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The signed Roman k-domatic number of digraphs

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Abstract

Let $k \geq 1$ be an integer. A *signed Roman k-dominating function* on a digraph D is a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{x \in N^-[v]} f(x) \geq k$ for every $v \in V(D)$, where $N^-[v]$ consists of v and all in-neighbors of v, and every vertex $u \in V(D)$ for which $f(u) = -1$ has an in-neighbor w for which $f(w) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed Roman k-dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called a *signed Roman k-dominating family* (of functions) on D. The maximum number of functions in a signed Roman k-dominating family on D is the *signed Roman k-domatic number* of G, denoted by $d_{sR}^k(D)$. In this paper we initiate the study of signed Roman k-domatic numbers in digraphs, and we present sharp bounds for $d_{sR}^k(D)$. In particular, we derive some Nordhaus-Gaddum type inequalities. In addition, we determine the signed Roman k-domatic number of some digraphs.

1 Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [3]. In this paper we continue the study of Roman dominating functions in graphs and digraphs. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* or *r-regular* if $d(v) = r$ for each vertex v of G. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, $K_{p,p}$ for the *complete bipartite graph* of order $2p$ with equal size of partite sets, and C_n for the *cycle* of length n.

If $k \geq 1$ is an integer, then the *signed Roman k-dominating function* (SRkDF) on a graph G is defined in [4] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N[v]} f(u) \geq k$ for each $v \in V(G)$, and every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an SRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *signed Roman k-domination number* of a graph G, denoted by $\gamma^k_{sR}(G)$, equals the minimum weight of an SRkDF on G. The special case $k = 1$ was introduced and investigated in [1]. For $\gamma^1_{sR}(G)$ we also write $\gamma_{sR}(G)$.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [2]. They have defined the domatic number $d(G)$ of a graph G by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition. The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. But Rall has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. (This was mentioned by Slater and Trees in [9].) Analogous to the fractional domatic number we may define the signed Roman k-domatic number.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed Roman k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called in [10] a *signed Roman k-dominating family* (of functions) on G. The maximum number of functions in a signed Roman k-dominating family (SRkD family) on G is the *signed Roman k-domatic number* of G, denoted by $d^k_{sR}(G)$. If $k = 1$, then we write $d^1_{sR}(G) = d_{sR}(G)$. This case was introduced and investigated in [6]. The signed Roman k-domatic number is well-defined and $d^k_{sR}(G) \geq 1$ for all graphs G with $\delta(G) \geq k - 1$, since the set consisting of any SRkDF forms an SRkD family on G.

Now let D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and *size* of the digraph D, respectively. We write $d^+_D(v) = d^+(v)$ for the *out-degree* of a vertex v and $d^-_D(v) = d^-(v)$ for its *in-degree*. The *minimum* and *maximum in-degree* are $\delta^-(D) = \delta^-$ and $\Delta^-(D) = \Delta^-$ and the *minimum* and *maximum out-degree* are $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$. The sets $N^+_D(v) = N^+(v) = \{x | (v, x) \in A(D)\}$ and $N^-_D(v) = N^-(v) = \{x | (x, v) \in A(D)\}$ are called the *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise, $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x, y) \in A(D)$, the vertex y is an out-neighbor of x and x is an in-neighbor of y, and we also say that x *dominates* y or y is *dominated* by x. A digraph D is *out-regular* or *r-out-regular* if $\delta^+(D) = \Delta^+(D) = r$. A digraph D is *in-regular* or *r-in-regular* if $\delta^-(D) = \Delta^-(D) = r$. A digraph D is *regular* or *r-regular* if $\delta^-(D) = \Delta^-(D) = \delta^+(D) = \Delta^+(D) = r$. The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc (u, v) belongs to $\overline{D}$ if and only if (u, v) does not belong to D.

If $k \geq 1$ is an integer, then the *signed Roman k-dominating function* (SRkDF) on a digraph D is defined in [11] as a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N^-[v]} f(u) \geq k$ for each $v \in V(D)$, and such that every vertex $u \in V(D)$ for which $f(u) = -1$ has an in-neighbor w for which $f(w) = 2$. The *weight* of an SRkDF f is the value $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed Roman k-domination number* of a digraph D, denoted by $\gamma^k_{sR}(D)$, equals the minimum weight of an SRkDF on D. A $\gamma^k_{sR}(D)$-*function* is an SRkDF on D with weight $\gamma^k_{sR}(D)$. If $k = 1$, then we write $\gamma^1_{sR}(D) = \gamma_{sR}(D)$. This case was introduced and studied in [8].

