AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **30** (2004), Pages 57–73

Balanced incomplete block designs with block size 9: Part III

R. Julian R. Abel

School of Mathematics
University of New South Wales
Sydney, N.S.W. 2052, Australia
julian@maths.unsw.edu.au

Iliya Bluskov*

Department of Mathematics and Computer Science
University of Northern British Columbia
Prince George, B.C., Canada, V2N 4Z9
bluskovi@unbc.ca

Malcolm Greig

Greig Consulting
317-130 East 11th St.
North Vancouver, BC, Canada V7L 4R3
greig@sfu.ca

Abstract

The necessary conditions for the existence of a balanced incomplete block design on v points, with index λ and block size k, are that:

$$\begin{aligned}\lambda(v-1) &\equiv 0 \bmod (k-1)\\ \lambda v(v-1) &\equiv 0 \bmod k(k-1)\end{aligned}$$

Earlier work has studied $k = 9$ with $\lambda \in \{1, 2, 3, 4, 6, 8, 9, 12\}$. In this article we show that the necessary conditions are sufficient for $\lambda = 9$ and every other λ not previously studied.

1 Introduction

In this article, we are concerned with the existence of $(v, 9, \lambda)$ balanced incomplete block designs (BIBDs).

* Research was partially supported by NSERC grant 228274-00.

Theorem 1.1 *The necessary conditions for the existence of a* $(v, 9, \lambda)$ *BIBD are that* $v \geq 9$ *and* $v \equiv 1, 9 \pmod{\frac{72}{\gcd(\lambda,72)}}$.

There are no known non-existence results for any $(v, 9, \lambda)$ BIBD.

The status of the case $\lambda = 1$ was summarized in [2], where we studied the cases $\lambda = 2$, 4, and 8. The cases $\lambda = 3$, 6 and 12 were studied in [3], and the case $\lambda = 9$ was studied in [5], where examples were constructed for all $v \equiv 1 \pmod 8$ except $v = 553$, mostly by "filling in the hole" of a near resolvable $(v, 8, 7)$ BIBD. For $v = 553$, one can take three copies of the $(553, 9, 3)$ BIBD given in [3].

We now summarize our results from [2, 3].

Theorem 1.2 *The necessary conditions for the existence of a* $(v, 9, \lambda)$ *BIBD are sufficient in the following cases:*

a. For $\lambda = 2$ *(necessary conditions:* $v \equiv 1, 9 \pmod{36}$*) with the possible exceptions of* $v =$ 189, 253, 505, 765, 837, 1197, 1837 *and* 1845*;*

b. For $\lambda = 3$ *(necessary conditions:* $v \equiv 1, 9 \pmod{24}$*) with the possible exceptions of* $v =$ 177, 345, *and* 385*;*

c. For $\lambda = 4$ *(necessary conditions:* $v \equiv 1, 9 \pmod{18}$*) with the possible exceptions of* $v =$ 315, 459 *and* 783*;*

d. For $\lambda = 6$ *(necessary conditions:* $v \equiv 1, 9 \pmod{12}$*) with the possible exception of* 213*;*

e. For $\lambda = 8$ *(necessary conditions:* $v \equiv 0, 1 \pmod 9$*);*

f. For $\lambda = 9$ *(necessary conditions:* $v \equiv 1 \pmod 8$*);*

g. For $\lambda = 12$ *(necessary conditions:* $v \equiv 1, 3 \pmod 6$ *with* $v \geq 9$*).*

The outline of this article is that we briefly give a number of general constructions in Section 2; in Section 3, we summarize the relevant published existence results on transversal designs, and establish that $TD_9(10, n)$ exist, except possibly for $n = 35$. Several of our theorems were given in [2, 3], so we state these without proof here.

In the next five sections we accomplish the basic objective of this article, which is to establish that the necessary conditions are sufficient when $\lambda = 18$, 24, 36, 72 and all values of λ not dividing 72.

Finally, in Section 9, we summarize the open cases for $(v, 9, \lambda)$ BIBDs.

2 Basic Constructions

The terminology and notation we will use is quite standard; e.g., see [6].

Theorem 2.1 (Wilson's Fundamental Construction, (WFC))
Suppose we have a "master" (K_1, λ_1)-GDD with g groups and a group type vector of $\{|G_j| : j = 1, \ldots, g\}$, and a weighting that assigns a positive weight of $w(x)$ to each point x. Let $W(B_i)$ be the weight vector of the elements of the i-th block. If, for every block B_i, we have an "ingredient" (K_2, λ_2)-GDD with a group size vector of $W(B_i)$, then there exists a $(K_2, \lambda_1\lambda_2)$-GDD with a group size vector of $\{\sum_{x \in G_j} w(x) : j = 1, \ldots, g\}$.

The following theorem establishes our most important ingredient GDDs.

Theorem 2.2 *For $(9, \lambda)$ GDDs, the following group types exist:*

a. If $\lambda = 1$, then types 8^9, 8^{10} exist;

b. If $\lambda = 2$, then types $8^9 4^1$, 4^{10}, 4^9 exist;

c. If $\lambda = 4$, then types 2^9, 2^{10}, $8^9 2^1$, $4^9 2^1$ exist;

d. If $\lambda = 8$, then types 1^9, 1^{10}, $8^9 1^1$, $4^9 1^1$, $2^9 1^1$ exist.

Proof: The designs with 9 groups are simply transversal designs, while type 1^{10} is a BIBD.

A $(9, 1)$ GDD of type 8^{10} is given by the difference set

$$(0, 49, 47, 73, 46, 55, 68, 4, 32)$$

which is developed over Z_{80}. The groups are points with the same residue modulo 10. If we identify the points 40 with 0, 50 with 10, 60 with 20 and 70 with 30, then we get a design that could be described as a $(9, 2)$ GDD of type $8^9 4^1$ missing a $(9, 1)$ GDD of type 8^9, and since a $TD(9, 8)$ exists, so does a $(9, 2)$ GDD of type $8^9 4^1$. Alternatively, identifying 20, 40, and 60 with 0, and 30, 50 and 70 with 10, then adjoining a $TD_3(9, 8)$ gives a $(9, 4)$ GDD of type $8^9 2^1$. Also, identifying $10, 20, \ldots, 70$ with 0 and adjoining a $TD_7(9, 8)$ gives a $(9, 8)$ GDD of type $8^9 1^1$.

Reducing our difference set (mod 40), gives the difference set

$$(0, 9, 7, 33, 6, 15, 28, 4, 32)$$

which yields a $(9, 2)$ GDD of type 4^{10} when developed over Z_{40}. If we identify the points 20 with 0 and 30 with 10, then adjoin a $TD_2(9, 4)$, we get a $(9, 4)$ GDD of type $4^9 2^1$. Alternatively, identifying points $10, 20, 30$ with 0 and adjoining a $TD_6(9, 4)$ gives a $(9, 8)$ GDD of type $4^9 1^1$.

Reducing our original difference set (mod 20), gives the difference set

$$(0, 9, 7, 13, 6, 15, 8, 4, 12)$$

which yields a $(9, 4)$ GDD of type 2^{10} when developed over Z_{20}. Further, identifying 10 with 0 and adjoining a $TD_4(9, 2)$ gives a $(9, 8)$ GDD of type $2^9 1^1$. ■

For the next theorem, it is important to remember that a PBD or a BIBD on v points can be considered as a GDD with group type 1^v.

