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On the order of almost regular graphs without a matching of given size

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Abstract

A graph G is almost regular, or more precisely, is a $(d, d+1)$-graph, if the degree of each vertex of G is either d or $d+1$. Let $p \geq 1$ and $d \geq 2$ be integers. If G is a $(d, d+1)$-graph of order n with at most p odd components and without a matching M of size $2|M| = n - p$, then we show in this paper that

(i) $n \geq (p+3)(d+1)+1$,

(ii) $n \geq (p+3)(d+1)+p+2$ when $d \geq 3$ is odd,

(iii) $n \geq (p+3)(d+1)+p+4$ when $d \geq 3$ is odd and G is connected,

(iv) $n \geq (p+3)(d+1)+2p+1 = 5p+10$ when $d = 2$ and G is connected.

The special case $p = 1$ of this result was recently proved by Volkmann (Australas. J. Combin. 29 (2004), 119–126). Furthermore, this theorem generalizes corresponding statements by C. Zhao (J. Combin. Math. Combin. Comput. 9 (1991), 195–198) and Wallis (Ars Combin. 11 (1981), 295–300) on almost regular graphs with no odd component and without a perfect matching. Examples will show that the given bounds are best possible.

We shall assume that the reader is familiar with standard terminology on graphs (see, e.g., Chartrand and Lesniak [2]). In this paper, all graphs are finite and simple. The vertex set of a graph G is denoted by $V(G)$. The *neighborhood* $N_G(x) = N(x)$ of a vertex x is the set of vertices adjacent with x, and the number $d_G(x) = d(x) = |N(x)|$ is the *degree* of x in the graph G. If $d \leq d_G(x) \leq d+1$ for each vertex x in a graph G, then we speak of an *almost regular graph* or more precisely of a $(d, d+1)$-*graph*. If M is a matching in a graph G with the property that every vertex (with exactly one exception) is incident with an edge of M, then M is a *perfect matching* (an *almost perfect matching*). We denote by K_n the complete graph of order n and by $K_{r,s}$ the complete bipartite graph with partite sets A and B, where $|A| = r$ and $|B| = s$. If G is a graph and $A \subseteq V(G)$, then we denote by $q(G - A)$ the number of odd components in the subgraph $G - A$.

The proof of our main theorem is based on the following generalization of Tutte's famous 1-factor theorem [3] by Berge [1] in 1958, and we call it the theorem of Tutte-Berge (for a proof see e.g., [4]).

Theorem of Tutte-Berge (Berge [1] 1958) Let G be a graph of order n. If M is a maximum matching of G, then

$$n - 2|M| = \max_{A \subseteq V(G)} \{q(G - A) - |A|\}.$$

Theorem 1 Let $p \geq 1$ and $d \geq 2$ be integers. If G is a $(d, d+1)$-graph of order n with at most p odd components and without a matching M of size $2|M| = n - p$, then

(i) $|V(G)| \geq (p+3)(d+1) + 1$,

(ii) $|V(G)| \geq (p+3)(d+1) + p + 2$ when $d \geq 3$ is odd,

(iii) $|V(G)| \geq (p+3)(d+1) + p + 4$ when $d \geq 3$ is odd and G is connected,

(iv) $|V(G)| \geq (p+3)(d+1) + 2p + 1 = 5p + 10$ when $d = 2$ and G is connected.

Proof In view of the hypotheses, we observe that n and p are of the same parity. Suppose to the contrary that there exists a $(d, d+1)$-graph G with at most p odd components and without a matching M of size $2|M| = n - p$ such that

(a) $|V(G)| \leq (p+3)(d+1)$,

(b) $|V(G)| \leq (p+3)(d+1) + p$ when $d \geq 3$ is odd,

(c) $|V(G)| \leq (p+3)(d+1) + p + 2$ when $d \geq 3$ is odd and G is connected,

(d) $|V(G)| \leq (p+3)(d+1) + 2p - 1 = 5p + 8$ when $d = 2$ and G is connected.