A set $\{f_1, f_2, \dots, f_d\}$ of distinct SRkDF on a digraph D with the property that $\sum_{i=1}^d f_i(v) \le k$ for each $v \in V(D)$, is called a *signed Roman k-dominating family* (of functions) on D. The maximum number of functions in a signed Roman k-dominating family (SRkD family) on D is the *signed Roman k-domatic number* of D, denoted by $d^k_{sR}(D)$. If $k = 1$, then we write $d^1_{sR}(G) = d_{sR}(G)$. This case was introduced and investigated in [7].

The signed Roman k-domination number exists when $\delta^- \ge \frac{k}{2} - 1$. However, for investigations of the signed Roman k-dominating number and the signed Roman k-domatic number it is reasonable to claim that $\delta^-(D) \ge k - 1$. Thus we assume throughout this paper that $\delta^-(D) \ge k - 1$. The signed Roman k-domatic number is well-defined and $d^k_{sR}(D) \ge 1$ for all digraphs D, since the set consisting of the SRkDF with constant value 1 forms an SRkD family on D.

Our purpose in this paper is to initiate the study of the signed Roman k-domatic number in digraphs. We first derive basic properties and bounds for the signed Roman k-domatic number of a digraph. In particular, we obtain the Nordhaus-Gaddum type result

$$d^k_{sR}(D) + d^k_{sR}(\overline{D}) \le n + 1,$$

and we discuss the equality in this inequality. In addition, we determine the signed Roman k-domatic number of some classes of digraphs. Some of our results are extensions of known properties of the signed Roman k-domatic number of graphs, given in [10].

We make use of the following results in this paper.

Proposition A. *([8])* Let D be a digraph of order n. Then $\gamma_{sR}(D) \le n$ with equality if and only if D is the disjoint union of isolated vertices and oriented triangles C_3.

Proposition B. *([11])* If D is a digraph of order n with minimum in-degree $\delta^-(D) \ge k - 1$, then $\gamma^k_{sR}(D) \le n$.

Proposition C. *([1, 4])* If K_n is the complete graph of order $n \ge k \ge 1$, then $\gamma^k_{sR}(K_n) = k$, unless $k = 1$ and $n = 3$ in which case $\gamma_{sR}(K_3) = 2$.

Proposition D. *([6, 10])* If K_n is the complete graph of order $n \ge k \ge 1$, then $d^k_{sR}(K_n) = n$, unless $k = 1$ and $n = 3$ in which case $d_{sR}(K_3) = 1$ and unless $n = k = 2$ in which case $d^2_{sR}(K_2) = 1$.

Proposition E. *([11])* If D is a digraph of order n with $\delta^-(D) \ge k + 1$, then $\gamma^k_{sR}(D) \le n - 1$.

Proposition F. *([11])* If D is an δ-out-regular digraph of order n with $\delta \geq k-1$, then

$$\gamma_{sR}^k(D) \geq \left\lceil \frac{kn}{\delta+1} \right\rceil.$$

Proposition G. *([4])* If $k \geq 2$, then $\gamma_{sR}^k(K_{k,k}) = 2k$.

Proposition H. *([10])* If $k \geq 4$ is an even integer, then $d_{sR}^k(K_{k,k}) = k$.

The *associated digraph* G^* of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N_{G^*}^-[v] = N_G[v]$ for each vertex $v \in V(G) = V(G^*)$, the following useful observation is valid.

Observation 1. If G^* is the associated digraph of the graph G, then $\gamma_{sR}^k(G^*) = \gamma_{sR}^k(G)$ and $d_{sR}^k(G^*) = d_{sR}^k(G)$.

Let K_n^* be the associated digraph of the complete graph K_n. Using Observation 1 and Propositions C, D, we obtain the signed Roman k-domination number and the signed Roman k-domatic number of the complete digraph K_n^*.

Corollary 2. If K_n^* is the complete digraph of order $n \geq k \geq 1$, then $\gamma_{sR}^k(K_n^*) = k$, unless $k = 1$ and $n = 3$ in which case $\gamma_{sR}(K_3^*) = 2$.

Corollary 3. If K_n^* is the complete digraph of order $n \geq k \geq 1$, then $d_{sR}^k(K_n^*) = n$, unless $k = 1$ and $n = 3$ in which case $d_{sR}(K_3^*) = 1$ and unless $n = k = 2$ in which case $d_{sR}^2(K_2^*) = 1$.

Let $K_{p,p}^*$ be the associated digraph of the complete bipartite graph $K_{p,p}$. Observation 1, Propositions G and H lead to the next results immediately.

Corollary 4. If $k \geq 2$, then $\gamma_{sR}^k(K_{k,k}^*) = 2k$.

Corollary 5. If $k \geq 4$ is an even integer, then $d_{sR}^k(K_{k,k}^*) = k$.

2 Bounds on the signed Roman k-domatic number

In this section we present basic properties of $d_{sR}^k(D)$ and sharp bounds on the signed Roman k-domatic number of a graph.

