The μ-way intersection problem for cubes

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Abstract

A collection of μ 3-cube decompositions of a graph G is said to have intersection size t if there is a set S, with $|S| = t$, of 3-cubes which is contained in every decomposition, and no 3-cube which is not in S occurs in more than one decomposition. We determine all integers n, t and μ, with $\mu \leq 12$, for which there exists a collection of μ 3-cube decompositions of K_n with intersection size t. Also, we determine all integers m, n, t and μ, with $\mu \leq 8$, for which there exists a collection of μ 3-cube decompositions of $K_{m,n}$ with intersection size t.

1 Introduction

A great deal of work has been done in recent years on the *intersection problem* for combinatorial designs. The question addressed in intersection problems is: given two designs based on the same underlying set of elements, how many blocks may they have in common? The intersection problem has been considered for many classes of designs, including Steiner triple systems [8], m-cycle systems [4] and Steiner quadruple systems [7]. For a fine survey on the intersection problem, the reader is directed to Billington [3], and the references therein.

There is no reason to restrict one's attention to the number of blocks that a pair of designs have in common: one may consider the intersection problem for a collection of μ designs (defined on the same underlying set). Indeed, this problem has already been considered for m-cycle systems [1] and in the case $\mu = 3$, for Steiner triple systems [10] and latin squares [6]. In this paper we consider the μ-way intersection problem for decompositions of the complete graph into cubes of dimension three. The case $\mu = 2$ was solved in [2]. The *3-cube*, one of the most loved graphs, is the graph C whose vertex set is $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and whose edge set consists of the edges of two 4-cycles (x_1, x_2, x_3, x_4) and (x_5, x_6, x_7, x_8) and the edges $\{x_1, x_5\}$,

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$\{x_2, x_6\}$, $\{x_3, x_7\}$ and $\{x_4, x_8\}$. We denote this graph by $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ and henceforth will refer to it simply as the *cube*. A C-decomposition of a graph G is a set D of cubes whose edge sets form a partition of the edge set of G. We need to take care when defining the intersection of μ C-decompositions if $\mu > 2$.

Definition 1.1 *A collection of μ C-decompositions of a graph G have intersection size t if there is a set S, with $|S| = t$, of 3-cubes which is common to each decomposition, and no 3-cube which is not in S occurs in more than one decomposition.*

In this paper, we determine all integers n, t and μ, with $\mu \leq 12$, for which there exists a collection of μ C-decompositions of K_n with intersection size t. Also, we determine all integers m, n, t and μ, with $\mu \leq 8$, for which there exists a collection of μ C-decompositions of $K_{m,n}$ with intersection size t. These values of μ, for which we solve the problem, are maximal in the sense that the range of expected intersection sizes is reduced for larger μ. We have shown by exhaustive computer search that for any (bipartite) graph G which can be decomposed into two 3-cubes, it is not possible to find ($\mu > 8$) $\mu > 12$ C-decompositions of G, having intersection size zero. This means that for $\mu > 12$ in the case $G = K_n$ and for $\mu > 8$ in the case $G = K_{m,n}$, there does not exist a collection of μ C-decompositions of G with intersection size $t = |E(G)|/12 - 2$.

For a graph G, let $I_\mu(G)$ denote the set of integers t for which there exists a collection of μ C-decompositions of G with intersection size t. We define $J_\mu(G)$ to be the set of expected intersection numbers. That is, $J_\mu(G) = \{0, 1, 2, \ldots, b\} \setminus \{b-1\}$, where $b = |E(G)|/12$ if there exists a C-decomposition of G, and $J_\mu(G) = \emptyset$ otherwise. Let $I_\mu(n), J_\mu(n), I_\mu(m,n)$ and $J_\mu(m,n)$ denote $I_\mu(K_n), J_\mu(K_n), I_\mu(K_{m,n})$ and $J_\mu(K_{m,n})$, respectively.

In [9], Maheo showed that there is a C-decomposition of K_n if and only if $n \equiv 1$ or 16 (mod 24). Thus for $2 \leq \mu \leq 12$, $J_\mu(n) = \{0, 1, 2, \ldots, b\} \setminus \{b-1\}$, where $b = n(n-1)/24$ if $n \equiv 1$ or 16 (mod 24), and $J_\mu(n) = \emptyset$ otherwise. In [5], it was shown that for $m \leq n$, there is a C-decomposition of $K_{m,n}$ if and only if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$. Thus for $2 \leq \mu \leq 8$, $J_\mu(m,n) = \{0, 1, 2, \ldots, b\} \setminus \{b-1\}$, where $b = mn/12$ if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$, and $J_\mu(m,n) = \emptyset$ otherwise.

Lemmas 1.1 and 1.2 follow immediately from the definitions of $J_\mu(n)$ and $J_\mu(m,n)$.

Lemma 1.1 For all m, n and for $2 \leq \mu \leq 8$, $I_\mu(m,n) \subseteq J_\mu(m,n)$.

Lemma 1.2 For all n and for $2 \leq \mu \leq 12$, $I_\mu(n) \subseteq J_\mu(n)$.

In Sections 2 and 3 respectively we will show that we have equality in the above two lemmas, except for the isolated case that $1 \notin I_\mu(6,6)$ for $\mu = 7$ and $\mu = 8$.

We will use the following notation. Given a graph G and subgraph H, let $G \setminus H$ be the graph with vertex set $V(G \setminus H) = V(G)$ and edge set $E(G \setminus H) = E(G) \setminus E(H)$. The graph $K_v \setminus K_u$ is called the *complete graph on v vertices with a hole of size u*, with the vertices of K_u forming the *hole*. Given graphs G_1 and G_2, with $E(G_1) \cap E(G_2) = \emptyset$, let $G_1 + G_2$ be the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. We will use the notation $G_1 + G_2$ only when $E(G_1) \cap E(G_2) = \emptyset$.

We require the following straightforward lemmas; proof of the first is obvious.

Lemma 1.3 If $G = G_1+G_2+\ldots+G_r$ and there is a collection of μ C-decompositions of G_i with an intersection of size t_i (for $i = 1, 2, \ldots, r$), then there is a collection of μ C-decompositions of G with intersection size $t_1 + t_2 + \ldots + t_r$.

Lemma 1.4 Let G_1 and G_2 be edge-disjoint graphs such that $|E(G_1)| \geq 36$ and $|E(G_2)| \geq 24$. Suppose $C|G_1$ and $C|G_2$ with $I_\mu(G_1) = J_\mu(G_1)$ and $I_\mu(G_2) = J_\mu(G_2)$. Then $I_\mu(G_1 + G_2) = J_\mu(G_1 + G_2)$.

Proof: Let $n_1 = |E(G_1)|/12$ and let $n_2 = |E(G_2)|/12$. Without loss of generality, assume that $n_1 \geq n_2$. Let $t \in J_\mu(G_1 + G_2)$.

If $t \leq n_1 - 2$, then there exists a collection of μ C-decompositions of G_1 with intersection size t and there exists a collection of μ C-decompositions of G_2 with intersection size zero. Thus by Lemma 1.3, there exists a collection of μ C-decompositions of $G_1 + G_2$ with intersection size t.

If $t = n_1 - 1$ and $n_2 \neq 2$, there exists a collection of μ C-decompositions of G_1 with intersection size $n_1 - 2$ and there exists a collection of μ C-decompositions of G_2 with intersection size 1. Thus there exists a collection of μ C-decompositions of $G_1 + G_2$ with intersection size t.

