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Partitions into three triangular numbers

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Abstract

A celebrated result of Gauss states that every positive integer can be represented as the sum of three triangular numbers. In this article we study $p_{3\Delta}(n)$, the number of partitions of the integer n into three triangular numbers, as well as $p^d_{3\Delta}(n)$, the number of partitions of n into three distinct triangular numbers.

Unlike $t(n)$, which counts the number of representations of n into three triangular numbers, $p_{3\Delta}(n)$ and $p^d_{3\Delta}(n)$ appear to satisfy very few arithmetic relations (apart from certain parity results). However, we shall show that, for all $n \geq 0$,

$$p_{3\Delta}(27n+12) = 3p_{3\Delta}(3n+1) \quad \text{and} \quad p^d_{3\Delta}(27n+12) = 3p^d_{3\Delta}(3n+1).$$

Two separate proofs of these results are given, one via generating function manipulations and the other by a combinatorial argument.

1 Introduction

In 1796, C. F. Gauss proved his now famous result that every integer can be written as the sum of three triangular numbers. That is, if $t(n)$ is the number of **representations** of n as a sum of three triangular numbers, then $t(n) \geq 1$ for all $n \geq 0$.

When we study $t(n)$, we find that it grows quite rapidly. Consider, for example, $t(30)$. Note that 30 can be written as

$$28+1+1,\ 1+28+1,\ 1+1+28,$$
$$21+6+3,\ 21+3+6,\ 6+21+3,\ 6+3+21,\ 3+6+21,\ 3+21+6,$$
$$15+15+0,\ 15+0+15,\ 0+15+15,$$

and

$$10+10+10.$$

So we see that $t(30)=13$.

Our goal in this paper is to study the **partitions** of n into three triangular numbers rather than the representations of n into three triangular numbers. For instance, the three representations $28+1+1$, $1+28+1$, and $1+1+28$ stem from one partition, $28+1+1$. Thus the integer 30 can be partitioned into three triangular numbers in only four ways. In this note, we will denote this partition function by $p_{3\Delta}(n)$, so that $p_{3\Delta}(30)=4$.

Unlike $t(n)$, $p_{3\Delta}(n)$ appears to satisfy very few arithmetic relations (apart from certain parity results). This claim is based on a great deal of numerical evidence. However, while searching for congruences satisfied by $p_{3\Delta}$, we did discover the following:

Theorem 1.
Let $p_{3\Delta}(n)$ be defined as above and let $p^d_{3\Delta}(n)$ be the number of partitions of n into three distinct triangular numbers. Then, for all $n\geq 0$,

$$p_{3\Delta}(27n+12)=3p_{3\Delta}(3n+1) \tag{1}$$

$$\text{and}\quad p^d_{3\Delta}(27n+12)=3p^d_{3\Delta}(3n+1). \tag{2}$$

Theorem 1 (1) is reminiscent of a result proven by the authors [2] involving $t(n)$. Namely, for all $n\geq 0$,

$$t(27n+12)=3t(3n+1). \tag{3}$$

Many other results of this type hold for $t(n)$, such as

$$t(27n+21)=5t(3n+2), \tag{4}$$

$$t(81n+3)=4t(9n), \tag{5}$$

$$\text{and}\quad t(81n+57)=4t(9n+6). \tag{6}$$

However, results corresponding to (4), (5), and (6) do not exist for $p_{3\Delta}$.

In Section 2, we develop the generating functions for $p_{3\Delta}(n)$ and $p^d_{3\Delta}(n)$ using techniques similar to those employed in [3]. Section 3 is then devoted to dissecting these generating functions and proving Theorem 1. Finally, we give a combinatorial proof of Theorem 1 in Section 4.

2 The generating functions

As defined by Ramanujan, let

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}.$$

It is clear that $\sum\limits_{n\geq 0} t(n)q^n = \psi(q)^3$. However, the generating functions for $p_{3\Delta}(n)$ and $p^d_{3\Delta}(n)$ are somewhat more complicated, as we see here.