Theorem 2.1. *If D is a digraph with $\delta^-(D) \geq k-1$, then*

$$d_{sR}^k(D) \leq \delta^-(D) + 1.$$

Moreover, if $d_{sR}^k(D) = \delta^-(D) + 1$, then for each SRkD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_{sR}^k(D)$ and each vertex v of minimum in-degree, $\sum_{x \in N^-[v]} f_i(x) = k$ for each function f_i and $\sum_{i=1}^d f_i(x) = k$ for all $x \in N^-[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an SRkD family on D such that $d = d^k_{sR}(D)$. If v is a vertex of minimum in-degree $\delta^-(D)$, then we deduce that

$$\begin{aligned} kd &\le \sum_{i=1}^{d} \sum_{x\in N^-[v]} f_i(x) = \sum_{x\in N^-[v]} \sum_{i=1}^{d} f_i(x) \\ &\le \sum_{x\in N^-[v]} k = k(\delta^-(D)+1) \end{aligned}$$

and thus $d^k_{sR}(D) \le \delta^-(D) + 1$.

If $d^k_{sR}(D) = \delta^-(D) + 1$, then the two inequalities occurring in the proof become equalities. Hence for the SRkD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each vertex v of minimum in-degree, $\sum_{x\in N^-[v]} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N^-[v]$. □

Example 2.2. If C^*_{3t} is the associated digraph of a cycle C_{3t} of length $3t$ with an integer $t \ge 1$, then $d^2_{sR}(C^*_{3t}) = 3$.

Proof. According to Theorem 2.1, $d^2_{sR}(C^*_{3t}) \le 3$. Let $C^*_{3t} = v_0v_1\ldots v_{3t-1}v_0$. Define the functions f_1, f_2, f_3 by

$$f_1(v_{3i}) = 2,\ \ f_1(v_{3i+1}) = 1,\ \ f_1(v_{3i+2}) = -1,$$

$$f_2(v_{3i}) = -1,\ \ f_2(v_{3i+1}) = 2,\ \ f_2(v_{3i+2}) = 1,$$

$$f_3(v_{3i}) = 1,\ \ f_3(v_{3i+1}) = -1,\ \ f_3(v_{3i+2}) = 2$$

for $0 \le i \le t-1$. It is easy to see that f_i is a signed Roman 2-dominating function on C^*_{3t} for $1 \le i \le 3$ and $\{f_1, f_2, f_3\}$ is a signed Roman 2-dominating family on C^*_{3t}. Therefore $d^2_{sR}(C^*_{3t}) \ge 3$ and so $d^2_{sR}(C^*_{3t}) = 3$. □

Example 2.3. Let $C_{3t} = v_0v_1\ldots v_{3t-1}v_0$ be a cycle with an integer $t \ge 1$. Add t new vertices $w_0, w_1, \ldots, w_{t-1}$ and join w_i to the three vertices v_{3i+2}, v_{3i+1} and v_{3i} for $i = 0, 1, \ldots, t-1$. If G is the resulting cubic graph, then let G^* be the associated digraph of G. We have $d^3_{sR}(G^*) = 4$.

Proof. According to Theorem 2.1, $d^3_{sR}(G^*) \le 4$. Define the functions f_1, f_2, f_3, f_4 by

$$f_1(w_i) = -1,\ \ f_1(v_{3i}) = 2,\ \ f_1(v_{3i+1}) = 1,\ \ f_1(v_{3i+2}) = 1,$$

$$f_2(w_i) = 1,\ \ f_2(v_{3i}) = -1,\ \ f_2(v_{3i+1}) = 2,\ \ f_2(v_{3i+2}) = 1,$$

$$f_3(w_i) = 1,\ \ f_3(v_{3i}) = 1,\ \ f_3(v_{3i+1}) = -1,\ \ f_3(v_{3i+2}) = 2,$$

$$f_4(w_i) = 2,\ \ f_4(v_{3i}) = 1,\ \ f_4(v_{3i+1}) = 1,\ \ f_4(v_{3i+2}) = -1$$

for $0 \le i \le t-1$. It is easy to see that f_i is a signed Roman 3-dominating function on G^* for $1 \le i \le 4$ and $\{f_1, f_2, f_3, f_4\}$ is a signed Roman 3-dominating family on G^*. Therefore $d^3_{sR}(G^*) \ge 4$ and so $d^3_{sR}(G^*) = 4$. □

Examples 2.2 and 2.3 show that Theorem 2.1 is sharp for $k = 2$ as well as for $k = 3$.

