, and positively homogeneous.

We now seek to establish what can be said about $\hat{f}$ when certain restrictive hypotheses are placed on f, such as, for instance, that f be sub-additive, or that f belong to $\mathcal{A}_\infty$, or that f belong to $\mathcal{M}_\infty$.

Let $(E \times R)$ be the product of E and the real axis; and for every $\varphi \in Q_+$ define $[\varphi]$ as the set of all points (x, y) in $(E \times R)$ such that $y \leqq \varphi(x)$. Furthermore, let (λ) be the set of all points (x, y) in $(E \times R)$ such that $y \geqq \lambda$. We have, obviously,

$$E_\lambda = pr_E([\varphi] \cap (\lambda)).$$

Now the following two relations are true:

$$[\varphi_1 \smile \varphi_2] = [\varphi_1] \cup [\varphi_2], \qquad [\varphi_1 \frown \varphi_2] = [\varphi_1] \cap [\varphi_2],$$

where $\smile$ and $\frown$ denote the operations sup and inf on the φ.

These formulas make it possible to transform every relation satisfied by f, which involves no operations other than intersection and union, into a relation satisfied by $f(E_\lambda)$ and involving the operations $\smile$ and $\frown$ on the φ.

If these relations are linear, then it is possible to integrate and to obtain relations satisfied by $\widehat{f}$. If, in particular, f is sub-additive, then

$$\widehat{f}(\varphi_1 \smile \varphi_2) \leqq \widehat{f}(\varphi_1) + \widehat{f}(\varphi_2).$$

If $f \in \mathcal{A}_\infty$, then $\widehat{f}$ is alternating of order ∞ on Q_+ (relative to the operation $\smile$ on Q_+).

If $f \in \mathcal{M}_\infty$, then $\widehat{f}$ is monotone of order ∞ on Q_+ (relative to the operation $\frown$ on Q_+.

But it is not true that every functional on Q_+ which is non-negative and increasing, and which satisfies one of the three preceding conditions is identical with the $\widehat{f}$ associated with an $f \in \mathfrak{I}$, where f is respectively sub-additive, of order $\mathcal{A}_\infty$ or $\mathcal{M}_\infty$.

This will be shown by examples, in which we shall choose E such that $\overline{\overline{E}} = 2$.

52. 1. **Study of an example.** — Let the points of E be x_1 and x_2. Every function $\varphi \in Q_+$ is defined by its values $y_i = \varphi(x_i)$ $(i = 1, 2)$. Thus Q_+ is isomorphic to the ordered cone R^2_+ of all couples (y_1, y_2). Every $f \in \mathfrak{I}$ such that $f(\emptyset) = 0$

is characterized by the three values $f(x_1)$, $f(x_2)$, and $f(x_1, x_2)$.

To say that $f \in \mathfrak{I}$ is the same as saying

$$f(x_1) \geqq 0, \qquad f(x_2) \geqq 0,$$
$$f(x_1, x_2) \geqq \sup (f(x_1), f(x_2)).$$

To say that $f \in \mathfrak{A}$ is the same as saying

$$f(x_1) \geqq 0, \qquad f(x_2) \geqq 0,$$
$$\sup (f(x_1), f(x_2)) \leqq f(x_1, x_2) \leqq f(x_1) + f(x_2).$$

To say that $f \in \mathfrak{M}$ is the same as saying

$$f(x_1) \geqq 0, \qquad f(x_2) \geqq 0,$$
$$f(x_1) + f(x_2) \leqq f(x_1, x_2).$$

In each of these three cases, the function $\widehat{f}(\varphi) = f(y_1, y_2)$ is a linear function in each of the regions $y_1 \leqq y_2$, $y_2 \leqq y_1$; it is defined by its values on the lines $y_1 = 0$, $y_2 = 0$, $y_1 = y_2$ and so it depends upon three parameters.