Theorem 2.3 (Breaking up blocks) *Suppose we have a (v, K, λ) PBD, and for each $k \in K$, we have a (k, K_1, μ) PBD; then we have a $(v, K_1, \mu\lambda)$ PBD.*

The following theorem suffices for filling in our groups.

Theorem 2.4 *Let a (k, λ) GDD with group type $G_1, G_2, \cdots, G_n$ on v points be given and suppose, for each $i > 1$, we have a $(G_i + w, k, \lambda)$ BIBD missing a (w, k, λ) BIBD, where $w \geq 0$. Then there exists a $(v + w, k, \lambda)$ BIBD missing a $(G_1 + w, k, \lambda)$ BIBD.*

Lemma 2.5 *If a symmetric (i.e., $v = b$) $(v + k + \lambda, k + \lambda, \lambda)$ BIBD exists, then a $(v + k + \lambda, k, k(k - 1))$ BIBD exists.*

Proof: Form the complementary design, a $(v + k + \lambda, v, k(k - 1)/\lambda)$ BIBD, then break the blocks using a (v, k, λ) BIBD, (which exists as the residual of the original symmetric design), to obtain the required BIBD. ■

In view of the previous lemma, it is worth reviewing which $(v, 9, \lambda)$ BIBDs exist as residuals. Perhaps surprisingly, these exist whenever λ is a divisor of 72. The symmetric designs on the right are complements of those on the left.

Table 1: Residual $(v, 9, \lambda)$ BIBDs

Symmetric $(v + k + \lambda, k + \lambda, \lambda)$	Residual (v, k, λ)	Symmetric $(v + k + \lambda, k + \lambda, \lambda)$	Residual (v, k, λ)
$(91, 10, 1)$	$(81, 9, 1)$	$(91, 81, 72)$	$(10, 9, 72)$
$(56, 11, 2)$	$(45, 9, 2)$	$(56, 45, 36)$	$(11, 9, 36)$
$(45, 12, 3)$	$(33, 9, 3)$	$(45, 33, 24)$	$(12, 9, 24)$
$(40, 13, 4)$	$(27, 9, 4)$	$(40, 27, 18)$	$(13, 9, 18)$
$(36, 15, 6)$	$(21, 9, 6)$	$(36, 21, 12)$	$(15, 9, 12)$
$(35, 17, 8)$	$(18, 9, 8)$	$(35, 18, 9)$	$(17, 9, 9)$

Corollary 2.6 *A $(v, 9, 72)$ BIBD exists for $v = 56$ (also for $v = 91, 45, 40, 36, 35$).*

In this article, we are dealing with some BIBDs for which λ is large enough to enable the "J-construction" found by Mullin and Stanton [13] to be used. This construction was also given in [10, Lemma 4.6] and is given in the next lemma.

Lemma 2.7 *Let $r = \lambda(v-1)/(k-1)$. If both a (v, k, λ) BIBD and a $(v, k+1, r-\lambda)$ BIBD exist, then a $(v + 1, k + 1, r)$ BIBD exists.*

Proof: We augment all the blocks of the first design with a new point, $\{\infty\}$, then adjoin the blocks of the second design formed on the finite points, to obtain the required BIBD. Note that the replication number for the final design is $rv/k = \lambda(v - 1)v/(k - 1)k$, the number of blocks in the first design. ■

Corollary 2.8 *The following BIBDs exist:* $(44, 9, 72)$, $(58, 9, 24)$, $(74, 9, 72)$ *and* $(86, 9, 72)$.

Proof: Apply Lemma 2.7, using the following BIBDs as ingredients: $(43, 8, 12)$, $(43, 9, 60)$, $(57, 8, 3)$, $(57, 9, 21)$, $(73, 8, 7)$, $(73, 9, 65)$, $(85, 8, 6)$ and $(85, 9, 66)$. These ingredient designs are either given in [1] or [3], or are multiples of designs given there. ■

Hanani [10, Lemmas 4.1–4.4] gives a very useful set of BIBDs obtainable from difference family constructions. These constructions start by building a generating base block from one or more cosets, possibly augmented with zero; the remaining base blocks are then produced by repeated multiplication by a primitive element of $GF(v)$. All the base blocks have full orbit.

Theorem 2.9 *Let v be a prime power with $v \geq k$.*

a. If $f = \gcd(v-1, k)$, then a $(v, k, k(k-1)/f)$ difference family exists.

b. If $f = \gcd(v-1, k-1)$, then a $(v, k, k(k-1)/f)$ difference family exists.

Theorem 2.10 *Let v be an odd prime power.*

a. If $f = \gcd(v-1, k)$ and k is odd, then a $(v, k, k(k-1)/2f)$ difference family exists.

b. If $f = \gcd(v-1, k-1)$ and k is even, then a $(v, k, k(k-1)/2f)$ difference family exists.

Corollary 2.11 *The following BIBDs are obtainable from difference families when $v \geq k$ and v is a prime power of the indicated form:*

a. $(18t+1, 9, 4)$, $(9t+1, 9, 8)$, $(8t+1, 9, 9)$, $(6t+1, 9, 12)$, $(4t+1, 9, 18)$, $(3t+1, 9, 24)$, $(2t+1, 9, 36)$ *and* $(t, 9, 72)$ *BIBDs;*

b. $(9t+1, 10, 10)$, $(10t+1, 10, 9)$ *and* $(18t+1, 10, 5)$ *BIBDs.*

2.1 Grouplet Divisible Designs

Grouplet divisible designs were discussed in more detail in [1]. Here we begin by giving a definition of a gDD.

Definition 2.12 *Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a triple, where $\mathcal{V}$ is a point set, $\mathcal{G}$ is a λ-resolution set of $\mathcal{V}$, say, $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n\}$, so each point of $\mathcal{V}$ appears in λ of the sets in $\mathcal{G}$, and $\mathcal{B}$ is a block set. Then $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ is a (K, λ) grouplet divisible design (a gDD) if each pair of points occurs λ times in $\mathcal{B} \cup \mathcal{G}$, and each block in $\mathcal{B}$ has a size in K. If the grouplets are of uniform size k_1, we will denote the gDD as a (K, λ) gDD(k_1).*

Remark 2.13 *Note that in the special case that the grouplet structure consists of λ identical parallel classes, then a gDD is equivalent to a GDD. In this case, it is usual to merely specify one of these parallel classes as the "group type".*

Remark 2.14 *We may construct a gDD with grouplets of size $k-1$ from a (v,k,λ) BIBD by deleting a point and using the blocks through that point to define the grouplets; We may construct a gDD with grouplets of size k from a (v,k,λ) BIBD by deleting a λ-resolution set, (assuming one exists), and using its blocks to define the grouplets.*

We now consider the use of gDDs as master designs in WFC. This will create designs that are some sort of hybrid between gDDs and GDDs. Since we are going to fill these inflated grouplets to get the designs of interest, as in [3], we present our construction in a combined theorem.

