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Independence and 2-domination in bipartite graphs

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Abstract

For a positive integer k, a set of vertices S in a graph G is said to be a k-dominating set if each vertex x in $V(G) - S$ has at least k neighbors in S. The order of a smallest k-dominating set of G is called the k-domination number of G and is denoted by $\gamma_k(G)$. In Blidia, Chellali and Favaron [*Australas. J. Combin.* 33 (2005), 317–327], they proved that a tree T satisfies $\alpha(T) \leq \gamma_2(T) \leq \frac{3}{2}\alpha(T)$, where $\alpha(G)$ is the independence number of a graph G. They also claimed that they characterized the trees T with $\gamma_2(T) = \frac{3}{2}\alpha(T)$. In this note, we will show that the second inequality is even valid for bipartite graphs. Further, we give a characterization of the bipartite graphs G satisfying $\gamma_2(G) = \frac{3}{2}\alpha(G)$ and point out that the characterization in the aforementioned paper of the trees with this property contains an error.

1 Introduction

Let G be a simple graph with vertex set $V(G)$. The *order* of G is $|G| := |V(G)|$. A vertex of degree one is called a *leaf*. The number of leaves of G is denoted by $l(G)$.

For a positive integer k, a set of vertices S in a graph G is said to be a *k-dominating set* if each vertex of G not contained in S has at least k neighbors in S. The order of a smallest k-dominating set of G is called the *k-domination number*, and it is denoted by $\gamma_k(G)$. By definition, a dominating set coincides with a 1-dominating set, and $\gamma_1(G)$ is the domination number $\gamma(G)$ of G.

A subset $I \subseteq V(G)$ of the vertex set of a graph G is called *independent* if every pair of vertices in I is not adjacent. The number $\alpha(G)$ represents the cardinality of a maximum independent set of G.

For each vertex x in a graph G, we introduce a new vertex x' and join x and x' by an edge. The resulting graph is called the *corona* of G. A graph is said to be a *corona graph* if it is the corona of some graph.

For graph-theoretic notation not explained in this paper, we refer the reader to [2].

A well-known upper bound for the domination number of a graph was given by Ore in 1962.

Theorem 1 ([4]) *If G is a graph without isolated vertices, then $\gamma(G) \leq |G|/2$.*

In 1982, Payan and Xuong [5] and independently, in 1985, Fink, Jacobson, Kinch and Roberts [3] characterized the graphs achieving equality in Ore's bound.

Theorem 2 ([3, 5]) *Let G be a connected graph. Then $\gamma(G) = |G|/2$ if and only if G is the corona graph of any connected graph J or G is isomorphic to the cycle C_4.*

In [1], Blidia, Chellali and Favaron studied the relationship between the 2-domination number and the independence number of a tree. In particular, they proved that the ratio $\gamma_2(T)/\alpha(T)$ for a tree T is contained in a small interval.

Theorem 3 ([1]) *For any tree, $\alpha(T) \leq \gamma_2(T) \leq \frac{3}{2}\alpha(T)$.*

They also proved that both the upper and lower bounds are sharp. Moreover, they tried to characterize all the trees that attain the equality for both bounds. They successfully characterized the trees T with $\gamma_2(T) = \alpha(T)$. However, their characterization of the trees T with $\gamma_2(T) = \frac{3}{2}\alpha(T)$ contains an error.

Let P_n denote the path of order n and let H_8 be the tree depicted in Figure 1. In [1], Blidia, Chellali and Favaron introduced the following operation.

Operation Ω: Given a tree T, introduce a path $x_1y_1y_2x_2$ of order four which is vertex-disjoint from T and join one inner vertex of T and y_1 by an edge.

Let $\mathcal{T}_0$ be the class of trees obtained from P_4 by recursively performing Ω. Note that by definition, each tree in $\mathcal{T}_0$ has its order divisible by four. The only trees in $\mathcal{T}_0$ of order four and eight are P_4 and H_8, respectively, while there exist two trees of order twelve, T_{12} and T'_{12} in Figure 1, in $\mathcal{T}_0$. They have claimed that a tree T satisfies $\gamma_2(T) = \frac{3}{2}\alpha(T)$ if and only if $T \in \mathcal{T}_0$.

However, the operation Ω can create a tree T with $\gamma_2(T) < \frac{3}{2}\alpha(T)$. For example, it is not difficult to see $\gamma_2(T_{12}) = 8$ and $\alpha(T_{12}) = 6$.

The main problem of the operation Ω is that it allows to join an inner vertex of P_4 with a vertex of degree two in a given tree.