Each function which is not of this type cannot belong to $\widehat{\mathfrak{I}}$. The following are three such functions on R^2_+ which are moreover increasing and positively homogeneous and respectively alternating of order ∞ for the operation $\smile$ and monotone of order ∞ for the operation $\frown$:

$$g = \frac{x^2 + xy + y^2}{x + y}, \qquad g = \frac{xy}{x + y}, \qquad \text{or} \qquad g = (xy)^{1/2}.$$

53. **Definition and properties of the classes I, A, M.** — Let E be locally compact. We denote by I the cone of the functions $f(\varphi)$ defined on the lattice cone Q_+ which are (a) positive, (b) increasing, and (c) positively homogeneous.

We denote by A (respectively M) the subcone of I made up of the functions on Q_+ which are alternating of order ∞ for the operation $\smile$ (monotone of order ∞ for the operation $\frown$).

We know already that $\widehat{\mathfrak{I}} \subset I$; $\widehat{\mathfrak{A}} \subset A$; $\widehat{\mathfrak{M}} \subset M$. When E is compact, these cones I, A, M, are locally compact under the topology of simple convergence.

We can easily extend the definition of each f belonging to one of these classes to the lattice cone SS_+, with preservation of the functional properties of f.

We shall now state without proof several results about the structure of these cones.

53. 1. **Extremal elements of** A. — *The extremal elements of* A *are the functions* $f \geqq 0$, *positively homogeneous on* Q_+, *and such that*

$$(f(\varphi_1) = f(\varphi_2)) \Longrightarrow (f(\varphi_1 \smile \varphi_2) = f(\varphi_1) = f(\varphi_2)).$$

An equivalent condition to this is the following:

$$f(\varphi_1 \smile \varphi_2) = \sup (f(\varphi_1), f(\varphi_2)).$$

It is immediate that each such function belongs to A, and that it is an extremal element of this cone. The converse is a little more difficult.

These extremal elements can still be characterized in another way. Let $(x_i)_{i \in I}$ be an arbitrary family of points of E, and let $(\lambda_i)_{i \in I}$ be some constants > 0. The function $f_I(\varphi) = \sup_{i \in I} \lambda_i \varphi(x_i)$, which is assumed $< +\infty$ for each φ, is an extremal element; and conversely each extremal element is of this form.

This last formula can also be written as

$$f_\Phi(\varphi) = \max_{x \in E} (\varphi(x) . \Phi(x)),$$

where $\Phi(x)$ is any function $\geqq 0$ and upper semi-continuous on E. There is a one-to-one correspondence between the extremal elements of A and the functions Φ to which they are associated.

For example, if $\Phi \equiv 1$, $f_\Phi(x)$ is the ordinary norm on Q_+.

When E is compact, it is immediate that each $f \in A$ admits an integral representation such that

$$f(\varphi) = \int f_\Phi(\varphi)\, d\mu(\Phi) \qquad \text{for each} \qquad \varphi \in SS_+,$$

where μ is a measure on the compact set of all Φ normalized by the condition

$$f_\Phi(1) = 1 \qquad \text{or} \qquad \max(\Phi(x)) = 1.$$

The topology on the set of these Φ is by definition the topology of simple convergence on the corresponding f_Φ. This topology can be interpreted as follows: each Φ is represented in $E \times R$ by the compact set $[\Phi]$ of points (x, y) where $0 \leqq y \leqq \Phi(x)$. The set of these $[\Phi]$ is a compact subset of the space $\mathcal{K}(E \times R)$ of subcompacts of $E \times R$. The topology thus induced on the set of normalized Φ is identical with the preceding topology.

The measure μ associated to each $f \in A$ is unique.

Example of elements of A:

$$f(\varphi) = \left[\int \varphi^{\alpha}\, d\nu\right]^{1/\alpha} \begin{cases}\text{for each } \alpha > 1 \text{ and} \\ \text{each measure } \nu \geq 0 \text{ on E.}\end{cases}$$

53. 2. **Extremal elements of M.** — We obtain a characterization of the extremal elements of M analogous with the preceding by changing the operation $\smile$ to $\frown$ and *sup* to *inf*.

We can write them in the form

$$f_I(\varphi) = \inf_{i \in I} \lambda_i \varphi(x_i).$$

(We will have $f \neq 0$ only if the $(x_i)_{i \in I}$ are taken on a compact set.)