If $t = n_1 - 1$ and $n_2 = 2$, there exists a collection of μ C-decompositions of G_1 with intersection size $n_1 - 3$ and there exists a collection of μ C-decompositions of G_2 with intersection size 2. Thus there exists a collection of μ C-decompositions of $G_1 + G_2$ with intersection size t.

Finally, if $t \geq n_1$, there exists a collection of μ C-decompositions of G_1 with intersection size n_1 and there exists a collection of μ C-decompositions of G_2 with intersection size $t-n_1$. Thus there exists a collection of μ C-decompositions of G_1+G_2 with intersection size t.

Therefore, $t \in I_\mu(G_1 + G_2)$ and thus, $I_\mu(G_1 + G_2) = J_\mu(G_1 + G_2)$. □

2 The complete bipartite graph

We begin with the following theorem that settles the special case of $K_{6,6}$.

Theorem 2.1 For $2 \leq \mu \leq 6$, $I_\mu(6, 6) = J_\mu(6, 6) = \{0, 1, 3\}$ and for $7 \leq \mu \leq 8$, $I_\mu(6, 6) = J_\mu(6, 6) \setminus \{1\} = \{0, 3\}$.

Proof: It has been shown by exhaustive computer search that $1 \notin I_7(6, 6)$. See the Appendix for proof that $0 \in I_8(6, 6)$ and $1 \in I_6(6, 6)$. Hence the result follows. □

Before we prove the main theorem for this section we need two lemmas.

Lemma 2.1 $I_\mu(9, 12) = J_\mu(9, 12)$.

Proof: See the Appendix for proof that $0, 1, \ldots 7 \in I_8(9, 12)$. Clearly, $9 \in I_8(9, 12)$, as there are nine cubes in a decomposition of $K_{9,12}$. $\square$

Lemma 2.2 $I_\mu(6, 12) = J_\mu(6, 12)$.

Proof: Since $0, 3 \in K_{6,6}$ and $K_{6,12} = K_{6,6} + K_{6,6}$, by Lemma 1.3 we have $0, 3, 6 \in I_8(6, 12)$. See the Appendix for proof that $1, 2$ and $4 \in I_8(6, 12)$. $\square$

Theorem 2.2 For $2 \leq \mu \leq 8$ and all m,n except $m = n = 6$, $I_\mu(m, n) = J_\mu(m, n)$.

Proof: In [5], it was shown that there is a C-decomposition of $K_{m,n}$ if and only if $m \equiv n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$ and $m \geq 4$. Under these conditions, we can assume without loss of generality that either

(C1) $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 6$, or

(C2) $m \equiv 3 \pmod 6$ and $n \equiv 0 \pmod{12}$.

Since $\{0, 3\} + \{0, 1, \ldots, xy/12 - 2, xy/12\} = \{0, 1, \ldots, xy/12 + 1, xy/12 + 3\}$ for $xy/12 \geq 4$, if $G = K_{6,6} + K_{x,y}$, there exists a C-decomposition of $K_{x,y}$ and $I_\mu(x, y) = J_\mu(x, y)$, then by Lemma 1.3 we have $I_\mu(G) = J_\mu(G)$. Notice that if $(x, y) \neq (6, 6)$ and there exists a C-decomposition of $K_{x,y}$, then $xy/12 \geq 6$. Now suppose there exists a C-decomposition of $K_{m,n}$ and $(m, n) \neq (6, 6)$. Then $K_{m,n} = K_{9,12} + \sum_{i=1}^{r} K_{6,6}$ or $K_{m,n} = K_{6,12} + \sum_{i=1}^{r} K_{6,6}$. Hence, since $I_\mu(9, 12) = J_\mu(9, 12)$ and $I_\mu(6, 12) = J_\mu(6, 12)$, the result follows by induction. $\square$

3 The complete graph

3.1 Small cases

In this subsection we show that $I_\mu(n) = J_\mu(n)$ for $n = 16$ and 25.

Lemma 3.1 $I_\mu(16) = J_\mu(16)$.

Proof: See the Appendix for proof that $0, 1, \ldots 8 \in I_{12}(16)$. Clearly, $10 \in I_{12}(16)$, as there are ten cubes in a decomposition of K_{16}. $\square$

Before proving that $I_\mu(25) = J_\mu(25)$, we need one more result.

Lemma 3.2 There are twelve C-decompositions of $K_{13} \setminus K_4$ having precisely zero common cubes, and twelve with precisely six common cubes.

Proof: See the Appendix for proof that $0 \in I_{12}(K_{13} \setminus K_4)$. Clearly, $6 \in I_{12}(K_{13} \setminus K_4)$, as there are six cubes in a decomposition of $K_{13} \setminus K_4$. $\square$

Lemma 3.3 $I_\mu(25) = J_\mu(25)$

Proof: Let V_1, V_2 and V_3 be three mutually disjoint vertex sets of sizes 12, 9 and 4 respectively. Let $G_1 \cong K_{16}$ have vertex set $V_1 \cup V_3$, let $G_2 \cong K_{9,12}$ have vertex set $V_1 \cup V_2$ (and the obvious bipartition), and let $G_3 \cong K_{13} \setminus K_4$ have vertex set $V_2 \cup V_3$ (with the vertices of V_3 in the hole). Then $K_{25} = G_1 + G_2 + G_3$.

Now using Lemma 1.3 with $t_1 \in I_\mu(16)$ (see Lemma 3.1), $t_2 \in \{0, 9\}$ (see Lemma 2.1) and $t_3 \in \{0, 6\}$ (see Lemma 3.2) it is straightforward to check that we have $I_\mu(25) = J_\mu(25)$. $\square$

3.2 Constructions

Lemma 3.4 For $2 \leq \mu \leq 12$ and $n \equiv 1 \pmod{24}$, $I_\mu(n) = J_\mu(n)$.

Proof: Let $n = 24r + 1$ and let $V_1, V_2, \ldots, V_r$ be r mutually disjoint vertex sets of size 24 and let $\infty \notin \bigcup_{i=1}^{r} V_i$. For each i, j with $1 \leq i < j \leq r$, let $G_{i,j} \cong K_{24,24}$ have vertex set $V_i \cup V_j$ (and the obvious vertex partition), and for each $i = 1, 2, \ldots, r$ let $G_i \cong K_{25}$ have vertex set $V_i \cup \{\infty\}$. Then $K_n = \sum_{1 \leq i < j \leq r} G_{i,j} + \sum_{1 \leq i \leq r} G_i$.

Since $I_\mu(24, 24) = J_\mu(24, 24)$ and $I_\mu(25) = J_\mu(25)$, we conclude from Lemma 1.4 that $I_\mu(n) = J_\mu(n)$. □

In the following lemma we use a similar construction to that described in [5].

Lemma 3.5 For $2 \leq \mu \leq 12$ and $n \equiv 16 \pmod{24}$, $I_\mu(n) = J_\mu(n)$.

Proof: Let $n = 24r + 16$, $A = \{x_1, x_2, \ldots, x_{16}\}$ and $V_1, V_2, \ldots, V_r$ be r mutually disjoint vertex sets of size 24 such that $V_i \cap A = \emptyset$ for all i. For each i, j with $1 \leq i < j \leq r$, let $G_{i,j} \cong K_{24,24}$ have vertex set $V_i \cup V_j$ (and the obvious vertex partition), for each $i = 1, 2, \ldots, r$ let $G_i \cong K_{25}$ have vertex set $V_i \cup \{x_1\}$, and let $G'_i \cong K_{15,24}$ have vertex set $V_i \cup \{x_2, x_3, \ldots, x_{16}\}$ (and the obvious vertex partition). Finally, let $G \cong K_{16}$ have vertex set A. Then

$$K_n = G + \sum_{1 \leq i < j \leq r} G_{i,j} + \sum_{1 \leq i \leq r} (G_i + G'_i).$$

Since $I_\mu(16) = J_\mu(16)$, $I_\mu(24, 24) = J_\mu(24, 24)$, $I_\mu(25) = J_\mu(25)$ and $I_\mu(15, 24) = J_\mu(15, 24)$, we conclude by Lemma 1.4 that $I_\mu(n) = J_\mu(n)$. □

Combining Lemmas 3.4 and 3.5 we have our main theorem.