Theorem 2.

$$\sum_{n\geq 0} p_{3\Delta}(n)q^n = \frac{1}{6}\left(\psi(q)^3 + 3\psi(q)\psi(q^2) + 2\psi(q^3)\right) \tag{7}$$

$$\textit{and} \quad \sum_{n\geq 0} p^d_{3\Delta}(n)q^n = \frac{1}{6}\left(\psi(q)^3 - 3\psi(q)\psi(q^2) + 2\psi(q^3)\right). \tag{8}$$

Proof: Let $\Delta_n = (n^2+n)/2$. Then $\sum_{n\geq 0} q^{\Delta_n} = \psi(q)$. In order to build the generating functions in (7) and (8), we mimic the approach utilized in [3]. We denote the generating function for the number of partitions of n of the form $n = \Delta_a + \Delta_b + \Delta_c$ by $F(\Delta_a + \Delta_b + \Delta_c, q)$, and use similar notation to define related generating functions. (So, for example, $F(\Delta_a + \Delta_a + \Delta_b, q)$ is the generating function for the number of partitions of n into twice one triangular number plus another (different) triangular number.) With the above notation in place, we have

$$\begin{aligned}
F(\Delta_a, q) &= \psi(q),\\
F(\Delta_a + \Delta_a, q) &= \psi(q^2),\\
F(\Delta_a + \Delta_a + \Delta_a, q) &= \psi(q^3),\\
F(\Delta_a + \Delta_b, q) &= \frac{1}{2}\left(F(\Delta_a, q)^2 - F(\Delta_a + \Delta_a, q)\right)\\
&= \frac{1}{2}\left(\psi(q)^2 - \psi(q^2)\right),\\
F(\Delta_a + \Delta_a + \Delta_b, q) &= F(\Delta_a + \Delta_a, q)F(\Delta_a, q) - F(\Delta_a + \Delta_a + \Delta_a, q)\\
&= \psi(q)\psi(q^2) - \psi(q^3),\\
\text{and} \quad F(\Delta_a + \Delta_b + \Delta_c, q) &= \frac{1}{3}\left(F(\Delta_a + \Delta_b, q)F(\Delta_a, q) - F(\Delta_a + \Delta_a + \Delta_b, q)\right)\\
&= \frac{1}{6}\left(\psi(q)^3 - 3\psi(q)\psi(q^2) + 2\psi(q^3)\right).
\end{aligned}$$

It follows that

$$\sum_{n\geq 0} p_{3\Delta}(n)q^n = F(\Delta_a+\Delta_b+\Delta_c,q)+F(\Delta_a+\Delta_a+\Delta_b,q)+F(\Delta_a+\Delta_a+\Delta_a,q)$$
$$= \frac{1}{6}\left(\psi(q)^3+3\psi(q)\psi(q^2)+2\psi(q^3)\right),$$

which is (7). Also,

$$\sum_{n\geq 0} p^d_{3\Delta}(n)q^n = F(\Delta_a+\Delta_b+\Delta_c,q) = \frac{1}{6}\left(\psi(q)^3-3\psi(q)\psi(q^2)+2\psi(q^3)\right),$$

which is (8). □

3 A generating function proof of Theorem 1

In order to prove Theorem 1, we must dissect the generating functions found in Theorem 2. To do so, we develop a large number of ancillary dissection results. As in Cooper and Hirschhorn [1], let

$$\phi(q)=\sum_{n=-\infty}^{\infty} q^{n^2},\quad X(q)=\sum_{n=-\infty}^{\infty} q^{3n^2+2n},\quad P(q)=\sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2},$$

$$A(q)=\sum_{n=-\infty}^{\infty} q^{9n^2+2n},\quad B(q)=\sum_{n=-\infty}^{\infty} q^{9n^2+4n},\quad C(q)=\sum_{n=-\infty}^{\infty} q^{9n^2+8n},$$

$$H(q)=\sum_{n=-\infty}^{\infty} q^{(9n^2+n)/2},\quad I(q)=\sum_{n=-\infty}^{\infty} q^{(9n^2+5n)/2},\quad J(q)=\sum_{n=-\infty}^{\infty} q^{(9n^2+7n)/2}.$$

We need the following collection of results.

Lemma 1.

