AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **64(2)** (2016), Pages 379–391

Thickly-resolvable block designs

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Abstract

We show that the necessary divisibility conditions for the existence of a σ-resolvable BIBD(v, k, λ) are sufficient for large v. The key idea is to form an auxiliary graph based on an $[r, k]$-configuration with $r = \sigma$, and then edge-decompose the complete λ-fold graph $K_v^{(\lambda)}$ into this graph. As a consequence, we initiate a similar existence theory for incomplete designs with index λ.

* Supported by NSERC grant 312595–2010

1 Introduction

A *balanced incomplete block design*, or BIBD(v, k, λ) is a pair $(V, \mathcal{B})$, where V is a set of v *points*, $\mathcal{B}$ is a family of k-element subsets of V called *blocks*, and such that any two distinct points appear together in exactly λ blocks. The parameters v, k, λ are often called the *order*, *block size* and *index*, respectively. BIBDs are also known as 2-designs, since they are pairwise balanced set systems.

A partition of the point set V of a design into blocks is called a *parallel class* or *resolution class*. A design is called *resolvable* if the block collection $\mathcal{B}$ can be completely partitioned into parallel classes. Existence questions for resolvable BIBDs have a long history and date back to 1850 with the famous Kirkman Schoolgirl problem. The 'asymptotic' (large order) existence question for resolvable BIBDs of index unity was settled by R.M. Wilson and D.K. Ray-Chaudhuri in 1973.

Theorem 1.1 ([12]) *Let k be an integer at least 2. For sufficiently large v, there exists a resolvable BIBD$(v, k, 1)$ if and only if $v \equiv k \pmod{k(k-1)}$.*

Then, in [11], J.X. Lu extended this result to the case of arbitrary index λ. In this more general case, the necessary congruence condition on v becomes

$$\begin{aligned} v &\equiv 0 \pmod{k}, \text{ and} \\ \lambda(v-1) &\equiv 0 \pmod{k-1}. \end{aligned}$$

The first condition is necessary for the existence of a parallel class, while the second condition, true in any BIBD(v, k, λ), arises from the integrality of the number of blocks at a point $x \in V$.

We are interested here in a parameterized weakening of resolvability. A *σ-parallel class* in $(V, \mathcal{B})$ is a sub-collection of blocks $\mathcal{A}$ such that every point in V belongs to exactly σ of the blocks in $\mathcal{A}$. In alternative language, $(V, \mathcal{A})$ forms as a 1-design with index σ, a parameter which might be called the 'thickness' of the class. A balanced incomplete block design $(V, \mathcal{B})$ is *σ-resolvable* if $\mathcal{B}$ admits a partition into σ-parallel classes. The necessary divisibility conditions for σ-resolvable designs are easy extensions of the above congruences. We have

$$\sigma v \equiv 0 \pmod{k}, \text{ and} \tag{1.1}$$
$$\lambda(v-1) \equiv 0 \pmod{\sigma(k-1)}. \tag{1.2}$$

These can be seen by noting that $\lambda(v-1)/\sigma(k-1)$ counts the parallel classes, while $\sigma v/k$ counts the blocks in each parallel class.

The integers v satisfying (1.1) and (1.2) are 'admissible' for σ, λ, k. In what follows, for convenience we put $a := \sigma(k-1)/\gcd(\sigma(k-1), \lambda)$ and $b := k/\gcd(k, \sigma)$. Then admissibility can be simply stated as $v \equiv 1 \pmod{a}$ and $v \equiv 0 \pmod{b}$. We may assume $\gcd(a, b) = 1$, or else there are no solutions v. With this assumption, admissible orders v are periodic with least period $\pi := ab$.

In the case of block size three, it was shown in [8] that (1.1–1.2) are sufficient for all positive integers v, σ, λ, except for $v = 6$, $\sigma = 1$ and $\lambda \equiv 2 \pmod 4$. A similar result was shown in [13] for the case $k = 4$, this time with only one exception: $v = 10$, $\lambda = \sigma = 2$. Other than these small block sizes and some other special cases, little else is known about σ-resolvable designs with $\sigma > 1$.

We remark that most authors use the parameter α instead of σ to denote the class thickness. We have chosen the latter to avoid conflict with the standard parameter $\alpha(K) = \gcd\{k-1 : k \in K\}$ used for a set of block sizes in a pairwise balanced design (and more generally for a family of graphs in a decomposition). We also suggest adoption of a generic term such as 'thickly-resolvable' BIBD.