Theorem 3.1 For $2 \leq \mu \leq 12$ and for all n, $I_\mu(n) = J_\mu(n)$.

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4 Appendix

In each case, a collection of μ C-decompositions is given by $D_i \cup I$ (for $i = 1, 2, \ldots, \mu$). Thus the (possibly empty) set of cubes I is the intersection.

$\boxed{K_{6,6}}$ Let the vertex set of $K_{6,6}$ be $\{0, 1, \ldots, 5\} \cup \{6, 7, \ldots, 11\}$, with the obvious vertex partition.

(1) $0 \in I_8(6, 6)$.

$D_1 = \{[0,6,1,7,8,2,9,3],[0,9,4,10,11,5,6,3],[1,8,5,10,11,4,7,2]\}$
$D_2 = \{[0,6,1,7,8,2,9,4],[0,9,3,10,11,5,6,4],[1,8,5,10,11,3,7,2]\}$
$D_3 = \{[0,6,1,7,8,2,9,5],[0,9,3,10,11,4,6,5],[1,8,4,10,11,3,7,2]\}$
$D_4 = \{[0,6,1,7,8,2,10,3],[0,9,4,10,11,1,8,5],[2,7,5,9,11,4,6,3]\}$
$D_5 = \{[0,6,1,7,8,2,10,4],[0,9,3,10,11,1,8,5],[2,7,5,9,11,3,6,4]\}$
$D_6 = \{[0,6,1,7,8,2,10,5],[0,9,3,10,11,1,8,4],[2,7,4,9,11,3,6,5]\}$
$D_7 = \{[0,6,1,7,8,2,11,3],[0,9,1,10,11,4,8,5],[2,7,5,9,10,4,6,3]\}$
$D_8 = \{[0,6,1,7,8,2,11,4],[0,9,1,10,11,3,8,5],[2,7,5,9,10,3,6,4]\}$

(2) $1 \in I_6(6, 6)$. Let $I = \{[0,6,1,7,8,2,9,3]\}$.

$D_1 = \{[0,9,4,10,11,5,6,3],[1,8,5,10,11,4,7,2]\}$ $D_2 = \{[0,9,4,10,11,5,7,2],[1,8,5,10,11,4,6,3]\}$
$D_3 = \{[0,9,4,10,11,5,8,1],[2,7,5,10,11,4,6,3]\}$ $D_4 = \{[0,9,5,10,11,4,6,3],[1,8,4,10,11,5,7,2]\}$
$D_5 = \{[0,9,5,10,11,4,7,2],[1,8,4,10,11,5,6,3]\}$ $D_6 = \{[0,9,5,10,11,4,8,1],[2,7,4,10,11,5,6,3]\}$

$\boxed{K_{6,12}}$ Let the vertex set of $K_{6,12}$ be $\{0, 1, \ldots, 5\} \cup \{6, 7, \ldots, 17\}$, with the obvious vertex partition.

(1) $1 \in I_8(6, 12)$. Let $I = \{[0,6,1,7,8,2,10,3]\}$.

$D_1 = \{[0,9,1,11,12,2,13,3],[0,10,4,14,15,5,6,3],[0,13,4,16,17,5,7,2],[1,8,5,14,15,4,11,2],$
$[1,12,5,16,17,4,9,3]\}$
$D_2 = \{[0,9,1,11,12,2,13,4],[0,10,4,14,15,5,7,2],[0,13,3,16,17,5,6,4],[1,8,5,12,15,4,9,3],$
$[1,14,5,16,17,3,11,2]\}$
$D_3 = \{[0,9,1,11,12,2,13,5],[0,10,4,13,14,5,6,3],[0,15,1,16,17,2,14,4],[1,8,4,12,17,5,9,3],$
$[2,7,4,11,16,5,15,3]\}$
$D_4 = \{[0,9,1,11,12,2,14,3],[0,10,4,13,15,5,6,3],[0,14,4,16,17,5,7,2],[1,8,5,16,17,4,9,3],$
$[1,12,5,13,15,4,11,2]\}$

$D_5 = \{[0,9,1,11,12,2,14,4],[0,10,4,13,14,5,9,3],[0,15,1,16,17,2,13,5],[1,8,5,12,17,4,6,3],$
$[2,7,5,11,16,4,15,3]\}$
$D_6 = \{[0,9,1,11,12,2,14,5],[0,10,4,13,15,5,7,2],[0,14,3,16,17,4,6,5],[1,8,4,12,13,5,9,3],$
$[1,15,4,16,17,3,11,2]\}$
$D_7 = \{[0,9,1,11,12,2,15,3],[0,10,4,13,14,5,7,2],[0,15,4,16,17,5,6,3],[1,8,5,13,14,4,9,3],$
$[1,12,5,16,17,4,11,2]\}$
$D_8 = \{[0,9,1,11,12,2,15,4],[0,10,4,13,14,5,8,1],[0,15,3,16,17,5,6,4],[1,12,5,16,17,3,11,2],$
$[2,7,5,13,14,4,9,3]\}$

(2) $2 \in I_8(6,12)$. Let $I = \{[0,6,1,7,8,2,10,3],[0,9,1,11,12,2,13,3]\}$.

$D_1 = \{[0,10,4,14,15,5,6,3],[0,13,4,16,17,5,7,2],[1,8,5,14,15,4,11,2],[1,12,5,16,17,4,9,3]\}$
$D_2 = \{[0,10,4,14,15,5,7,2],[0,13,4,16,17,5,6,3],[1,8,5,14,15,4,9,3],[1,12,5,16,17,4,11,2]\}$
$D_3 = \{[0,10,4,14,15,5,8,1],[0,13,4,16,17,5,9,3],[1,12,5,16,17,4,7,2],[2,11,5,14,15,4,6,3]\}$
$D_4 = \{[0,10,4,14,15,5,9,3],[0,13,4,16,17,5,8,1],[1,12,5,14,15,4,7,2],[2,11,5,16,17,4,6,3]\}$
$D_5 = \{[0,10,4,14,15,5,11,2],[0,13,4,16,17,5,12,1],[1,8,5,14,15,4,6,3],[2,7,5,16,17,4,9,3]\}$
$D_6 = \{[0,10,4,14,15,5,12,1],[0,13,4,16,17,5,11,2],[1,8,5,16,17,4,6,3],[2,7,5,14,15,4,9,3]\}$
$D_7 = \{[0,10,4,14,16,5,6,3],[0,13,4,15,17,5,7,2],[1,8,5,14,16,4,11,2],[1,12,5,15,17,4,9,3]\}$
$D_8 = \{[0,10,4,14,16,5,7,2],[0,13,4,15,17,5,6,3],[1,8,5,14,16,4,9,3],[1,12,5,15,17,4,11,2]\}$

(3) $4 \in I_8(6,12)$. Let $I = \{[0,10,4,14,15,5,6,3],[0,13,4,16,17,5,7,2],[1,8,5,14,15,4,11,2],$
$[1,12,5,16,17,4,9,3]\}$

$D_1 = \{[0,6,1,7,8,2,10,3],[0,9,1,11,12,2,13,3]\}$ $D_2 = \{[0,6,1,7,8,2,13,3],[0,9,1,11,12,2,10,3]\}$
$D_3 = \{[0,6,1,7,12,2,10,3],[0,8,2,9,11,3,13,1]\}$ $D_4 = \{[0,6,1,7,12,2,13,3],[0,8,2,9,11,3,10,1]\}$
$D_5 = \{[0,6,1,11,12,2,10,3],[0,7,3,8,9,1,13,2]\}$ $D_6 = \{[0,6,1,11,12,2,13,3],[0,7,3,8,9,1,10,2]\}$
$D_7 = \{[0,6,2,8,11,1,10,3],[0,7,1,9,12,3,13,2]\}$ $D_8 = \{[0,6,2,8,11,1,13,3],[0,7,1,9,12,3,10,2]\}$

$\boxed{K_{9,12}}$ Let the vertex set of $K_{9,12}$ be $\{0,1,\ldots,8\} \cup \{9,10,\ldots,20\}$, with the obvious vertex partition.

