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The out-arc 5-pancyclic vertices in strong tournaments*

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Abstract

An arc in a tournament T with $n \geq 3$ vertices is called k-pancyclic, if it belongs to a cycle of length l for all $k \leq l \leq n$. In this paper, the result that each s-strong ($s \geq 3$) tournament T contains at least $s+2$ out-arc 5-pancyclic vertices is obtained. Furthermore, our proof yields a polynomial algorithm to find $s+2$ out-arc 5-pancyclic vertices of T.

1 Introduction

Let D be a digraph with vertex-set $V(D)$ and arc-set $A(D)$. D is said to be *strong*, if for all $x, y \in V(D)$, there is a path from x to y. A directed path from x to y in D is called an (x, y)-path. D is called k*-strong* if $|V(D)| \geq k+1$ and $D - X$ is strong for any set $X \subseteq V(D)$ with $|X| < k$. If D is k-strong, but not $(k+1)$-strong, then $\sigma(D) = k$ is defined as the *strong connectivity* of D.

A *tournament* is an orientation of edges of a complete graph. An l-cycle is a cycle of length l. An arc or a vertex is said to be k*-pancyclic* in a digraph D, if it belongs to an l-cycle for all $k \leq l \leq |V(D)|$. For $k = 3$, we also say that the arc or the vertex is *pancyclic*. An arc leaving from a vertex x in a digraph is called an *out-arc* of x. If all out-arcs of a vertex x are k-pancyclic, then we say that x is an out-arc k-pancyclic vertex. For $k = 3$, we also say that the vertex is an out-arc pancyclic vertex.

In 1980, Thomassen [8] proved that every strong tournament contains a vertex x such that each out-arc of x is contained in a Hamiltonian cycle. In 2000, Yao, Guo and Zhang [9] extended the result of Thomassen and proved that every strong

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tournament T contains an out-arc pancyclic vertex. At present, there are many results on out-arc pancyclic vertices in strong tournaments, see[2, 3–7, 9, 10].

In [3], Feng et al. proved that every s-strong ($s \geq 3$) tournament T contains at least $s+1$ out-arc 4-pancyclic vertices. In [4], the author and Li proved that each s-strong ($s \geq 4$) tournament T contains at least $s+2$ out-arc 5-pancyclic vertices. In this paper, we prove that each s-strong ($s \geq 3$) tournament T contains at least $s+2$ out-arc 5-pancyclic vertices. Furthermore, our proof yields a polynomial algorithm to find $s+2$ out-arc 5-pancyclic vertices of T. This result is not true for $s = 1$, since there is an infinite family of 1-strong tournaments with exactly one out-arc 5-pancyclic vertex.

Example. Let $n \geq 5$ be an integer and let T be a tournament with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the arc set $\{v_iv_j \mid 2 \leq i < j \leq n\} \cup \{v_{n-1}v_1, v_nv_1\} \cup \{v_1v_j \mid 2 \leq j \leq n-2\}$. It is easy to check that v_n is unique out-arc 5-pancyclic vertex of T (see Figure 1).

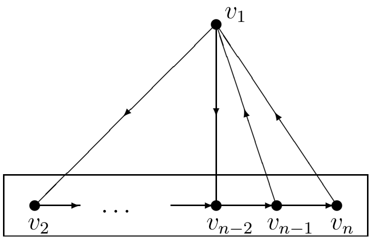

Figure 1. A strong tournament T containing exactly one out-arc 5-pancyclic vertex, where $T[v_2, \ldots, v_n]$ is a transitive tournament.

Problem. Is it true for $s = 2$ that each s-strong tournament T contains at least $s+2$ out-arc 5-pancyclic vertices?

2 Preliminaries

This paper will generally follow the notation and terminology defined in [1].

Let $D = (V(D), A(D))$ be a digraph without multiple arcs and loops. We denote the number of vertices of D by $|V(D)|$. A subdigraph induced by a subset $X \subseteq V(D)$ is denoted by $D[X]$. We also write $D - X$ for $D[V(D) - X]$.

If (u, v) is an arc in D, then we say that u *dominates* v and write $u \to v$. We also say that uv is an *out-arc* of u. For disjoint subsets X and Y of $V(D)$, if every vertex of X dominates every vertex of Y, we say X dominates Y and write $X \to Y$. $X \Rightarrow Y$ means that there is no arc from a vertex in Y to a vertex in X.

For a vertex x in D, the set of all vertices dominating x (dominated by x, respectively) is denoted by $N_D^-(x)$ ($N_D^+(x)$, respectively). Furthermore, $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ are the *out-degree* and *in-degree* of x in D, respectively. We use

$\delta^+(D)$ =min$\{d_D^+(x) : x \in V(D)\}$ to stand for the *minimum out-degree* of D. When there is no confusion possible, we use $N^+(x)$, $N^-(x)$, $d^+(x)$ and $d^-(x)$ instead of $N_D^+(x)$, $N_D^-(x)$, $d_D^+(x)$ and $d_D^-(x)$, respectively.