$$\psi(q)=P(q^3)+q\psi(q^9), \tag{i}$$

$$X(q)=A(q^3)+qB(q^3)+q^5C(q^3), \tag{ii}$$

$$P(q)=H(q^3)+qI(q^3)+q^2J(q^3), \tag{iii}$$

$$H(q)H(q^2)+qI(q)I(q^2)+q^2J(q)J(q^2)=\phi(q^3)P(q), \tag{iv}$$

$$H(q^2)I(q)+qI(q^2)J(q)+qJ(q^2)H(q)=X(q)P(q), \tag{v}$$

$$H(q)I(q^2)+qI(q)J(q^2)+J(q)H(q^2)=2\psi(q^3)X(q), \tag{vi}$$

$$P(q)P(q^2)=\phi(q^9)P(q^3)+qX(q^3)P(q^3)+2q^2\psi(q^9)X(q^3), \tag{vii}$$

$$A(q)H(q)+qB(q)J(q)+q^2C(q)J(q)=P(q)P(q^2), \tag{viii}$$

$$A(q)I(q)+B(q)H(q)+q^2C(q)J(q)=2\psi(q^3)P(q^2), \tag{ix}$$

$$A(q)J(q)+B(q)I(q)+qC(q)H(q)=2\psi(q^6)P(q), \tag{x}$$

and

$$X(q)P(q)=P(q^3)P(q^6)+2q\psi(q^9)P(q^6)+2q^2\psi(q^{18})P(q^3) \tag{xi}$$

Proof: First, (i), (ii) and (iii) are straightforward 3-dissections. Next, let

$$F_1(q)=H(q)H(q^2)+qI(q)I(q^2)+q^2J(q)J(q^2),$$

the left hand side of (iv). Then $F_1(q)$ equals

$$\sum_{m,n=-\infty}^{\infty} q^{(9m^2+m)/2+(9n^2+n)}+\sum_{m,n=-\infty}^{\infty} q^{(9m^2+5m)/2+(9n^2+5n)}+\sum_{m,n=-\infty}^{\infty} q^{(9m^2+7m)/2+(9n^2+7n)}.$$

Thus,

$$\begin{aligned}
q^3F_1(q^{72}) &= \sum q^{(18m+1)^2+2(18n+1)^2}+\sum q^{(18m-5)^2+2(18n-5)^2}+\sum q^{(18m+7)^2+2(18n+7)^2}\\
&= \sum_{a\equiv b \pmod 3} q^{(6a+1)^2+2(6b+1)^2}\\
&= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s+1)^2+2(6s-6r+1)^2}\\
&= q^3\sum_{r,s=-\infty}^{\infty} q^{216r^2+108s^2+36s}\\
&= q^3\phi(q^{216})P(q^{72}),
\end{aligned}$$

so that $F_1(q)=\phi(q^3)P(q)$. This yields (iv). The proofs of (v) and (vi) are similar. Indeed, let

$$F_2(q)=H(q^2)I(q)+qI(q^2)J(q)+qJ(q^2)H(q).$$

Then

$$q^{27}F_2(q^{72})=\sum_{a-b\equiv -1 \pmod 3} q^{(6a-1)^2+(6b+1)^2}$$

$$= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s-5)^2+2(6s-6r+1)^2}$$
$$= q^{27} \sum_{r,s=-\infty}^{\infty} q^{216r^2-144r+108s^2-36s}$$
$$= q^{27} X(q^{72}) P(q^{72}),$$

which implies $F_2(q) = X(q)P(q)$.

Next, let

$$F_3(q) = H(q)I(q^2) + qI(q)J(q^2) + J(q)H(q^2).$$

Then

$$q^{51} F_3(q^{72}) = \sum_{a-b\equiv 1 \pmod 3} q^{(6a+1)^2+2(6b+1)^2}$$
$$= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s+7)^2+2(6s-6r+1)^2}$$
$$= q^{51} \sum_{r,s=-\infty}^{\infty} q^{216r^2+144r+108s^2+108s}$$
$$= 2q^{51}\psi(q^{216})X(q^{72}),$$

which means $F_3(q) = 2\psi(q^3)X(q)$. Thanks to (iii), (iv), (v) and (vi), we know

$$\begin{aligned} P(q)P(q^2) &= \left(H(q^3) + qI(q^3) + q^2J(q^3)\right)\left(H(q^6) + q^2I(q^6) + q^4J(q^6)\right) \\ &= \left(H(q^3)H(q^6) + q^3I(q^3)I(q^6) + q^6J(q^3)J(q^6)\right) \\ &\quad + q\left(H(q^6)I(q^3) + q^3I(q^6)J(q^3) + q^3J(q^6)H(q^3)\right) \\ &\quad + q^2\left(H(q^3)I(q^6) + q^3I(q^3)J(q^6) + J(q^3)H(q^6)\right) \\ &= \phi(q^9)P(q^3) + qX(q^3)P(q^3) + 2q^2\psi(q^9)X(q^3). \end{aligned}$$

This is (vii). The proofs of (viii)–(xi) follow using similar techniques and are omitted here. □

With this machinery in hand, we now prove two additional theorems in preparation for our proof of Theorem 1.