Our main result is the existence of σ-resolvable BIBD(v, k, λ) for all large admissible v.

Theorem 1.2 *Let $k \geq 2$, $\sigma \geq 1$ and $\lambda \geq 0$ be integers. For sufficiently large v, there exists a σ-resolvable BIBD(v, k, λ) if and only if* (1.1) *and* (1.2) *hold.*

Our proof combines similar existence theories for combinatorial configurations, resolvable graph decompositions, and frames. The background for these structures is covered in the next section. Then, in Section 3, we construct examples of σ-resolvable BIBD(v, k, λ) for v in each admissible congruence class modulo a large multiple of π. These fibers are extended with a recursion similar to that used in [3] for resolvable group divisible designs. In Section 4, we discuss an important connection with 'incomplete' designs of arbitrary index.

2 Background

2.1 Combinatorial configurations

An (n_r, m_k)-*configuration* is a triple $(U, \mathcal{A}, \iota)$, where U is an n-set of *points*, $\mathcal{A}$ is an m-set of *lines*, and $\iota \subseteq U \times \mathcal{A}$ is a relation called *incidence* such that

- every line is incident with exactly k points,
- every point is incident with exactly r lines, and
- every pair of distinct points are together incident with at most one line.

It is clear by counting flags in ι that $nr = mk$ is necessary. Such configurations are equivalent to linear k-uniform (pointwise) r-regular hypergraphs. In design theoretic language, these are 1-designs with block size k and replication number r which are simultaneously 2-packings of index one. As an example, we can take the union of any r parallel classes of an affine plane of order q to obtain constructions with $n = q^2$, $k = q$.

Although 'geometric' (n_r, m_k)-configurations require embeddings in the Euclidean (or projective) plane, we stress here that our use requires only abstract points and lines. That is, we consider 'combinatorial' configurations. Since these structures are closed under disjoint unions, the asymptotic existence (in n) is not hard; see [1] for a (constructive) proof of the following.

Theorem 2.1 *Given integers $k \geq 2$ and $r \geq 1$, there exists a combinatorial (n_r, m_k)-configuration with $m = \dfrac{nr}{k}$ for all sufficiently large integers $n \equiv 0 \pmod{k/\gcd(k,r)}$.*

2.2 Graph decompositions

A BIBD is equivalent to an edge-decomposition of the λ-fold complete graph $K_v^{(\lambda)}$ into copies of cliques K_k. Let us replace K_k by an arbitrary simple undirected graph G. A *G-decomposition* (or *G-design*) of *order* v and *index* λ is a pair $(V, \mathcal{B})$, where V is a set of points, $\mathcal{B}$ is a collection of graphs on vertices in V, each isomorphic to G, and such that every unordered pair of points of V is an edge of exactly λ graphs in $\mathcal{B}$. The elements of $\mathcal{B}$ are called *G-blocks* or just *blocks*. Sticking with BIBD notation, we abbreviate a G-design of order v and index λ as simply a (v, G, λ)-design.

A *parallel class* in a G-design $(V, \mathcal{B})$ is a collection of vertex-disjoint copies of G which spans V. As before, a graph design is *resolvable* if its blocks can be completely partitioned into parallel classes. We make use of the following existence result for resolvable graph designs.

Theorem 2.2 ([6]) *Let G be a simple graph with n vertices, $e > 0$ edges, and vertex degrees $d_1, \dots, d_n$. There exists a resolvable (v, G, λ)-design for all sufficiently large v satisfying $v \equiv 0 \pmod{n}$ and $\lambda(v-1) \equiv 0 \pmod{\alpha^*}$, where α^*, a function of G alone, is the least positive integer A such that the ordered pair $[A, An/2e]$ is an integral linear combination of $[d_i, 1]$, $i = 1, \dots, n$.*

The conditions in Theorem 2.2 are necessary for similar reasons as in the case of BIBDs. We offer a quick justification for the second numerical condition. By dividing, the number of parallel classes is

$$\frac{\lambda}{e}\binom{v}{2} \Big/ \frac{v}{n} = \frac{\lambda(v-1)n}{2e}.$$

This must also equal the common number of blocks at each point, and so this many vertex degrees in G-blocks must combine somehow to produce $\lambda(v-1)$, the degree in $K_v^{(\lambda)}$.