For a strong digraph $D = (V(D), A(D))$, a set $S \subseteq V(D)$ is called a *separating set* of D if $D - S$ is not strong. Furthermore, S is called a *minimum separating set* of D if $D - S$ is not strong and $\sigma(D) = |S|$. A vertex $u \in V(D)$ is called a *cut vertex* of D if $D - \{u\}$ is not strong. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. If D is strong, then D is the only strong component. If D is not strong, then we can partition D into strong components $T_1, T_2, \ldots, T_r$ $(r \geq 2)$, such that $T_1 \Rightarrow T_2 \Rightarrow \ldots \Rightarrow T_r$.

Let D be a digraph, let x and y be distinct vertices of D and let P be an (x, y)-path in D. We say that D' is the digraph obtained from D by *path-contracting* P into a new vertex w, not contained in D, if the following holds: $V(D') = (V(D) \setminus V(P)) \cup \{w\}$, $N_{D'}^+(w) = N_D^+(y) \cap (V(D) \setminus V(P))$, $N_{D'}^-(w) = N_D^-(x) \cap (V(D) \setminus V(P))$ and an arc with both end-points in $V(D) \setminus V(P)$ belongs to D' if and only if it belongs to D. It is easy to see that uwv is a path in D', if and only if uPv is a path in D. Analogously, there exists an l-cycle containing w in D' if and only if there is an $(l - 1 + |V(P)|)$-cycle containing P in D.

In the proofs of our main results, the following results are very useful.

Lemma 2.1 (Yeo [10]). *Let D be a strong digraph, containing a vertex x, such that $D - x$ is a tournament and $d_D^+(x) + d_D^-(x) \geq |V(D)|$. Then there is an l-cycle containing x in D, for all* $2 \leq l \leq |V(D)|$.

Lemma 2.2 (Yeo [10]). *Let T be a 2-strong tournament, containing an arc $e = xy$, such that $d^+(y) \geq d^+(x)$. Then e is pancyclic in T.*

By Lemma 2.2, we can obtain the following lemma.

Lemma 2.3 (Yao, Guo and Zhang [9]). *In a 2-strong tournament, all out-arcs of the vertices with minimum out-degree are pancyclic.*

Lemma 2.4 (Guo, Li and Li [6]). *Let T be an s-strong tournament and let $P = x_1x_2 \ldots x_t$ be a path in T. Let D be the digraph obtained from T by path-contracting P to w. If $s \geq t \geq 2$, then D is strong.*

Lemma 2.5 (Yeo [10]). *Let D be a strong digraph, and let $S = \{x\}$ be a separating set in D, such that $T = D - S$ is a tournament. Then x is pancyclic in D.*

Lemma 2.6. *Let T be a 3-strong tournament and xyz be a path of length 2. If $T - \{x, y, z\}$ is not strong, then xy and yz are 5-pancyclic in T.*

Proof. Let D be the digraph obtained from T by contracting the path xyz into a vertex w. By Lemma 2.4, we have that D is strong. Since $S = \{w\}$ be a separating set in D and $D - S$ is a tournament, we have that w is pancyclic in D by Lemma 2.5. In other words, xy and yz is 5-pancyclic in T. □

Lemma 2.7. *Let T be a 3-strong tournament and let $x_1x_2x_3x_4$ be a path of length 3. Let D be the digraph obtained from T by contracting the path $x_1x_2x_3x_4$ into a vertex z. If $T-\{x_1,x_2,x_3\}$ and $T-\{x_2,x_3,x_4\}$ are strong, then D is strong.*

Proof. Assume that $T-\{x_1,x_2,x_3\}$ and $T-\{x_2,x_3,x_4\}$ are strong. For any vertex $y \in D-z = T-\{x_1,x_2,x_3,x_4\}$, there is an (x_4,y)-path in $T-\{x_1,x_2,x_3\}$ and a (y,x_1)-path in $T-\{x_2,x_3,x_4\}$. In other words, there is a (z,y)-path and a (y,z)-path in D. Therefore, D is strong. □

Lemma 2.8 (Feng, Li and Li [3]). *Let T be a 3-strong tournament and $u_1, x \in V(T)$ with $d^+(u_1) = \delta^+(T)$ and $x \in N^+(u_1)$. If there is $y \in N^+(u_1)$ with $y \to x$, then all out-arcs of x are 4-pancyclic in T.*

Lemma 2.9. *Let T be a 3-strong tournament and let u be a vertex with minimum out-degree in T. Then there are at least $|N^+(u)|-1$ out-arc 4-pancyclic vertices of T in $N^+(u)$.*

Proof. Note that $T[N^+(u)]$ is a tournament. There is a Hamilton path, say $v_{i1}v_{i2}\dots v_{it}$, in $T[N^+(u)]$. By Lemma 2.8, each out-arc of v_{ij} is 4-pancyclic for all $1 < j \leq t$. Thus, $v_{i2}, v_{i3}, \dots, v_{it}$ are $|N^+(u)|-1$ out-arc 4-pancyclic vertices of T in $N^+(u)$. □