Theorem 3.
Let $u(n)$ be defined by

$$\sum_{n\geq 0} u(n)q^n = \psi(q)\psi(q^2).$$

Then, for all $n \geq 0$, $u(27n+12) = 3u(3n+1)$.

Proof: From (i) we have

$$\begin{aligned}\sum_{n\geq 0} u(n)q^n &= \psi(q)\psi(q^2)\\ &= \left(P(q^3)+q\psi(q^9)\right)\left(P(q^6)+q^2\psi(q^{18})\right)\\ &= \left(P(q^3)P(q^6)+q^3\psi(q^9)\psi(q^{18})\right)+q\psi(q^9)P(q^6)+q^2\psi(q^{18})P(q^3).\end{aligned}$$

Thanks to this dissection, we immediately see that

$$\sum_{n\geq 0} u(3n+1)q^n = \psi(q^3)P(q^2).$$

Also, from the work above and (vii), we know

$$\begin{aligned}\sum_{n\geq 0} u(3n)q^n &= P(q)P(q^2)+q\psi(q^3)\psi(q^6)\\ &= \phi(q^9)P(q^3)+qX(q^3)P(q^3)+2q^2\psi(q^9)X(q^3)+q\psi(q^3)\psi(q^6).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} u(9n+3)q^n &= X(q)P(q)+\psi(q)\psi(q^2)\\ &= X(q)P(q)+\sum_{n\geq 0} u(n)q^n.\end{aligned}$$

This fact, combined with (xi), implies that

$$\begin{aligned}\sum_{n\geq 0} u(9n+3)q^n-\sum_{n\geq 0} u(n)q^n &= X(q)P(q)\\ &= P(q^3)P(q^6)+2q\psi(q^9)P(q^6)+2q^2\psi(q^{18})P(q^3).\end{aligned}$$

Hence,

$$\sum_{n\geq 0} u(27n+12)q^n-\sum_{n\geq 0} u(3n+1)q^n = 2\psi(q^3)P(q^2) = 2\sum_{n\geq 0} u(3n+1)q^n$$

or

$$u(27n+12) = 3u(3n+1).$$

□

Theorem 4.
Let $v(n)$ be defined by

$$\sum_{n\geq 0} v(n)q^n = \psi(q^3).$$

Then, for all $n \geq 0$, $v(27n+12) = 3v(3n+1)$.

Proof: Since

$$\sum_{n\geq 0} v(n)q^n = \psi(q^3),$$

and $\psi(q^3)$ is a power series in q^3, we know

$$\sum_{n\geq 0} v(3n+1)q^n = 0.$$

Also, by (i), we have

$$\sum_{n\geq 0} v(3n)q^n = \psi(q) = P(q^3) + q\psi(q^9),$$

so that

$$\sum_{n\geq 0} v(9n+3)q^n = \psi(q^3).$$

By the same argument then,

$$\sum_{n\geq 0} v(27n+12)q^n = 0.$$

□

We are now prepared to prove Theorem 1.

Proof of Theorem 1: Using the notation in the previous theorems, we see that

$$\sum_{n\geq 0} p_{3\Delta}(n)q^n = \frac{1}{6}\sum_{n\geq 0}\left(t(n) + 3u(n) + 2v(n)\right)q^n,$$

which means

$$p_{3\Delta}(n) = \frac{1}{6}\left(t(n) + 3u(n) + 2v(n)\right).$$

Similarly,

$$p^d_{3\Delta}(n) = \frac{1}{6}\left(t(n) - 3u(n) + 2v(n)\right).$$

Theorem 1 then follows from (3) and Theorems 3 and 4. □

4 A combinatorial proof of Theorem 1

We start by noting that there is a one-to-one correspondence between partitions of $3n+1$ into three triangular numbers and partitions of $24n+11$ into three odd squares. A similar correspondence can be made between the partitions of $27n+12$ into three triangular numbers and $216n+99$ into three odd squares. Hence, proving (1) is equivalent to proving that the number of partitions of $216n+99$ into three odd squares equals three times the number of partitions of $24n+11$ into three odd squares.

In order to prove the result, we shall establish a one-to-three correspondence between the two sets of partitions.