Theorem 2.15 *Suppose there exists a (k,λ) gDD(k_1) on v points, a $TD_\mu(k,m)$, and a (mk_1+w,k,μ) BIBD missing a (w,k,μ) BIBD as a subdesign, where $w \geq 0$. Then there exists a $(mv+w,k,\lambda\mu)$ BIBD missing a $(w,k,\lambda\mu)$ BIBD as a subdesign.*

Corollary 2.16 *A $(v,9,\lambda)$ BIBD exists in the following cases:*

a. $\lambda = 18$ *and* $v \in \{245, 365\}$;

b. $\lambda = 24$ *and* $v \in \{52, 66, 106, 114, 132\}$;

c. $\lambda = 36$ *and* $v = 119$;

d. $\lambda = 72$ *and* $v \in \{50, 68, 98\}$.

Proof: For $(66,9,24)$ and (132,9,24) BIBDs, note that a $(33,9,3)$ BIBD can be formed by residualizing a symmetric ($v = b$) BIBD; hence it can be constructed so as to contain a λ-resolution set. For a $(114,9,24)$ BIBD, note that a $(57,9,3)$ BIBD with a 3-resolution set is given in [3]. A base block in a $(v,9,9)$ BIBD difference family over a group of order v generates a 9-resolution set. Some required difference families of this form (for $v = 17$, 25, 49 and 73) are given by Corollary 2.11.a. For $(52,9,24)$ and $(106,9,24)$ BIBDs, the grouplets obtained from the master BIBD have size $k-1 = 8$, and are obtained as indicated in Remark 2.14 by deleting a point from an $(18,9,8)$ or $(36,9,8)$ BIBD. With these examples of gDDs, we may construct the required BIBDs using Theorem 2.15 with the following designs:

result BIBD	master BIBD	ingredient	w	filler BIBD
$(245,9,18)$	$(49,9,9)$	$TD_2(9,5)$	0	$(45,9,2)$
$(365,9,18)$	$(73,9,9)$	$TD_2(9,5)$	0	$(45,9,2)$
$(52,9,24)$	$(18,9,8)-1$	$TD_3(9,3)$	1	$(25,9,3)$
$(66,9,24)$	$(33,9,3)$	$TD_8(9,2)$	0	$(18,9,8)$
$(106,9,24)$	$(36,9,8)-1$	$TD_3(9,3)$	1	$(25,9,3)$
$(114,9,24)$	$(57,9,3)$	$TD_8(9,2)$	0	$(18,9,8)$
$(132,9,24)$	$(33,9,3)$	$TD_8(9,4)$	0	$(36,9,8)$
$(119,9,36)$	$(17,9,9)$	$TD_4(9,7)$	0	$(63,9,4)$
$(50,9,72)$	$(25,9,9)$	$TD_8(9,2)$	0	$(18,9,8)$
$(68,9,72)$	$(17,9,9)$	$TD_8(9,4)$	0	$(36,9,8)$
$(98,9,72)$	$(49,9,9)$	$TD_8(9,2)$	0	$(18,9,8)$

■

3 Transversal Designs

An important element in our constructions is a truncated $TD_\lambda(10, n)$; for $\lambda = 1$, there are tables showing whether a $TD(10, n)$ is known for all values of n of potential interest [4]. However, there is no such table for $\lambda > 1$, although [8] and [9] do give some information. (The situation for block size 9 is quite different, and summarized in Theorem 3.1 [3, 7].)

Theorem 3.1 *A $TD_\lambda(9, n)$ with $\lambda > 1$ is known, except where noted below:*

a. Let $E = \{2, 3, 6, 14, 34, 38, 39, 50, 51, 54, 62, 74, 75\}$. If $n \notin E$, then a $TD_2(9, n)$ exists.

b. If $n \notin \{5, 45, 60\}$, then a $TD_3(9, n)$ exists.

c. If $n \notin \{6, 14\}$, then a $TD_5(9, n)$ exists.

In [3] we established an existence result for $TD_3(10, n)$.

Theorem 3.2 *A $TD_3(10, n)$ exists for $n \notin \{5, 6, 14, 20, 35, 45, 55, 56, 60, 78, 84, 85, 102\}$.*

As a consequence, we can establish an existence result for $TD_9(10, n)$ after a standard preliminary lemma [7, Lemma 1.4].

Lemma 3.3 *If a $TD_\lambda(10, m)$ and a $TD_\mu(10, n)$ exist, then a $TD_{\lambda\mu}(10, mn)$ exists.*

Theorem 3.4 *Let $n \neq 35$. Then a $TD_9(10, n)$ exists.*

Proof: The values $6 = 2 \cdot 3$, $14 = 2 \cdot 7$, $20 = 2 \cdot 10$, $28 = 4 \cdot 7$, $56 = 8 \cdot 7$, $60 = 2 \cdot 30$ and $84 = 2 \cdot 42$ follow from Lemma 3.3 with $\lambda = \mu = 3$. We can take the value 5 from [8, Table II.4.3]. The values $45 = 5 \cdot 9$, $55 = 5 \cdot 11$, $78 = 6 \cdot 13$, $85 = 5 \cdot 17$ and $102 = 6 \cdot 17$ follow from Lemma 3.3 using $\lambda = 9$ and $\mu = 1$. ∎

4 BIBDs with block size 9 and index 18

Our object in this section is to construct $(v, 9, 18)$ BIBDs. The necessary conditions reduce to $v \equiv 1 \pmod 4$ and $v \geq 9$. We begin with a couple of direct constructions.

Example 4.1 *The following blocks generate a $(77, 9, 18)$ BIBD when developed over Z_{77}. This design has a multiplier, 67, of order 3. The first block is invariant under the multiplier; each of the remaining 6 blocks has to be multiplied by 1, 67 and 23.*

$$\begin{array}{ll}
(1, 67, 23, 2, 57, 46, 4, 37, 15)\,, & (27, 2, 3, 70, 12, 18, 22, 31, 5)\,, \\
(36, 2, 62, 4, 7, 6, 37, 8, 74)\,, & (58, 59, 37, 20, 33, 39, 7, 19, 9)\,, \\
(66, 25, 3, 31, 32, 6, 46, 19, 28)\,, & (65, 43, 58, 36, 5, 10, 73, 8, 26)\,, \\
(1, 68, 33, 52, 58, 73, 46, 36, 66)\,. &
\end{array}$$

Example 4.2 *The following blocks generate a* $(213, 9, 18)$ *BIBD when developed over* Z_{213}. *This design has a multiplier,* 37*, of order* 7. *The first block is invariant under the multiplier, and has to be repeated four times. Each of the remaining blocks has to be multiplied by* 1*,* 37*,* 91*,* 172*,* 187*,* 103 *and* 190.

$$\begin{gathered}(1, 37, 91, 172, 187, 103, 190, 71, 142)\,,\\ (80, 44, 27, 4, 91, 160, 173, 53, 144)\,,\\ (1, 70, 64, 59, 86, 114, 7, 168, 184)\,,\\ (1, 135, 207, 160, 137, 121, 26, 49, 128)\,,\\ (139, 207, 37, 38, 130, 66, 208, 92, 141)\,,\\ (1, 101, 55, 4, 50, 149, 171, 125, 153)\,,\\ (185, 114, 126, 30, 129, 88, 7, 113, 191)\,,\\ (212, 14, 164, 147, 129, 166, 177, 150, 155)\,.\end{gathered}$$

We now look at the smaller designs.

