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Total and connected domination in digraphs

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Abstract

Let $D = (V, A)$ be a digraph. A subset S of V is called a *dominating set of* D if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that $(u, v) \in A$. A dominating set S is called a *total dominating set* if the induced subdigraph $\langle S \rangle$ has no isolated vertices. It is called an open dominating set if for every vertex $v \in V$, there exists a vertex $u \in S$ such that $(u, v) \in A$. A dominating set S is called a weakly connected (respectively unilaterally connected, strongly connected) dominating set if the induced subdigraph $\langle S \rangle$ is weakly connected (respectively unilaterally connected, strongly connected). In this paper we introduce the domination parameters corresponding to total, open and connected domination in digraphs and obtain several results on these parameters.

1 Introduction

Throughout this paper $D = (V, A)$ is a finite directed graph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) and $G = (V, E)$ is a finite undirected graph with neither loops nor multiple edges. For basic terminology on graphs and digraphs, we refer to Chartrand and Lesniak [1].

Let $G = (V, E)$ be a graph. A subset S of E is called an independent set of edges if no two edges in S are adjacent. The maximum cardinality of an independent set of edges in G is called the edge independence number of G and is denoted by $\beta_1(G)$.

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$ or simply γ.

Cockayne *et al.,* [3] introduced the concept of total domination in graphs. Let $G = (V, E)$ be a graph without isolated vertices. A dominating set S of V is called a total dominating set of G if the induced subgraph $\langle S \rangle$ has no isolated vertices, or equivalently, if every vertex in V is adjacent to at least one vertex in S. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by γ_t.

Walikar and Sampathkumar [8] introduced the concept of connected domination for graphs. Let $G = (V, E)$ be a connected graph. A dominating set S is called a connected dominating set of G if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by γ_c. For undirected graphs various types of domination have been studied by different authors by imposing conditions on the dominating set. More than 75 models of dominating and related types of sets are given in the Appendix of Haynes et al. [5].

Let $D = (V, A)$ be a digraph. For any vertex $u \in V$, the sets $O(u) = \{v/(u, v) \in A\}$ and $I(u) = \{v/(v, u) \in A\}$ are called the outset and inset of u. The indegree and outdegree of u are defined by $id(u) = |I(u)|$ and $od(u) = |O(u)|$. The minimum indegree, the minimum outdegree, the maximum indegree and the maximum outdegree of D are denoted by $\delta^-, \delta^+, \Delta^-$ and Δ^+ respectively. A subset S of V is called a dominating set of the digraph D if for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that $(v, u) \in A$. The domination number γ of D is the minimum cardinality of a dominating set of D.

Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogue on digraphs have not received much attention. A survey of results on domination in directed graphs by Ghoshal, Laskar and Pillone is found in chapter 15 of Haynes *et al.,* [4], but most of the results in this survey chapter deal with the concepts of kernels and solutions in digraphs and on domination in tournaments. Among other results in this survey, the authors added a few new results of their own, including the following, that have some relation to the results presented in this paper.

Theorem 1.1. ([4], Page 423) *For any digraph D on n vertices, $\frac{n}{1+\Delta^+(D)} \leq \gamma(D) \leq n - \Delta^+(D)$, where $\Delta^+(D)$ denotes the maximum outdegree.*

Observation 1.2. ([4], Page 423) *For any digraph D on n vertices, which has a hamiltonian circuit, $\gamma(D) \leq \lceil \frac{n}{2} \rceil$.*

Theorem 1.3. ([4], Page 423) *For a strongly connected digraph D on n vertices, $\gamma(D) \leq \lceil \frac{n}{2} \rceil$.*

The first Ph.D. dissertation dedicated to the study of the domination numbers in digraphs is by Lee [7]. He has, among other results, the following theorem which bears some similarity to the results presented in this paper.

Theorem 1.4. [7] *Let D be a digraph of order n and minimum indegree $\delta^- \geq 1$. Then, we have $1 \leq \gamma(D) \leq \frac{\delta^-+1}{2\delta^-+1}n$.*

Chartrand *et al.,*[2] have obtained several results on the sum of the in-domination and out-domination numbers of a digraph.

In this paper we introduce the analogue of total and connected domination in digraphs and present several results on the corresponding domination parameters.

2 Open and total domination in digraphs

The concept of total domination can be extended to digraphs in two different ways.