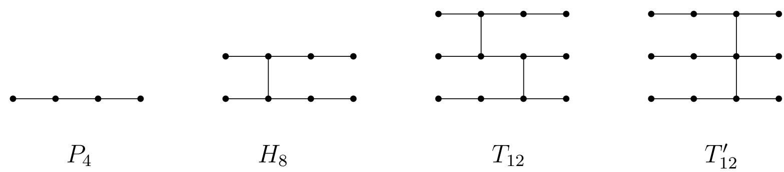

Figure 1: P_4, H_8, T_{12} and T'_{12}

In this note, we will show that, if G is a bipartite graph, the inequality $\gamma_2(G) \le \frac{3}{2}\alpha(G)$ is also valid. In addition, we present a characterization of the bipartite graphs G with $\gamma_2(G) = \frac{3}{2}\alpha(G)$. Hereby, the characterization of the trees with this property follows directly from the latter.

2 Characterization of bipartite graphs G with $\gamma_2(G) = \frac{3}{2}\alpha(G)$

Before presenting the main theorem, we notice the following two observations, where the first one is immediate.

Observation 1 *If a connected graph G is the corona of a corona graph or the corona of the cycle C_4, then $\gamma_2(G) = \frac{3}{2}\alpha(G) = \frac{3}{4}|G|$.*

Observation 2 *If G is the corona graph of a connected graph H of order at least two, then $\gamma_2(G) \le \frac{3}{4}|G|$ with equality if and only if H is either the corona of a connected graph or H is isomorphic to the cycle C_4.*

Proof. Let L be the set of leaves of G and let D be a minimum dominating set of $H = G - L$. Then, since G is a corona graph, $D \cup L$ is a minimum 2-dominating set of G and hence we obtain with Ore's inequality

$$\gamma_2(G) = \gamma(H) + |L| \le \frac{|G - L|}{2} + |L| = \frac{3}{4}|G|.$$

In view of Theorem 2, equality holds if and only if H is the corona of a connected graph or if $H \cong C_4$. $\square$

Theorem 4 *If G is a connected bipartite graph of order at least 3, then $\gamma_2(G) \le \frac{3}{2}\alpha(G)$ and equality holds if and only if G is the corona of the corona of a connected bipartite graph or G is the corona of the cycle C_4.*

Proof. Let L be the set of leaves in G, and let I be a maximum independent set of G. Since $|G| \ge 3$, we can assume, without loss of generality, that $L \subseteq I$ and thus it follows that $l(G) \le \alpha(G)$. Since G is bipartite, evidently $2\alpha(G) \ge |G|$.

Let A and B be the partition sets of G. Define $A_1 := A - L$ and $B_1 := B - L$ and assume, without loss of generality, that $|A_1| \le |B_1|$. Then $|A_1| \le \frac{|G|-l(G)}{2}$. Since every vertex in B_1 has at least two neighbors in $A_1 \cup L$, we see that the latter is a 2-dominating set of G and hence

$$\gamma_2(G) \le |A_1 \cup L| \le \frac{|G| - l(G)}{2} + l(G) = \frac{|G| + l(G)}{2}.$$

Combining this inequality with $l(G) \le \alpha(G)$ and $|G| \le 2\alpha(G)$, we obtain the desired bound

$$\gamma_2(G) \le \frac{|G| + l(G)}{2} \le \frac{2\alpha(G) + \alpha(G)}{2} = \frac{3}{2}\alpha(G).$$

Thus G is a bipartite graph with $\gamma_2(G) = \frac{3}{2}\alpha(G)$ if and only if $|G| = 2\alpha(G)$, $l(G) = \alpha(G)$ and $\gamma_2(G) = \frac{|G|+l(G)}{2}$. The facts that $l(G) = \alpha(G)$ and $|G| = 2\alpha(G) = 2l(G)$ show that G is a corona graph. Furthermore, the identity $\gamma_2(G) = \frac{|G|+l(G)}{2}$ leads to $\gamma_2(G) = \frac{3}{4}|G|$ and, in view of Observation 2, it follows that G is either the corona of the corona of a connected bipartite graph or G is the corona of the cycle C_4.

Conversely, if G is either the corona of the corona of a bipartite graph or G is the corona of the cycle C_4, then Observation 1 implies that $\gamma_2(G) = \frac{3}{2}\alpha(G)$. $\square$

Corollary 1 *If T is a tree of order at least* 3*, then $\gamma_2(G) \le \frac{3}{2}\alpha(G)$ with equality if and only if T is the corona of the corona of a tree.*

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