Lemma 4.3 *If* $v \equiv 1 \pmod 4$*, and either* $v < 333$ *or* $v = 365$*, then a* $(v, 9, 18)$ *BIBD exists.*

Proof: If $v \equiv 1 \pmod 8$, then a $(v, 9, 9)$ BIBD exists by Theorem 1.2.f; take two copies of this design.

If $v \equiv 1, 9 \pmod{12}$ and $v \neq 213$, then a $(v, 9, 6)$ BIBD exists by Theorem 1.2.d; take three copies of this design.

So now we must deal with $v \equiv 5 \pmod{24}$ and $v = 213$. The values 29, 53, 101, 125, 149, 173, 197, 269, 293 and 317 are all prime powers, and a design is given by Corollary 2.11.a. Designs for 77 and 213 are given by Examples 4.1 and 4.2. Grouplet constructions for 245 and 365 are given in Corollary 2.16.a. Finally, for 221 a $TD(13, 17)$ exists, so we can use a $(13, 9, 18)$ BIBD to break the blocks of the TD, then we can use a $(17, 9, 18)$ BIBD to fill the groups. ∎

The following is a simple consequence of Theorem 3.4.

Lemma 4.4 *If* $m \geq 83$ *and* $m \neq 91$*, then we can write* $m = 9n + r$ *such that a* $TD_9(10, n)$ *exists with* $n \geq 9$ *and* $0 \leq r \leq n$ *and* $r \neq 1$.

Lemma 4.5 *If* $v \equiv 1 \pmod 4$*,* $v \geq 333$ *and* $v \neq 365$*, then a* $(v, 9, 18)$ *BIBD exists.*

Proof: Express $m = (v - 1)/4$ as $9n + r$ where n, r satisfy the conditions given in Lemma 4.4, and take the $(\{9, 10\}, 9)$ GDD of type $n^9 r^1$ provided by that lemma. Give every point a weight of 4 in an application of WFC (Theorem 2.1); the required ingredients, namely $(9, 2)$ GDDs of types 4^9 and 4^{10} exist by Theorem 2.2.b. This gives a $(9, 18)$ GDD of type $4n^9 4r^1$; now fill the groups of this design using one extra point (i.e. $w = 1$) in Theorem 2.4. ∎

Combining Lemmas 4.3 and 4.5, we have the result for $(v, 9, 18)$ BIBDs:

Theorem 4.6 *The necessary conditions for the existence of a* $(v, 9, 18)$ *BIBD, viz. that* $v \geq 9$ *and* $v \equiv 1 \pmod 4$*, are sufficient.*

5 BIBDs with block size 9 and index 24

Our object in this section is to construct $(v, 9, 24)$ BIBDs. The necessary conditions reduce to $v \equiv 1 \pmod 3$ and $v \geq 9$. We begin with some direct constructions.

Example 5.1 *The following blocks generate a* $(48, 9, 24)$ *BIBD when developed over* $Z_{47} \cup \{\infty\}$.

$$\begin{array}{ll}
(1,13,20,16,6,2,4,28,\infty)\,, & (1,13,22,35,19,9,44,2,\infty)\,,\\
(1,29,20,42,38,5,21,25,\infty)\,, & (1,3,8,14,18,7,21,45,15)\,,\\
(1,25,20,31,36,34,40,15,23)\,, & (1,15,22,28,11,10,16,19,17)\,,\\
(1,28,3,22,42,5,25,30,33)\,, & (1,2,33,43,21,20,9,3,27)\,,\\
(1,26,3,16,34,2,35,9,30)\,, & (1,24,46,45,33,17,43,35,9)\,,\\
(1,38,46,8,24,45,22,3,13)\,, & (1,14,31,34,33,21,43,18,32)\,,\\
(1,21,22,45,19,13,32,39,37)\,, & (1,32,21,6,4,39,44,25,26)\,,\\
(1,3,31,22,42,18,10,32,28)\,, & (1,28,37,12,0,31,13,16,20)\,.
\end{array}$$

Example 5.2 *The following blocks generate a* $(70, 9, 24)$ *BIBD when developed over* Z_{70}. *This design has a multiplier,* 11, *of order* 3. *The first block is invariant under the multiplier and has to be repeated twice. Each of the remaining blocks has to be multiplied by* 1, 11 *and* 51.

$$\begin{array}{ll}
(1,11,51,2,22,32,4,44,64)\,, & (13,49,5,45,39,24,58,50,4)\,,\\
(41,1,13,44,65,3,55,46,39)\,, & (2,57,61,9,47,59,10,0,24)\,,\\
(38,40,34,13,57,31,68,3,60)\,, & (32,1,19,20,15,6,49,8,35)\,,\\
(39,5,47,43,69,55,2,21,60)\,, & (20,52,67,31,51,66,58,48,5)\,.
\end{array}$$

Example 5.3 *The following blocks generate a* $(78, 9, 24)$ *BIBD when developed over* $Z_{77} \cup \{\infty\}$. *This design has a multiplier,* 67, *of order* 3. *The first two blocks are invariant under the multiplier; each of the remaining* 8 *blocks has to be multiplied by* 1, 67 *and* 23.

$$\begin{array}{ll}
(1,67,23,2,57,46,4,37,15)\,, & (43,32,65,9,64,53,18,51,29)\,,\\
(1,2,4,8,16,32,64,51,\infty)\,, & (76,9,41,77,3,71,58,45,12)\,,\\
(9,32,30,13,11,36,52,70,58)\,, & (17,7,14,27,69,55,1,71,9)\,,\\
(60,36,10,63,26,15,48,22,31)\,, & (61,19,64,4,71,31,66,60,9)\,,\\
(38,51,63,39,37,68,14,57,44)\,, & (49,3,12,34,56,4,30,31,26)\,.
\end{array}$$

Example 5.4 *The following blocks generate a* $(84, 9, 24)$ *BIBD when developed over* $Z_{74} \cup (I_{10} \times \{\infty\})$. *This design has a multiplier,* 47, *of order* 3. *The first block is invariant under the multiplier; each of the remaining* 10 *blocks has to be multiplied by* 1, 47 *and* 63. *The design is completed by forming a* $(10, 9, 24)$ *BIBD on the infinite points.*

$$\begin{array}{ll}
(1,47,63,2,20,52,4,40,30)\,, & (61,63,32,4,5,6,70,8,\infty_1)\,,\\
(17,39,35,46,30,6,49,19,\infty_2)\,, & (1,22,71,38,32,50,14,41,\infty_3)\,,\\
(67,57,10,51,21,18,7,29,\infty_4)\,, & (31,54,62,48,66,63,13,8,\infty_5)\,,\\
(37,49,14,4,19,13,7,18,\infty_6)\,, & (20,35,1,14,3,11,55,57,\infty_7)\,,\\
(58,26,25,19,5,62,7,8,\infty_8)\,, & (47,2,12,73,28,21,72,42,\infty_9)\,,\\
(25,31,6,4,42,70,68,8,\infty_{10})\,. &
\end{array}$$