Theorem 2.4. *If D is a digraph of order n, then*

$$\gamma^k_{sR}(D) \cdot d^k_{sR}(D) \le kn.$$

Moreover, if $\gamma^k_{sR}(D) \cdot d^k_{sR}(D) = kn$, then for each SRkD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d^k_{sR}(D)$, each function f_i is a $\gamma^k_{sR}(D)$-function and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an SRkD family on D such that $d = d^k_{sR}(D)$. Then

$$\begin{aligned} d \cdot \gamma^k_{sR}(D) &= \sum_{i=1}^{d} \gamma^k_{sR}(D) \le \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) \\ &= \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(D)} k = kn. \end{aligned}$$

If $\gamma^k_{sR}(D) \cdot d^k_{sR}(D) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the SRkD family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i, $\sum_{v \in V(D)} f_i(v) = \gamma^k_{sR}(D)$. Thus each function f_i is a $\gamma^k_{sR}(D)$-function, and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(D)$. □

Corollaries 2 and 3 demonstrate that Theorems 2.1 and 2.4 are both sharp.

Let G^* be the associated digraph of the graph G of order n. Since $\delta^-(G^*) = \delta(G)$, $\gamma^k_{sR}(G^*) = \gamma^k_{sR}(G)$ and $d^k_{sR}(G^*) = d^k_{sR}(G)$, Theorems 2.1 and 2.4 lead to $d^k_{sR}(G) \le \delta(G) + 1$ and $\gamma^k_{sR}(G) \cdot d^k_{sR}(G) \le kn$ immediately. These known bounds can be found in [10].

Using the upper bound on the product $\gamma^k_{sR}(D) \cdot d^k_{sR}(D)$ in Theorem 2.4, we obtain a sharp upper bound on the sum of these two parameters.

Theorem 2.5. *If D is a digraph of order $n \ge 1$ and $\delta^-(D) \ge k - 1$, then*

$$\gamma^k_{sR}(D) + d^k_{sR}(D) \le n + k.$$

If $\gamma^k_{sR}(D) + d^k_{sR}(D) = n + k$, then

(a) $\gamma^k_{sR}(D) = k$ and $d^k_{sR}(D) = n$ (in this case $D = K^*_n$ unless $k = 1$ and $n = 3$ or $k = n = 2$) or

(b) $\gamma^k_{sR}(D) = n$ and $d^k_{sR}(D) = k$ (in this case D is the disjoint union of isolated vertices and oriented triangles when $k = 1$, $k \ne 2$ and $k - 1 \le \delta^-(D) \le k$ when $k \ge 3$).

Proof. If $d^k_{sR}(D) \le k$, then Proposition B implies $\gamma^k_{sR}(D) + d^k_{sR}(D) \le n + k$ immediately. Let now $d^k_{sR}(D) \ge k$. It follows from Theorem 2.4 that

$$\gamma^k_{sR}(D) + d^k_{sR}(D) \le \frac{kn}{d^k_{sR}(D)} + d^k_{sR}(D).$$

According to Theorem 2.1, we have $k \le d^k_{sR}(D) \le n$. Using these bounds, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, we obtain

$$\gamma^k_{sR}(D) + d^k_{sR}(D) \le \frac{kn}{d^k_{sR}(D)} + d^k_{sR}(D) \le \max\{n + k, k + n\} = n + k,$$

and the desired bound is proved.

Now assume that $\gamma^k_{sR}(D) + d^k_{sR}(D) = n + k$. The above inequality leads to

$$n + k = \gamma^k_{sR}(D) + d^k_{sR}(D) \le \frac{kn}{d^k_{sR}(D)} + d^k_{sR}(D) \le n + k.$$

This implies that $d^k_{sR}(D) = n$ and $\gamma^k_{sR}(D) = k$ or $d^k_{sR}(D) = k$ and $\gamma^k_{sR}(D) = n$.

(a) If $d^k_{sR}(D) = n$ and $\gamma^k_{sR}(D) = k$, then $\delta^-(D) = n - 1$, by Theorem 2.1 and thus D is the complete digraph. In view of Corollaries 2 and 3, the digraph D is isomorphic to K^*_n unless $n = 3$ and $k = 1$ or $n = k = 2$.

(b) If $d^k_{sR}(D) = k$ and $\gamma^k_{sR}(D) = n$, then it follows from Proposition E that $k - 1 \le \delta^-(D) \le k$.

If $k = 1$, then Proposition A shows that D consists of the disjoint union of isolated vertices and oriented triangles.

If $k = 2$, then suppose that $\{f_1, f_2\}$ is an SR2D family on D. By Theorem 2.4 f_1 and f_2 are $\gamma^2_{sR}(D)$-functions and $f_1(v) + f_2(v) = 2$ for all $v \in V(D)$. This yields to the contradiction that $f_1(v) = f_2(v) = 1$ for each $v \in V(D)$, and thus $k = 2$ is not possible in that case. □

Corollaries 2 and 3 imply that $\gamma^k_{sR}(K^*_n) + d^k_{sR}(K^*_n) = n + k$, unless $k = 1$ and $n = 3$ or $k = n = 2$. Therefore Theorem 2.5 is sharp.