3 Main results

Lemma 3.1. *Let T be an s-strong ($s \geq 3$) tournament and M be the set of vertices with minimum out-degree in T. If $|M| \geq 2$, then T contains at least $s+2$ out-arc 5-pancyclic vertices.*

Proof. Let $u \in M$ be a vertex with minimum out-degree of T. By Lemma 2.3, u is an out-arc pancyclic vertex. Let $N^+(u) = \{v_1, v_2, \dots, v_t\}$ with $d^+(v_1) \leq d^+(v_2) \leq \dots \leq d^+(v_t)$, then $s \leq t$. By Lemma 2.9, $N^+(u)$ contains at least $|N^+(u)|-1 = t-1$ out-arc 4-pancyclic vertices of T. Thus, there are at least $(t-1)+|\{u\}| = t \geq s$ out-arc 4-pancyclic vertices in T. Clearly, they are out-arc 5-pancyclic vertices of T. Below, we will seek for another two out-arc 5-pancyclic vertices of T. The following two cases are considered.

Case 1. There are exactly $|N^+(u)|-1$ out-arc 5-pancyclic vertices of T in $N^+(u)$.

Suppose that $v_i \in N^+(u)$ is not an out-arc 5-pancyclic vertex of T. Then we have $v_i \to v_j$ for all $v_j \in N^+(u) \setminus \{v_i\}$ by Lemma 2.8. Let x be an arbitrary out-neighbor of v_i and let D_1 be the digraph obtained from T by contracting the path uv_ix into a vertex z_1. By Lemma 2.4, we have that D_1 is strong.

When $x \in N^+(u)$, we get

$$\begin{aligned} d^+_{D_1}(z_1) + d^-_{D_1}(z_1) &= d^+(x) + d^-(u) = d^+(x) + (n-1-d^+(u)) \\ &= n-1+(d^+(x)-d^+(u)) > n-2 = |V(D_1)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_1 in D_1, for all $2 \leq l \leq |V(D_1)|$. Therefore, $v_i x$ is 4-pancyclic in T.

When $x \notin N^+(u)$, we have $x \to u$. If there exists some $v_k \in N^+(u) \setminus \{v_i\}$ such that $x \to v_k$, let D_2 be the digraph obtained from T by contracting the path uv_ixv_k into a vertex z_2. If $T - \{u, v_i, x\}$ or $T - \{v_i, x, v_k\}$ is not strong, then, by Lemma 2.6, $v_i x$ is 5-pancyclic. Assume that $T - \{u, v_i, x\}$ and $T - \{v_i, x, v_k\}$ are both strong. By Lemma 2.7, D_2 is strong. Note that $v_i \to v_k$. We have

$$\begin{aligned} d^+_{D_2}(z_2) + d^-_{D_2}(z_2) &= d^+(v_k) + d^-(u) - 1 = d^+(v_k) + (n - 1 - d^+(u)) - 1 \\ &= n - 2 + (d^+(v_k) - d^+(u)) \\ &> n - 3 = |V(D_2)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_2 in D_2, for all $2 \leq l \leq |V(D_2)|$. That is, $v_i x$ is 5-pancyclic in T. Thus, v_i is an out-arc 5-pancyclic vertex of T, a contradiction. So we have $v_j \to x$ for all $v_j \in N^+(u) \setminus \{v_i\}$.

If $d^+(x) > d^+(u)$, then let D_3 be the digraph obtained from T by contracting the path uv_ix into a vertex z_3. By Lemma 2.4, we have that D_3 is strong.

$$\begin{aligned} d^+_{D_3}(z_3) + d^-_{D_3}(z_3) &= d^+(x) - 1 + d^-(u) - 1 = d^+(x) - 1 + (n - 1 - d^+(u)) - 1 \\ &= n - 2 + (d^+(x) - d^+(u) - 1) \\ &\geq n - 2 = |V(D_3)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_3 in D_3, for all $2 \leq l \leq |V(D_3)|$. That is, $v_i x$ is 4-pancyclic in T. Thus, v_i is an out-arc 4-pancyclic vertex of T, a contradiction. Therefore $d^+(x) = d^+(u)$. Now, x is an out-arc pancyclic vertex of T by Lemma 2.3, and $|N^+(x) \setminus N^+(u)| \geq 3$. By Lemma 2.9, there exists a vertex $w \in N^+(x) \setminus (\{u\} \cup N^+(u))$ such that w is even out-arc 4-pancyclic.

In this case, x and w are two desired vertices, and we are done.

Case 2. All vertices in $N^+(u)$ are out-arc 5-pancyclic vertices of T.

In this case, there are at least $|N^+(u)| + |\{u\}| = t + 1 \geq s + 1$ out-arc 5-pancyclic vertices in T. We only need one other out-arc 5-pancyclic vertex. Recall that $|M| \geq 2$.