Example 5.5 *The following blocks generate a* $(88, 9, 24)$ *BIBD when developed over* $Z_{79} \cup (I_9 \times \{\infty\})$. *This design has a multiplier,* 23, *of order* 3. *The first block is invariant under the multiplier, and has to be repeated twice; each of the remaining* 10 *blocks has to be multiplied by* 1, 23 *and* 55. *The design is completed by forming a* $(9, 9, 24)$ *BIBD on the infinite points.*

$$\begin{array}{ll}
(1,23,55,2,46,31,4,13,62)\,, & (60,73,3,78,5,8,22,56,9)\,,\\
(1,26,3,30,54,60,7,66,\infty_1)\,, & (24,30,3,4,31,50,19,65,\infty_2)\,,\\
(53,59,20,41,71,23,68,76,\infty_3)\,, & (59,78,6,51,49,42,7,48,\infty_4)\,,\\
(22,32,46,78,11,29,53,5,\infty_5)\,, & (63,33,65,4,16,6,50,9,\infty_6)\,,\\
(4,50,47,3,62,55,51,22,\infty_7)\,, & (58,48,24,39,13,51,73,7,\infty_8)\,,\\
(50,64,9,45,5,34,7,67,\infty_9)\,. &
\end{array}$$

Example 5.6 *The following blocks generate a* $(94, 9, 24)$ *BIBD when developed over* $Z_{93} \cup \{\infty\}$. *This design has a multiplier,* 67, *of order* 3. *The first block is invariant under the multiplier, and has to be repeated four times. Each of the remaining blocks has to be multiplied by* 1, 67 *and* 25.

$$\begin{array}{ll}
(1,67,25,32,5,56,63,36,87)\,, & (42,9,13,29,80,91,10,66,\infty)\,,\\
(0,1,92,75,38,2,5,39,16)\,, & (81,85,35,16,27,90,83,28,71)\,,\\
(55,61,76,47,17,31,63,68,65)\,, & (86,22,40,74,28,85,1,62,46)\,,\\
(4,49,72,44,74,3,9,60,54)\,, & (71,50,22,3,34,63,81,17,89)\,,\\
(5,21,74,75,77,92,59,11,25)\,, & (22,11,52,50,9,31,38,49,78)\,,\\
(31,58,21,6,33,50,90,46,85)\,. &
\end{array}$$

Example 5.7 *The following blocks generate a* $(96, 9, 24)$ *BIBD when developed over* $Z_{95} \cup \{\infty\}$. *This design has a multiplier,* 11, *of order* 3. *The first block is short and invariant under the multiplier, and has to be repeated twice. Each of the remaining blocks has to be multiplied by* 1, 11 *and* 26.

$$\begin{array}{ll}
(1,11,26,2,22,52,4,44,9)\,, & (70,2,53,58,15,86,49,18,\infty)\,,\\
(51,32,43,85,24,60,76,22,80)\,, & (90,91,31,50,56,13,39,51,5)\,,\\
(25,91,78,13,34,8,0,57,67)\,, & (36,21,67,91,54,18,24,92,69)\,,\\
(80,76,70,88,24,22,43,17,62)\,, & (21,49,57,35,7,6,37,8,58)\,,\\
(1,9,43,51,20,79,4,30,41)\,, & (52,17,56,28,84,6,20,81,36)\,,\\
(42,78,58,48,46,23,52,16,83)\,. &
\end{array}$$

Example 5.8 *The following blocks generate a* $(102, 9, 24)$ *BIBD when developed over* $Z_{93} \cup (I_9 \times \{\infty\})$. *This design has a multiplier,* 67, *of order* 3. *The first block is short, and invariant under the multiplier, and has to be repeated twice; the other twelve blocks have to be multiplied by* 1, 67 *and* 25. *The design is completed by forming a* $(9, 9, 24)$ *BIBD on the infinite points.*

$$\begin{array}{ll}
(1,67,25,32,5,56,63,36,87)\,, & (1,45,52,13,44,67,82,28,\infty_1)\,,\\
(28,17,39,1,22,51,38,58,\infty_2)\,, & (1,36,88,4,73,43,31,15,\infty_3)\,,\\
(43,72,88,21,71,56,28,50,\infty_4)\,, & (23,70,64,42,5,47,56,48,\infty_5)\,,\\
(51,53,3,4,19,44,43,77,\infty_6)\,, & (10,48,58,70,5,59,52,20,\infty_7)\,,\\
(1,41,2,77,31,6,34,40,\infty_8)\,, & (29,39,91,4,82,6,19,76,\infty_9)\,,\\
(31,2,12,49,5,22,32,84,9)\,, & (1,2,3,81,29,63,34,37,42)\,,\\
(70,2,90,91,48,36,7,8,77)\,. &
\end{array}$$

Example 5.9 *The following blocks generate a* $(112, 9, 24)$ *BIBD when developed over* Z_{112}. *This design has a multiplier,* 65, *of order* 3. *The first block is invariant under the multiplier. Each of the remaining blocks has to be multiplied by* 1, 65 *and* 81.

$$
\begin{array}{ll}
(1,65,81,2,18,50,4,36,100)\,, & (51,27,56,12,1,63,102,55,9)\,, \\
(97,66,65,11,42,38,24,21,9)\,, & (68,59,52,41,74,29,10,6,101)\,, \\
(90,42,74,45,2,46,95,36,79)\,, & (89,79,11,102,55,99,15,86,80)\,, \\
(102,64,24,87,109,7,58,84,39)\,, & (95,20,7,54,109,30,84,75,82)\,, \\
(26,75,17,47,63,49,51,42,25)\,, & (26,96,45,110,44,59,65,63,71)\,, \\
(24,37,42,72,84,80,106,31,61)\,, & (84,98,45,11,30,4,28,40,88)\,, \\
(26,12,21,62,25,6,1,102,24)\,. &
\end{array}
$$

Example 5.10 *The following blocks generate a* $(124, 9, 24)$ *BIBD when developed over* Z_{124}. *This design has a multiplier,* 33, *of order* 5. *The first block is invariant under the multiplier. Each of the remaining blocks has to be multiplied by* 1, 33, 97, 101 *and* 109.

$$
\begin{array}{ll}
(1,33,97,101,109,31,62,93,0)\,, & (112,65,14,56,5,87,20,32,9)\,, \\
(112,53,6,96,82,107,15,8,9)\,, & (20,62,82,81,107,26,50,74,9)\,, \\
(62,103,68,53,21,115,56,36,49)\,, & (89,2,26,75,36,39,85,53,99)\,, \\
(88,2,5,103,44,38,46,8,71)\,, & (67,91,1,65,94,123,30,116,45)\,, \\
(55,66,75,27,21,40,99,43,115)\,. &
\end{array}
$$

Example 5.11 *The following blocks generate a* $(142, 9, 24)$ *BIBD when developed over* Z_{142}. *This design has a multiplier,* 37, *of order* 7. *The first block is invariant under the multiplier, and has to be repeated five times. Each of the remaining blocks has to be multiplied by* 1, 37, 91, 101, 45, 103 *and* 119.

$$
\begin{array}{ll}
(1,37,91,101,45,103,119,71,0)\,, & (25,37,61,118,90,129,85,8,100)\,, \\
(87,97,3,24,94,4,124,109,21)\,, & (128,69,3,4,123,109,7,20,9)\,, \\
(1,65,3,4,140,54,7,50,110)\,, & (54,2,57,32,28,115,125,8,39)\,, \\
(131,2,72,142,81,125,21,87,9)\,. &
\end{array}
$$

We now look at the smaller designs.

