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Separating pairs of points in the plane by monotone subsequences

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Abstract

Let S be a finite set of points in $\mathbf{R}^2$. Let k be a positive integer. A pair of points $\{a, b\}$ of S is called k-linked if there exists a weakly monotone sequence with $k+1$ points of S in which a and b are two endpoints. Let $f(n,k)$ be the maximum integer t such that every n-set $S \subset \mathbf{R}^2$ has t k-linked pairs. It is known that $f(n,k) = 0$ if and only if $n \le k^2$. Let $t(n,k)$ be $(1/2) \cdot \sum_{i=1}^{k}(\lfloor (n+i-1)/k \rfloor - k + 1)(\lfloor (n+i-1)/k \rfloor - k)$ for $n \ge k^2+1$. It is known that $f(n,2) = t(n,2)$ for $n \ge 5$. In this paper, it is shown that $f(n,3) = t(n,3)$ for $n \ge 10$. For $k \ge 4$, it is shown that there exists a positive constant c_k depending only on k such that $t(n,k) - c_k \le f(n,k) \le t(n,k)$ for $n \ge k^2+1$.

1 Introduction

Let S be a set in the Euclidean d-dimensional space $\mathbf{R}^d$. We say that S is an n-set if $|S| = n$. A sequence of points $v_1, v_2, \ldots, v_l \subset \mathbf{R}^d$ is called *weakly monotone* or simply *monotone* if it is weakly monotone in each of its coodinates. We say that v_1 and v_l are endpoints of the sequence. Let k be a positive integer. A pair of points $\{a, b\}$ of S is called *k-linked* if there exists a monotone sequence in S with $k+1$ points containing a and b as its endpoints. Throughout the paper, we concentrate on the case $d = 2$.

For a positive integer n, let $f(n,k)$ be the largest integer t such that every n-set $S \subset \mathbf{R}^2$ has t k-linked pairs. In this paper, we study $f(n,k)$. Obviously $f(n,1) = \binom{n}{2}$

for all $n \geq 1$. The Erdös-Szekeres theorem on monotone subsequences implies that $f(n,k) = 0$ if and only if $n \leq k^2$ [2]. Several proofs of the Erdös-Szekeres theorem are reviewed in [3]. Alon, Füredi and Katchalski introduced a notion of separation for a pair of points in $\mathbf{R}^d$ [1]. A 2-subset $\{a,b\}$ is called *separated* if it is not 2-linked in our definition. In [1], it is proved that $f(n,2) = \binom{n}{2} - (\lfloor n^2/4 \rfloor + n - 2)$ for $n \geq 2$, and the corresponding problem in the case of higher dimensions is asymptotically solved.

2 Main Results

First, we show an upper bound of $f(n,k)$. Let us partition n into k parts as equal as possible. Precisely, set $n = n_1 + n_2 + \cdots + n_k$ such that $n_i = \lfloor (n+i-1)/k \rfloor$ for $1 \leq i \leq k$. For $v \in \mathbf{R}^d$, let us denote the coordinates of v by $(x(v), y(v))$. Put $S = \{v_1, \ldots, v_n\}$ such that $x(v_i) = i$ and $(y(v_1), y(v_2), \ldots, y(v_n)) = (n_1, n_1 - 1, \ldots, 1, n_1 + n_2, n_1 + n_2 - 1, \ldots, n_1 + 1, \ldots, n, n-1, \ldots, n - n_k + 1)$. Let $t(n,k)$ be the number of k-linked pairs of S. It is easily checked that $t(n,k) = 0$ for $n \leq k^2$ and $t(n,k) = (1/2) \cdot \sum_{i=1}^{k} (n_i - k + 1)(n_i - k)$ for $n \geq k^2 + 1$. By the definition, $f(n,k) \leq t(n,k)$ holds. It is plausible that the above arrangement is a best one.

Conjecture 1. *Let n and k be positive integers. Then $f(n,k) = t(n,k)$.*

We propose another conjecture, which is slightly stronger than Conjecture 1. For a finite set $S \subset \mathbf{R}^2$ and for a point $v \in S$, we denote the number of points of S being k-linked with v by $n_k(v)$.