Example 2.6. If C^*_{3t} is the associated digraph of the cycle C_{3t} of length $3t$ with an integer $t \ge 1$, then $d^3_{sR}(C^*_{3t}) = 3$.

Proof. According to Theorem 2.1, $d^3_{sR}(C^*_{3t}) \le 3$. Let $C^*_{3t} = v_0v_1, \dots v_{3t-1}v_0$. Define the functions f_1, f_2, f_3 by

$$f_1(v_{3i+1}) = -1, \ \ f_1(v_{3i+2}) = 2, \ \ f_1(v_{3i}) = 2,$$

$$f_2(v_{3i+1}) = 2, \ \ f_2(v_{3i+2}) = -1, \ \ f_2(v_{3i}) = 2,$$

$$f_3(v_{3i+1}) = 2, \ \ f_3(v_{3i+2}) = 2, \ \ f_3(v_{3i}) = -1$$

for $0 \le i \le t - 1$. It is easy to see that f_i is a signed Roman 3-dominating function on C^*_{3t} of weight $3t$ for $1 \le i \le 3$ and $\{f_1, f_2, f_3\}$ is a signed Roman 3-dominating family on C^*_{3t}. Therefore $d^3_{sR}(C^*_{3t}) \ge 3$ and so $d^3_{sR}(C^*_{3t}) = 3$. □

Example 2.7. Let $C_{3t} = v_0v_1, \dots v_{3t-1}v_0$ be a cycle of length $3t$ with an integer $t \geq 1$. Add t new vertices $w_0, w_1, \dots, w_{t-1}$ and join w_i to the three vertices v_{3i+2}, v_{3i+1} and v_{3i} for $i = 0, 1, \dots, t-1$. If H is the resulting cubic graph, then let H^* be the associated digraph of H. Then we have $d^4_{sR}(H^*) = 4$.

Proof. According to Theorem 2.1, $d^4_{sR}(H^*) \leq 4$. Define the functions f_1, f_2, f_3, f_4 by

$$f_1(w_i) = -1,\ \ f_1(v_{3i}) = 2,\ \ f_1(v_{3i+1}) = 2,\ \ f_1(v_{3i+2}) = 1,$$
$$f_2(w_i) = 1,\ \ f_2(v_{3i}) = -1,\ \ f_2(v_{3i+1}) = 2,\ \ f_2(v_{3i+2}) = 2,$$
$$f_3(w_i) = 2,\ \ f_3(v_{3i}) = 1,\ \ f_3(v_{3i+1}) = -1,\ \ f_3(v_{3i+2}) = 2,$$
$$f_4(w_i) = 2,\ \ f_4(v_{3i}) = 2,\ \ f_4(v_{3i+1}) = 1,\ \ f_4(v_{3i+2}) = -1$$

for $0 \leq i \leq t-1$. It is easy to see that f_i is a signed Roman 4-dominating function on H^* for $1 \leq i \leq 4$ and $\{f_1, f_2, f_3, f_4\}$ is a signed Roman 4-dominating family on H^*. Therefore $d^4_{sR}(H^*) \geq 4$ and so $d^4_{sR}(H^*) = 4$. □

It follows from Proposition F that $\gamma^3_{sR}(C^*_{3t}) \geq 3t$ and so $\gamma^3_{sR}(C^*_{3t}) = 3t$ by Proposition B. For the digraph H^* in Example 2.7, it follows from Proposition F that $\gamma^4_{sR}(H^*) \geq 4t$ and so $\gamma^4_{sR}(H^*) = 4t = n(H^*)$ by Proposition B.

Thus Examples 2.6, 2.7 and Corollaries 4 and 5 show that Case (b) in Theorem 2.5 is possible for $\delta^- = k-1$ as well as for $\delta^- = k$.

For some regular digraphs we will improve the upper bound given in Theorem 2.1.

Theorem 2.8. *Let D be a δ-out-regular digraph of order n with $\delta \geq k-1$ such that $n = p(\delta+1)+r$ with integers $p \geq 1$ and $1 \leq r \leq \delta$ and $kr = t(\delta+1)+s$ with integers $t \geq 0$ and $1 \leq s \leq \delta$. Then $d^k_{sR}(D) \leq \delta$.*

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be an SRkD family on D such that $d = d^k_{sR}(D)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} k = kn.$$