Lemma 5.12 *If* $9 \le v < 144$ *and* $v \equiv 0, 1 \pmod 3$, *or* $v \in \{192, 193, 195, 196\}$, *then a* $(v, 9, 24)$ *BIBD exists.*

Proof: If $v \equiv 1, 3 \pmod 6$, then a $(v, 9, 12)$ BIBD exists by Theorem 1.2.g; note that this includes the cases $v = 193,\ 195$. Take two copies of this design.

If $v \equiv 0, 1 \pmod 9$, then a $(v, 9, 8)$ BIBD exists by Theorem 1.2.e; take three copies of this design.

So now we must deal with $v \equiv 4, 6, 12, 16 \pmod{18}$. The values $v \le 43$ are all given in [10]. Using Mullin and Stanton's J-construction gives a construction for $v = 58$, see Corollary 2.8.

Designs for 48, 70, 78, 84, 88, 94, 96, 102, 112, 124 and 142 are given by Examples 5.1–5.11. Grouplet constructions for 52, 66, 106, 114 and 132 are given in Corollary 2.16.b.

For 60, delete a point from a $(61,10,3)$ BIBD [12], then break the blocks using $(k,9,8)$ BIBDs for $k \in \{9,10\}$ in Theorem 2.3. For 76, a $(76,19,6)$ BIBD exists (it is the residual of a $(101,25,6)$ BIBD which is given by a difference set of the 25th roots in Z_{101}). Break the blocks of this BIBD using a $(19,9,4)$ BIBD in Theorem 2.3. For 130, a $TD(10,13)$ exists, so we can use a $(10,9,24)$ BIBD to break the blocks of the TD, then we can use a $(13,9,24)$ BIBD to fill the groups. Similarly for 192, a $TD(12,16)$ exists, so we can use a $(12,9,24)$ BIBD to break the blocks of the TD, then we can use a $(16,9,24)$ BIBD to fill the groups. For 120, since a $TD(13,13)$ exists, we can spike one block of a $TD(9,13)$ to get a $(\{9,10,12^*\},1)$ GDD of type $13^9 1^3$ so we can use $(k,9,24)$ BIBDs for $k \in \{9,10,12\}$ to break the blocks of the TD, then we can use a $(13,9,24)$ BIBD to fill the groups. Similarly, for 196, a $TD(16,16)$ exists; we can spike one block of a $TD(12,16)$ to obtain a $(\{12,13,16^*\},1)$ GDD of type $16^{12}1^4$, so we can use $(k,9,24)$ BIBDs for $k \in \{12,13,16\}$ to break the blocks of the TD, then we can use a $(16,9,24)$ BIBD to fill the groups.

Finally, for $v=138$ we may take a $TD_3(10,7)$ and give three points in one group a weight of 1 and all other points a weight of 2 in WFC (Theorem 2.1). Since $(9,8)$ GDDs of types 2^{10} and $2^9 1^1$ exist (Theorem 2.2) this gives a $(9,24)$ GDD of type $14^9 11^1$. Now use an extra point to fill the groups. ■

The following is a fairly simple consequence of Theorem 3.2.

Lemma 5.13 *If* $v \geq 144$, $v \equiv 0,1 \pmod 3$ *and* $v \notin \{192,193,195,196\}$, *then a* $(v,9,24)$ *BIBD exists.*

Proof: Write v in the form $9n+r+w$ where a $TD_3(10,n)$ exists, $w=0$ or 1, and $9-w \leq r \leq n$. If $n \equiv 0 \pmod 3$, we can take $w=0$ or 1, although in some cases (e.g. $v=151$, $n=r=15$, $w=1$) only one of these options may be available. If $n \equiv 1 \pmod 3$, we require $w=0$, and if $n \equiv 2 \pmod 3$, we require $w=1$.

It is easy to check that if $v \equiv 0,1 \pmod 3$ $v \geq 144$, and $v \notin \{192,193,195,196\}$, then v can be written as $9n+r+w$ for some n,r,w where the above conditions are satisfied. By truncating one group of a $\mathrm{TD}_3(10,n)$ given by Theorem 3.2, we now obtain a $(\{9,10\},3)$ GDD of type $n^9 r^1$ with $9-w \leq r \leq n$. Give every point in this GDD a weight of 1, and apply WFC (Theorem 2.1), using Theorem 2.2.b for the required ingredients, namely $(9,8)$ GDDs of types 1^9 and 1^{10}. This gives a $(9,24)$ GDD of type $n^9 r^1$. Finally, fill in the groups of this design using w extra points in Theorem 2.4. ■

Combining Lemmas 5.12 and 5.13, we have the following result for $(v,9,24)$ BIBDs:

Theorem 5.14 *The necessary conditions for the existence of a* $(v,9,24)$ *BIBD, viz. that* $v \geq 9$ *and* $v \equiv 0,1 \pmod 3$, *are sufficient.*

6 BIBDs with block size 9 and index 36

Our object in this section is to construct $(v,9,36)$ BIBDs. The necessary conditions reduce to $v \equiv 1 \pmod 2$ and $v \geq 9$. We begin with a couple of direct constructions.

Example 6.1 *The following blocks generate a* $(95, 9, 36)$ *BIBD when developed over* Z_{95}*. This design has a multiplier,* 6*, of order* 9*. The first block is invariant under the multiplier and has to be repeated twice. Each of the remaining blocks has to be multiplied by* 1*,* 6*,* 36*,* 26*,* 61*,* 81*,* 11*,* 66 *and* 16*.*

$$\begin{array}{ll}(1,6,36,26,61,81,11,66,16)\,, & (53,36,3,47,5,21,7,72,9)\,,\\ (10,2,3,72,49,40,85,93,9)\,, & (49,2,36,4,5,93,45,8,78)\,,\\ (26,75,3,12,5,23,7,37,9)\,, & (46,90,3,4,5,6,10,51,38)\,.\end{array}$$

Example 6.2 *The following blocks generate a* $(203, 9, 36)$ *BIBD when developed over* $Z_7 \times Z_{29}$*. This design has a multiplier,* $(2, 16)$*, of order* 21*. The first block remains invariant when multiplied by* $(1, 16)$ *and should be multiplied by* $(2^i, 1)$ *for* $0 \le i \le 2$*; the next two blocks remain invariant when multiplied by* $(2, 1)$ *and should be multiplied by* $(1, 16^j)$ *for* $0 \le j \le 6$*. Each of the last* 4 *blocks has to be multiplied by the* 21 *values* $(2^i, 16^j)$ *for* $0 \le i \le 2$ *and* $0 \le j \le 6$*.*

$$\begin{array}{c}((0,0),(1,0),(2,1),(2,16),(2,24),(2,7),(2,25),(2,23),(2,20))\,,\\ ((0,0),(0,2),(0,10),(1,13),(1,14),(2,13),(2,14),(4,13),(4,14))\,,\\ ((0,0),(0,19),(0,27),(3,10),(3,17),(5,10),(5,17),(6,10),(6,17))\,,\\ ((0,0),(1,25),(2,12),(2,13),(3,22),(4,24),(5,4),(5,8),(5,16))\,,\\ ((0,0),(2,5),(2,17),(3,24),(4,25),(4,27),(4,28),(6,4),(6,13))\,,\\ ((0,0),(1,8),(1,12),(2,12),(2,23),(3,6),(4,25),(5,20),(6,18))\,,\\ ((0,0),(0,7),(0,16),(1,6),(1,21),(1,24),(2,10),(4,9),(6,22))\,.\end{array}$$

We now look at the smaller designs.