(1) $0 \in I_{12}(9,12)$.
$D_1 = \{[0,9,1,10,11,2,12,3],[0,12,4,13,14,5,9,3],[3,16,8,18,20,4,11,5],[0,15,1,16,17,2,13,5],$
$[0,18,1,19,20,2,14,6],[1,11,6,17,20,7,9,8],[2,10,6,16,19,8,12,7],[3,15,7,17,19,5,10,4],$
$[4,14,8,15,18,7,13,6]\}$
$D_2 = \{[0,9,1,10,11,2,12,4],[0,12,3,13,14,5,9,4],[0,15,1,16,17,2,13,6],[0,18,1,19,20,2,14,3],$
$[1,11,3,17,20,7,10,5],[2,10,8,16,19,6,9,7],[3,15,4,16,18,8,19,5],[4,17,7,18,20,8,12,6],$
$[5,11,8,13,15,6,14,7]\}$
$D_3 = \{[0,9,1,10,11,2,12,5],[0,12,3,13,14,4,9,5],[0,15,1,16,17,2,13,4],[0,18,1,19,20,2,14,7],$
$[1,11,6,17,20,3,15,5],[2,10,8,16,19,3,18,5],[3,14,6,16,17,8,9,7],[4,10,7,11,19,6,12,8],$
$[4,15,7,18,20,8,13,6]\}$
$D_4 = \{[0,9,1,10,11,2,12,6],[0,12,3,13,14,4,9,6],[0,15,1,16,17,2,13,7],[0,18,1,19,20,2,14,5],$
$[1,11,3,17,20,7,14,8],[2,10,8,16,19,3,15,6],[3,16,5,18,20,4,17,6],[4,10,5,11,18,7,9,8],$
$[4,13,5,15,19,8,12,7]\}$
$D_5 = \{[0,9,1,10,11,2,12,7],[0,12,3,13,14,4,9,7],[0,15,1,16,17,2,13,8],[0,18,1,19,20,2,14,8],$
$[1,11,3,17,20,5,14,6],[2,10,3,16,19,6,15,4],[3,18,5,19,20,4,17,7],[4,10,8,11,13,5,9,6],$
$[5,12,8,15,16,6,18,7]\}$
$D_6 = \{[0,9,1,10,11,2,12,8],[0,12,3,13,14,4,9,8],[0,15,1,16,17,2,14,3],[0,18,1,19,20,2,13,5],$
$[1,11,5,17,20,6,9,7],[2,10,5,16,19,7,12,6],[3,10,6,15,19,4,17,8],[3,11,7,18,20,4,16,8],$
$[4,13,7,15,18,6,14,5]\}$
$D_7 = \{[0,9,1,10,11,2,13,3],[0,12,1,14,15,2,16,3],[0,13,4,16,17,5,9,6],[1,11,4,15,20,5,10,6],$
$[0,18,1,19,20,2,17,3],[2,10,7,14,19,8,11,6],[3,9,7,12,18,8,13,6],[4,12,5,18,20,8,15,7],$
$[4,14,8,17,19,5,16,7]\}$
$D_8 = \{[0,9,1,10,11,2,13,4],[0,12,1,14,15,2,16,4],[0,13,3,16,17,5,9,6],[1,11,3,15,20,5,10,6],$
$[0,18,1,19,20,2,17,4],[2,10,7,14,19,8,13,6],[3,12,4,18,20,7,9,8],[3,14,8,17,19,5,15,7],$
$[5,12,8,16,18,6,11,7]\}$

$D_9 = \{[0,9,1,10,11,2,13,5],[0,12,1,14,15,2,16,5],[2,10,7,14,19,8,15,4],[0,18,1,19,20,2,17,5],$
$[1,11,3,15,20,4,10,6],[0,13,3,16,17,4,9,6],[3,12,4,18,20,7,16,8],[3,14,8,17,19,6,11,7],$
$[5,9,8,12,18,7,13,6]\}$

$D_{10} = \{[0,9,1,10,11,2,13,6],[0,12,1,14,15,2,16,6],[2,10,7,14,19,8,11,5],[0,18,1,19,20,2,17,6],$
$[1,11,3,15,20,4,10,5],[0,13,3,16,17,4,9,5],[3,12,4,14,17,7,15,8],[3,18,4,19,20,8,16,7],$
$[5,12,8,13,18,6,9,7]\}$

$D_{11} = \{[0,9,1,10,11,2,13,7],[0,12,1,14,15,2,16,7],[2,10,5,14,19,6,11,8],[0,18,1,19,20,2,17,7],$
$[1,11,3,15,20,4,10,8],[0,13,3,16,17,4,9,8],[3,12,4,14,17,5,15,6],[3,18,4,19,20,6,16,5],$
$[5,9,6,13,18,7,12,8]\}$

$D_{12} = \{[0,9,1,10,11,2,13,8],[0,12,1,14,15,2,16,8],[2,10,5,14,19,6,11,7],[0,18,1,19,20,2,17,8],$
$[1,11,3,15,20,4,10,7],[0,13,3,16,17,4,9,7],[3,12,4,14,17,5,16,6],[3,18,4,19,20,6,15,5],$
$[5,9,6,13,18,8,12,7]\}$

(2) $1 \in I_8(9,12)$. Let $I = \{[0,9,1,10,11,2,12,3]\}$.

$D_1 = \{[2,10,6,16,19,8,12,7],[0,15,1,16,17,2,13,5],[0,18,1,19,20,2,14,6],[1,11,6,17,20,7,9,8],$
$[0,12,4,13,14,5,9,3],[3,15,7,17,19,5,10,4],[3,16,8,18,20,4,11,5],[4,14,8,15,18,7,13,6]\}$

$D_2 = \{[2,10,7,16,19,8,11,6],[3,9,7,15,18,8,12,6],[0,18,1,19,20,2,14,3],[1,11,4,17,20,5,10,6],$
$[0,12,4,13,14,5,9,6],[0,15,1,16,17,2,13,3],[4,14,8,15,18,7,13,5],[4,16,5,19,20,8,17,7]\}$

$D_3 = \{[2,10,4,16,19,6,18,5],[3,9,6,15,16,8,11,7],[0,18,1,19,20,2,14,4],[1,11,4,17,20,5,15,8],$
$[0,12,4,13,14,5,9,7],[0,15,1,16,17,2,13,6],[3,13,5,17,18,8,10,7],[3,14,8,19,20,6,12,7]\}$