Suppose that

$$24n + 11 = k^2 + l^2 + m^2$$

with k, l and m odd and positive. Considering this equation modulo 6 gives

$$k^2 + l^2 + m^2 \equiv -1 \pmod 6.$$

The only solutions of this are (permutations of)

$$k \equiv \pm 1, \quad l \equiv \pm 1, \quad m \equiv 3 \pmod 6.$$

Allowing k and l to go negative, we can assume without loss of generality that

$$k \equiv 1, \quad l \equiv 1, \quad m \equiv 3 \pmod 6$$

and that

$$k \geq l \text{ and } m > 0.$$

Now set

$$x_1 = 2k + 2l - m, \quad y_1 = 2k - l + 2m, \quad z_1 = -k + 2l + 2m,$$
$$x_2 = 2k + 2l + m, \quad y_2 = 2k - l - 2m, \quad z_2 = -k + 2l - 2m,$$
$$x_3 = 3k, \quad y_3 = 3l, \quad z_3 = 3m.$$

That is,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix},$$
$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}.$$

Then

$$x_1^2+y_1^2+z_1^2=x_2^2+y_2^2+z_2^2=x_3^2+y_3^2+z_3^2=216n+99$$

and, modulo 6,

$$(x_1,y_1,z_1)\equiv(1,1,1),\quad (x_2,y_2,z_2)\equiv(1,1,1),\quad (x_3,y_3,z_3)\equiv(3,3,3).$$

It is clear that the partition given by (x_3,y_3,z_3) is different from the other two. We now show that the partitions given by (x_1,y_1,z_1) and (x_2,y_2,z_2) are different from one another. If indeed they are the same then one of the following six situations pertains.

$$\begin{aligned}
x_1=x_2,&\quad y_1=y_2,\quad z_1=z_2,\\
x_1=x_2,&\quad y_1=z_2,\quad z_1=y_2,\\
x_1=y_2,&\quad y_1=x_2,\quad z_1=z_2,\\
x_1=y_2,&\quad y_1=z_2,\quad z_1=x_2,\\
x_1=z_2,&\quad y_1=x_2,\quad z_1=y_2,\\
\text{or } x_1=z_2,&\quad y_1=y_2,\quad z_1=x_2.
\end{aligned}$$

In every one of these cases it follows that $m=0$, but this is false as m is odd.

So we see that for each partition of $24n+11$ into three odd squares, there are three partitions of $216n+99$ into three odd squares.

Corresponding to the partition of $24n+11$ given by $\mathbf{v}=\begin{pmatrix}k\\l\\m\end{pmatrix}$, we have the three partitions of $216n+99$ given by $A\mathbf{v}$, $B\mathbf{v}$ and $C\mathbf{v}$, where A, B and C are the three matrices defined above. We now show that the three partitions of $216n+99$ are uniquely determined, that is, if $\mathbf{v}\neq\mathbf{v}'$ where $\mathbf{v}'=\begin{pmatrix}k'\\l'\\m'\end{pmatrix}$ and $(k',l',m')\equiv(1,1,3)$ (mod 6) then $\{A\mathbf{v},B\mathbf{v},C\mathbf{v}\}\cap\{A\mathbf{v}',B\mathbf{v}',C\mathbf{v}'\}=\{\}$.

Suppose

$$C\mathbf{v}=C\mathbf{v}'.$$

Then

$$\begin{pmatrix}3k\\3l\\3m\end{pmatrix}=\begin{pmatrix}3k'\\3l'\\3m'\end{pmatrix}$$

and it follows that $\mathbf{v}=\mathbf{v}'$.

Suppose

$$A\mathbf{v}=A\mathbf{v}'.$$

Then

$$A^2\mathbf{v} = A^2\mathbf{v}',$$

that is,

$$9\mathbf{v} = 9\mathbf{v}'$$

so $\mathbf{v} = \mathbf{v}'$. A similar result holds if $B\mathbf{v} = B\mathbf{v}'$. We simply multiply by the transpose of B, and note that $B^T B = 9I$.

Next, suppose

$$A\mathbf{v} = B\mathbf{v}'.$$

Then $A^2\mathbf{v} = AB\mathbf{v}'$, or,

$$\begin{pmatrix} 9k \\ 9l \\ 9m \end{pmatrix} = \begin{pmatrix} 7k + 4l - 4m \\ 4k + l + 8m \\ 4k - 8l - m \end{pmatrix}.$$

Modulo 6, this become

$$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

clearly false. A similar result holds (by symmetry) if $B\mathbf{v} = A\mathbf{v}'$.

We need now show that each partition of $216n + 99$ into three odd squares gives rise to a partition of $24n + 11$ into three odd squares.