Or else, by designating by ψ an element of SS_+ (hence zero outside of a compact),

$f_\psi(\varphi) =$ max. of numbers $k \geq 0$ such that $k\psi(x) \leq \varphi(x)$ for each $x \in E$, which amounts to saying that

$$f_\psi(\varphi) = \min_{x \in E} \frac{\varphi(x)}{\psi(x)}$$

with the convention

$$\frac{\varphi(x)}{\psi(x)} = +\infty \qquad \text{when} \qquad \psi(x) = 0.$$

There is a one-to-one correspondence between the $\psi \in SS_+$ and the extreme elements of M.

For example, if E is compact and if $\psi \equiv 1$, we obtain $f_\psi =$ minimum of φ on E.

Example of an element of M.

$$f(\varphi) = \left[\int \varphi^{\alpha}\, d\nu\right]^{1/\alpha}$$

for each $\alpha = 1/p$ with p a positive integer, and for each $\nu \geq 0$ on E.

The normalized set (by max $\psi = 1$) of $\psi \in SS_+$ is locally compact by the topology of simple convergence on the corresponding f. For each $f \in M$ there exists on this set one and only one measure μ such that

$$f(\varphi) = \int f_\psi(\varphi)\, d\mu(\psi) \qquad \text{for each} \qquad \varphi \in SS_+.$$

In other words

$$f(\varphi) = \int d\mu',$$

where μ' is the product of μ with the function of ψ: $\min\left(\frac{\varphi}{\psi}\right)$ which is $\neq 0$ only when ψ has its support contained in the support of φ.

Let us remark that, since $f(\varphi)$ is a function $\geqq 0$, monotone of order ∞ and continuous on the right on the lattice SS_+, there exists, according to the theorem of A. Revuz mentioned previously, one measure $\nu \geqq 0$ and only one on the locally compact space SS_+ such that

$f(\varphi) = \nu$-measure of the set of $\varphi' \leqq \varphi$.

The measure μ is obtained from ν by the following relation:

$$\mu(A) = \nu(B)$$

where A is an arbitrary compact subset of the set of normalized ψ and B is the set of $\psi' \in SS_+$ of the form

$$\psi' = \theta\psi \qquad \text{where} \qquad 0 \leqq \theta \leqq 1 \qquad \text{and} \qquad \psi \in A.$$

Conversely, ν is also determined as soon as μ is known.

53. 3. **Extremal elements of A ∩ M.** — The elements f of $A \cap M$ are characterized by the following relations:

(a) $f \geqq 0$;
(b) $f(\lambda\varphi) = \lambda f(\varphi)$ for $\lambda \geqq 0$;
(c) $f(\varphi_1 \smile \varphi_2) + f(\varphi_1 \frown \varphi_2) = f(\varphi_1) + f(\varphi_2)$.

The extremal elements of the cone $A \cap M$ are, up to a constant factor, the $f_a(\varphi) = \varphi(a)$, where $a \in E$.

For each $f \in A \cap M$ there exists a unique measure $\mu \geqq 0$ on E such that

$$f(\varphi) = \int f_a(\varphi)\, d\mu(a) = \int \varphi(a)\, d\mu.$$

In other words, the cone $A \cap M$ is identical with the cone of Radon measures on E.

53. 4. **Study of the cone I.** — We want to show that the elements of I are closely related to the elements of A and M and more precisely with extreme elements of these cones.

53. 5. THEOREM. — (*a*) *The superior envelope (supposed finite) and the inferior envelope of any family of functions* $f(\varphi)$ *belonging to* I *also belongs to* I.

(*b*) *Any element of* I *is the superior (inferior) envelope of a family of* extremal *elements of* M *(respectively* A).