$D_4 = \{[0,12,4,13,14,5,9,8],[0,15,1,16,17,2,13,7],[0,18,1,19,20,2,14,7],[1,11,4,17,20,6,14,3],$
$[2,10,4,16,19,8,15,3],[3,9,6,13,18,7,10,5],[4,18,6,19,20,8,16,5],[5,11,7,15,17,8,12,6]\}$

$D_5 = \{[0,12,4,13,14,5,10,2],[2,16,8,19,20,4,17,5],[0,18,4,19,20,6,9,3],[1,11,4,14,17,7,15,6],$
$[1,13,6,19,20,8,10,7],[3,13,7,14,15,5,9,8],[0,15,1,16,17,2,18,3],[5,11,6,16,18,8,12,7]\}$

$D_6 = \{[0,12,4,13,14,5,10,6],[0,15,1,16,17,2,13,8],[0,18,1,19,20,3,14,2],[1,11,4,17,20,5,9,7],$
$[2,10,7,16,18,8,11,6],[3,9,6,17,19,8,15,5],[3,13,7,15,16,5,18,4],[4,14,7,19,20,8,12,6]\}$

$D_7 = \{[0,12,4,13,14,5,10,7],[0,15,1,16,17,2,14,3],[0,18,1,19,20,2,13,3],[1,11,4,17,20,5,9,6],$
$[2,10,6,16,19,8,11,7],[3,9,7,15,18,8,17,5],[4,14,6,18,20,8,12,7],[4,15,8,16,19,6,13,5]\}$

$D_8 = \{[0,12,4,13,14,5,10,8],[0,15,1,16,17,2,14,4],[0,18,1,19,20,2,13,5],[1,11,5,17,20,8,9,6],$
$[2,10,6,16,19,7,12,8],[3,9,4,15,16,7,18,5],[3,13,6,19,20,7,11,4],[3,14,7,17,18,6,15,8]\}$

(3) $2 \in I_8(9,12)$. Let $I = \{[0,9,1,10,11,2,12,3],[0,12,4,13,14,5,9,3]\}$.

$D_1 = \{[0,15,1,16,17,2,13,5],[0,18,1,19,20,2,14,6],[1,11,6,17,20,7,9,8],[4,14,8,15,18,7,13,6],$
$[2,10,6,16,19,8,12,7],[3,16,8,18,20,4,11,5],[3,15,7,17,19,5,10,4]\}$

$D_2 = \{[0,15,1,16,17,2,13,6],[0,18,1,19,20,2,14,4],[2,10,8,16,19,6,9,7],[1,11,4,17,20,7,10,5],$
$[3,15,4,16,19,8,18,5],[3,17,7,18,20,8,12,6],[5,11,8,13,15,6,14,7]\}$

$D_3 = \{[0,15,1,16,17,2,13,7],[0,18,1,19,20,2,14,7],[5,10,6,13,18,7,9,8],[1,11,4,17,20,6,14,8],$
$[2,10,4,16,19,8,15,3],[3,17,6,18,20,5,19,4],[5,11,7,15,16,8,12,6]\}$

$D_4 = \{[0,15,1,16,17,2,13,8],[0,18,1,19,20,2,14,8],[3,15,6,16,18,8,9,7],[1,11,4,17,20,6,14,7],$
$[2,10,4,16,19,7,15,5],[3,17,6,19,20,5,18,4],[5,10,8,11,13,6,12,7]\}$

$D_5 = \{[0,15,1,16,17,2,14,4],[0,18,1,19,20,2,13,6],[2,10,6,16,19,8,9,7],[1,11,6,17,20,7,12,8],$
$[3,15,7,17,19,4,10,5],[3,16,8,18,20,5,11,4],[5,13,8,15,18,7,14,6]\}$

$D_6 = \{[0,15,1,16,17,2,14,6],[0,18,1,19,20,2,13,5],[4,14,7,19,20,8,9,6],[1,11,4,17,20,7,15,3],$
$[2,10,4,16,19,8,18,3],[5,10,6,11,16,7,12,8],[5,15,8,17,18,6,13,7]\}$

$D_7 = \{[0,15,1,16,17,2,14,7],[0,18,1,19,20,2,13,7],[5,10,6,13,15,7,9,8],[1,11,4,17,20,6,15,3],$
$[2,10,4,16,19,8,14,6],[3,16,8,18,19,5,20,4],[5,11,8,17,18,7,12,6]\}$

$D_8 = \{[0,15,1,16,17,2,14,8],[0,18,1,19,20,2,13,8],[5,10,8,11,13,6,9,7],[1,11,4,17,20,6,15,5],$
$[2,10,4,16,19,7,14,6],[3,15,7,17,18,8,12,6],[3,16,5,19,20,7,18,4]\}$

(4) $3 \in I_8(9,12)$. Let $I = \{[0,9,1,10,11,2,12,3],[0,12,4,13,14,5,9,3],[0,15,1,16,2,17,2,13,5]\}$.

$D_1 = \{[0,18,1,19,20,2,14,6],[1,11,6,17,20,7,9,8],[3,16,8,18,20,4,11,5],[2,10,6,16,19,8,12,7],$
$[3,15,7,17,19,5,10,4],[4,14,8,15,18,7,13,6]\}$

$D_2 = \{[0,18,1,19,20,2,14,7], [1,11,6,17,20,4,15,3], [4,14,6,16,17,8,9,7], [2,10,8,16,19,4,18,3],$
$[5,10,7,11,19,6,12,8], [5,15,7,18,20,8,13,6]\}$
$D_3 = \{[0,18,1,19,20,2,14,8], [1,11,6,17,20,4,16,3], [4,14,7,17,18,6,12,8], [2,10,8,16,19,6,9,7],$
$[3,15,7,18,19,4,10,5], [5,11,8,15,20,7,13,6]\}$
$D_4 = \{[0,18,1,19,20,3,17,6], [1,11,6,14,20,7,9,8], [3,15,8,16,19,5,11,4], [2,10,6,16,19,8,13,7],$
$[2,14,7,18,20,4,10,5], [4,15,7,17,18,6,12,8]\}$
$D_5 = \{[0,18,1,19,20,3,17,7], [1,11,6,14,20,4,10,2], [4,14,7,16,17,8,9,6], [2,16,8,18,19,3,15,4],$
$[5,10,7,15,19,8,12,6], [5,11,7,18,20,8,13,6]\}$
$D_6 = \{[0,18,1,19,20,3,17,8], [1,11,6,14,20,4,16,2], [3,15,8,16,19,6,9,7], [2,10,7,18,19,4,15,5],$
$[4,14,7,17,18,8,12,6], [5,10,8,11,20,6,13,7]\}$
$D_7 = \{[0,18,1,19,20,5,11,6], [1,14,6,17,20,7,9,8], [4,11,8,16,18,7,12,6], [2,10,7,16,19,4,17,3],$
$[2,14,8,18,20,4,15,3], [5,10,6,15,19,8,13,7]\}$
$D_8 = \{[0,18,1,19,20,5,11,7], [1,14,6,17,20,2,10,4], [3,16,7,17,20,6,9,8], [2,16,8,18,19,4,15,3],$
$[4,11,8,14,18,6,12,7], [5,10,7,15,19,8,13,6]\}$

(5) $4 \in I_8(9,12)$. Let $I = \{[0,9,1,10,11,2,12,3], [0,12,4,13,14,5,9,3], [0,15,1,16,2,17,2,13,5], [0,18,1,19,20,2,14,6]\}$.