Proposition F implies

$$\begin{aligned} \omega(f_i) &\geq \gamma^k_{sR}(D) \geq \left\lceil \frac{kn}{\delta+1} \right\rceil = \left\lceil \frac{kp(\delta+1)+kr}{\delta+1} \right\rceil \\ &= kp + \left\lceil \frac{kr}{\delta+1} \right\rceil = kp + \left\lceil \frac{t(\delta+1)+s}{\delta+1} \right\rceil = kp+t+1 \end{aligned}$$

for each $i \in \{1, 2, \dots, d\}$. If we suppose to the contrary that $d \geq \delta+1$, then the above inequality chains lead to the contradiction

$$\begin{aligned} kn &\geq \sum_{i=1}^{d} \omega(f_i) \geq d(kp+t+1) \geq (\delta+1)(kp+t+1) \\ &= kp(\delta+1) + (\delta+1)(t+1) = kp(\delta+1) + t(\delta+1) + \delta + 1 \\ &= kp(\delta+1) + kr - s + \delta + 1 > kp(\delta+1) + kr = k(p(\delta+1)+r) = kn. \end{aligned}$$

Thus $d \leq \delta$, and the proof is complete. □

Corollary 5 shows that Theorem 2.8 is sharp, and Corollary 3 demonstrates that Theorem 2.8 is not valid in general. A digraph without directed cycles of length 2 is called an *oriented graph*. An oriented graph D is called a *tournament* when either $(u,v) \in A(D)$ or $(v,u) \in A(D)$ for each pair of distinct vertices $u,v \in V(D)$. By D^{-1} we denote the digraph obtained by reversing all arcs of D.

Theorem 2.9. *If T is a δ-regular tournament of order n such that $\delta^-(T) \geq k$, then $d^k_{sR}(T) \leq \delta$.*

Proof. Since T is a δ-regular tournament, we observe that $n = 2\delta + 1$. Since $n = p(\delta+1) + r = (\delta+1) + \delta$ and $kr = k\delta = t(\delta+1) + s = (k-1)(\delta+1) + (\delta - k + 1)$ and $s = \delta - k + 1 \geq 1$, it follows from Theorem 2.8 that $d^k_{sR}(D) \leq \delta$. □

Corollary 2.10. *If D is an oriented graph of order n such that $\delta^-(D), \delta^-(D^{-1}) \geq k$, then*

$$d^k_{sR}(D) + d^k_{sR}(D^{-1}) \leq n.$$

Proof. If D is not a tournament or D is a non-regular tournament, then $\delta^-(D) + \delta^-(D^{-1}) \leq n - 2$, and hence we deduce from Theorem 2.1 that

$$d^k_{sR}(D) + d^k_{sR}(D^{-1}) \leq (\delta^-(D) + 1) + (\delta^-(D^{-1}) + 1) \leq n.$$

Let now D be a δ-regular tournament. Then D^{-1} is also a δ-regular tournament such that $n = 2\delta + 1$. Thus it follows from Theorem 2.9 that

$$d^k_{sR}(D) + d^k_{sR}(D^{-1}) \leq \delta + \delta = 2\delta = n - 1.$$

This completes the proof. □

The proof of Corollary 2.10 also implies the next result immediately.

Corollary 2.11. *If T is δ-regular tournament of order n such that $\delta^-(T) \geq k$, then $d^k_{sR}(T) + d^k_{sR}(T^{-1}) \leq n - 1$.*

3 Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph or digraph and its complement. In their classical paper [5], Nordhaus and Gaddum discussed this problem for the chromatic number of graphs. We present such inequalities for the signed Roman k-domatic number of digraphs.

Theorem 3.1. *If D is a digraph of order n such that $\delta^-(D), \delta^-(\overline{D}) \geq k - 1$, then*

$$d^k_{sR}(D) + d^k_{sR}(\overline{D}) \leq n + 1.$$

Furthermore, if $d^k_{sR}(D) + d^k_{sR}(\overline{D}) = n + 1$, then D is in-regular.

Proof. It follows from Theorem 2.1 that

$$\begin{aligned} d_{sR}^k(D)+d_{sR}^k(\overline{D}) &\le (\delta^-(D)+1)+(\delta^-(\overline{D})+1) \\ &= (\delta^-(D)+1)+(n-\Delta^-(D)-1+1)\le n+1. \end{aligned}$$

If D is not in-regular, then $\Delta^-(D)-\delta^-(D)\ge 1$, and hence the above inequality chain implies the better bound $d_{sR}^k(D)+d_{sR}^k(\overline{D})\le n$. □

For tournaments of odd order we improve Theorem 3.1.

Theorem 3.2. *If T is a tournament of odd order $n\ge 3$ such that $\delta^-(T),\delta^-(\overline{T})\ge k$, then*

$$d_{sR}^k(T)+d_{sR}^k(\overline{T})\le n-1.$$

Proof. If T is not regular, then $\delta^-(T)\le (n-3)/2$ and $\delta^-(\overline{T})\le (n-3)/2$. Hence Theorem 2.1 implies that

$$d_{sR}^k(T)+d_{sR}^k(\overline{T})\le (\delta^-(T)+1)+(\delta^-(\overline{T})+1)\le \frac{n-3}{2}+\frac{n-3}{2}+2=n-1.$$

Let now T be a δ-regular tournament. Then $\overline{T}$ is also a δ-regular tournament such that $n=2\delta+1$. Thus it follows from Theorem 2.9 that

$$d_{sR}^k(T)+d_{sR}^k(\overline{T})\le \delta+\delta=2\delta=n-1.$$

□

In [7], we have proved the following Nordhaus-Gaddum type inequality for regular digraphs.

