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Partitions with bounded differences between largest and smallest parts

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Abstract

We give a simple formal proof of a formula for the generating function of partitions with bounded differences between largest and smallest part.

1 Introduction

In [3] Breuer and Kronholm gave in effect two proofs for an explicit formula for the generating function for partitions where the difference between largest and smallest part is bounded by a given integer t. Their first proof is geometric, involving counting lattice points within a polyhedral region; their second proof constructs an explicit bijection. In this paper we give another proof, a formal calculation involving elementary q-series manipulation, involving no results deeper than the q-binomial theorem.

The results of [3] imply a theorem of Andrews, Beck and Robbins [2] on partitions where the difference between largest and smallest part is a fixed integer t. They use formal q-series methods which go beyond ours, for instance Heine's transformation for basic hypergeometric series.

2 The main result

Recall that a partition λ of an integer n is a finite sequence $(\lambda_1, \dots, \lambda_k)$ where the integers λ_i satisfy $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Its parts are $\lambda_1, \dots, \lambda_k$. We write $|\lambda| = \lambda_1 + \cdots + \lambda_k$. It is convenient to allow trailing zeros in partition notation: we regard $(\lambda_1, \dots, \lambda_k, 0)$ as the same partition as $(\lambda_1, \dots, \lambda_k)$.

We use standard q-series notation. For integers $n \ge 0$ we define

$$(a)_n = \prod_{j=0}^{n-1}(1 - aq^j).$$

For integers $n \geq k \geq 0$ define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}} = \frac{(q^{n-k+1})_k}{(q)_k}.$$

Let $\mathcal{P}_t$ be the generating function for nonempty partitions where the difference between the largest and smallest part is $\leq t$. Write

$$P_t(q) = \sum_{\lambda \in \mathcal{P}_t} q^{|\lambda|}$$

for the generating function of $\mathcal{P}_t$. As Breuer and Kronholm [3] point out, $P_0(q)$ is not a rational function, but for $t \geq 1$, $P_t(q)$ is a rational function. We give an alternative proof of this theorem.

Theorem 1 *[3] For $t \geq 1$*

$$P_t(q) = \frac{1}{1-q^t}\left(\frac{1}{(q)_t} - 1\right)$$

where $(a)_t = \prod_{j=0}^{t-1}(1 - aq^j)$.

Proof The set $\mathcal{P}_t$ is the disjoint union of sets $\mathcal{P}_{t,r,m}$ for r, $m \geq 1$ where $\mathcal{P}_{t,r,m}$ is the set of $\lambda \in \mathcal{P}_t$ with r parts and smallest part m (and so largest part $\leq m + t$). Then

$$P_t(q) = \sum_{r,m=1}^{\infty} P_{t,r,m}(q)$$

where $P_{t,r,m}(q) = \sum_{\lambda \in \mathcal{P}_{t,r,m}} q^{|\lambda|}$. Each element of $\mathcal{P}_{t,r,m}$ has the form $\lambda = (\lambda_1, \ldots, \lambda_r)$ where $m + t \geq \lambda_1 \geq \cdots \geq \lambda_r = m$. Then $\mu = (\lambda_1 - m, \ldots, \lambda_{r-1} - m)$ is a partition of $|\lambda| - rm$ with at most $r - 1$ parts and greatest part $\leq t$. The generating function for such partitions is the q-binomial coefficient $\begin{bmatrix} r+t-1 \\ t \end{bmatrix}_q$ [1, Theorem 3.1] and so

$$P_{t,r,m}(q) = q^{rm}\begin{bmatrix} r+t-1 \\ t \end{bmatrix}_q.$$

Therefore

$$\begin{aligned} P_t(q) &= \sum_{r,m=1}^{\infty} q^{rm}\begin{bmatrix} r+t-1 \\ t \end{bmatrix}_q = \sum_{r=1}^{\infty} \frac{q^r}{1-q^r}\frac{(q^r)_t}{(q)_t} \\ &= \frac{1}{(q)_t}\sum_{r=1}^{\infty} q^r (q^{r+1})_{t-1}. \end{aligned}$$

At this point we use the q-binomial theorem in the form [1, Theorem 3.3]

$$(x)_n = \sum_{j=0}^{n} (-1)^j q^{j(j-1)/2}\begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j}.$$

We get

$$\begin{aligned}
P_t(q) &= \frac{1}{(q)_t}\sum_{j=0}^{t-1}\sum_{r=1}^{\infty}(-1)^j q^{(j+1)r} q^{j(j+1)/2}\begin{bmatrix} t-1 \\ j \end{bmatrix}_q \\
&= \frac{1}{(q)_t}\sum_{j=0}^{t-1}(-1)^j \frac{q^{j+1}}{1-q^{j+1}} q^{j(j+1)/2}\begin{bmatrix} t-1 \\ j \end{bmatrix}_q \\
&= \frac{1}{(q)_t(1-q^t)}\sum_{j=0}^{t-1}(-1)^j q^{(j+1)(j+2)/2}\begin{bmatrix} t \\ j+1 \end{bmatrix}_q \\
&= \frac{1}{(q)_t(1-q^t)}\sum_{k=1}^{t}(-1)^{k-1} q^{k(k+1)/2}\begin{bmatrix} t \\ k \end{bmatrix}_q \\
&= \frac{1-(q)_t}{(q)_t(1-q^t)} = \frac{1}{1-q^t}\left(\frac{1}{(q)_t}-1\right).
\end{aligned}$$

at the last stage using the q-binomial theorem again. □

3 Remarks

In [2, Theorem 1] Andrews, Beck and Robbins prove a formula for $\tilde{P}_t(q) = \sum_{\lambda\in\tilde{P}_t} q^{|\lambda|}$ where $\tilde{P}_t$ is the set of partitions in which the difference between largest and smallest part is exactly t, valid when $t \geq 2$. As pointed out in in [3], $\tilde{P}_t(q) = P_t(q) - P_{t-1}(q)$ and so this formula follows immediately from Theorem 1.

Andrews, Beck and Robbins [2, Theorem 3] also give a generalization to partitions with a set of specified distances. The author is uncertain whether the methods of the present paper can be extended to prove such generalizations.

References

[1] George E. Andrews, *The Theory of Partitions*, Addison-Wesley, 1976.

[2] George E. Andrews, Matthias Beck and Neville Robbins, Partitions with fixed differences between largest and smallest parts, *Proc. Amer. Math. Soc.* **143** (2015), 4283–4289.

[3] Felix Breuer and Brandt Kronholm, A polyhedral model of partitions with bounded differences and a bijective proof of a theorem of Andrews, Beck, and Robbins, `arXiv:1505.00250`, 2015.

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