Definition 2.1. *Let $D = (V, A)$ be a digraph in which $id(v) > 0$ for all $v \in V$. A subset S of V is called an open dominating set of D if for every v in V there exists $u \in S$ such that $(u, v) \in A$. The minimum cardinality of an open dominating set is called the open domination number of D and is denoted by $\gamma_0(D)$.*

Definition 2.2. *Let $D = (V, A)$ be a digraph in which $id(v) + od(v) > 0$ for all $v \in V$. A subset S of V is called a total dominating set of D if S is a dominating set of D and the induced subdigraph $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of D is called the total domination number of D and is denoted by $\gamma_t(D)$.*

Remark 2.3. *If D is a directed graph which contains at least one open dominating set, then we have $\gamma_t(D) \leq \gamma_0(D)$. This inequality can be strict. For the digraph given in Figure 1, $\gamma_t = 4$ and $\gamma_0 = 6$.*

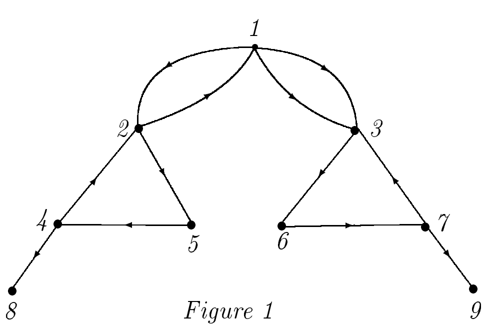

Figure 1

In fact $\{2, 3, 4, 7\}$ is a minimum total dominating set and $\{2, 3, 4, 5, 6, 7\}$ is a minimum open dominating set for D.

Remark 2.4. *If S is an open dominating set of a digraph D, then $\langle S \rangle$ contains a directed cycle.*

We now proceed to characterize weakly connected digraphs of order n with (i) $\gamma_0(D) = n$, (ii) $\gamma_t(D) = n$.

Theorem 2.5. *If $D = (V, A)$ is a weakly connected digraph of order n with $\delta^-(D) > 0$, then $\gamma_0(D) = n$ if and only if D is a dicycle.*

Proof. If D is a dicycle, then $\gamma_0(D) = n$ is immediate. Conversely, suppose that $\gamma_0(D) = n$. If there exists a vertex v with $od(v) = 0$, then $V - \{v\}$ is an open dominating set of D, a contradiction to the hypothesis. This yields $\delta^+(D) > 0$. Now let $P = (x_1 x_2 \ldots x_p)$ be a longest directed path of D. Since $od(x_p) > 0$, there exists an index i with $1 \leq i \leq p - 1$ such that $(x_p, x_i) \in A$.

Assume first that $(x_p, x_1) \notin A$. Then $i > 1$, and since P is a longest directed path and $\delta^-(D) > 0$, it follows that $V - \{x_p\}$ is an open dominating set of D, which is a contradiction.

Assume second that $i = 1$. Then $(x_p, x_1) \in A$. Since D is weakly connected and since P is a longest directed path in D, it follows easily that $V = \{x_1, x_2, \ldots, x_p\}$. If D contains a further arc, say, without loss of generality, (x_1, x_i) with $3 \leq i \leq p = n$, then $V - \{x_{i-1}\}$ is an open dominating set of D, a contradiction. This shows that D is a dicycle, and the proof is complete. □

Theorem 2.6. *Let $D = (V, A)$ be a weakly connected digraph of order $n \geq 3$. Then $\gamma_t(D) = n$ if and only if there exists a subdigraph $D_1 = (V_1, A_1)$ such that all vertices of $V - V_1$ have indegree zero and for each vertex $u \in V_1$, there exists at least one vertex $v \in V - V_1$ such that $(v, u) \in A$ and $od(v) = 1$.*

Proof. Suppose $\gamma_t(D) = n$. Let $V_1 = \{v \in V \,|\, id(v) > 0\}$, and let $D_1 = \langle V_1 \rangle$. The definition of V_1 yields $id(x) = 0$ for each $x \in V - V_1$. Now let $u \in V_1$, and suppose that $(v, u) \notin A$ for all $v \in V - V_1$. Then $V - \{u\}$ is a dominating set of D. In the case that $V - \{u\}$ does not contain an isolated vertex, this is a contradiction to $\gamma_t(D) = n$. If $V - \{u\}$ contains an isolated vertex w, then $(w, u) \in A$, $(u, w) \in A$ and $od(w) = id(w) = 1$. Since D is weakly connected and $n \geq 3$, we arrive at the contradiction that $V - \{w\}$ is a total dominating set of D.