Conjecture 2. *Let $n = mk+1$ with $m \geq k \geq 1$. Then every n-set $S \subset \mathbf{R}^2$ contains a point v such that $n_k(v) \geq m - k + 1$.*

Note that Conjecture 2 implies Conjecture 1. Indeed, assume that Conjecture 2 holds for a fixed k and for all m with $m \geq k$. Let $S \subset \mathbf{R}^2$ be an n-set with $mk+1 \leq n \leq (m+1)k$. Let us take a point $v \in S$ with $n_k(v) \geq m - k + 1$. Set $S' = S - v$. Then by induction on n, the number of k-linked pairs in S' is at least $t(n-1,k)$. Hence, the number of k-linked pairs in S is at least $m - k + 1 + t(n-1,k) = t(n,k)$.

Conjecture 2 is true for $m = k$ [2], and for $k = 2$ with all $m \geq 2$ [1]. In this paper, we prove:

Theorem 3. *Let $n = mk + 1$ with $k \geq 3$ and $m \geq (k-1)^2$. Then every n-set $S \subset \mathbf{R}^2$ contains a point v such that $n_k(v) \geq m - k + 1$.*

For $k = 3$, Theorem 3 with the result of Erdös-Szekeres implies the following result.

Corollary 4. *$f(n,3) = t(n,3)$ for all $n \geq 1$.*

For $k \geq 4$, $f(n,k) - f(n-1,k) \geq t(n,k) - t(n-1,k)$ holds for $n \geq (k-1)^2 k + 1$ by Theorem 3. Hence, we have the following corollary.

Corollary 5. *For $k \geq 4$, there exists a positive constant c_k depending only on k such that $t(n,k) - c_k \leq f(n,k) \leq t(n,k)$.*

3 Proof of Theorem 3

Let S be an n-set in the plane such that $n_k(v) \leq m - k$ for all $v \in S$. We may assume $x(a) \neq x(b)$ and $y(a) \neq y(b)$ for any pair $\{a, b\}$ of S, because otherwise we may slightly change a position of some point without increasing the number of k-linked pairs. Let $v_0 \in S$ be the leftmost point of S. Precisely, $x(v_0) \leq x(v)$ holds for all $v \in S$. Set $S^+ = \{v \in S : y(v) > y(v_0)\}$ and $S^- = \{v \in S : y(v) < y(v_0)\}$. For a pair $\{a, b\}$ of S, let us denote the largest integer t such that a and b are t-linked with each other by $d(a, b)$. Set $S_i = \{v \in S : d(v, v_0) = i\}$, $S_i^+ = S_i \cap S^+$ and $S_i^- = S_i \cap S^-$ for $i \geq 1$. We cannot have $u, v \in S_i^+$ with $x(u) < x(v)$ and $y(u) < y(v)$ since then the monotone sequence ending at $(x(u), x(v))$ could be extended to include $(x(v), y(v))$, contradicting its assumed maximality. Thus each S_i^+ is the range of a monotone sequence with the x components increasing and the y components decreasing. Similarly each S_i^- is the range of a monotone sequence with the x components and the y components both increasing. Let v_i^+ and v_i^- be the rightmost points of S_i^+ and S_i^-, respectively for each i. Moreover, let w be the rightmost point of $\bigcup_{i=1}^{k-1} S_i$. Without loss of generality, we may assume $w = v_\alpha^-$ for some α with $1 \leq \alpha \leq k - 1$. Because S_i^- is monotone, v_i^- is k-linked with at least $|S_i^-| - k$ points in S_i^- for each i. Since $n_k(v_i^-) \leq m - k$, we have

$$|S_i^-| \quad \leq \quad m \tag{1}$$

for each i. We can make a better estimate for $n_k(w)$, because w is k-linked with at least $|S_i^+| - (k - 1)$ points in S_i^+ for all i with $1 \leq i \leq k - 1$. It follows that $m - k \geq n_k(w) \geq \sum_{i=1}^{k-1}(|S_i^+| - (k - 1)) + |S_\alpha^-| - k$. Hence, we have

$$|S_\alpha^-| + \sum_{i=1}^{k-1} |S_i^+| \quad \leq \quad m + (k - 1)^2. \tag{2}$$