$D_1 = \{[1,11,6,17,20,7,9,8], [2,10,6,16,19,8,12,7], [3,15,7,17,19,5,10,4],$
$[3,16,8,18,20,4,11,5], [4,14,8,15,18,7,13,6]\}$
$D_2 = \{[1,11,6,17,20,7,12,8], [2,10,6,16,19,8,9,7], [3,15,7,17,19,5,10,4],$
$[3,16,8,18,20,4,11,5], [4,14,8,15,18,7,13,6]\}$
$D_3 = \{[1,11,6,17,20,7,13,8], [2,10,6,16,19,8,9,7], [3,15,7,17,19,5,10,4],$
$[3,16,8,18,20,4,11,5], [4,14,8,15,18,7,12,6]\}$
$D_4 = \{[1,11,6,17,20,8,9,7], [2,10,6,16,19,7,12,8], [3,15,7,16,19,5,11,4],$
$[3,17,8,18,20,4,10,5], [4,14,8,15,18,7,13,6]\}$
$D_5 = \{[1,11,6,17,20,8,12,7], [2,10,6,16,19,7,9,8], [3,15,7,16,19,5,11,4],$
$[3,17,8,18,20,4,10,5], [4,14,8,15,18,7,13,6]\}$
$D_6 = \{[1,11,6,17,20,8,13,7], [2,10,6,16,19,7,9,8], [3,15,7,16,19,5,11,4],$
$[3,17,8,18,20,4,10,5], [4,14,8,15,18,7,12,6]\}$
$D_7 = \{[1,11,7,17,20,4,15,3], [2,10,8,16,19,4,18,3], [4,14,7,16,17,8,9,6],$
$[5,10,6,11,19,7,12,8], [5,15,6,18,20,8,13,7]\}$
$D_8 = \{[1,11,7,17,20,4,16,3], [2,10,6,16,19,7,9,8], [3,15,8,18,19,4,10,5],$
$[4,14,8,17,18,7,12,6], [5,11,6,15,20,8,13,7]\}$

(6) $5 \in I_8(9,12)$. Let $I = \{[0,9,4,13,18,5,11,8], [0,17,1,20,19,2,18,3], [1,10,5,14,19,6,12,8], [2,11,6,15,20,7,9,8], [3,12,4,17,16,7,10,8]\}$.

$D_1 = \{[0,10,2,12,15,3,9,1], [0,11,3,14,16,1,13,2], [4,14,6,16,19,7,13,5], [4,15,7,18,20,5,17,6]\}$
$D_2 = \{[0,10,2,12,15,3,13,1], [0,11,3,14,16,1,9,2], [4,14,6,16,19,7,17,5], [4,15,7,18,20,5,13,6]\}$
$D_3 = \{[0,10,3,11,12,2,9,1], [0,14,3,15,16,2,13,1], [4,14,7,15,16,6,13,5], [4,18,7,19,20,6,17,5]\}$
$D_4 = \{[0,10,3,11,12,2,13,1], [0,14,3,15,16,2,9,1], [4,14,7,15,16,6,17,5], [4,18,7,19,20,6,13,5]\}$
$D_5 = \{[0,10,3,11,16,2,9,1], [0,12,2,14,15,1,13,3], [4,14,7,15,20,6,13,5], [4,16,6,18,19,5,17,7]\}$
$D_6 = \{[0,10,3,11,16,2,13,1], [0,12,2,14,15,1,9,3], [4,14,7,15,20,6,17,5], [4,16,6,18,19,5,13,7]\}$
$D_7 = \{[0,10,3,15,16,2,9,1], [0,11,1,12,14,3,13,2], [4,14,7,19,20,6,13,5], [4,15,5,16,18,7,17,6]\}$
$D_8 = \{[0,10,3,15,16,2,13,1], [0,11,1,12,14,3,9,2], [4,14,7,19,20,6,17,5], [4,15,5,16,18,7,13,6]\}$

(7) $6 \in I_8(9,12)$. Let $I = \{[0,18,1,19,20,2,14,6], [1,11,6,17,20,7,9,8], [2,10,6,16,19,8,12,7], [3,15,7,17,19,5,10,4], [3,16,8,18,20,4,11,5], [4,14,8,15,18,7,13,6]\}$.

$D_1 = \{[0,9,1,10,11,2,12,3], [0,12,4,13,14,5,9,3], [0,15,1,16,17,2,13,5]\}$
$D_2 = \{[0,9,1,10,11,2,13,3], [0,12,4,13,14,3,9,5], [0,15,1,16,17,2,12,5]\}$
$D_3 = \{[0,9,1,10,12,4,13,3], [0,11,3,14,17,2,9,5], [0,13,2,15,16,5,12,1]\}$
$D_4 = \{[0,9,1,15,17,5,12,2], [0,10,3,14,16,1,13,5], [0,11,3,12,13,2,9,4]\}$
$D_5 = \{[0,9,1,15,17,5,13,2], [0,10,3,14,16,1,12,5], [0,11,2,12,13,3,9,4]\}$
$D_6 = \{[0,9,2,11,12,4,13,3], [0,10,3,14,16,1,9,5], [0,13,1,15,17,5,12,2]\}$

$D_7 = \{[0,9,2,15,16,5,12,1],[0,10,3,12,13,1,9,4],[0,11,3,14,17,2,13,5]\}$
$D_8 = \{[0,9,2,15,16,5,13,1],[0,10,1,12,13,3,9,4],[0,11,3,14,17,2,12,5]\}$

(8) $7 \in I_8(9,12)$. Let $I = \{[0,12,4,13,14,8,15,1]$, $[0,15,2,16,17,3,13,5]$, $[0,18,1,19,20,2,17,7]$, $[1,11,8,16,20,5,18,4]$, $[2,10,6,14,19,8,17,4]$, $[3,9,7,16,20,8,13,6]$, $[3,14,7,18,19,5,15,6]\}$.

$D_1 = \{[0,9,1,10,11,2,12,3],[4,9,5,10,11,6,12,7]\}$
$D_2 = \{[0,9,1,10,11,2,12,7],[3,10,4,11,12,5,9,6]\}$
$D_3 = \{[0,9,1,10,11,6,12,3],[2,9,4,11,12,5,10,7]\}$
$D_4 = \{[0,9,1,10,11,6,12,7],[2,9,4,11,12,5,10,3]\}$
$D_5 = \{[0,9,5,10,11,2,12,3],[1,9,4,10,12,6,11,7]\}$
$D_6 = \{[0,9,5,10,11,2,12,7],[1,9,4,10,12,6,11,3]\}$
$D_7 = \{[0,9,5,10,11,6,12,3],[1,9,4,10,12,2,11,7]\}$
$D_8 = \{[0,9,5,10,11,6,12,7],[1,9,4,10,12,2,11,3]\}$

$\boxed{K_{16}}$ Let the vertex set of K_{16} be $\{0,1,\ldots,15\}$.

(1) $0 \in I_{12}(16)$.