Altogether we have shown that $V - V_1 \neq \emptyset$ and for each $u \in V_1$ there exists a vertex $v \in V - V_1$ such that $(v, u) \in A$.

Now, let $u \in V_1$, and suppose that $od(v) \geq 2$ for each $v \in V - V_1$ with $(v, u) \in A$. Since every vertex $x \in V_1$ has an in-neighbor in $V - V_1$, we observe that $V - \{u\}$ has no isolated vertices, and thus $V - \{u\}$ is a total dominating set of D. This contradiction shows that for each vertex $u \in V_1$, there exists at least one vertex $v \in V - V_1$ such that $(v, u) \in A$ and $od(v) = 1$.

Conversely, if D satisfies the conditions of the theorem, then any dominating set of D contains all the vertices of $V - V_1$, and $D - \{u\}$ contains an isolated vertex for each $u \in V_1$. Hence it follows that V is the only total dominating set of D and thus $\gamma_t(D) = n$. □

Example 2.7. *An example of a digraph D for which $\gamma_t(D) = n$ is given in Figure 2.*

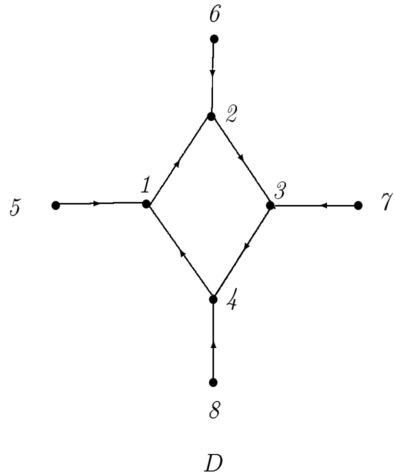

Figure 2

Theorem 2.8. *Let $D = (V, A)$ be a digraph with $id(v) > 0$ for all $v \in V$. Then $\gamma_t(D) \leq n - \Delta^+(D) + 1$.*

Proof. If $\Delta^+(D) = n-1$, then $\gamma_t(D) = n-\Delta^+(D)+1 = 2$. Suppose $\Delta^+(D) < n - 1$. Let v be a vertex with $od(v) = \Delta^+(D)$ and let $X = V - \{\{v\} \cup O(v)\}$. Let U be the set of all isolated vertices in the induced digraph $\langle X \rangle$. If $U = \emptyset$, $X \cup \{v, x\}$ where $x \in O(v)$ is a total dominating set of cardinality $n - \Delta^+(D) + 1$. If $U \neq \emptyset$, for each $u \in U$, choose $x(u) \in O(v)$ such that $(x(u), u) \in A$. Then $\{v\} \cup (X - U) \cup \{x(u) : u \in U\}$ is a total dominating set of D and hence $\gamma_t(D) \leq 1 + |U| + |X - U| = 1 + |X| = n - \Delta^+(D)$. □

Remark 2.9. *The bound given in Theorem 2.8 is sharp.*

(a) *For any digraph D with $\Delta^+(D) = n - 1$, we have $\gamma_t(D) = n - \Delta^+(D) + 1 = 2$.*

(b) *Let $D = (V, A)$ be a digraph with the vertex set*

$$V = \{v, x_1, x_2, \ldots, x_p\} \cup \{a_1, b_1, a_2, b_2, \ldots, a_r, b_r\}$$

such that $(x_1, v) \in A$, $(v, x_i) \in A$ for $1 \leq i \leq p$ and $(a_j, b_j) \in A$ as well as $(b_j, a_j) \in A$ for $1 \leq j \leq r$. Then $\Delta^+(D) = p$, $n(D) = 2r+p+1$ and $\delta^-(D) > 0$. Since $V - \{x_2, x_3, \ldots, x_p\}$ is a minimum total dominating set, we deduce that $\gamma_t(D) = 2r + 2 = n(D) - \Delta^+(D) + 1$.

In the case that D is weakly connected in Theorem 2.8, we obtain a slightly better upper bound for $\gamma_t(D)$.

Theorem 2.10. *Let $D = (V, A)$ be a weakly connected digraph of order n with $\delta^-(D) > 0$. If $\Delta^+(D) \le n-2$, then $\gamma_t(D) \le n - \Delta^+(D)$.*

Proof. Let v be a vertex with $od(v) = \Delta^+(D)$, and let $X = V - \{\{v\} \cup O(v)\}$. Let U be the set of isolated vertices in the induced subdigraph $\langle X \rangle$. If $U \neq \emptyset$, then the desired result follows from the proof of Theorem 2.8. Thus assume in the following that $U = \emptyset$.