If there exists some $y \in M$ and $y \notin \{u\} \cup N^+(u)$, then y is another desired vertex by Lemma 2.3, we are done. Suppose now that each vertex in M is in $N^+(u) \cup \{u\}$.

Recalling $d^+(v_1) \leq d^+(v_2) \leq \ldots \leq d^+(v_t)$, we have $v_1 \in M$. In other words, for any vertex z in T, we have $d^+(z) \geq d^+(u) = d^+(v_1)$. If there exists a vertex $v_j \in N^+(u)$ such that $v_j \to v_1$, then $|N^+(v_1) \setminus N^+(u)| \geq 2$. By Lemma 2.9, there is a vertex in $N^+(v_1) \setminus N^+(u)$ which is a desired vertex, we are done. Therefore, suppose that $v_1 \to v_j$ for all $v_j \in N^+(u) \setminus \{v_1\}$. Thus, $|N^+(v_1) \setminus N^+(u)| = 1$. Let $w \in N^+(v_1) \setminus N^+(u)$. We will prove that w is a desired out-arc 5-pancyclic vertex. Let $x \in N^+(w)$ be arbitrary.

Subcase 2.1: $x \notin N^+(u) \cup \{u\}$.

In this case, we have $d^+(x) > d^+(v_1)$. Let D_1 be the digraph obtained from T by contracting the path v_1wx into a vertex z_1. By Lemma 2.4, we have that D_1 is strong. And we get

$$\begin{aligned} d^+_{D_1}(z_1) + d^-_{D_1}(z_1) &\geq d^+(x) - 1 + d^-(v_1) - 1 = d^+(x) - 2 + (n - 1 - d^+(v_1)) \\ &= n - 2 + (d^+(x) - d^+(v_1) - 1) \geq n - 2 = |V(D_1)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_1 in D_1, for all $2 \leq l \leq |V(D_1)|$. Therefore, wx is 4-pancyclic in T.

Subcase 2.2: $x \in N^+(u)$.

Without loss of generality, suppose $x = v_j$ for some $j \neq 1$. Let D_2 be the digraph obtained from T by contracting the path v_1wv_j into a vertex z_2. By Lemma 2.4, we have that D_2 is strong. Note that $v_1 \to v_j$. We get

$$\begin{aligned} d^+_{D_2}(z_2) + d^-_{D_2}(z_2) &= d^+(v_j) + d^-(v_1) = d^+(v_j) + (n - 1 - d^+(v_1)) \\ &= n - 1 + (d^+(v_j) - d^+(v_1)) > n - 2 = |V(D_2)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_2 in D_2, for all $2 \leq l \leq |V(D_2)|$. Therefore, wv_j is 4-pancyclic in T.

Subcase 2.3: $x = u$.

If $T - \{w, u, v_k\}$ is not strong for some $v_k \in N^+(u)$, then wu is 5-pancyclic in T by Lemma 2.6. Thus, w is an out-arc 5-pancyclic vertex of T, we are done. Therefore, suppose that $T - \{w, u, v_k\}$ is strong for each $v_k \in N^+(u)$. Let D_3 be the digraph obtained from T by contracting the path v_1wuv_2 into a vertex z_3. By Lemma 2.7, D_3 is strong. And we have

$$\begin{aligned} d^+_{D_3}(z_3) + d^-_{D_3}(z_3) &\geq d^+(v_2) - 1 + d^-(v_1) - 1 = d^+(v_2) - 1 + (n - 1 - d^+(v_1)) - 1 \\ &= n - 3 + (d^+(v_2) - d^+(v_1)) \geq n - 3 = |V(D_3)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_3 in D_3, for all $2 \leq l \leq |V(D_3)|$. That is, wu is 5-pancyclic in T.

Thus, w is an out-arc 5-pancyclic vertex of T, we are done. ■

Lemma 3.2. *Let T be an s-strong ($s \geq 3$) tournament and M be the set of vertices with minimum out-degree in T. If $|M| = 1$, then T contains at least $s + 2$ out-arc 5-pancyclic vertices.*

Proof. Let $u \in M$. Then u is an out-arc pancyclic vertex of T and we have $d^+(y) > d^+(u)$ for all $y \neq u$. Let $N^+(u) = \{v_1, v_2, \ldots, v_t\}$ with $d^+(v_1) \leq d^+(v_2) \leq \ldots \leq d^+(v_t)$. Then $s \leq t$. When $d^+(v_1) = d^+(v_2)$, we always assume $v_1 \to v_2$.

By the proof of Lemma 3.1, we get that all vertices in $N^+(u)$ are out-arc 5-pancyclic vertices of T (Otherwise, we can get a vertex $x \neq u$ such that $d^+(x) =$

$d^+(u)$. Thus $|M| \geq 2$, a contradiction.). If $t \geq s+1$, then there are at least $t + |\{u\}| = t+1 \geq s+2$ out-arc 5-pancyclic vertices in T, we are done. Therefore $t = s$ and there are at least $t + |\{u\}| = t+1 \geq s+1$ out-arc 5-pancyclic vertices in T. Below, we will seek for another out-arc 5-pancyclic vertex of T. The following two cases are considered.