$D_1 = \{[0,1,2,3,4,5,6,7],[0,2,4,6,7,5,3,1],[0,5,8,9,10,11,1,4],[0,8,2,11,12,3,9,6],$
$[0,13,1,14,15,2,10,3],[1,9,5,12,15,7,10,14],[3,6,8,11,13,15,12,9],$
$[4,12,7,14,15,10,13,5],[2,7,11,12,14,8,4,13],[6,10,8,13,14,9,15,11]\}$
$D_2 = \{[0,1,2,3,4,5,6,8],[0,2,4,6,7,5,3,9],[0,8,2,11,12,7,10,3],[0,5,8,9,10,11,1,12],$
$[0,13,1,14,15,2,7,3],[1,3,6,10,15,13,11,14],[2,9,15,12,14,13,10,8],$
$[4,7,11,9,13,6,15,5],[1,4,12,6,9,10,5,14],[4,11,12,14,15,8,13,7]\}$
$D_3 = \{[0,1,2,3,4,5,6,9],[0,2,4,6,7,5,3,8],[0,5,8,9,10,11,1,7],[0,8,2,11,12,4,7,3],$
$[0,13,1,14,15,2,9,5],[1,3,6,10,12,13,15,14],[2,10,5,12,14,9,13,11],$
$[3,10,13,14,15,12,6,7],[1,4,14,6,15,10,8,11],[4,11,7,13,15,9,12,8]\}$
$D_4 = \{[0,1,2,3,4,5,6,10],[0,2,4,6,7,5,3,11],[0,5,8,9,10,11,1,13],[0,8,2,11,12,3,7,4],$
$[0,13,2,14,15,3,9,1],[1,3,6,7,10,14,8,12],[2,10,5,12,15,7,14,6],$
$[4,8,7,13,15,10,9,11],[1,4,9,6,12,14,15,13],[5,9,14,13,15,12,11,8]\}$
$D_5 = \{[0,1,2,3,4,5,6,11],[0,2,4,6,7,5,3,10],[0,5,8,9,10,11,1,14],[0,8,2,11,12,3,7,9],$
$[0,13,2,14,15,1,9,3],[1,3,6,7,10,13,12,15],[2,10,4,12,15,5,14,11],$
$[4,7,11,13,15,14,8,6],[1,4,9,6,12,8,13,14],[5,9,10,12,13,15,8,7]\}$
$D_6 = \{[0,1,2,3,4,5,6,12],[0,2,4,6,7,5,3,13],[0,5,8,9,10,11,1,3],[0,8,2,11,12,7,9,13],$
$[0,13,1,14,15,2,7,4],[1,4,10,12,15,8,13,5],[2,10,15,12,14,6,3,11],$
$[3,7,11,8,14,10,9,12],[1,6,15,9,10,8,14,5],[4,9,6,11,13,14,7,15]\}$
$D_7 = \{[0,1,2,3,4,5,6,13],[0,2,4,6,7,5,3,12],[0,5,8,9,10,11,1,6],[0,8,2,11,12,4,7,9],$
$[0,13,1,14,15,2,10,5],[1,3,9,4,15,8,13,11],[3,6,14,7,10,8,12,13],$
$[3,11,8,14,15,6,7,10],[1,7,15,9,12,11,14,2],[4,10,9,14,15,12,5,13]\}$
$D_8 = \{[0,1,2,3,4,5,6,14],[0,2,4,6,7,5,3,15],[0,5,8,9,10,11,1,15],[0,8,2,11,12,3,7,6],$
$[0,13,1,14,15,2,9,11],[1,4,13,6,7,10,3,9],[2,10,8,12,14,9,13,5],$
$[4,7,12,9,15,13,10,5],[1,3,6,10,12,11,8,14],[4,8,7,11,12,15,14,13]\}$
$D_9 = \{[0,1,2,3,4,5,6,15],[0,2,4,6,7,5,3,14],[0,5,8,9,10,11,2,7],[0,8,1,11,12,3,6,7],$
$[0,13,1,14,15,2,9,10],[1,3,7,4,15,13,8,11],[3,9,4,10,11,12,8,6],$
$[4,12,6,13,14,5,9,11],[1,7,13,10,12,15,14,2],[5,10,12,13,15,8,14,9]\}$
$D_{10} = \{[0,1,2,3,4,5,7,6],[0,2,4,7,8,5,3,1],[0,5,6,9,10,11,1,4],[0,6,2,11,12,8,9,3],$
$[0,13,1,14,15,2,10,6],[1,9,5,12,15,7,10,13],[3,7,13,8,10,14,5,15],$
$[3,13,9,14,15,11,12,4],[2,8,10,12,14,11,9,15],[4,8,7,11,13,14,12,6]\}$
$D_{11} = \{[0,1,2,3,4,5,7,8],[0,2,4,6,7,9,1,3],[0,5,2,8,9,3,10,6],[0,10,1,11,12,4,7,6],$
$[0,13,1,14,15,2,6,5],[1,8,5,12,15,9,10,7],[3,4,9,13,14,15,11,7],$
$[4,11,10,13,14,12,15,6],[2,11,3,12,14,13,15,8],[5,9,14,11,13,12,10,8]\}$
$D_{12} = \{[0,1,2,3,4,5,7,9],[0,2,4,6,7,8,1,3],[0,5,2,9,10,3,11,1],[0,8,3,12,13,4,14,1],$
$[0,11,5,14,15,4,10,2],[1,6,10,7,15,5,8,11],[3,4,12,13,15,7,14,8],$
$[6,7,12,11,15,13,9,10],[2,6,8,12,13,14,9,5],[6,9,15,12,13,11,14,10]\}$

(2) $1 \in I_{12}(16)$. Let $I = \{[0,1,2,3,4,5,6,7]\}$.

$$\begin{aligned}
D_1 = \{&[0,2,4,6,7,5,3,1],[0,5,8,9,10,11,1,4],[0,8,2,11,12,3,9,6],\\
&[0,13,1,14,15,2,10,3],[1,9,5,12,15,7,10,14],[2,7,11,12,14,8,4,13],\\
&[3,6,8,11,13,15,12,9],[4,12,7,14,15,10,13,5],[6,10,8,13,14,9,15,11]\}\\
D_2 = \{&[0,2,4,6,7,5,3,8],[0,5,8,9,10,11,1,3],[0,8,2,11,12,4,9,6],\\
&[0,13,1,14,15,2,7,9],[1,4,10,6,9,11,7,13],[1,10,5,12,15,8,14,3],\\
&[2,10,9,12,14,15,5,13],[3,6,14,11,13,15,12,8],[4,13,10,14,15,11,12,7]\}\\
D_3 = \{&[0,2,4,6,7,5,3,9],[0,5,8,9,10,11,1,12],[0,8,2,11,12,3,10,4],\\
&[0,13,1,14,15,2,7,8],[1,3,6,10,15,11,8,13],[1,4,14,6,9,15,3,13],\\
&[2,9,11,12,14,5,13,7],[4,8,10,9,13,12,15,14],[5,10,14,12,15,7,11,6]\}\\
D_4 = \{&[0,2,4,6,7,5,3,10],[0,5,8,9,10,11,1,13],[0,8,2,11,12,3,9,4],\\
&[0,13,2,14,15,4,10,1],[1,3,6,9,12,13,8,10],[1,4,14,6,7,8,15,13],\\
&[2,7,9,12,15,11,14,5],[3,11,8,14,15,6,12,7],[5,9,15,10,13,11,12,14]\}\\
D_5 = \{&[0,2,4,6,7,5,3,11],[0,5,8,9,10,11,1,6],[0,8,2,11,12,3,9,13],\\
&[0,13,1,14,15,2,7,10],[1,3,10,4,15,6,8,11],[1,9,5,10,12,4,14,2],\\
&[3,13,6,14,15,8,12,9],[4,8,14,13,15,7,12,5],[7,9,10,13,14,11,12,15]\}\\
D_6 = \{&[0,2,4,6,7,5,3,12],[0,5,8,9,10,11,1,7],[0,8,2,11,12,4,9,13],\\
&[0,13,1,14,15,2,10,4],[1,3,10,12,15,14,6,8],[1,4,13,6,9,11,8,3],\\
&[2,7,15,12,14,8,10,5],[3,11,14,13,15,6,9,5],[7,11,15,13,14,12,9,10]\}\\
D_7 = \{&[0,2,4,6,7,5,3,13],[0,5,8,9,10,11,1,14],[0,8,2,11,12,3,9,7],\\
&[0,13,2,14,15,1,7,8],[1,3,6,9,10,15,11,12],[1,4,10,6,12,13,5,15],\\
&[2,10,8,12,15,9,11,14],[3,10,13,11,14,7,15,4],[4,8,13,9,12,6,14,5]\}\\
D_8 = \{&[0,2,4,6,7,5,3,14],[0,5,8,9,10,11,1,15],[6,10,7,12,15,14,8,13],\\
&[0,8,2,11,12,3,9,14],[0,13,1,14,15,2,10,5],[1,3,6,9,12,10,13,5],\\
&[1,4,8,6,7,9,12,11],[2,7,15,12,14,13,11,4],[3,11,9,13,15,8,10,4]\}\\
D_9 = \{&[0,2,4,6,7,5,3,15],[0,5,8,9,10,11,2,7],[0,8,1,11,12,3,6,9],\\
&[0,13,1,14,15,2,9,4],[1,3,9,10,12,14,5,15],[1,4,8,7,15,13,6,14],\\
&[2,10,4,12,14,8,11,13],[3,10,6,11,13,5,12,7],[8,12,10,13,15,11,14,9]\}\\
D_{10} = \{&[0,2,4,6,7,5,8,1],[0,5,3,8,9,10,1,11],[0,10,2,11,12,3,9,4],\\
&[0,13,1,14,15,2,12,5],[1,4,13,9,15,3,6,12],[2,7,9,8,14,10,6,15],\\
&[3,11,12,13,14,6,8,7],[4,10,13,14,15,11,5,9],[7,11,14,12,15,13,8,10]\}\\
D_{11} = \{&[0,2,4,6,7,5,8,9],[0,5,3,8,9,10,1,12],[0,10,2,11,12,3,9,5],\\
&[0,13,1,14,15,2,7,11],[1,4,3,6,8,14,13,10],[1,9,13,11,15,14,12,4],\\
&[2,8,6,12,14,7,15,10],[3,11,6,14,15,8,13,5],[4,9,11,10,13,15,12,7]\}\\
D_{12} = \{&[0,2,4,6,7,5,8,10],[0,5,3,8,9,10,1,6],[0,10,2,11,12,3,9,1],\\
&[0,13,1,14,15,2,7,12],[1,4,9,8,15,11,5,14],[2,8,13,12,14,11,10,4],\\
&[3,4,15,6,11,13,5,12],[3,13,6,14,15,9,11,7],[7,8,12,9,13,15,10,14]\}
\end{aligned}$$