If there is an arc $(x, v) \in A$ with $x \in X$, then $X \cup \{v\}$ is a total dominating set of D, and we deduce that $\gamma_t(D) \le n - \Delta^+(D)$. If there is no such arc, then the weakly connectivity of D implies that there is an arc between $O(v)$ and X.

Case 1: There is an arc $(y, x) \in A$ with $y \in O(v)$ and $x \in X$.

Since $U = \emptyset$, the induced subdigraph $\langle X \rangle$ contains an arc (x, x_1) or an arc (x_1, x).

In the case that $\langle X \rangle$ contains an arc (x, x_1), let $P = (xx_1x_2 \dots x_p)$ be a longest directed path in $\langle X \rangle$ with the initial vertex x. If $\langle X - \{x_p\} \rangle$ does not contain an isolated vertex, then $\{v, y\} \cup (X - \{x_p\})$ is a total dominating set, and this leads to the desired bound. If $\langle X - \{x_p\} \rangle$ contains an isolated vertex w, then since P is a longest directed path and since $d^-(w) > 0$, there exists an arc $(u, w) \in A$ with $u \in O(v)$. Now $\{v, u\} \cup (X - \{w\})$ is a total dominating set, and we are done.

Next assume that $\langle X \rangle$ does not contain an arc (x, x_1). If $\langle X - \{x\} \rangle$ does not contain an isolated vertex, then $\{v, y\} \cup (X - \{x\})$ is a total dominating set, and we are done. If $\langle X - \{x\} \rangle$ contains an isolated vertex w, then since $\langle X \rangle$ does not contain an arc (x, x_1) and since $d^-(w) > 0$, we observe that there exists an arc $(u, w) \in A$ with $u \in O(v)$. Now $\{v, u\} \cup (X - \{w\})$ is total dominating set, and we are done.

Case 2: There is no arc (y, x) with $y \in O(v)$ and $x \in X$.

The weakly connectivity of D implies that there exists an arc $(x, y) \in A$ with $x \in X$ and $y \in O(v)$. Since $U = \emptyset$, the induced digraph $\langle X \rangle$ contains an arc (x, x_1) or an arc (x_1, x).

If $\langle X \rangle$ does not contain an arc (x, x_1), then let $P = (xx_1x_2 \dots x_p)$ be a longest directed path in $\langle X \rangle$ with the initial vertex x. If $\langle X - \{x_p\} \rangle$ does not contain an isolated vertex, then $\{v, y\} \cup (X - \{x_p\})$ is a total dominating set, and this yields the desired bound. If $\langle X - \{x_p\} \rangle$ contains an isolated vertex w, it follows that there is an arc $(u, w) \in A$ with $u \in O(v)$, a contradiction to our assumption. Using an argument similar to one in Case 1, we also obtain the desired result in the case that $\langle X \rangle$ does not contain an arc (x, x_1). □

Corollary 2.11. [3] *Let $G = (V, E)$ be a graph of order $n \ge 2$.*

(a) If G has no isolated vertices, then $\gamma_t(G) \le n - \Delta(G) + 1$.

(b) If G is connected and $\Delta(G) \le n-2$, then $\gamma_t(G) \le n - \Delta(G)$.

Proof. Define the digraph D on the vertex set V by replacing each edge of G by two arcs in opposite direction and apply Theorem 2.8 and Theorem 2.10, respectively. □

We now present simple lower bounds for $\gamma_t(D)$ and $\gamma_0(D)$.

Theorem 2.12. *Let D be a digraph of order $n \ge 2$ and maximum outdegree $\Delta^+ \ge 1$.*

(i) If D has no isolated vertices, then $\gamma_t(D) \geq \frac{2n}{2\Delta^+ + 1}$.

(ii) If $\delta^-(D) > 0$, then $\gamma_0(D) \geq \frac{n}{\Delta^+}$.

Proof. (i) Let S be a minimum total dominating set of D. Then the induced subdigraph $\langle S \rangle$ contains at least $\lceil |S|/2 \rceil$ arcs. Since each vertex has at most Δ^+ out-neighbors, we conclude that

$$n \leq |S| + |S|\Delta^+ - \lceil |S|/2 \rceil \leq |S| + |S|\Delta^+ - |S|/2 = \gamma_t(D)(\Delta^+ + 1/2),$$

and this leads to the desired lower bound for $\gamma_t(D)$.