Theorem 3.3. *Let D be an δ-regular digraph of order n. Then $d_{sR}(D)+d_{sR}(\overline{D})\le n+1$ with equality if and only if $D=K_n^*$ or $\overline{D}=K_n^*$ and $n\ne 3$.*

As a supplement to Theorem 3.3, we present the following result for $k\ge 2$.

Theorem 3.4. *Let $k\ge 2$ be an integer, and let D be a δ-regular digraph such that $\delta\ge k-1$ and $\overline{\delta}=\delta^-(\overline{D})\ge k-1$. Then there is only a finite number of digraphs D such that*

$$d_{sR}^k(D)+d_{sR}^k(\overline{D})=n(D)+1.$$

Proof. Let $n(G)=n$. The strategy of our proof is as follows. For a fixed integer $k\ge 2$, we show that $d_{sR}^k(D)+d_{sR}^k(\overline{D})\le n$ or $n\le k^3+\frac{5}{2}k^2-2k+1$. Together with Theorem 3.1 this implies the desired result.

Since D is δ-regular, $\overline{D}$ is $\overline{\delta}$-regular such that $\delta+\overline{\delta}+1=n$. Assume, without loss of generality, that $\overline{\delta}\le\delta$.

Let $k\overline{\delta} = t(\delta+1)+s$ with integers $t \geq 0$ and $0 \leq s \leq \delta$. If $s \neq 0$, then we deduce from Theorem 2.8 that $d^k_{sR}(D) \leq \delta$, and Theorem 2.1 yields to

$$d^k_{sR}(D) + d^k_{sR}(\overline{D}) \leq \delta + (\overline{\delta}+1) = n.$$

If $s = 0$, then the condition $\overline{\delta} \leq \delta$ shows that

$$k\overline{\delta} = t(\delta+1) \text{ with } 1 \leq t \leq k-1 \tag{1}$$

and thus

$$\delta = \frac{k\overline{\delta}}{t} - 1. \tag{2}$$

Let now

$$n = p(\overline{\delta}+1) + r \text{ with integers } p \geq 1 \text{ and } 0 \leq r \leq \overline{\delta} \tag{3}$$

and when $r \neq 0$

$$kr = a(\overline{\delta}+1) + b \text{ with integers } a \geq 0 \text{ and } 0 \leq b \leq \overline{\delta}. \tag{4}$$

If $b, r \neq 0$, then we conclude from Theorem 2.8 that $d^k_{sR}(\overline{D}) \leq \overline{\delta}$, and we obtain by Theorem 2.1

$$d^k_{sR}(D) + d^k_{sR}(\overline{D}) \leq (\delta+1) + \overline{\delta} = n.$$

Now let $r \neq 0$ and $b = 0$. Then (3) and (4) yield to

$$kr = a(\overline{\delta}+1) \text{ with } 1 \leq a \leq k-1$$

and thus

$$\overline{\delta} = \frac{kr}{a} - 1. \tag{5}$$

In view of (2), we obtain

$$\delta = \frac{k}{t}\left(\frac{kr}{a} - 1\right) - 1$$

and so

$$n = \delta + \overline{\delta} + 1 = \frac{k}{t}\left(\frac{kr}{a} - 1\right) + \frac{kr}{a} - 1. \tag{6}$$

According to (3) and (5), we have

$$n = p(\overline{\delta}+1) + r = \frac{pkr}{a} + r. \tag{7}$$

Combining (6) and (7), we find that

$$r\left(\frac{pk}{a} + 1\right) = \frac{kr}{a}\left(\frac{k}{t} + 1\right) - \frac{k}{t} - 1$$

and therefore

$$1 + \frac{k}{t} = r\left(\frac{k^2}{at} + \frac{k}{a} - \frac{pk}{a} - 1\right) = \frac{kr}{a}\left(\frac{k}{t} + 1 - p\right) - r. \tag{8}$$