Suppose

$$216n + 99 = x^2 + y^2 + z^2$$

with x, y and z odd. Modulo 54, this becomes

$$x^2 + y^2 + z^2 \equiv 45 \pmod{54}.$$

Consideration of all possibilities yields the 240 solutions (not counting permutations),

$$\begin{aligned}(x, y, z) \equiv\ & (\pm1, \pm1, \pm23),\ (\pm1, \pm5, \pm17),\ (\pm1, \pm7, \pm7),\ (\pm1, \pm11, \pm25),\\ & (\pm1, \pm13, \pm19),\ (\pm5, \pm5, \pm7),\ (\pm5, \pm11, \pm13),\ (\pm5, \pm19, \pm19),\\ & (\pm5, \pm23, \pm25),\ (\pm7, \pm11, \pm19),\ (\pm7, \pm13, \pm23),\ (\pm7, \pm17, \pm25),\\ & (\pm11, \pm11, \pm17),\ (\pm11, \pm23, \pm23),\ (\pm13, \pm13, \pm25),\ (\pm13, \pm17, \pm17),\\ & (\pm17, \pm19, \pm23),\ (\pm19, \pm25, \pm25),\ (\pm3, \pm3, \pm9),\ (\pm3, \pm3, \pm27),\\ & (\pm3, \pm9, \pm15),\ (\pm3, \pm15, \pm27),\ (\pm3, \pm9, \pm21),\ (\pm3, \pm21, \pm27),\\ & (\pm9, \pm15, \pm15),\ (\pm15, \pm15, \pm27),\ (\pm9, \pm15, \pm21),\ (\pm15, \pm21, \pm27),\\ & (\pm9, \pm21, \pm21),\ (\pm21, \pm21, \pm27).\end{aligned}$$

If we allow x, y and z to go negative, we can assume without loss of generality that, modulo 54, one of the following 30 possibilities holds.

$$\begin{aligned}(x,y,z)\equiv{}&(-23,1,1),\ (-5,1,-17),\ (1,7,7),\ (1,25,-11),\ (13,1,19),\ (7,-5,-5),\\ &(-11,-5,13),\ (-5,19,19),\ (25,-23,-5),\ (19,-11,7),\ (7,13,-23),\\ &(-17,7,25),\ (-17,-11,-11),\ (-11,-23,-23),\ (25,13,13),\ (13,-17,-17),\\ &(-23,19,-17),\ (19,25,25),\ (3,3,9),\ (3,3,27),\ (3,-15,9),\ (3,-15,27),\\ &(3,21,9),\ (3,21,27),\ (-15,-15,9),\ (-15,-15,27),\ (-15,21,9),\\ &(-15,21,27),\ (21,21,9),\ \text{or } (21,21,27).\end{aligned}$$

In the first eighteen cases, if we apply the matrix $A^{-1}=\frac{1}{9}A$ to the vector $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, we obtain a vector $\begin{pmatrix}k\\l\\m\end{pmatrix}$ that satisfies

$$k^2+l^2+m^2=24n+11,\quad (k,l,m)\equiv(1,1,3)\pmod 6.$$

If $k<l$, simply switch the second and third coordinates of $\mathbf{w}$, and then $k>l$. If $m<0$, apply B^{-1} instead of A^{-1}, and then $m>0$. In the latter twelve cases, if we apply the matrix $C^{-1}=\frac{1}{9}C$ to $\mathbf{w}$, we obtain a vector $\begin{pmatrix}k\\l\\m\end{pmatrix}$ that satisfies

$$k^2+l^2+m^2=24n+11,\quad (k,l,m)\equiv(1,1,3)\pmod 6.$$

If $k<l$, switch the first two coordinates of $\mathbf{w}$, and then $k>l$. If $m<0$, change the sign of the third coordinate of $\mathbf{w}$ and then $m>0$.

This establishes the desired one-to-three correspondence, and completes the proof of (1). To prove (2), we need only replace non-strict inequalities where they occur with strict inequalities, and the proof goes through as before.

References

[1] S. Cooper and M. D. Hirschhorn, Results of Hurwitz type for three squares, *Discrete Math.* **274** (2004), 9–24.

[2] M. D. Hirschhorn and J. A. Sellers, On representations of a number as a sum of three triangles, *Acta Arithmetica* **77** (1996), 289–301.

[3] M. D. Hirschhorn and J. A. Sellers, On a problem of Lehmer on partitions into squares, *The Ramanujan Journal*, to appear.

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