Observe that, for d-regular graphs, we have simply $\alpha^* = d$, because the sum of degrees is $nd = 2e$.

2.3 Frames and subdesigns

A *group divisible design*, or (k, λ)-GDD (or simply GDD), is a triple $(X, \Pi, \mathcal{B})$, where X is a set of *points*, Π is a partition of X into sets called *groups*, $\mathcal{B}$ is a collection of k-subsets of X called *blocks*, and such that

- any two different points from the same group appear together in no block; while
- any two points from different groups appear together in exactly λ blocks.

If all groups have the same size, then the GDD is called *uniform*. In general, the list of group sizes of a GDD is its *type*. Such a list is usually abbreviated with 'exponential notation', so that for instance a uniform GDD with u groups of size g has type g^u. We note that a BIBD(v, k, λ) is equivalent to a (k, λ)-GDD of type 1^v. Also, a transversal design TD(k, n) is equivalent to a $(k, 1)$-GDD of type n^k.

Group divisible designs played a central role in R.M. Wilson's asymptotic existence theory for BIBDs, [14]. Roughly speaking, the groups of a GDD can be 'filled' with compatible BIBDs to produce larger BIBDs. This is by now quite standard. When we wish to produce resolvable BIBDs, we require some extra structure.

A set of blocks in a GDD which touches, exactly once each, only those points not in a given group is called a *near parallel class* or *frame class*. If the blocks of a GDD admit a (multiset) partition into near parallel classes, then such a GDD is called a *frame*. As usual, k is the block size and λ is the index.

For our purposes, we define a *σ-frame* similarly, except that the blocks of the underlying GDD resolve into *near σ-classes*, each of which covers, exactly σ times each, the points outside a given group (and covers points of that group zero times).

The necessary divisibility conditions on uniform σ-frames of type g^u, block size k and index λ are

$$\sigma g(u-1) \equiv 0 \pmod{k}, \text{ and} \tag{2.1}$$

$$\lambda g \equiv 0 \pmod{\sigma(k-1)}. \tag{2.2}$$

We have (2.1) because $\sigma g(u-1)/k$ counts the number of blocks per class. For (2.2), note that there are $\lambda g(u-1)/\sigma(k-1)$ classes which touch a given group (by analyzing neighborhoods) and $\lambda gu/\sigma(k-1)$ classes in total (dividing blocks by blocks per class). It follows that every group is missed exactly $\lambda g/\sigma(k-1)$ times, and this must be an integer. The sufficiency of these conditions for large u is recorded here for later use.

Theorem 2.3 *Let $k \geq 2$, $g, \sigma \geq 1$ and $\lambda \geq 0$ be integers satisfying* (2.2). *There exists a σ-frame of type g^u with block size k and index λ for all sufficiently large u satisfying* (2.1).

Proof: This is an application of [10, Theorem 13.1] on edge-coloured graph decompositions. The method is nearly identical to [2, Theorem 7.1], which handles the case $\sigma = 1$. Each graph in our family is a complete bidirected graph on k vertices, say $\{1, \ldots, k\}$, together with an extra vertex ∞ of indegree k and outdegree 0. Let $s = \lambda g/\sigma(k-1)$ and use $g^2 + gs$ edge colours. The 'first' g^2 colours are simply the ordered pairs $\{1, \ldots, g\}^2$. For each of the g^k vertex labellings $f : \{1, \ldots, k\} \to \{1, \ldots, g\}$, we include in our family the K_{k+1} on $\{1, \ldots, k\} \cup \{\infty\}$

where arc (i, j) is given colour $(f(i), f(j))$ for $1 \leq i, j \leq k$. Arcs to ∞ use the 'last' gs colours, for which we use ordered pairs $\{1, \ldots, g\} \times \{1^*, \ldots, s^*\}$. For each of our graphs described so far, we make a choice $t \in \{1, \ldots, s\}$ and colour arc (i, ∞) by $(f(i), t^*)$.

Now, suppose the above family of graphs decomposes the large bidirected graph on u vertices having λ edges of each of the first colours and σ edges of each of the last colours. Each graph in the decomposition 'lifts' to its induced vertex labelling, viewed as positions within the groups of size g. The first colours ensure that we have a GDD of index λ just as in [10, §14]. The arcs to ∞ ensure that the blocks missing each group (where ∞ is placed) resolve into exactly m near σ-classes; this is very similar to colouring schemes in [2] and [10, §15].