Set $T = \bigcup_{i \geq k} S_i$. Since T is the set of points k-linked with v_0, we have

$$|T| \quad \leq \quad m - k. \tag{3}$$

<u>Case 1.</u> $x(v_j^-) > x(v_{j+1}^-)$ for some j with $1 \leq j \leq k - 2$.
First, we assume that $j \neq \alpha$. In this case, we have $n_k(v_j^-) \geq |S_j^-| - k + |S_{j+1}^-| - (k - 1)$. Then we have

$$|S_j^-| + |S_{j+1}^-| \quad \leq \quad m + (k - 1). \tag{4}$$

By adding (1) for all i with $1 \leq i \leq k - 1$ and $i \neq j, j + 1, \alpha$, (2), (3) and (4), we have

$$\begin{aligned} n - 1 \quad &\leq \quad (k - 4)m + m + (k - 1)^2 + m - k + m + (k - 1) \\ &= \quad (k - 1)m + (k - 1)^2 - 1. \end{aligned}$$

This contradicts the assumption that $m \geq (k-1)^2$. If $j = \alpha$, then (2) can be replaced by

$$|S_\alpha^-| + |S_{\alpha+1}^-| + \sum_{i=1}^{k-1} |S_i^+| \quad \leq \quad m + k(k-1).$$

In the same manner as in the case $j \neq \alpha$, we have $m \leq (k-1)^2 - 1$, a contradiction.
Case 2. $x(v_i^-) < x(v_{i+1}^-)$ for all i with $1 \leq i \leq k-2$.
In this case, $\alpha = k-1$ and $w = v_{k-1}^-$. By the definition of S_i^-, we have $y(v_i^-) > y(v_{i+1}^-)$ for all i with $1 \leq i \leq k-2$. We claim that $|S_1^-| \geq k$. Indeed, by adding (1) for all i with $2 \leq i \leq k-2$, (2) and (3), we have

$$\begin{aligned} n - 1 - |S_1^-| &\leq (k-3)m + m + (k-1)^2 + m - k \\ &= (k-1)m + (k-1)^2 - k. \end{aligned}$$

Hence, we have $|S_1^-| - k \geq n - 1 - (k-1)m - (k-1)^2 = m - (k-1)^2 \geq 0$, as required. Let us partition S^+ into $U_1 \cup U_2 \cup \cdots \cup U_{k-1} \cup P \cup Q$ as follows:

$$\begin{aligned} U_i &= \{v \in S_i^+ : x(v) < x(v_1^-)\} \text{ for } 1 \leq i \leq k-1, \\ P &= \{v \in S^+ \cap T : x(v) < x(v_1^-)\}, \\ Q &= \{v \in S^+ : x(v) > x(v_1^-)\}. \end{aligned}$$

Let z be the leftmost point of S_1^-. Since $|S_1^-| \geq k$, all the points of Q are k-linked with z. Therefore $n_k(z) \geq |S_1^-| - k + |Q|$ holds. Hence, we have

$$|S_1^-| + |Q| \quad \leq \quad m. \tag{5}$$

Next, we consider $n_k(w)$. Since $v_1^-, v_2^-, \ldots, v_{k-1}^-(= w)$ is monotone decreasing and U_i is monotone decreasing for each i with $1 \leq i \leq k-1$, at least $|U_i| - 1$ points of U_i are k-linked with w. Therefore $n_k(w) \geq |S_{k-1}^-| - k + \sum_{i=1}^{k-1}(|U_i| - 1)$ holds. Hence, we have

$$|S_{k-1}^-| + \sum_{i=1}^{k-1} |U_i| \quad \leq \quad m + k - 1. \tag{6}$$

Lastly, note that $n_k(v_0) = |T| \geq |T \cap S^-| + |P|$. Hence, we have

$$|T \cap S^-| + |P| \quad \leq \quad m - k. \tag{7}$$

By adding (1) for all i with $2 \leq i \leq k-2$, (5), (6) and (7), we have

$$\begin{aligned} n - 1 &\leq (k-3)m + m + (m + k - 1) + (m - k) \\ &= mk - 1, \end{aligned}$$

a contradiction. This completes the proof. □

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