Lemma 6.3 *If* v *is odd, and either* $9 \le v < 144$ *or* $v \in \{167, 203\}$*, then a* $(v, 9, 36)$ *BIBD exists.*

Proof: If $v \equiv 1 \pmod 4$ and $v \ge 9$, then a $(v, 9, 18)$ BIBD exists by Theorem 4.6; take two copies of this design.

If $v \equiv 1, 3 \pmod 6$ and $v \ge 9$, then a $(v, 9, 12)$ BIBD exists by Theorem 1.2.g; take three copies of this design.

So now we must deal with $v \equiv 11 \pmod{12}$. The values 11, 23, 47, 59, 71, 83, 107, 131 and 167 are all primes and a design is given by Corollary 2.11.a. Designs for 35, 95 and 203 are given respectively by Hanani [10], Examples 6.1 and 6.2. A grouplet construction for 119 is given in Corollary 2.16.c. For 143, a $TD(11, 13)$ exists, so we can use a $(11, 9, 36)$ BIBD to break the blocks of the TD, then we can use a $(13, 9, 36)$ BIBD to fill the groups. ■

The following is a simple consequence of Theorem 3.4.

Lemma 6.4 *If* $m > 72$*,* $m \equiv 5 \pmod 6$ *and* $m \notin \{83, 101\}$*, then we can write* $m = 9n + r$ *such that a* $TD_9(10, n)$ *exists with* $n \ge 8$*,* $0 \le r \le n$ *and* $r = 0$ *or* $r \ge 4$*.*

Lemma 6.5 *If* $v > 144$ *and* $v \equiv 11 \pmod{12}$ *and* $v \notin \{167, 203\}$*, then a* $(v, 9, 36)$ *BIBD exists.*

Proof: Take a $(\{9,10\},9)$ GDD of type n^9r^1 (where $9n+r=(v-1)/2=m$); this GDD is provided by Lemma 6.4. Now give every point in this GDD a weight of 2 and apply WFC (Theorem 2.1); the required ingredients for this construction, namely $(9,4)$-GDDs of types 2^9 and 2^{10} exist by Theorem 2.2.c. This gives a $(9,36)$ GDD of type $2n^92r^1$. Finally fill this design using one extra point, i.e., use $w=1$ in Theorem 2.4. ■

Taking three copies of the $(v,9,12)$ given by Theorem 1.2.g, or two copies of the $(v,9,18)$ given by Theorem 4.6 leaves only $v\equiv 11 \pmod{12}$, and combining Lemmas 6.3 and 6.5, we have the following result for $(v,9,36)$ BIBDs:

Theorem 6.6 *The necessary conditions for the existence of a* $(v,9,36)$ *BIBD, viz. that* $v\geq 9$ *and* $v\equiv 1 \pmod 2$*, are sufficient.*

7 BIBDs with block size 9 and index 72

Our object in this section is to construct $(v,9,72)$ BIBDs. The necessary conditions reduce to $v\geq 9$. We begin with a direct construction.

Example 7.1 *For a* $(104,9,72)$ *on* $Z_{103}\cup\{\infty\}$*, start with five copies of a* $(103,9,12)$ *BIBD on the finite points, then generate* 19 *extra base blocks obtained by multiplying all blocks below except the first by* 1, 46 *and* 56*; (the first block is invariant under this multiplication).*

$$
\begin{array}{ll}
(1,46,56,2,92,9,4,81,18)\,, & (84,19,79,74,95,30,2,85,13)\,,\\
(94,76,95,99,73,80,49,24,17)\,, & (90,15,12,98,94,97,45,83,56)\,,\\
(25,23,69,75,15,4,17,76,\infty)\,, & (89,71,57,8,74,1,35,12,\infty)\,,\\
(91,64,12,30,1,96,65,17,\infty)\,. &
\end{array}
$$

We now look at the smaller designs.

Lemma 7.2 *A* $(v,9,72)$ *BIBD exists for* $v=62$, 80 *and* 92.

Proof: For each of these values of v, we can construct a $(v,\{9,10\},9)$ PBD. For $v=62$, combine the blocks of a $(61,10,3)$ difference family given in [12] with those of a $(61,9,6)$ difference family given in [3]; then add an infinite point to one base block of size 9. For $v=80$, delete a point from a $(81,10,9)$ BIBD; this BIBD is obtainable by the difference family construction given by Corollary 2.11.b. For $v=92$, we first add an infinite point to one base block of the $(91,9,4)$ difference family on Z_{91}, which is given in [2]. Combining this with 5 copies of a $(91,10,1)$ BIBD on all points except the infinite one now gives a $(92,\{9,10\},9)$ PBD.

Now take the $(v,\{9,10\},9)$ PBD we have just constructed and break the blocks with $(9,9,8)$ and $(10,9,8)$ BIBDs in Theorem 2.3 for the desired BIBDs. ■

Lemma 7.3 *If* $9\leq v\leq 107$*, then a* $(v,9,72)$ *BIBD exists.*

Proof: If $v \equiv 1 \pmod 2$ and $v \geq 9$, then a $(v, 9, 36)$ BIBD exists by Theorem 4.6; take two copies of this design.

If $v \equiv 0, 1 \pmod 3$ and $v \geq 9$, then a $(v, 9, 24)$ BIBD exists by Theorem 5.14; take three copies of this design.

So now we must deal with $v \equiv 2 \pmod 6$. The values $v \leq 43$ are all given in [10]. Using Mullin and Stanton's J-construction gives constructions for 44, 74 and 86, see Corollary 2.8. Grouplet constructions for 50, 68 and 98 are given in Corollary 2.16.d, while Corollary 2.6 gives a $(56, 9, 72)$ BIBD. Designs for 62, 80, 92 and 104 are given by Lemma 7.2 and Example 7.1. ■

Lemma 7.4 *If $v \geq 9$ then a $(v, \{9, 10, 11, \ldots, 107\}, 1)$ PBD exists.*

Proof: See Ling and Colbourn [11]. ■

Lemma 7.5 *If $v > 107$, then a $(v, 9, 72)$ BIBD exists.*

Proof: Take the PBD provided by Lemma 7.4 and break each block with a $(k, 9, 72)$ BIBD in an application Theorem 2.3. ■

Combining Lemmas 7.3 and 7.5, we have the result for $(v, 9, 72)$ BIBDs:

Theorem 7.6 *The necessary condition for the existence of a $(v, 9, 72)$ BIBD, viz. that $v \geq 9$, is sufficient.*