Proof. (*a*) The first part of the theorem is immediate since homogeneity and monotony are preserved by the operations *sup* and *inf.*

(*b*) Now let $f \in \mathrm{I}$; for each $\varphi_0 \in \mathrm{Q}_+$ where $\varphi_0 \not\equiv 0$, the function

$$f_{\varphi_0}(\varphi) = \min\left(\frac{\varphi_0(x)}{\varphi(x)}\right),$$

with the convention of section 53. 2. is an extremal element of M; the same is true of

$$g = f(\varphi_0) f_{\varphi_0}(\varphi).$$

Now, $f(\varphi) \geqq f(\varphi_0) f_{\varphi_0}(\varphi)$ for each φ. In fact, if we set

$$\lambda = f_{\varphi_0}(\varphi) = \min\left(\frac{\varphi}{\varphi_0}\right),$$

we have $\varphi \geqq \lambda\varphi_0$ and hence $f(\varphi) \geqq \lambda f(\varphi_0)$, which is exactly the required relation.

Hence, not only is $f(\varphi)$ the superior envelope of a family of extremal elements of M, but for each φ_0 there is one of these elements, namely $f(\varphi_0) f_{\varphi_0}(\varphi)$, which is equal to $f(\varphi)$ for $\varphi = \varphi_0$.

(*c*) For each $\varepsilon > 0$ and for each $\varphi_0 \in \mathrm{Q}_+$ (with $\varphi_0 \not\equiv 0$), let $\varepsilon(x)$ be a continuous positive function on E such that $\varepsilon(x) \leqq \varepsilon$ and let $\varphi_\varepsilon = \sup(\varphi_0, \varepsilon(x))$.

Let us show that

$$f(\varphi) \leqq \max\left(\frac{\varphi}{\varphi_\varepsilon}\right) f(\varphi_\varepsilon),$$

where

$$f(\varphi_\varepsilon) = \sup f(\varphi') \quad \text{for all} \quad \varphi' \leqq \varphi_\varepsilon.$$

In fact, if

$$\lambda = \max\left(\frac{\varphi}{\varphi_\varepsilon}\right),$$

we have $\varphi \leqq \lambda\varphi_\varepsilon$ and so $f(\varphi) \leqq \lambda f(\varphi_\varepsilon)$ which is exactly the required equality.

Now if ε is small enough,

$$\max\left(\frac{\varphi_0}{\varphi_\varepsilon}\right) = 1;$$

hence,

$$\max\left(\frac{\varphi}{\varphi_\varepsilon}\right).f(\varphi_\varepsilon),$$

which is an extremal element of A, takes the value $f(\varphi_\varepsilon)$ for $\varphi = \varphi_0$.

Hence, if for each φ_0 and each ε we can choose $\varepsilon(x)$ such that $(f(\varphi_\varepsilon) - f(\varphi_0))$ is arbitrarily small — a restriction that we have not indicated in the wording of the theorem in order not to complicate it — we have proved the last part of the theorem.

This restriction, in the case where E is compact, concerns only functions φ which can take zero values on E; it is equivalent in a way to the continuity on the right of f. Note here that when E is compact, or more generally when E is the denumerable union of compacts, this restrictive condition is satisfied for each $f \in$ I which is sub-additive (for the operation $\smile$), for example for each $f \in$ A.

53. 6. **Extremal elements of I.** — When $\overline{\overline{E}}$ is finite, we can give a complete characterization of the extremal elements of I. The study of an f, $f = f(y_1, y_2, \ldots y_n)$, of I amounts indeed to the study of its trace on the simplex σ defined by $y_i \geqq 0$ and $\Sigma y_i = 1$. This trace is locally Lipschitzian on the interior of the simplex; in order that this be the trace of an extremal f, it is necessary and sufficient that almost everywhere the graph of this trace has a tangent hyperplane which passes through any one of the n faces of the simplex.

For example, and this is valid for any space E, each f which is the superior or inferior envelope of a *finite* family of elements of A or M is an extremal element of I. From this it follows that the set of extremal elements of I is *everywhere dense on* I.

53. 7. Primitive elements of I and the operations sup, inf, and $\int$. — Let us call each multiple λf_a (where $\lambda > 0$) of the function $f_a(\varphi) \equiv \varphi(a)$ a *primitive element of* I.

The preceding shows that we can generate A, M, A $\cap$ M, and I by starting from the primitive elements and applying the following operations: superior envelope and inferior envelope of a family of functions f, and the operation $\int f_t \, d\mu(t)$, where μ is a non-negative measure on a set of elements of I.