Case 1: $d^+(v_t) \geq d^+(v_1)+2$ or there exists some $v_i \in N^+(u) \setminus \{v_1\}$ such that $v_1 \to v_i$.

Let w be a vertex with minimum out-degree in $N^+(v_1) \setminus N^+(u)$. Below, we will show that w or another vertex is a desired vertex. Let $x \in N^+(w)$ be arbitrary. By Lemma 2.2, we only need to consider the case when $d^+(x) < d^+(w)$.

Subcase 1.1: $x = u$.

If $T - \{w, u, v_k\}$ is not strong for some $k \in \{1, 2, \ldots, t\}$, we have that wu is 5-pancyclic in T by Lemma 2.6. Suppose now that $T - \{w, u, v_k\}$ is strong for each $k \in \{1, 2, \ldots, t\}$.

If $d^+(v_t) \geq d^+(v_1) + 2$, let D be the digraph obtained from T by contracting the path v_1wuv_t into a vertex z. By Lemma 2.7, we get that D is strong. And

$$\begin{aligned} d_D^+(z) + d_D^-(z) &\geq d^+(v_t) - 2 + d^-(v_1) - 2 \\ &= d^+(v_t) - 2 + (n - 1 - d^+(v_1)) - 2 \\ &= n - 3 + (d^+(v_t) - d^+(v_1) - 2) \geq n - 3 = |V(D)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z in D, for all $2 \leq l \leq |V(D)|$. That is, wu is 5-pancyclic in T.

If there exists some $v_i \in N^+(u) \setminus \{v_1\}$ such that $v_1 \to v_i$, let D_1 be the digraph obtained from T by contracting the path v_1wuv_i into a vertex z_1. By Lemma 2.7, we get that D_1 is strong. And

$$\begin{aligned} d_{D_1}^+(z_1) + d_{D_1}^-(z_1) &\geq d^+(v_i) - 1 + d^-(v_1) - 1 = d^+(v_i) - 1 + (n - 1 - d^+(v_1)) - 1 \\ &= n - 3 + (d^+(v_i) - d^+(v_1)) \geq n - 3 = |V(D_1)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_1 in D_1, for all $2 \leq l \leq |V(D_1)|$. That is, wu is 5-pancyclic in T.

Altogether, wu is 5-pancyclic in T.

Subcase 1.2: $x \in N^+(u)$.

Let $x = v_k$ for some $k \neq 1$. Let D_2 be the digraph obtained from T by contracting the path v_1wv_k into a vertex z_2. By Lemma 2.4, D_2 is strong.

If $d^+(v_k) > d^+(v_1)$, then

$$\begin{aligned} d_{D_2}^+(z_2) + d_{D_2}^-(z_2) &\geq d^+(v_k) - 1 + d^-(v_1) - 1 = d^+(v_k) - 1 + (n - 1 - d^+(v_1)) - 1 \\ &= n - 2 + (d^+(v_k) - d^+(v_1) - 1) \geq n - 2 = |V(D_2)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_2 in D_2, for all $2 \le l \le |V(D_2)|$. That is, wv_k is 4-pancyclic.

If $d^+(v_k) = d^+(v_1)$, let D_3 be the digraph obtained from T by contracting the path uv_1wv_k into a vertex z_3. Then

$$\begin{aligned} d^+_{D_3}(z_3) + d^-_{D_3}(z_3) &\ge d^+(v_k) - 1 + d^-(u) - 1 = d^+(v_k) - 1 + (n - 1 - d^+(u)) - 1 \\ &= n - 3 + (d^+(v_k) - d^+(u)) \ge n - 3 = |V(D_3)|. \end{aligned}$$

If D_3 is strong, then, by Lemma 2.1, there is an l-cycle containing z_3 in D_3, for all $2 \le l \le |V(D_3)|$. That is, wv_k is 5-pancyclic in T.

Suppose that D_3 is not strong. If $s \ge 4$, then D_3 is strong by Lemma 2.4, a contradiction. So $s = 3$. Recalling that $s = t$, we have $|N^+(u)| = 3$ and $N^+(u) = \{v_1, v_2, v_3\}$. We will show that wv_2 and wv_3 are 5-pancyclic in T. By the above discussion, we only need to consider the case $d^+(v_j) = d^+(v_1)$ for $j = 2, 3$.

When $d^+(v_2) = d^+(v_1)$, we have $v_1 \to v_2$ by the assumption (since we always assume $v_1 \to v_2$ when $d^+(v_1) = d^+(v_2)$). Let D_4 be the digraph obtained from T by contracting the path v_1wv_2 into a vertex z_4. By Lemma 2.4, D_2 is strong. We get

$$\begin{aligned} d^+_{D_4}(z_4) + d^-_{D_4}(z_4) &= d^+(v_2) + d^-(v_1) = d^+(v_2) + (n - 1 - d^+(v_1)) \\ &= n - 1 + (d^+(v_2) - d^+(v_1)) > n - 2 = |V(D_4)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_4 in D_4, for all $2 \le l \le |V(D_4)|$. That is, wv_2 is 4-pancyclic in T.