By the hypotheses and the theorem of Tutte-Berge, there exists a non-empty set $A \subseteq V(G)$ such that $q(G - A) \geq |A| + p + 2$. We call an odd component of $G - A$ large if it has more than d vertices and small otherwise. If we denote by α and β the number of large and small components, respectively, then we deduce that

$$\alpha + \beta = q(G - A) \geq |A| + p + 2, \tag{1}$$

$$|V(G)| \geq |A| + \beta + \alpha(d+1), \tag{2}$$

$$|V(G)| \geq |A| + \beta + \alpha(d+2) \text{ when } d \geq 3 \text{ is odd.} \tag{3}$$

Since G is a $(d, d+1)$-graph, it is easy to verify that there are at least d edges of G joining each small component of $G - A$ with A. Therefore it follows from the hypothesis that G has at most p odd components that

$$\alpha - p + d\beta \leq |A|(d+1) \tag{4}$$

$$d\beta \leq |A|(d+1) \text{ when } \alpha = 0. \tag{5}$$

Assumption (a) and inequality (2) lead to

$$(p+3)(d+1) \geq |V(G)| \geq |A| + \beta + \alpha(d+1) \geq 1 + \alpha(d+1)$$

and this immediately yields $\alpha \leq p+2$. Assumptions (b) and (c) and inequality (3) show that

$$(p+3)(d+1) + p + 2 \geq |V(G)| \geq |A| + \beta + \alpha(d+2) \geq 1 + \alpha(d+2).$$

This inequality chain leads to $(p+3-\alpha)(d+2) \geq 2$ and so we obtain $\alpha \leq p+2$ also in these cases.

Now we investigate the case $\alpha = 0$. From the inequalities (1) and (5) we deduce that $d(|A| + p + 2) \leq |A|(d+1)$ and thus we have $d(p+2) \leq |A|$. Combining this with (1) and (2), we deduce that

$$\begin{aligned} |V(G)| &\geq |A| + \beta \geq |A| + |A| + p + 2 \\ &\geq d(p+2) + d(p+2) + p + 2 \\ &= dp + 3d + p + 3 + dp + d - 1 \\ &= (p+3)(d+1) + d(p+1) - 1. \end{aligned}$$

Therefore we arrive at a contradiction to each of the assumptions (a), (b), (c), and (d).

Consequently, it remains to consider the case $\alpha \geq 1$. We note that inequality (4) is equivalent with

$$\beta \leq |A| + \frac{\beta + p - \alpha}{d+1}. \tag{6}$$

Firstly, we prove (i) and (ii). We have seen above that assumptions (a), (b), and (c) yield

$$\alpha \leq p + 2 \tag{7}$$

and hence we conclude from (1) that

$$\beta \geq |A|. \tag{8}$$

Next we distinguish two cases.

Case 1. Assume that $\beta + p - \alpha \leq d$. Because of (6), we conclude that $\beta \leq |A|$, and therefore (8) yields $\beta = |A|$. In view of (1) and (7), it follows that $\alpha = p+2$. Let U be a small component of $G - A$. Since $N(x) \subseteq V(U) \cup A$ for $x \in V(U)$, we observe that $|A| + |V(U)| \geq d + 1$. In addition, we deduce from $\beta = |A|$ and inequality (6) that $\beta \geq \alpha - p = 2$, and so we obtain instead of (2) the estimate

$$\begin{aligned}|V(G)| &\geq |A| + |V(U)| + \beta - 1 + \alpha(d+1)\\ &\geq d + 1 + 1 + (p+2)(d+1)\\ &= (p+3)(d+1) + 1.\end{aligned}$$

This is a contradiction to assumption (a). In the case that $d \geq 3$ is odd, we arrive similar the following contradiction to assumption (b):

$$\begin{aligned}|V(G)| &\geq |A| + |V(U)| + \beta - 1 + \alpha(d+2)\\ &\geq d + 1 + 1 + (p+2)(d+1) + p + 2\\ &= (p+3)(d+1) + p + 3.\end{aligned}$$