Obtaining the decomposition for all sufficiently large admissible u requires some congruence calculations. Details can be found in the third author's dissertation and are similar to those in [2]. □

Let $(V, \mathcal{B})$ be a BIBD, say of index λ. A *subdesign* is a BIBD $(V_0, \mathcal{B}_0)$, also of index λ, such that $V_0 \subseteq V$ and $\mathcal{B}_0 \subseteq \mathcal{B}$ (as multisets). If our original BIBD is σ-resolvable, by 'subdesign' we mean also that some partition of $\mathcal{B}$ into σ-parallel classes induces a partition of $\mathcal{B}_0$ into σ-parallel classes.

To close this section, we offer two very standard frame constructions adapted to σ-frames. These essentially appear (with minor modifications) as Theorems IV.5.11 and IV.5.12 in [4]. We omit the proofs but refer the reader to the book [7] for detailed proofs of similar results. In the next two results and what follows, all our frames are assumed to have block size k, index λ, and thickness σ.

Lemma 2.4 (Filling in groups) *Suppose there is a σ-frame with group sizes g_i, $i = 1, 2, \ldots, u$. Suppose also that there exists a σ-resolvable BIBD$(g_i + h, k, \lambda)$ with a subdesign of order $h \geq 1$ for $i = 1, 2, \ldots, u - 1$, and a σ-resolvable BIBD$(g_u + h, k, \lambda)$ (with no condition needed on subdesigns). Then there exists a σ-resolvable BIBD$(g_1 + \cdots + g_u + h, k, \lambda)$. Furthermore, it contains the last ingredient, a σ-resolvable BIBD$(g_u + h, k, \lambda)$, as a subdesign.*

Lemma 2.5 (Wilson's fundamental construction) *Let $(X, \Pi, \mathcal{B})$ be a GDD, and let $w : X \to \mathbb{N} \cup \{0\}$ be a weight function on X. Suppose, for each block $B \in \mathcal{B}$, that there exists a σ-frame with group sizes $w(x)$ for $x \in B$. Then there exists a σ-frame with group sizes $\sum_{x \in G} w(x)$ for each group $G \in \Pi$.*

3 Proof of the main result

We divide the proof of Theorem 1.2 into three broad steps. First, given k, σ, λ, we apply a trick to construct one example of our design. Next, we use uniform frames to get a sparse but 'numerically exhaustive' family of examples. Finally, we use non-uniform frames to obtain eventual periodicity. Recall in what follows that $a := \sigma(k-1)/\gcd(\sigma(k-1), \lambda)$ and $b := k/\gcd(k, \sigma)$.

3.1 An auxiliary graph

We begin with the key observation that leads to our first example of a general thickly-resolvable design.

Using Theorem 2.1, take an (n_r, m_k)-configuration with $r = \sigma$ and $n = p_{k,\sigma}\frac{k}{\gcd(k,\sigma)} = p_{k,\sigma}b$, where $p_{k,\sigma}$ is a prime chosen greater than a and b. Starting from this configuration, replace each line of size k with a clique K_k on the incident points. Let $G_{k,\sigma}$ denote the resulting graph, which is regular of degree $\sigma(k-1)$.

By Theorem 2.2, there exists a resolvable $(z, G_{k,\sigma}, \lambda)$-design for sufficiently large integers z satisfying

$$z \equiv 0 \pmod{p_{k,\sigma}b} \quad \text{and} \quad z \equiv 1 \pmod{a}. \tag{3.1}$$

Upon breaking up $G_{k,\sigma}$-blocks into K_k, a parallel class of such blocks becomes a σ-parallel class of cliques K_k. It follows that, under the same conditions $z \gg 0$ and (3.1), we have the existence of a σ-resolvable BIBD(z, k, λ). This trick may be worth further analysis and development.

Figure 1 shows an example $(15_2, 10_3)$-configuration, drawn as on the cylinder. This can be turned into a graph $G_{3,2}$ upon replacing lines by triangles, or extending adjacencies vertically to the torus.

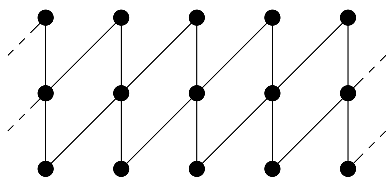

Figure 1: A $(15_2, 10_3)$-configuration

3.2 Constructions in each fiber

We next use σ-frames to construct instances of σ-resolvable BIBD(v, k, λ) in each admissible congruence class modulo arbitrarily large periods.