When $d^+(v_3) = d^+(v_1)$, we have $d^+(v_3) = d^+(v_2) = d^+(v_1)$. If $v_1 \to v_3$, then by a similar argument, wv_3 is 4-pancyclic in T. Therefore, suppose that $v_3 \to v_1$.

If $T - \{v_1, w, v_3\}$ is not strong, then wv_3 is 5-pancyclic in T by Lemma 2.6. Suppose now that $T - \{v_1, w, v_3\}$ is strong. Recalling that D_3 is not strong, therefore, $T - \{u, v_1, w\}$ is not strong by Lemma 2.7. Let $T_1 \Rightarrow T_2 \Rightarrow \ldots \Rightarrow T_r$ be r strong components of $T - \{u, v_1, w\}$. If $v_3 \notin V(T_r)$, then it is easy to check that wv_3 is pancyclic. So assume that $v_3 \in V(T_r)$. Recalling that $d^+(y) > d^+(u)$ for all $y \ne u$, we have $|V(T_r)| \ge 3$. Since $T - \{v_1, w\}$ is strong, there is an arc from u to $V(T_1)$. So $v_2 \in V(T_1)$. Note that $d^+(v_3) = d^+(v_2) = d^+(v_1)$. Then $r = 2$ and $V(T_1) = \{v_2\}$ and $d^+(v_2) = d^+(v_3) = |V(T_r)|$. Since $T_1 \Rightarrow T_r$ and $\{u, w\} \to v_3$, we have $|V(T_r)| \le d^+(v_2) = d^+(v_3) = d^+_{T_r}(v_3) + 1$. Taking into account that T_r is strong and $|V(T_r)| \ge 3$ the vertex v_3 has positive in-degree and positive out-degree in T_r. Thus we have $|V(T_r)| \le d^+_{T_r}(v_3) + 1 \le |V(T_r)| - 2 + 1 = |V(T_r)| - 1$, a contradiction.

Thus, wx is 5-pancyclic.

Subcase 1.3: $x \notin N^+(u) \cup \{u\}$.

In this case, we have $x \to \{u, v_1\}$.

If $d^+(x) \le d^+(u) + 1$, then, since $d^+(x) > d^+(u)$, we have $d^+(x) = d^+(u) + 1$. We will show x is a desired vertex.

Let $y \in N^+(x)$ be arbitrary. Recalling that $d^+(y) > d^+(u)$ when $y \ne u$, that is, $d^+(y) \ge d^+(u) + 1 = d^+(x)$ when $y \ne u$. By Lemma 2.2, xy is pancyclic when $y \ne u$.

For the arc xu, let D_5 be the digraph obtained from T by contracting the path xuv_1 into a vertex z_5. By Lemma 2.4, D_5 is strong. Note that $x \to v_1$. We have

$$\begin{aligned} d^+_{D_5}(z_5) + d^-_{D_5}(z_5) &= d^+(v_1) + d^-(x) = d^+(v_1) + (n-1-d^+(x)) \\ &= n-1+(d^+(v_1)-d^+(x)) > n-2 = |V(D_5)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_5 in D_5, for all $2 \le l \le |V(D_5)|$. That is, xu is 4-pancyclic in T. Thus, x is a desired vertex, we are done.

Suppose now that $d^+(x) \ge d^+(u) + 2$. Let D_6 be the digraph obtained from T by contracting the path uv_1wx into a vertex z_6. Then

$$\begin{aligned} d^+_{D_6}(z_6) + d^-_{D_6}(z_6) &\ge d^+(x) - 2 + d^-(u) - 2 = d^+(x) - 2 + (n-1-d^+(u)) - 2 \\ &= n-3+(d^+(x)-d^+(u)-2) \ge n-3 = |V(D_6)|. \end{aligned}$$

If D_6 is strong, then, by Lemma 2.1, there is an l-cycle containing z_6 in D_6, for all $2 \le l \le |V(D_6)|$. That is, wx is 5-pancyclic in T. Suppose that D_6 is not strong.

If $T-\{v_1,w,x\}$ is not strong, then wx is 5-pancyclic in T by Lemma 2.6. Suppose now that $T-\{v_1,w,x\}$ is strong. Note that D_6 is not strong. Therefore $T-\{u,v_1,w\}$ is not strong by Lemma 2.7. Let $T_1 \Rightarrow T_2 \Rightarrow \cdots \Rightarrow T_r$ be r strong components of $T-\{u,v_1,w\}$.