Case 2. Assume that $\beta + p - \alpha \geq d + 1$. This condition implies $\beta \geq d + 1 - p + \alpha$ and hence it follows from (4) that

$$\begin{aligned}|A| &\geq \frac{\alpha - p + d\beta}{d+1}\\ &\geq \frac{\alpha - p + d(d+1-p+\alpha)}{d+1}\\ &= \frac{d^2 + d(1+\alpha-p) + \alpha - p}{d+1}. \qquad (9)\end{aligned}$$

Subcase 2.1. Assume that $\alpha \geq p + 1$. In this case, we deduce from inequality (9) that $|A| \geq d + 1$. Thus (2) yields

$$\begin{aligned}|V(G)| &\geq |A| + \beta + \alpha(d+1)\\ &\geq 2(d+1) + 1 + (p+1)(d+1)\\ &= (p+3)(d+1) + 1,\end{aligned}$$

a contradiction to assumption (a). If $d \geq 3$ is odd, then we deduce from (3) the following contradiction to (b):

$$\begin{aligned}|V(G)| &\geq |A| + \beta + \alpha(d+2)\\ &\geq 2(d+1) + 1 + (p+1)(d+2)\\ &= (p+3)(d+1) + p + 2.\end{aligned}$$

Subcase 2.2. Assume that $1 \leq \alpha \leq p$. If we define $\alpha = p + 1 - s$, then we obtain by (1) the inequality $\beta \geq |A| + s + 1$ and (4) leads to $|A| \geq d(1+s) + 1 - s$. Since $1 \leq s \leq p$, we conclude from (2) that

$$\begin{aligned}|V(G)| &\geq |A| + \beta + \alpha(d+1)\\ &\geq 2|A| + s + 1 + (p+1-s)(d+1)\\ &\geq 2d(s+1) + 2 - 2s + s + 1 + (p+1-s)(d+1)\\ &= (p+3+s)(d+1) - 3s + 1\\ &\geq (p+3)(d+1) + 1,\end{aligned}$$

a contradiction to assumption (a). If $d \geq 3$ is odd, then we deduce from (3) analogously the following contradiction to (b):

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \\ &\geq (p+3+s)(d+1) + p - 4s + 2 \\ &\geq (p+3)(d+1) + p + 2. \end{aligned}$$

Secondly, we prove (iii) and (iv). If G is connected, then we use instead of (4) the better bound

$$\alpha + d\beta \leq |A|(d+1) \tag{10}$$

or the equivalent inequality

$$\beta \leq |A| + \frac{\beta - \alpha}{d+1}. \tag{11}$$

Case 3. Assume that $\beta - \alpha \leq d$. We observe that (11) yields $\beta \leq |A|$, and according to (1), we conclude that $\alpha \geq p+2$.

Subcase 3.1. Assume that $d \geq 3$ is odd. Because of (7) and (8), we note that $\beta = |A|$ and $\alpha = p+2$. Therefore (11) implies $\beta \geq \alpha = p+2$. If U is a small component of $G - A$, then we arrive at the following contradiction to assumption (c):

$$\begin{aligned} |V(G)| &\geq |A| + |V(U)| + \beta - 1 + \alpha(d+2) \\ &\geq d+1+1+(p+2)(d+1)+p+2 \\ &= (p+3)(d+1)+p+3. \end{aligned}$$

Subcase 3.2. Assume that $d = 2$. If we define $\alpha = p+2+s$, then we obtain by (1) the inequality $\beta \geq |A| - s$ and (10) leads to $|A| \geq \alpha - ds = p+2+s-ds$. Now (2) yields the following contradiction to assumption (d):

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+1) \\ &\geq |A| + |A| - s + (p+2+s)(d+1) \\ &\geq 2(p+2+s-ds) - s + (p+2)(d+1) + 3s \\ &= (p+3)(d+1) + 2p + 1. \end{aligned}$$

Case 4. Assume that $\beta - \alpha \geq d+1$. This implies $\beta \geq d + \alpha + 1$ and so (10) yields

$$|A| \geq \frac{\alpha + d\beta}{d+1} \geq \frac{\alpha + d(\alpha + d + 1)}{d+1}. \tag{12}$$