These equalities show that

$$\frac{k^2}{at} + \frac{k}{a} - \frac{pk}{a} - 1 > 0 \text{ and } \frac{k}{t} + 1 - p > 0$$

and hence

$$\frac{k^2}{at} + \frac{k}{a} - \frac{pk}{a} - 1 \geq \frac{1}{at}. \tag{9}$$

and

$$\frac{k}{t} + 1 - p \geq \frac{1}{t}.$$

We deduce from the last inequality that

$$p \leq \frac{k-1}{t} + 1 \leq k. \tag{10}$$

Using (9) and the first equality in (8), we obtain

$$1 + \frac{k}{t} \geq \frac{r}{at}$$

and thus

$$r \leq at + ak. \tag{11}$$

In view of (5), it follows that

$$\overline{\delta} + 1 = \frac{kr}{a} \leq kt + k^2. \tag{12}$$

If $t = 1$, then we deduce from (3), (10), (11), $a \leq k - 1$ and the last inequality leads to the desired bound as follows

$$\begin{aligned} n &= p(\overline{\delta} + 1) + r \leq k(kt + k^2) + at + ak \\ &\leq k(k + k^2) + (k - 1) + k(k - 1) \\ &= k^3 + 2k^2 - 1 \leq k^3 + \frac{5}{2}k^2 - 2k + 1. \end{aligned}$$

If $t \geq 2$, then the first inequality of (10) leads to $p \leq \frac{k+1}{2}$. Applying this bound, (3), (11), (12), $t \leq k - 1$ and $a \leq k - 1$, we arrive at the desired bound

$$\begin{aligned} n &= p(\overline{\delta} + 1) + r \leq \frac{k+1}{2}(kt + k^2) + at + ak \\ &\leq \frac{k+1}{2}(k(k-1) + k^2) + (k-1)^2 + k(k-1) \\ &= k^3 + \frac{5}{2}k^2 - \frac{7}{2}k + 1 \leq k^3 + \frac{5}{2}k^2 - 2k + 1. \end{aligned}$$

It remains the case that $r = 0$ and thus $n = p(\overline{\delta} + 1)$ with an integer $p \geq 2$. Since $n = \delta + \overline{\delta} + 1$, we deduce that

$$\delta + 1 = (p - 1)\overline{\delta} + p.$$

Using this identity and (1), we obtain

$$k\overline{\delta} = t(\delta + 1) = t(p-1)\overline{\delta} + tp$$

and thus

$$tp = \overline{\delta}(k - t(p-1)).$$

It follows that $t(p-1) \le k-1$ and so $tp \ge \overline{\delta}$ and $p \le k$. Therefore $\overline{\delta} \le tp \le k(k-1)$ and consequently,

$$n = p(\overline{\delta} + 1) \le k(k(k-1)+1) = k^3 - k^2 + k \le k^3 + \frac{5}{2}k^2 - 2k + 1.$$

This completes the proof. □

Example 3.5. Let $k \ge 3$ be an integer and let D be the disjoint union of two copies of the complete digraph K_k^*. Then $d_{sR}^k(D) = k$.

Proof. The digraph $D = K_k^* \cup K_k^*$ is k-regular of order $2k$. Since $2k = p(\delta+1)+r = (k+1)+(k-1)$ and $kr = k(k-1) = t(k+1)+s = (k-2)(k+1)+2$ and $s = 2 \le k$, it follows from Theorem 2.8 that $d_{sR}^k(D) \le k$.

Now let $\{v_0, v_1, \ldots, v_{k-1}\}$ be the vertex set of one copy of K_k^* and $\{w_0, w_1, \ldots, w_{k-1}\}$ the vertex set of the other copy of K_k^*. Define the functions $f_1, f_2, \ldots, f_k$ by $f_1(v_0) = f_1(v_{k-1}) = f_1(w_0) = f_1(w_{k-1}) = 2$, $f_1(v_1) = f_1(w_1) = -1$ and $f_1(v_i) = f_1(w_i) = 1$ for $2 \le i \le k-2$ and for $2 \le j \le k$ and $0 \le i \le k-1$

$$f_j(v_i) = f_{j-1}(v_{i+j-1}) \text{ and } f_j(w_i) = f_{j-1}(w_{i+j-1}),$$

where the indices are taken modulo k. It is easy to see that f_i is a signed Roman k-dominating function on D for $1 \le i \le k$ and $\{f_1, f_2, \ldots, f_k\}$ is a signed Roman k-dominating family on D. Hence $d_{sR}^k(D) \ge k$ and thus $d_{sR}^k(D) = k$. □

Example 3.5 also demonstrates the sharpnes of Theorem 2.8

Conjecture 3.6. *Let $k \ge 2$ be an integer. If D is a δ-regular digraph of order n such that $\delta, \overline{\delta} \ge k-1$, then*

$$d_{sR}^k(D) + d_{sR}^k(\overline{D}) \le n.$$

If $k \ge 4$ is an even integer, then Corollary 5 and Example 3.5 show that

$$d_{sR}^k(K_{k,k}^*) + d_{sR}^k(\overline{K_{k,k}^*}) = 2k = n(K_{k,k}^*).$$

Thus Conjecture 3.6 would be tight, at least for $k \ge 4$ even.

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