If $x \notin V(T_r)$, then it is easy to check that wx is pancyclic in T. Suppose that $x \in V(T_r)$. Recalling that $d^+(y) > d^+(u)$ for all $y \neq u$, we have $|V(T_r)| \ge 3$. Therefore, $T-\{u,v_1,w,x\}$ is not strong.

Let D_7 be the digraph obtained from T by contracting the path v_1wxu into a vertex z_7. If $T-\{w,x,u\}$ is not strong, then wx is 5-pancyclic in T by Lemma 2.6. Suppose now that $T-\{w,x,u\}$ is strong. Recalling that $T-\{v_1,w,x\}$ is strong, we have D_7 is strong by Lemma 2.7. Note that $D_7 - z_7 = T-\{u,v_1,w,x\}$ is not strong. We have that z_7 is pancyclic in D_7 by Lemma 2.5. That is, wx is 6-pancyclic in T. Since $d^+(x) \ge d^+(u)+2$, there is a vertex $y \notin N^+(u) \cup \{u\}$ such that $x \to y \to u$. Clearly, $wxyuv_1w$ is a 5-cycle containing wx. Thus, wx is 5-pancyclic in T.

Altogether, we have that w or another vertex is a desired vertex.

Case 2: $d^+(v_t) \le d^+(v_1)+1$ and $v_i \to v_1$ for all $i \in \{2,3,\dots,t\}$.

In this case, there must exist i,j such that $|i-j|=1$ and $d^+(v_i) = d^+(v_j)$ and $v_i \to v_j$. Without loss of generality, assume that $j = i+1$. Let $w \in N^+(v_i) \setminus N^+(u)$ be a vertex with minimum out-degree. Below, we will prove that w or another vertex is a desired vertex. Let $x \in N^+(w)$ be arbitrary. By Lemma 2.2, we only need to consider the case when $d^+(x) < d^+(w)$.

Subcase 2.1: $x = u$.

If $T-\{w,u,v_k\}$ is not strong for some $v_k \in N^+(u)$, then wu is pancyclic in T by Lemma 2.6. Suppose now that $T-\{w,u,v_k\}$ is strong for all $v_k \in N^+(u)$. Let D be the digraph obtained from T by contracting the path v_iwuv_j into a vertex z. We get that D is strong by Lemma 2.7. And

$$\begin{aligned} d_D^+(z)+d_D^-(z) &\geq d^+(v_j)-1+d^-(v_i)-1=d^+(v_j)-1+(n-1-d^+(v_i))-1 \\ &= n-3+(d^+(v_j)-d^+(v_i))\geq n-3=|V(D)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z in D, for all $2\leq l\leq |V(D)|$. That is, wu is 5-pancyclic in T.

Subcase 2.2: $x\in N^+(u)$.

Let $x=v_k$ for some $k\neq i$. Let D_1 be the digraph obtained from T by contracting the path uv_iwv_k into a vertex z_1. Then

$$\begin{aligned} d_{D_1}^+(z_1)+d_{D_1}^-(z_1) &\geq d^+(v_k)-1+d^-(u)-1=d^+(v_k)-1+(n-1-d^+(u))-1 \\ &= n-3+(d^+(v_k)-d^+(u))\geq n-3=|V(D_1)|. \end{aligned}$$

If D_1 is strong, then, by Lemma 2.1, there is an l-cycle containing z_1 in D_1, for all $2\leq l\leq |V(D_1)|$. That is, wv_k is 5-pancyclic in T.

Suppose that D_1 is not strong. If $s\geq 4$, then D_1 is strong by Lemma 2.4, a contradiction. So $s=3$. Now $|N^+(u)|=3$ and $N^+(u)=\{v_1,v_2,v_3\}$. Since $d^+(v_i)=d^+(v_{i+1})$ and $v_i\to v_{i+1}$, we have $i\in\{1,2\}$. Note that we assume $v_i\to v_1$ for all $i\in\{2,3,\ldots,t\}$. So $i=2$. We will show that wv_1 and wv_3 are 5-pancyclic in T.

Let D_2 be the digraph obtained from T by contracting the path v_2wv_1 into a vertex z_2. Then D_2 is strong by Lemma 2.4. Note that $v_2\to v_1$ and $d^+(v_2)\leq d^+(v_1)+1$.

$$\begin{aligned} d_{D_2}^+(z_2)+d_{D_2}^-(z_2) &= d^+(v_1)+d^-(v_2)=d^+(v_1)+(n-1-d^+(v_2)) \\ &= n-2+(d^+(v_1)+1-d^+(v_2))\geq n-2=|V(D_2)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_2 in D_2, for all $2\leq l\leq |V(D_2)|$. That is, wv_1 is 4-pancyclic in T.