Subcase 4.1. Assume that $\alpha \geq p+1$. In view of (12), it follows that $|A| \geq d+p+1$. In the case that $d \geq 3$ is odd, we conclude from (3) that

$$\begin{aligned} |V(G)| &\geq |A| + \beta + \alpha(d+2) \\ &\geq d+p+1+d+p+1+1+(p+1)(d+2) \\ &= (p+3)(d+1) + 3p + 2, \end{aligned}$$

a contradiction to assumption (c). If $d=2$, then (2) leads to the following contradiction to (d):

$$\begin{aligned} |V(G)| &\geq |A|+\beta+\alpha(d+1) \\ &\geq d+p+1+d+p+1+1+(p+1)(d+1) \\ &= (p+3)(d+1)+2p+1. \end{aligned}$$

Subcase 4.2. Assume that $1 \leq \alpha \leq p$. If we define $\alpha = p+1-s$, then (1) leads to $\beta \geq |A|+s+1$ and (10) yields $|A| \geq d(1+s)+p+1-s$. In the case that $d \geq 3$ is odd, we conclude from (3) that

$$\begin{aligned} |V(G)| &\geq |A|+\beta+\alpha(d+2) \\ &\geq 2|A|+s+1+(p+1-s)(d+2) \\ &\geq 2d(s+1)+2p+2-2s+s+1+(p+1-s)(d+2) \\ &= (p+3+s)(d+1)-4s+3p+2 \\ &\geq (p+3)(d+1)+3p+2, \end{aligned}$$

a contradiction to assumption (c). If $d=2$, then (2) leads to the following contradiction to (d):

$$\begin{aligned} |V(G)| &\geq |A|+\beta+\alpha(d+1) \\ &\geq 2|A|+s+1+(p+1-s)(d+1) \\ &\geq 2d(1+s)+2p+2-2s+s+1+(p+1-s)(d+1) \\ &= 5p+10. \end{aligned}$$

Since we have discussed all possible cases, the proof of Theorem 1 is complete. □

The following examples will show that the different bounds in Theorem 1 are best possible.

Example 2 *Case 1.* Let $d \geq 2$ be even. Let $H_1, H_2, \ldots, H_{p+1}$ be $p+1$ copies of the complete graph K_{d+1}. In addition, let $K_{d+1,d+2}$ be the complete bipartite graph with the partite sets $\{x_1, x_2, \ldots, x_{d+1}\}$ and $\{y_1, y_2, \ldots, y_{d+2}\}$. If we delete in the graph $K_{d+1,d+2}$ the edges $x_1y_1, x_2y_2, \ldots, x_{d+1}y_{d+1}$ and x_1y_{d+2}, then we denote the resulting graph by F. If w is an arbitrary vertex of H_{p+1}, then we define the graph G as the disjoint union of $H_1, H_2, \ldots, H_{p+1}$ and F together with the edge wx_1. It is straightforward to verify that G is a $(d, d+1)$-graph of order $n = |V(G)| = (p+3)(d+1)+1$ with p odd components and without a matching M of size $2|M| = n-p$. Consequently, Condition (i) is best possible.

Case 2. Let $d \geq 3$ be odd. Let $H_1, H_2, \ldots, H_{p+1}$ be $p+1$ copies of the complete graph K_{d+2}. In addition, let $K_{d+1,d+2}$ be the complete bipartite graph with the partite sets $\{x_1, x_2, \ldots, x_{d+1}\}$ and $\{y_1, y_2, \ldots, y_{d+2}\}$. If we delete in the graph $K_{d+1,d+2}$ the edges $x_1y_1, x_2y_2, \ldots, x_{d+1}y_{d+1}$ and x_1y_{d+2}, then we denote the resulting graph by F. If u and w are two arbitrary vertices of H_{p+1}, then let $H'_{p+1} = H_{p+1} - uw$. We define the graph G as the disjoint union of $H_1, H_2, \ldots, H_p$, H'_{p+1}, and F together with the edge wx_1. It is easy to see that G is a $(d, d+1)$-graph of order $n = |V(G)| = (p+3)(d+1)+p+2$ with p odd components and without a matching M of size $2|M| = n-p$. Thus Condition (ii) is best possible.

