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The maximum degree in a vertex-magic graph

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1 Introduction

Let G be a finite (non-empty) graph, let E and V be the sets of edges and vertices of G, with $|E|$ and $|V|$ their respective cardinalities, and let $|G| = |E| + |V|$. A *total labelling* of G is a bijection $\lambda : E \cup V \to \{1, \dots, |G|\}$, and the associated *weight* $w_\lambda(v)$ of a vertex v in G is

$$w_\lambda(v) = \lambda(v) + \sum_{e \in E(v)} \lambda(e),$$

where $E(v)$ is the set of edges that have v as an end-point. The total labelling λ of G is *vertex-magic* if every vertex has the same weight, and the graph G is *vertex-magic* if a vertex-magic total labelling of G exists. Magic labellings of graphs were introduced by Sedláček [5] in 1963, and *vertex-magic total labellings* first appeared in 2002 in [4]. For a dynamic survey of various forms of graph labellings see [1]; for details of vertex-magic graphs, see [6].

The degree of a vertex v is the number of edges that have v as an endpoint. In [4] the authors assert that, roughly speaking, a graph is likely to be vertex-magic if and only if there is not much variation among the degrees of its vertices. This is reinforced in [3] with the suggestion that there might be a general principle to the effect that "if a graph G contains a vertex whose degree is high relative to the degrees of all the other vertices of G, then G is not vertex-magic". As evidence of this, we note the following result (see [2]): *if T is a vertex-magic tree with n vertices, then the degree of any vertex is at most* $\left(\sqrt{32n+33} - 7\right)/2$. Now in any tree, $|V| = |E| + 1$, so that this result implies that *in any vertex-magic tree, the degree d of any vertex satisfies*

$$d + 7/2 \le \sqrt{\frac{32|E|^2 + 97|E| + 65}{4|V|}}. \tag{1}$$

Here we establish the following result which holds for all graphs.

Theorem 1. *Let G be a vertex-magic graph with C components. Then the degree d of any vertex of G satisfies*

$$d + 2 \le \sqrt{\frac{7|E|^2 + (6C+5)|E| + C^2 + 3C}{|V|}}. \tag{2}$$

In particular,

$$d + 2 \le \sqrt{\frac{14|E|^2 + 16|E| + 4}{|V|}}, \tag{3}$$

while if G is connected, then

$$d + 2 \le \sqrt{\frac{7|E|^2 + 11|E| + 4}{|V|}}. \tag{4}$$

If G is a tree with n vertices, then $|V| = n$ and $|E| = n - 1$, and it is clear that (4) is better than (1) for all sufficiently large n. A calculation shows this to be so whenever $n \ge 43$.

2 A preliminary result

Our first result shows that in any vertex-magic graph, the degree of any vertex is bounded by a given function of $|V|$ and $|E|$.

Theorem 2. *Suppose that a graph G is vertex-magic. Then the degree d of any vertex satisfies*

$$|V|d^2 + (5|V| - 2)d + |V| \le |V|^2 + 2|E|^2 + 4|E|\,|V| + 2|E|. \tag{5}$$

Proof We suppose that λ is a vertex-magic labelling of G in which each vertex has weight h. Let v be a vertex of degree d, and let $v_1, \dots, v_{|V|-1}$ be the $|V| - 1$ vertices other than v. Let $e_1, \dots, e_d$ be the d edges that have v as an end-point, and let the other $|E| - d$ edges be $e'_1, \dots, e'_{|E|-d}$. Then

$$h = w_\lambda(v) = \lambda(v) + \sum_i \lambda(e_i),$$
$$(|V| - 1)h = \sum_i w_\lambda(v_i) = \sum_i \lambda(v_i) + 2\sum_j \lambda(e'_j) + \sum_k \lambda(e_k),$$

and hence

$$(|V|-1)\Big[\lambda(v)+\sum_i \lambda(e_i)\Big] = \sum_i \lambda(v_i) + 2\sum_j \lambda(e'_j) + \sum_k \lambda(e_k)$$
$$= (1+2+\cdots+|G|) - \lambda(v) + \sum_j \lambda(e'_j).$$

It follows that

$$|V|\lambda(v) + (|V|-1)\sum_i \lambda(e_i) - \sum \lambda(e'_i) = \tfrac{1}{2}|G|(|G|+1). \qquad (6)$$

Let $\sigma_a^b = (a+1)+\cdots+b$. Among all the total labellings of G, the left hand side of (6) takes its smallest value when

$$\lambda(v) = 1, \quad \sum_i \lambda(e_i) = \sigma_1^{d+1}, \quad \sum_j \lambda(e'_j) = \sigma_{|V|+d}^{|G|},$$

and if we use these values in (6) we obtain (5). □

3 The proof of Theorem 1

We now weaken (5) slightly to obtain a more manageable inequality from which Theorem 1 will follow. First, Theorem 1 holds when $|V|=1$ because in this case $|E|=0$, and the single vertex has degree zero. We may now assume that $|V|\geq 2$, so that $4|V|\leq 5|V|-2$. If we now add $3|V|$ to both sides of (5) we obtain

$$|V|(d+2)^2 \leq |V|^2 + 2|E|^2 + 4|E|\,|V| + 2|E| + 3|V|. \qquad (7)$$

Now if G is connected, then $|V|\leq|E|+1$. Thus for a general graph with C components, we must have $|V|\leq|E|+C$, and this with (7) gives (2). If G is connected, then $C=1$ and (2) becomes (4). In all cases, a vertex-magic graph can have at most one isolated vertex so $C\leq|E|+1$, and with this, (2) becomes (3). This completes the proof of Theorem 1. □

We can interpret Theorem 1 in the following way. Suppose that we list the degrees of the vertices in a vertex-magic graph in descending order, say $d_1\geq d_2\geq\cdots\geq d_{|V|}$. Then $\sum_j d_j = 2|E|$ so that, from Theorem 1,

$$d_1\sqrt{|V|} \leq \sqrt{14|E|^2+16|E|+4} \leq 6|E| \leq 3d_1 + 3(|V|-1)d_2,$$

and we conclude that

$$\frac{d_2}{d_1} \geq \frac{\sqrt{|V|}-3}{3|V|-3}.$$

In this sense, d_1 cannot be significantly larger than d_2.

Finally, we compare our results with other known results. In [3] the authors show that (a) the *wheel* W_n with n spokes, where $n\geq 12$, (b) the *fan* F_n, where $n\geq 11$, and (c) the *friendship graph* $\mathcal{F}_n$, where $n\geq 4$, are not vertex-magic. In fact, all of these results can be derived directly from the more precise Theorem 2.

References

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