(ii) Let S be a minimum open dominating set of D. Then the induced subdigraph $\langle S \rangle$ contains at least $|S|$ arcs. Since each vertex has at most Δ^+ out-neighbors, we deduce that

$$n \leq |S| + |S|\Delta^+ - |S| = \gamma_0(D)\Delta^+,$$

and this yields the desired lower bound for $\gamma_0(D)$. □

The following examples show that the presented lower bounds are best possible.

Example 2.13. *(i) Let $D_1 = (V_1, A_1)$ be a digraph with the vertices $v_1, v_2, \ldots, v_{2p}$ such that $(v_{2j-1}, v_{2j}) \in A_1$ for $1 \leq j \leq p$ and the vertices $u_1^i, u_2^i, \ldots, u_{r-1}^i$ for odd i with $1 \leq i \leq 2p-1$ and $w_1^i, w_2^i, \ldots, w_r^i$ for even i with $2 \leq i \leq 2p$ such that $(v_{2j-1}, u_k^{2j-1}) \in A_1$ for $1 \leq j \leq p$ and $1 \leq k \leq r-1$ as well as $(v_{2j}, w_k^{2j}) \in A_1$ for $1 \leq j \leq p$ and $1 \leq k \leq r$. Then we observe that $n(D_1) = p(2r+1)$, $\Delta^+(D_1) = r$, $\gamma_t(D_1) = 2p$, and therefore*

$$\gamma_t(D_1) = 2p = \frac{2n(D_1)}{2\Delta^+(D_1) + 1}.$$

(ii) Let $C = (v_1 v_2 \ldots v_p v_1)$ be a cycle and let $u_1^i, u_2^i, \ldots, u_r^i$ be further vertices for $1 \leq i \leq p$. Now let D_2 be the digraph consisting of C, the vertices u_j^i and the arcs (v_i, u_j^i) for $1 \leq i \leq p$ and $1 \leq j \leq r$. Then $n(D_2) = p(r+1)$, $\Delta^+(D_2) = r+1$, $\gamma_0(D_2) = p$, and therefore $\gamma_0(D_2) = n(D_2)/\Delta^+(D_2)$.

3 Connected domination in digraphs

Since there are three types of connectedness for digraphs, we have three types of connected domination in digraphs.

Definition 3.1. *Let $D = (V, A)$ be a digraph such that its underlying graph is connected. A dominating set S of D is called a strongly connected dominating set if $\langle S \rangle$ is a strongly connected subdigraph of D. The minimum cardinality of a strongly connected dominating set in D is called the sc-domination number of D and is denoted by $\gamma_{sc}(D)$. The concepts of weakly connected and unilaterally connected domination and the corresponding parameters $\gamma_{wc}(D)$ and $\gamma_{uc}(D)$ can be similarly defined.*

Remark 3.2. *Obviously* $\gamma \leq \gamma_{wc} \leq \gamma_{uc} \leq \gamma_{sc}$ *and these inequalities can be strict. For the digraph* D *given in Figure 1,* $\gamma = 4$*,* $\gamma_{wc} = 5$*,* $\gamma_{uc} = 6$ *and* $\gamma_{sc} = 7$*.* $S = \{2,3,4,7\}$*,* $S_{uc} = \{1,2,3,4,6,7\}$*,* $S_{sc} = \{1,2,3,4,5,6,7\}$ *and* $S_{wc} = \{1,2,3,4,7\}$ *are respectively the* γ*-set,* γ_{uc}*-set,* γ_{sc}*-set and* γ_{wc}*-set of* D*.*

Theorem 3.3. *Let* D *be an orientation of a connected graph* G *with* $n \geq 3$*. Then* $\gamma_{wc}(D) = n - l(D)$ *where* $l(D)$ *is the maximum number of leaves with outdegree 0 in any spanning tree of* D*.*

Proof. Let T be a spanning tree of D with $l(D)$ leaves of outdegree 0. Then $V(T) - S$ where S is the set of all leaves of outdegree 0 in T is a weakly connected dominating set of D and hence $\gamma_{wc}(D) \leq n - l(D)$.