Proposition 3.1 *Suppose $x \equiv 1 \pmod{a}$ and $x \equiv 0 \pmod{b}$. Let P be given with $ab \mid P$. There exists a σ-resolvable BIBD(v, k, λ) for some arbitrarily large integer $v \equiv x \pmod{P}$.*

Proof: Let $P = AB$, where prime factors of P common with b occur in B and all other prime factors, including divisors of a, occur in A. In particular, $\gcd(A, B) = 1$.

Using the auxiliary graph, start with a σ-resolvable BIBD(z, k, λ) for z satisfying (3.1). Take a σ-frame of type $(z-1)^u$ and index λ, noting that the group size satisfies (2.2). Add one point and fill each group using Lemma 2.4, identifying subdesigns of order 1. This gives a σ-resolvable BIBD(v, k, λ) for $v = u(z-1)+1$. It remains to analyze the constructible values of v, which we first do separately modulo A and B.

Modulo A, the constructible values of u exhaust all possibilities, while those for z are precisely the values $1 \pmod{a}$. Taking $u \equiv 1 \pmod{A}$ and $z \equiv x \pmod{A}$, we see that $v \equiv x \pmod{A}$.

Modulo B, the constructible values of u are $1 \pmod{b}$, by (2.1) and $\gcd(k, \sigma(z-1)) = \gcd(k, \sigma)$. The constructible values of z are the multiples of $p_{k,\sigma}b$. Take $u \equiv 1 \pmod{B}$ and $z \equiv x \pmod{B}$, the latter of which is possible since $\gcd(p_{k,\sigma}, B) = 1$. We have $v \equiv x \pmod{B}$.

The Chinese remainder theorem permits such a choice of u, z modulo $P = AB$. □

3.3 Recursion and proof

Here, we adapt the recursive construction strategy of [3]. The broad idea is to construct and fill a large non-uniform σ-frame, making use of subdesigns.

To this end, our first step is to note the 'eventual periodicity' of a thickly-resolvable design with a fixed subdesign. Recall the parameter

$$\pi = ab = \frac{\sigma k(k-1)}{\gcd(\sigma(k-1), \lambda)\gcd(k, \sigma)},$$

noting that σ-frames are numerically admissible whenever the group sizes are multiples of π.

Lemma 3.2 *Let $\pi \mid P$ and suppose there exists a σ-resolvable BIBD$(P+z, k, \lambda)$ containing as a subdesign a σ-resolvable BIBD(z, k, λ). Then for some s_0 and all integers $s \geq s_0$, there exists a σ-resolvable BIBD$(sP+z, k, \lambda)$ containing as a subdesign a σ-resolvable BIBD(z, k, λ).*

PROOF: There exist σ-frames of type P^s for sufficiently large s. Take one, add z additional points, and apply Lemma 2.4. □

The last tool we employ is an existence result for certain non-uniform frames.

Lemma 3.3 *For some positive integer m, there exists a σ-frame of type $g^m h_1^1 h_2^1$ for all sufficiently large integers g and any nonnegative integers $h_1, h_2 \leq g$ with $\pi \mid g, h_1, h_2$.*

PROOF: Using Theorem 2.3, choose m such that there exists σ-frames of type π^{m+i} for $i = 0, 1, 2$. There exists TD$(m+2, g/\pi)$ for all large integers g a multiple of π. Truncate the last two groups down to size h_1/π and h_2/π and apply Wilson's fundamental construction, Lemma 2.5, with constant weight π. This yields the required frame. □

We are now able to complete the proof of the main theorem.

Proof of Theorem 1.2: Begin by constructing a σ-resolvable BIBD(z, k, λ) using the auxiliary graph trick of Section 3.1. Let t be large enough so that, by Theorem 2.3, there exists a σ-frame of type $(z-1)^t$. Put $P = (z-1)(t-1)$ and conclude that there exists a σ-resolvable BIBD$(P+z, k, \lambda)$ containing a subdesign of order z.

Now consider an admissible integer x; namely, suppose $x \equiv 1 \pmod{a}$ and $x \equiv 0 \pmod{b}$. By Proposition 3.1, there exists a σ-resolvable BIBD(w, k, λ) for some $w \equiv x \pmod{P}$. We may assume $w \gg z$. It suffices to close the fiber $x \pmod{P}$; for this, we construct σ-resolvable BIBD$(nP+w, k, \lambda)$ for all sufficiently large integers n.