8 BIBDs with block size 9 and other indices

Fix v and let λ_v be the smallest λ satisfying the integrality conditions for the existence of a $(v, 9, \lambda)$ BIBD. Now λ_v will be a divisor of 72, and all the divisors of 72 have been examined. If a $(v, 9, \lambda_v)$ BIBD is known, then a $(v, 9, t\lambda_v)$ BIBD can be constructed by taking t copies of the known $(v, 9, \lambda_v)$ BIBD. However, if a $(v, 9, \lambda_v)$ BIBD is not known, yet a $(v, 9, 2\lambda_v)$ BIBD and a $(v, 9, 3\lambda_v)$ BIBD are known, then a $(v, 9, t\lambda_v)$ BIBD can be constructed for any $t > 1$. To do this we take one copy of the $(v, 9, 3\lambda_v)$ BIBD if t is odd, and none if t is even, then make up the remainder of the index using the appropriate number of copies of the $(v, 9, 2\lambda_v)$ BIBD. So our only possible new problems for missing $(v, 9, \lambda)$ BIBDs will occur if we do not have both a $(v, 9, 2\lambda_v)$ BIBD and a $(v, 9, 3\lambda_v)$ BIBD. Note that the only values of λ_v (dividing 72) for which we have exceptions are 1, 2, 3, 4, 6. From this, we see by examining our exception lists that the only value of v occurring in two lists is $v = 505$. Here we are missing designs for $\lambda = 1$ and 2, but have designs for $\lambda = 3$ and 4. These latter designs can be combined to yield a $(505, 9, \lambda)$ BIBD for any index except $\lambda = 5$.

Example 8.1 *The following blocks generate a $(64, 10, 5)$ BIBD when developed over $Z_3 \times Z_{21} \cup \{\infty\}$. The automorphisms are: $T_1(x, y) = (x + 1, 4y)$ (of order 3), and $T_2(x, y) = (x, y + 1)$ (of order 21). Substituting $(a, b, c) = (1, 8, 15)$, $(3, 10, 17)$ and $(5, 10, 15)$ in the first block generates three blocks, and applying the automorphisms, the first two of these generate 7 blocks each, and the third generates 21 blocks. The last three base blocks each generate 63 blocks.*

$$(\infty,(0,a),(0,b),(0,c),(1,4a),(1,4b),(1,4c),(2,16a),(2,16b),(2,16c))\,,$$
$$((0,0),(0,1),(0,3),(0,6),(0,14),(1,3),(1,4),(1,13),(1,19),(2,19))\,,$$
$$((0,0),(0,2),(0,6),(0,19),(1,14),(1,19),(2,1),(2,2),(2,11),(2,16))\,,$$
$$((0,0),(0,9),(0,12),(0,19),(0,20),(1,2),(1,5),(1,6),(1,8),(2,5))\,.$$

Lemma 8.2 *A* $(505,9,5)$ *BIBD exists.*

Proof: Removing a point from the design in Example 8.1 gives a $(63,\{9,10\},5)$ PBD. Give all points of this design a weight of 8 in WFC (Theorem 2.1), using the ingredients given by Theorem 2.2.a. This gives a $(9,5)$ GDD of type 8^{63}; now fill the groups using an extra point in Theorem 2.4. ∎

9 Summary

We now summarize our knowledge of the existence problem for $(v,9,\lambda)$ BIBDs.

There are no known definite exceptions to the necessary conditions given in Theorem 1.1. These conditions are sufficient with the possible exception of 106 parameter sets, consisting of 91 values with $\lambda=1$, and the 15 sets listed in Theorem 1.2 when $\lambda>1$. We list these exceptions in Tables 2 and 3.

Table 2: Number of Points in Unconstructed $(v,9,1)$ BIBDs

145	153	217	225	289	297	361	369	505	793
865	873	945	1017	1081	1305	1441	1513	1585	1593
1665	1729	1809	1881	1945	1953	2025	2233	2241	2305
2385	2449	2457	2665	2737	2745	2881	2889	2961	3025
3097	3105	3241	3321	3385	3393	3601	3745	3753	3817
4033	4257	4321	4393	4401	4465	4473	4825	4833	4897
4905	5401	5473	5481	6049	6129	6625	6705	6769	6777
6913	7345	7353	7425	9505	10017	10665	12529	12537	13185
13753	13833	13969	14113	14473	14553	14625	14689	15049	15057
16497									

Table 3: Unconstructed $(v,9,\lambda)$ BIBDs with $\lambda>1$

(177,9,3)	(189,9,2)	(213,9,6)	(253,9,2)	(315,9,4)
(345,9,3)	(385,9,3)	(459,9,4)	(505,9,2)	(765,9,2)
(783,9,4)	(837,9,2)	(1197,9,2)	(1837,9,2)	(1845,9,2)

References

[1] R.J.R. Abel, I. Bluskov and M. Greig, Balanced Incomplete Block Designs with block size 8, *J. Combin. Des.* **9** (2001), 233–268.

[2] R.J.R. Abel, I. Bluskov and M. Greig, Balanced Incomplete Block Designs with block size 9, *Des. Codes Cryptogr.* **26** (2002), 33–59.

[3] R.J.R. Abel, I. Bluskov and M. Greig, Balanced Incomplete Block Designs with block size 9: Part II, *Discrete Math.*, **279** (2004), 5–32.

[4] R.J.R. Abel, A.E. Brouwer, C.J. Colbourn and J.H. Dinitz, Mutually Orthogonal Latin Squares (MOLS), in: *C.R.C. Handbook of Combinatorial Designs*, (C.J. Colbourn and J.H. Dinitz, eds.), CRC Press, Boca Raton FL, 1996, 111–142.

[5] R.J.R. Abel, N.J. Finizio, M. Greig and S.J. Lewis, Pitch tournament designs and other BIBDs — existence results for the case $v = 8n + 1$, *Congr. Numer.* **138** (1999), 175–192.

[6] T. Beth, D. Jungnickel and H. Lenz, *Design Theory, 2nd. ed.*, Cambridge University Press, Cambridge, England, (1999).

[7] C.J. Colbourn, Transversal designs of block size eight and nine, *European J. Combin.* **17** (1996), 1–14.

[8] C.J. Colbourn, Orthogonal arrays of index more than one, in: *C.R.C. Handbook of Combinatorial Designs*, (C.J. Colbourn and J.H. Dinitz, eds.), CRC Press, Boca Raton FL, 1996, 172–182.

[9] C.J. Colbourn and W. de Launey, Difference matrices, in: *C.R.C. Handbook of Combinatorial Designs*, (C.J. Colbourn and J.H. Dinitz, eds.), CRC Press, Boca Raton FL, 1996, 287–297.

[10] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[11] A.C.H. Ling and C.J. Colbourn, Concerning the generating set of $Z_{\geq n}$, *Congr. Numer.* **114** (1996), 65–72.

[12] L.B. Morales, Constructing difference families through an optimization approach: six new BIBDs, *J. Combin. Des.* **8** (2000), 261–273.

[13] R.C. Mullin and R.G. Stanton, Classification and embedding of BIBD's, *Sankhyā Ser. A* **30** (1968), 91–100.

(Received 30 Nov 2002; revised 17 Apr 2003)