Now let S be any γ_{wc}-set of D. Since $\langle S \rangle$ is weakly connected it has a spanning tree T_s. Further since S is a dominating set of D, for each $u \in V - S$, we can choose $x(u) \in S$ such that $(x(u), u) \in A$. Let T be the spanning tree of D with arc set $A(T) = A(T_s) \cup \{(x(u), u)/u \in V - S\}$. Clearly T has $n - \gamma_{wc}(D)$ leaves with outdegree 0, so that $l(D) \geq n - \gamma_{wc}(D)$. Hence $\gamma_{wc}(D) \geq n - l(D)$ and thus $\gamma_{wc}(D) = n - l(D)$. □

Corollary 3.4. *For any connected graph* G*, there is an orientation* D *of* G *such that* $\gamma_{wc}(D) \leq n - \Delta(G)$*.*

Proof. Since G has a spanning tree with at least $\Delta(G)$ leaves, the result follows. □

Corollary 3.5. *For any graph* G*, there is an orientation* D *of* G *such that*

$$diam(G) - 1 \leq \gamma_{wc}(D).$$

Proof. Let T be a spanning tree of G having maximum number of leaves, say, $p(T)$. Let D be an orientation of G such that all leaves of T have outdegree 0 in T. Then $\gamma_{wc}(D) = n - l(D) = n - p(T)$. Now, $p(T) \leq n - diam(T) + 1$ and $diam(T) \geq diam(G)$ and hence it follows that $\gamma_{wc}(D) \geq diam(G) - 1$. □

Lemma 3.6. *Let* T *be a tree, and let* $\Omega(T)$ *be the set of leaves. There exists a maximum matching* M *of* T *such that* $V(T) - \Omega(T) \subseteq V(M)$*.*

Proof. Let M be a maximum matching of T such that $|(V(T) - \Omega(T)) \cap V(M)|$ is maximum. Suppose that there exists a vertex $u \notin \Omega(T) \cap V(M)$. Now let P be an M-alternating path of maximum length with the initial vertex u. Since M is a maximum matching, P is not an M-augmenting path, and thus the terminal vertex of P is an M-saturated leaf. It follows that $M' = (M - E(P)) \cup (E(P) - M)$ is also a maximum matching such that $u \in V(M')$ and $|(V(T) - \Omega(T)) \cap V(M')| > |(V(T) - \Omega(T)) \cap V(M)|$, which is a contradiction and the proof is complete. □

Lemma 3.7. *Let* T *be a tree, and let* $\Omega(T)$ *be the set of leaves. If* M *is an arbitrary maximum matching of* T*, then* $V(M) \cap \Omega(T) \neq \emptyset$*.*

Proof. Suppose to the contrary that $V(M) \cap \Omega(T) = \emptyset$. Let u be an arbitrary leaf, and let P be an M-alternating path of maximum length with the initial vertex u. It follows that the terminal vertex of P is also a leaf and hence P is an M-augmenting path, a contradiction to the hypothesis that M is a maximum matching. □

Theorem 3.8. *For any connected graph G, there is an orientation D such that $\gamma_{wc}(D) \leq 2\beta_1(G) - 1$.*

Proof. Choose a spanning tree T of G having maximum number of leaves, say $p(T)$. Let $\Omega(T)$ be the set of leaves of T. It follows from Lemmas 3.6 and 3.7 that there exists a maximum matching M of T such that $V(T) - \Omega(T) \subseteq V(M)$ and $V(M) \cap \Omega(T) \neq \emptyset$. Hence $n - p(T) \leq 2\beta_1(T) - 1$.

Now, let D be an orientation of G such that all leaves of T have outdegree 0 in T. Then $\gamma_{wc}(D) \leq n - p(T) \leq 2\beta_1(T) - 1 \leq 2\beta_1(G) - 1$. □

Hedetniemi and Laskar [6] have proved that $\gamma_c(G) \leq 2\beta_1(G)$ for any connected graph G. However Lemmas 3.6 and 3.7 show the slightly better bound:

Theorem 3.9. *For any connected graph G, we have $\gamma_c(G) \leq 2\beta_1(G) - 1$.*

The following are some interesting problems for further investigation.

Problem 3.10. *Characterize the class of digraphs D for which $\gamma_t(D) = \gamma_0(D)$.*

Problem 3.11. *Characterize the class of digraphs D for which $\gamma_t(D) = n - \Delta^+(D) + 1$.*

Problem 3.12. *Characterize the class of digraphs D for which $\gamma_t(D) = \frac{2n}{2\Delta^+ + 1}$.*

Problem 3.13. *Characterize the class of digraphs D for which $\gamma_0(D) = \frac{n}{\Delta^+}$.*

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