Let s_0 be chosen as in Lemma 3.2 and m be chosen as in Lemma 3.3. We use these two results in what follows. For large n, it is possible to write $n = sm + t$, where $s \geq t \geq \max\{s_0, w - z\}$. Take a σ-frame of type $(sP)^m(tP)^1(w-z)^1$ and add z points. Using Lemma 2.4, fill all but the last group with examples of order $sP + z$ or $tP + z$ containing a subsystem of order z. On the last group of the frame, of size $w - z$, together with the additional z points, we include the above example of order w. The result of this filled frame is a σ-resolvable BIBD$((sm+t)P + w, k, \lambda)$, as required. □

4 Designs with holes

Here, we prefer some slightly different (and more general) terminology for block designs. Let v be a positive integer, $K \subseteq \{2, 3, 4, \dots\}$, and λ a nonnegative integer. A *pairwise balanced design* PBD$_\lambda(v, K)$ is a pair $(V, \mathcal{B})$, where

- V is a a set of v points;
- $\mathcal{B} \subseteq \cup_{k \in K} \binom{V}{k}$ is a family of of subsets of V, called blocks; and
- any two distinct points appear together in exactly λ blocks.

A BIBD(v, k, λ) is just a pairwise balanced design with constant block size $K = \{k\}$. Next, let $v \geq w$ be positive integers, with k, λ as before. An *incomplete pairwise balanced design* IPBD$_\lambda((v; w), \{k\})$ is a triple $(V, W, \mathcal{B})$ where

- V is a set of v points and $W \subset V$ is a *hole* of size w;
- $\mathcal{B}$ is a collection of k-subsets of V called blocks;
- no two distinct points of W appear in a block; and
- any two distinct points not both in W appear together in exactly λ blocks.

An equivalent notion is a $\mathrm{PBD}_\lambda(v, \{k, w^*\})$, where the star indicates that there is exactly one block of size w if $w \neq k$. It is reasonable to think of these objects as $\mathrm{BIBD}(v, k, \lambda)$ missing a $\mathrm{BIBD}(w, k, \lambda)$. It is important to note, though, that neither BIBD need exist on its own.

In the preceding notation, it is common to omit the subscript λ if it equals one. In that case, the necessary conditions, which are easy to see from counting incidences in various ways, are

$$v - 1 \equiv w - 1 \equiv 0 \pmod{k-1}, \tag{4.1}$$
$$v(v-1) - w(w-1) \equiv 0 \pmod{k(k-1)}, \text{ and} \tag{4.2}$$
$$v \geq (k-1)w + 1. \tag{4.3}$$

Something approximating an asymptotic existence theory for $\lambda = 1$ has been shown by E.R. Lamken and the first two authors.

Theorem 4.1 ([5]) *Let $k \in \mathbb{Z}$, $k \geq 2$. For every real number $\epsilon > 0$, there exists an IPBD$((v; w), \{k\})$ for all sufficiently large admissible v and w satisfying* (4.1)*,* (4.2)*, and $v \geq (k - 1 + \epsilon)w$.*

An important starting point for the proof of Theorem 4.1 is the existence of examples with equality in (4.3).

Proposition 4.2 *Let $k \geq 3$. There exists an IPBD$((v; w), \{k\})$ for $v = (k-1)w+1$ if and only if there exists a resolvable BIBD$(v - w, k - 1, 1)$.*

The basic idea is that, for an IPBD, say $(V, W, \mathcal{B})$, the hole W being of maximal size forces every block to intersect W; consequently, the removal of hole points induces parallel classes on the remaining $v - w$ points into blocks which are reduced in size by one. Conversely, extension of parallel classes in a resolvable $\mathrm{BIBD}(v-w, k-1, 1)$ by $w = \frac{v-w-1}{k-2}$ new points results in the desired IPBD. This is reminiscent of the equivalence between projective and affine planes. This also reveals the difficulty in constructing IPBDs with mixed block sizes K, since resolvable designs in this general context are presently not well understood.

Now, consider IPBDs with block size k and arbitrary index λ. Conditions (4.1-4.2) weaken to acquire a factor λ on the left-hand sides; this is straightforward. The inequality (4.3) is noteworthy in that it does not weaken for general λ.

