

The Dedekind/Peano Axioms

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Richard Dedekind (1831–1916) published in 1888 a paper entitled *Was sind und was sollen die Zahlen?* variously translated as *What are numbers and what should they be?* or *The Nature of Meaning of Numbers*. In definition 71 of that work, he characterized the set of natural numbers \mathbf{N} as a set whose elements are called numbers, with one particular number 1 called the *base* or *initial* number, equipped with a function, called here the *successor function* $\mathbf{N} \rightarrow \mathbf{N}$ mapping a number n to its *successor* n' , such that

- (1). 1 is not the successor of any number.
- (2). The successor function is a one-to-one function, that is, if $n \neq m$, then $n' \neq m'$.
- (3). If a subset S of \mathbf{N} contains 1 and is closed under the successor function (i.e., $n \in S$ implies $n' \in S$), then that subset S is all of \mathbf{N} .

This characterization of \mathbf{N} by Dedekind has become to be known as Dedekind/Peano axioms for the natural numbers.

The principle of mathematical induction. The third axiom is recognizable as what is commonly called mathematical induction, a principle of proving theorems about natural numbers. In order to use mathematical induction to prove that some statement is true for all natural numbers n , you first prove the base case, that it's true for $n = 1$. Next you prove the inductive step, and for that, you assume that it's true for an arbitrary number n and prove, from that assumption, that it's true for the next number n' . Once you've done those two things, then you conclude that the statement is true for all natural numbers n .

This is what axiom 3 says, but in set-theoretical terms. Let S be the set of all numbers for which the statement is true. When you prove the base case that the statement is true, you've shown $1 \in S$. When you prove the inductive step, you've shown that S is closed under the successor function. Axiom 3 then says $S = \mathbf{N}$, so the statement is true for all natural numbers.

An example. Let's prove that $n^2 \geq 3n - 2$ for all natural numbers $n \geq 1$.

The base case is when $n = 1$, and the statement says $1^2 \geq 3 \cdot 1 - 2$, which we know to be true. Generally, the base case is very easy to verify, and it's enough to say "clearly the statement is true for 1." On rare occasions, the base case requires a substantial proof, but I doubt we'll see any of those occasions.

For the inductive step, we assume that the statement is true for a number n , that is, that $n^2 \geq 3n - 2$, and prove that it's true for the next number n' , that is $(n')^2 \geq 3(n') - 2$. We're

not in position to prove this from the Dedekind/Peano axioms yet, but after a number of definitions and theorems about addition, multiplication, and order, we could. And the proof would go as follows. First we would rewrite our goal $(n')^2 \geq 3(n') - 2$ in terms of addition and multiplication, and it would say $(n + 1)(n + 1) \geq 3(n + 1) - 2$. We'd use distributivity of multiplication over addition to again rewrite our goal as $n^2 + 2n + 1 \geq 3n + 1$. But $n^2 \geq 3n - 2$ by assumption, so, if we could show $2n + 1 \geq 3$, then we would reach our goal by adding the two inequalities, that is $n^2 \geq 3n - 2$ and $2n + 1 \geq 3$ together imply the goal $n^2 + 2n + 1 \geq 3n + 1$. With just a little more algebra, we can show the remaining inequality $2n + 1 \geq 3$, because that's equivalent to $2n \geq 2$, which follows from $n \geq 1$. That finishes the inductive step.

Therefore, by the principle of mathematical induction, we may conclude that $n^2 \geq 3n - 2$ for all natural numbers $n \geq 1$.

If you look through the proof given above, there were a lot of statements about numbers that were used. All of them can be proved, and should be proved, before accepting this proof as valid.

Our first theorems. Let's prove something that follows directly from the axioms.

Definition. A predecessor of a number n is a number m whose successor is n . Thus, if $m' = n$, then the successor of m is n while the predecessor of n is m .

Theorem 1. A number can have at most one predecessor

Proof. This follows directly from axiom 2. If a number n had two predecessors m_1 and m_2 , then $m_1' = n = m_2'$, so by axiom 2, m_1 would equal m_2 . Thus, n can have only one predecessor. Q.E.D.

Theorem 2. Every number except 1 has a predecessor.

Proof. We'll prove the logically equivalent statement

For each number n , either $n = 1$ or n has a predecessor

by induction.