More precisely, let $\mathfrak{L}$ be the class of primitive elements (which are indeed the extremal elements of A $\cap$ M).

The class $\mathfrak{L}_{\text{sup}}$ of elements obtained from $\mathfrak{L}$ by the operation *sup* is made up of the extremal elements of A.

The class $\mathfrak{L}_{\text{inf,}}$ is made up of the extremal elements of M.

The class $\mathfrak{L}_{\int}$ is identical with A $\cap$ M.

The classes $\mathfrak{L}_{\text{sup, inf}}$ and $\mathfrak{L}_{\text{inf, sup}}$ are identical to I.

The classes $\mathfrak{L}_{\text{sup}, \int}$ and $\mathfrak{L}_{\text{inf}, \int}$ are identical respectively to A and M.

One could show that the class $\mathfrak{L}_{\int, \text{sup}} \left(\mathfrak{L}_{\int, \text{inf}}\right)$ is identical to the class of positive, increasing, positively homogeneous and V-sub-additive (V-super-additive) (1) functions defined on Q_+. It would be interesting to characterize also these classes in terms of the operations $\smile$ and $\frown$.

For example, we can see easily that if $f \in \mathfrak{L}_{\int, \text{sup}}$, we always have

$$2f(\varphi_1 \smile \varphi_2 \smile \varphi_3) \leq f(\varphi_1 \smile \varphi_2) + f(\varphi_2 \smile \varphi_3) + f(\varphi_3 \smile \varphi_1)$$

as well as other inequalities of the same type.

Let us add that we cannot form, with our three operations, classes other than those which we have pointed out above.

(1) For the definition, see section 54.

53. 8. REMARK. — We obtain an analogous classification of the positive and increasing set functions by using the operations sup, inf, and $\int$.

For example, if we consider the functions $f(X)$ defined on the set 2^E of all subsets of a set E, the primitive functions are the functions $f_u(X)$, where u is an ultra-filter on E, with

$$f_u(X) = \begin{cases} 1 & \text{if } X \in \text{ultra-filter } u, \\ 0 & \text{otherwise.} \end{cases}$$

Our three operations lead to the classes $\mathcal{A} \cap \mathcal{M}$, $\mathcal{A}$, $\mathcal{M}$, $\mathcal{J}$, and to other classes which we have not studied.

Various problems related to these operations could be considered. For example, if we apply the operation sup or inf to a family of positive capacities on the set $\mathcal{K}(E)$ (where E is compact), for which the K-borelian sets are capacitable, to what extent is it the same for the function thus obtained?

54. Relation between the alternating functions of order 2 and the pseudo-norms. — Let us first prove a lemma relative to the V-sub-additive or V-super-additive functions on a cone.

Let $\mathcal{C}$ be a lattice cone, that is, a convex cone such that its natural order structure is a lattice structure.

A real function f on $\mathcal{C}$ is called V-sub-additive (V-super-additive) if

(a) $\qquad f(\lambda x) = \lambda f(x) \qquad$ for each $\quad \lambda \geq 0$;

(b) $\qquad f(a+b) \leq f(a) + f(b)$,

(respectively $f(a+b) \geq f(a) + f(b)$).

54. 1. THEOREM. — *If the function f on $\mathcal{C}$ is positively homogeneous and if it satisfies*

(1) $\qquad f(a \smile b) + f(a \frown b) \leq f(a) + f(b)$

or

(2) $\qquad f(a \smile b) + f(a \frown b) \geq f(a) + f(b)$,

then f is respectively V-sub-additive or V-super-additive.