Lemma 4.3 *In an IPBD$_\lambda((v; w), \{k\})$, it is necessary that $v \geq (k - 1)w + 1$.*

Proof: Suppose our IPBD is $(V, W, \mathcal{B})$. Every point of $V \setminus W$ is incident with exactly $\lambda(v - 1)/(k - 1)$ blocks. Exactly λ such blocks intersect a given point of W. But recall that no block in $\mathcal{B}$ can intersect two distinct points of W. It follows that $\lambda(v - 1)/(k - 1) \geq \lambda w$, which is equivalent to the claimed bound. □

Our main purpose here is to motivate σ-resolvable designs as the needed generalization in Proposition 4.2 to construct IPBDs with arbitrary index λ. The following equivalence is basically folklore, but we sketch a proof for completeness.

Proposition 4.4 *Let $k \geq 3$. There exists an IPBD$_\lambda((v;w),\{k\})$ for $v=(k-1)w+1$ if and only if there exists a λ-resolvable BIBD$(v-w,k-1,\lambda)$.*

Proof: Consider such an IPBD. From the proof of Lemma 4.3, we see that every point of the hole touches exactly λ blocks. It follows that truncating all hole points, which reduces each block in size by one, induces parallel classes of thickness λ. The result of this truncation is then a λ-resolvable BIBD$(v-w,k-1,\lambda)$. Conversely, given such a BIBD, we compute that it has $(v-w-1)/(k-2)=w$ parallel classes of thickness λ. Extend each class with the addition of a new point, one new point per class. Every pair of points, old with new, appears in exactly λ blocks, as desired. So we obtain an IPBD$_\lambda((v;w),\{k\})$ for $v=(k-1)w+1$. □

Suppose we are given any w with $\lambda(w-1) \equiv 0 \pmod{k-1}$. Put $n=(k-2)w+1$. Observe that $n \equiv 1 \pmod{k-2}$ and also $\lambda n \equiv \lambda(1-w) \equiv 0 \pmod{k-1}$. From our main result, Theorem 1.2, we have λ-resolvable BIBD$(n,k-1,\lambda)$ for such n, provided it is large.

We put $v=n+w=(k-1)w+1$ and obtain an IPBD$_\lambda((v;w),\{k\})$ from Proposition 4.4.

Many steps in the proof of Theorem 4.1 can be easily extended to index λ. For instance, the existence of uniform incomplete group divisible designs with many groups, whose proof uses edge-coloured decompositions, extends to λ as in [10, §15]. Also, the existence of various GDDs used in the proof can extend to index λ because Wilson's fundamental construction allows it. However, $\lambda=1$ is critical for the equivalence between an IPBD$((v;w),\{k\})$ and a k-GDD of type $(k-1)^{\frac{v-w}{k-1}}(w-1)^1$, and this is used heavily in [5]. These other details lead outside the scope of this paper, and so they are deferred.

5 Discussion

Theorem 1.2 settles the existence of σ-resolvable BIBD(v,k,λ) for large v given arbitrary parameters k,σ,λ. Perhaps this motivates some new exact (or almost exact) results for explicit parameters. The trick of using configurations and graph decompositions might potentially play a role in computer search for the challenging small examples.

The third author's dissertation outlines two extensions of Theorem 1.2. First, consider σ-resolvable group divisible designs. Adapting our auxiliary graph trick requires constructions of resolvable 'G-GDDs' appearing in [3]. Second, consider σ-resolvable G-decompositions, where G is a graph of order k. In this case, the necessary divisibility conditions become somewhat delicate, since σ-fold sums of degrees in G must be considered. Building the auxiliary graph uses a lemma on placements of G on the lines of a large (n_r, m_k)-configuration so that all possible σ-fold degree sums occur. An extension of Theorem 2.3 from blocks to graphs is also used.

For a list Σ of positive integers summing to the replication number, one can call a design Σ-resolvable if its blocks resolve into thick parallel classes whose thicknesses are governed by the list Σ. A related concept is that of 'uniformly resolvable block designs', where there is a mixture of block sizes overall but in which each parallel class has only blocks of a common size.

Finally, now that an existence theory has emerged for 't-designs', [9], we are interested in whether it can accommodate general hypergraph decompositions, edge-colours, and resolvability. In this setting, we are looking for t-designs which partition into s-designs for some $s < t$; the index of the smaller designs takes the role of our thickness parameter σ.

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(Received 16 May 2015; revised 20 Nov 2015)