The base case holds since if $n = 1$ then either $n = 1$ or n has a predecessor.

Now assume the statement is true for n . We'll show it's true for n' , that is, either $n' = 1$ or n' has a predecessor. But n' has the predecessor n . Q.E.D.

Variants of mathematical induction. Sometimes you don't want to prove a statement for all natural numbers, but only for those for some number k and beyond. Then you would use a variant of mathematical induction where the base case is $n = k$ rather than $n = 1$. Here's an example.

The factorial of a number n , written $n!$, is defined as the product of the natural numbers from 1 to n , so, for instance $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. The factorial function grows very fast, much faster than any polynomial. For instance, it grows faster than n^5 . But it doesn't start out larger than n^5 . In fact, by direct computation, you can show $n! < n^5$ for $n < 8$, but $8! = 40320$ while $8^5 = 32768$. Here's a proof that $n! > n^5$ for $n \geq 8$. The base case is $n = 8$, and for that $8! > 8^5$. Now for the inductive step. Assume $n! > n^5$ for some $n > 8$. We need to show that $(n + 1)! > (n + 1)^5$. But $(n + 1)! = (n + 1)n!$, so $(n + 1)! > (n + 1)^5$ will follow from $n! > (n + 1)^4$. But our assumption is that $n! > n^5$, so that last statement will follow from $n^5 > (n + 1)^4$. Unfortunately, that will require another proof by induction,

one that logically should be completed before this proof. I'll omit that proof, but here is the statement the relevant theorem: For $n \geq 4$, $n^5 > (n + 1)^4$.

Another variant of mathematical induction is where the inductive hypotheses is strengthened. Instead of assuming that the statement is true for just the preceding number, you may assume that it's true for all small numbers. For each n , if you can show that the statement being true for all $k < n$ implies that the statement is true for n , then you may conclude that the statement holds for all natural numbers. Note that when $n = 1$, since there are no smaller natural numbers, you have to prove the statement directly for $n = 1$.

For an example, we'll prove that every number $n > 1$ is a product of one or more prime numbers. Of course, we'll have to assume a few things about prime numbers since we haven't proved anything yet about them. Logically, this theorem should be placed after the definition of prime number (a number greater than 1 that is only divisible by itself and 1), and earlier theorems about divisibility and multiplication. Here's the proof. Let $n > 1$. If n is prime, then it is a product of one prime, namely itself. Otherwise, $n = km$ where neither k nor m equals 1. Then, since k and m are each smaller than n but greater than 1, by this variant of mathematical induction, each is a product one or more prime numbers. Multiply those two products of prime numbers together to conclude that n is product of prime numbers.

(Note that if you define an empty product to be 1, then every $n \geq 1$ is a product of zero or more numbers, each of which is prime.)

Of course, this strengthened form of mathematical induction needs to be justified. We can do that from the Dedekind/Peano axioms, but not yet, because we haven't even got a definition for the ordering, $m < n$, on natural numbers. Indeed, we haven't even got a definition for any of the operations of addition, subtraction, multiplication, or division.

Defining functions by induction, also called recursion. Dedekind and Peano both accepted the principle of mathematical induction for defining functions on the natural numbers \mathbf{N} . Both gave arguments that this method should work based on intuitive properties of functions on sets. We'll assume that the recursive definition of functions is valid. It is possible to show that it's valid using some basic axioms of set theory, or some basic assumptions of logic, but we won't go into that here. Alternatively, this property could be stated as an axiom for \mathbf{N} .

To define a function $f : \mathbf{N} \rightarrow S$ from the natural numbers to a set S , it is only necessary to (1) say what element a of S that $f(1)$ should be, and (2), say what $f(n')$ should be in terms of n and $f(n)$. In other words, if (1) a is a specified element of S , and (2) $g : \mathbf{N} \times S \rightarrow S$ is a function that is already defined that takes two arguments, a number k and an element s of S , and yields an element $g(k, s)$ of S , then the new function f can be defined *recursively* by the two equations

- (1). $f(1) = a$, and
- (2). $f(n') = g(n, f(n))$.

There are, as you can imagine, variants of recursive definition just as there are variants of mathematical induction. The one just mentioned isn't the simplest or the strongest, but it's good enough to use for many definitions.