Proof. The proof is based on the proof of the special case where $\mathcal{C}$ is finite dimensional (hence isomorphic to R_+^n), and where f possesses continuous second derivatives for $x \neq 0$.

$$\text{Let} \quad f(x) = f(x_1, \ldots, x_n).$$

If $a = \{(x_1 + h_1), x_2, \ldots, x_n\}$,
$b = \{x_1, (x_2 + h_2), \ldots, (x_n + h_n)\}$, where $h_i \geqq 0$,

we have

$$a \smile b = \{(x_i + h_i)\},$$
$$a \frown b = \{(x_i)\}.$$

When $h_i \to 0$, condition (1) implies that

$$\sum h_1 h_i f''_{x_1 x_i} \leqq 0;$$

hence,

$$f''_{x_1 x_i} \leqq 0 \quad \text{for} \quad i \neq 1,$$

and, more generally,

$$f''_{x_i x_j} \leqq 0 \quad \text{for} \quad i \neq j.$$

Now since f is homogeneous of order 1, we have

$$\sum_j x_j f''_{x_i x_j} = 0, \qquad \text{for each } i.$$

Therefore the terms F of second degree in the development of f in the neighborhood of the point x satisfy

$$2F = \sum_{i \neq j} - x_i x_j f''_{x_i x_j}(x) \left[\frac{dx_i}{x_i} - \frac{dx_j}{x_j}\right]^2 \geqq 0.$$

It follows that f is locally convex on $\mathcal{C}$ and hence also globally, and this is known to be equivalent to saying that f is V-sub-additive. For the V-super-additivity, it is sufficient to change f into $-f$.

Let us notice that the converse of that theorem is false. For example, in $\mathcal{C} = R^3_+$, the function

$$f = \frac{z(x + y)}{x + y + z}$$

is V-super-additive (and it is increasing also), but it does not satisfy the inequality (2). The function

$$f = \frac{z^2}{x + y + z} + x + y$$

is V-sub-additive and increasing, but it does not satisfy the inequality (1).

In order to verify this, it is sufficient to take $a = (0, 1, 1)$ and $b = (1, 0, 1)$.

54. 2. Application. — Let E be a locally compact space, f an element of $\mathcal{J}_r$, and $\hat{f}$ the function on Q_+ which is associated to it. Recall that $\hat{f}$ is said to be a pseudo-norm on Q_+ if we have

$$(1) \qquad \hat{f}(\varphi_1 + \varphi_2) \leqq \hat{f}(\varphi_1) + \hat{f}(\varphi_2).$$

Theorem. — *In order that $\hat{f}$ be a pseudo-norm on Q_+, it is necessary and sufficient that f be an alternating capacity of order $\mathcal{A}_2$.*

Proof. Since f is increasing, it is equivalent to say that f is alternating of order $\mathcal{A}_2$ or to say that we have

$$f(X_1 \cup X_2) + f(X_1 \cap X_2) \leqq f(X_1) + f(X_2).$$

Now this relation is equivalent to

$$\hat{f}(\varphi_1 \smile \varphi_2) + \hat{f}(\varphi_1 \frown \varphi_2) \leqq \hat{f}(\varphi_1) + \hat{f}(\varphi_2).$$

According to the preceding theorem this relation implies that $\hat{f}$ is a pseudo-norm. Conversely, let us assume that $\hat{f}$ is a pseudo-norm. It is immediate that the relation (1) above can be extended to the functions $\varphi \in SS_+$. Therefore, if φ_1 and φ_2 are the characteristic functions of the compacts X_1 and X_2, we have

$$\hat{f}(\varphi_1 + \varphi_2) \leqq \hat{f}(\varphi_1) + \hat{f}(\varphi_2) = f(X_1) + f(X_2).$$

Now $(\varphi_1 + \varphi_2) = 2$ on $(X_1 \cap X_2)$, is $\geqq 1$ on $X_1 \cup X_2$, and $= 0$ elsewhere. Hence by using the definition of section 48. 1.:

$$\hat{f}(\varphi_1 + \varphi_2) = f(X_1 \cup X_2) + f(X_1 \cap X_2).$$

The desired relation follows immediately.

In the same way, we could prove that *in order for a positive capacity f to be monotone of order $\mathcal{M}_2$, it is necessary and sufficient that the associated function $\hat{f}$ satisfy the relation*

$$\hat{f}(\varphi_1 + \varphi_2) \geqq f(\varphi_1) + f(\varphi_2).$$

An immediate application of this theorem is the following:

If a capacity f is only sub-additive, its extension $\hat{f}$ is not necessarily a pseudo-norm.

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This bibliography contains several works which have not been referred to in the text, but which are closely related to the theory of capacities and will be useful to anyone working in this field.

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