Case 3. Let $d \geq p+3$ be odd. Let $H_1, H_2, \ldots, H_{p+3}$ be $p+3$ copies of the graph $K_{d+2} - M'$, where M' is an almost perfect matching of the complete graph K_{d+2}, and let u be a further vertex. We denote the vertex sets of H_i by $V(H_i) = \{x_1^i, x_2^i, \ldots, x_{d+2}^i\}$ such that $d_{H_i}(x_{d+2}^i) = d+1$ for $i = 1, 2, \ldots, p+3$. We define the graph G as the disjoint union of $H_1, H_2, \ldots, H_{p+3}$ and the vertex u together with the edges

$$ux_1^i, ux_2^i, \ldots, ux^i_{\lfloor \frac{d}{p+3} \rfloor}$$

for $i = 1, 2, \ldots, p+2$ and

$$ux_1^{p+3}, ux_2^{p+3}, \ldots, ux^{p+3}_{d-(p+2)\lfloor \frac{d}{p+3} \rfloor}.$$

Since $\lfloor \frac{d}{p+3} \rfloor \geq 1$ and $d-(p+2)\lfloor \frac{d}{p+3} \rfloor \geq 1$, we observe that G is a connected $(d, d+1)$-graph of order $n = |V(G)| = (p+3)(d+1)+p+4$ without a matching M of size $2|M| = n-p$. This shows that Condition (iii) is best possible.

Case 4. Let $d = 2$. Let $C_{2p+4} = x_1 x_2 \ldots x_{2p+4} x_1$ be a cycle of length $2p+4$, and let $H_1, H_2, \ldots, H_{p+2}$ be $p+2$ cycles of length three. If $y_i \in V(H_i)$ for $i = 1, 2, \ldots, p+2$, then let G be the disjoint union of $H_1, H_2, \ldots, H_{p+2}$ and C_{2p+4} together with the edges $y_i x_{2i}$ for $i = 1, 2, \ldots, p+2$. The resulting $(2,3)$-graph G is connected of order $n = |V(G)| = 5p+10$ without a matching M of size $2|M| = n-p$. This implies that Condition (iv) is also best possible.

The special case $p = 1$ in Theorem 1 leads to the recent result by Volkmann [5].

Corollary 3 (Volkmann [5] 2004) Let $d \geq 2$ be an integer, and let G be a $(d, d+1)$-graph with exactly one odd component and without any almost perfect matching. Then

(i) $|V(G)| \geq 4(d+1)+1$,

(ii) $|V(G)| \geq 4(d+1)+3$ when $d \geq 3$ is odd or $d = 2$ and G is connected,

(iii) $|V(G)| \geq 4(d+1)+5$ when $d \geq 3$ is odd and G is connected.

Corollary 4 (Zhao [8] 1991) Let $d \geq 2$ be an integer. If a $(d, d+1)$-graph G has no odd component and no perfect matching, then

$$|V(G)| \geq 3d+4.$$

Proof Suppose to the contrary that there exists a graph G with no odd component and no perfect matching of size $|V(G)| \leq 3d+3$.

If d is even, then the disjoint union $H = G \cup K_{d+1}$ is a $(d, d+1)$-graph with exactly one odd component, but H has no almost perfect matching. Because of $|V(H)| \leq 4(d+1)$, this is a contradiction to inequality (i) in Corollary 3.

If d is odd, then the disjoint union $H = G \cup K_{d+2}$ is a $(d, d+1)$-graph with exactly one odd component, but H has no almost perfect matching. Because of $|V(H)| \leq 4(d+1)+1$, this is a contradiction to inequality (ii) in Corollary 3. □

Corollary 5 (Wallis [6] 1981) Let $d \geq 3$ be an integer. If a d-regular graph G has no odd component and no perfect matching, then $|V(G)| \geq 3d+4$.

Note that each 1-regular and 2-regular graph without an odd component has a perfect matching. Furthermore, if d is odd or $d = 4$ in Corollary 5, then Wallis [6], [7] has presented the better bounds $|V(G)| \geq 3d+7$ or $|V(G)| \geq 3d+10 = 22$, respectively.

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