Let D_3 be the digraph obtained from T by contracting the path v_2wv_3 into a vertex z_3. Then D_3 is strong by Lemma 2.4. Note that $v_2\to v_3$ and $d^+(v_2)=d^+(v_3)$.

$$\begin{aligned} d_{D_3}^+(z_3)+d_{D_3}^-(z_3) &= d^+(v_3)+d^-(v_2)=d^+(v_3)+(n-1-d^+(v_2)) \\ &= n-1+(d^+(v_3)-d^+(v_2))>n-2=|V(D_3)|. \end{aligned}$$

By Lemma 2.1, there is an l-cycle containing z_3 in D_3, for all $2\leq l\leq |V(D_3)|$. That is, wv_3 is 4-pancyclic in T.

Subcase 2.3: $x\notin N^+(u)\cup\{u\}$.

By a similar argument as in Subcase 1.3 (only using v_i instead of v_1), we get wx is 5-pancyclic or another vertex is a desired vertex. Thus, w or another vertex is a desired vertex. Altogether, the proof of Lemma 3.2 is complete. ■

Combining Lemma 3.1 and Lemma 3.2, we get the following Theorem 3.3.

Theorem 3.3. *Let T be an s-strong ($s \geq 3$) tournament. Then T contains at least $s+2$ out-arc 5-pancyclic vertices.*

The proofs of Lemma 3.1 and Lemma 3.2 gives rise to the following polynomial algorithm to find at least $s+2$ out-arc 5-pancyclic vertices in an s-strong tournament.

Algorithm:

Input: An s-strong tournament T.

Output: $s+2$ out-arc 5-pancyclic vertices.

1. For each vertex v in T, track record its out-neighbors $N^+(v)$ and compute its out-degree $d^+(v)$.

2. Let M be the set of vertices with minimum out-degree in T. Decide $|M| \geq 2$ or $|M| = 1$ and choose a vertex $u \in M$. Let $N^+(u) = \{v_1, v_2, \ldots, v_t\}$ with $d^+(v_1) \leq d^+(v_2) \leq \cdots \leq d^+(v_t)$.

3. If $|M| \geq 2$, then go to Step 4. If $|M| = 1$, then go to Step 5.

4. If there exists a vertex $v_i \in N^+(u)$ and a vertex $x \in M \setminus (N^+(u) \cup \{u\})$ satisfying that $v_i \to \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_t, x\}$ and $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_t\} \to x$, then there are two vertices in $N^+(x) \setminus \{u\}$, say w_1 and w_2, such that $w_1 \to w_2$. Now, $u, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_t, x, w_2$ are at least $s+2$ out-arc 5-pancyclic vertices of T.

Otherwise, all vertices in $N^+(u)$ are out-arc 5-pancyclic vertices. If there exists a vertex $y \in M \setminus (N^+(u) \cup \{u\})$, then $u, v_1, v_2, \ldots, v_t, y$ are at least $s+2$ out-arc 5-pancyclic vertices of T. Otherwise, we have that $v_1 \in M$. If $|N^+(v_1) \setminus N^+(u)| > 2$, assume that $w_1, w_2 \in N^+(v_1) \setminus N^+(u)$ and $w_1 \to w_2$, then $u, v_1, v_2, \ldots, v_t, w_2$ are at least $s+2$ out-arc 5-pancyclic vertices of T. If $|N^+(v_1) \setminus N^+(u)| = 1$, assume that $w \in N^+(v_1) \setminus N^+(u)$, then $u, v_1, v_2, \ldots, v_t, w$ are at least $s+2$ out-arc 5-pancyclic vertices of T.

5. If $d^+(v_t) \geq d^+(v_1)+2$ or there exists some $v_i \in N^+(u) \setminus \{v_1\}$ such that $v_1 \to v_i$, then go to Step 6. If $d^+(v_t) \leq d^+(v_1)+1$ and $v_i \to v_1$ for all $i \in \{2, 3, \ldots, t\}$, then go to Step 7.

6. Let w be a vertex in $N^+(v_1) \setminus N^+(u)$ with minimum out-degree. If there exists a vertex $x \in N^+(w) \setminus (N^+(u) \cup \{u\})$ and $d^+(w) > d^+(x) = d^+(u)+1$, then $u, v_1, v_2, \ldots, v_t, x$ are at least $s+2$ out-arc 5-pancyclic vertices of T. Otherwise, $u, v_1, v_2, \ldots, v_t, w$ are at least $s+2$ out-arc 5-pancyclic vertices of T.

7. Let v_i be a vertex with $v_i \to v_{i+1}$ and $d^+(v_i) = d^+(v_{i+1})$ and let w be a vertex in $N^+(v_i) \setminus N^+(u)$ with minimum out-degree. If there exists a vertex $x \in N^+(w) \setminus (N^+(u) \cup \{u\})$ and $d^+(w) > d^+(x) = d^+(u)+1$, then $u, v_1, v_2, \ldots, v_t, x$ are at least $s+2$ out-arc 5-pancyclic vertices of T. Otherwise, $u, v_1, v_2, \ldots, v_t, w$ are at least $s+2$ out-arc 5-pancyclic vertices of T.

The correctness of this algorithm follows from the proofs of Lemma 3.1 and Lemma 3.2. We see that this algorithm can be implemented in time $O(n^2)$.

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