(3) $2 \in I_{12}(16)$. Let $I = \{[0,1,2,3,4,5,6,7],[0,2,4,6,7,5,3,1]\}$.

$$\begin{aligned}
D_1 = \{&[0,5,8,9,10,11,1,4],[4,12,7,14,15,10,13,5],[3,6,8,11,13,15,12,9],[1,9,5,12,15,7,10,14],\\
&[2,7,11,12,14,8,4,13],[0,13,1,14,15,2,10,3],[0,8,2,11,12,3,9,6],[6,10,8,13,14,9,15,11]\}\\
D_2 = \{&[0,5,8,9,10,11,1,12],[3,10,8,11,15,13,14,6],[0,13,1,14,15,2,10,4],[1,4,13,9,15,8,6,10],\\
&[2,7,8,12,14,11,13,5],[0,8,2,11,12,3,9,4],[3,6,12,13,14,9,15,7],[5,9,7,10,15,11,12,14]\}\\
D_3 = \{&[0,5,8,9,10,11,1,13],[3,6,13,11,14,12,5,15],[0,13,2,14,15,3,10,1],[1,4,10,9,12,8,6,11],\\
&[2,7,13,12,15,8,14,4],[0,8,2,11,12,3,9,7],[4,9,14,11,13,15,10,8],[5,9,12,10,14,6,15,7]\}\\
D_4 = \{&[0,5,8,9,10,11,1,14],[3,6,9,11,13,15,12,14],[0,13,2,14,15,1,10,3],[1,4,10,9,12,8,13,5],\\
&[2,7,11,12,15,14,8,10],[0,8,2,11,12,3,9,13],[4,9,15,11,12,7,8,6],[4,13,6,14,15,7,10,5]\}\\
D_5 = \{&[2,7,9,12,14,8,11,4],[0,8,2,11,12,3,9,14],[3,6,14,10,13,11,15,8],[1,4,10,9,12,8,6,13],\\
&[0,5,8,9,10,11,1,15],[3,11,7,14,15,12,10,13],[0,13,1,14,15,2,10,5],[4,9,5,13,15,6,12,7]\}\\
D_6 = \{&[0,5,8,9,10,11,2,7],[3,6,11,13,14,12,15,4],[0,13,1,14,15,2,10,6],[1,4,8,12,15,10,14,5],\\
&[2,9,13,12,14,15,7,11],[0,8,1,11,12,3,9,4],[3,10,9,11,15,13,6,8],[5,9,12,10,13,14,7,8]\}\\
D_7 = \{&[2,7,11,9,14,13,15,5],[0,8,1,11,12,3,9,6],[0,13,1,14,15,2,10,7],[1,4,8,12,15,9,10,14],\\
&[0,5,8,9,10,11,2,12],[3,10,5,13,14,4,12,11],[3,6,8,11,15,10,13,4],[6,13,9,14,15,12,7,8]\}
\end{aligned}$$

$D_8 = \{[0,5,8,9,10,11,2,13],[3,6,12,11,14,9,5,13],[4,10,8,13,15,5,14,6],[1,4,8,12,15,11,6,10],$
$[2,7,10,9,12,15,14,4],[0,8,1,11,12,3,9,7],[0,13,1,14,15,3,10,2],[7,8,15,13,14,11,9,12]\}$
$D_9 = \{[4,11,14,12,15,6,13,5],[3,10,4,13,14,5,9,7],[0,8,1,11,12,3,9,13],[1,4,8,12,15,14,6,10],$
$[0,5,8,9,10,11,2,14],[2,7,11,9,12,15,3,6],[0,13,1,14,15,2,10,8],[7,8,13,10,12,11,15,9]\}$
$D_{10} = \{[0,5,8,9,10,11,2,15],[3,6,13,11,14,15,12,8],[0,13,1,14,15,3,10,4],[1,4,9,12,15,8,6,11],$
$[2,7,11,9,10,12,4,13],[2,12,5,13,14,6,10,7],[0,8,1,11,12,3,9,14],[5,9,10,14,15,7,8,13]\}$
$D_{11} = \{[0,5,8,9,10,11,3,6],[4,13,7,14,15,12,11,6],[0,13,1,14,15,2,10,9],[1,4,8,12,15,10,13,3],$
$[2,7,8,11,14,12,6,13],[3,9,5,10,14,11,15,8],[0,8,1,11,12,2,9,4],[5,12,9,13,14,10,7,15]\}$
$D_{12} = \{[0,5,8,9,10,11,3,12],[3,6,15,9,14,13,12,4],[0,13,1,14,15,3,10,5],[1,4,8,12,15,10,6,14],$
$[2,7,15,11,14,8,13,9],[2,10,7,13,15,8,11,4],[0,8,1,11,12,2,9,6],[5,9,7,12,13,10,14,11]\}$

(4) $3 \in I_{12}(16)$. Let $I = \{[0,1,2,3,4,5,6,7],\ [0,2,4,6,7,5,3,1],\ [0,5,8,9,10,11,1,4]\}$.

$D_1 = \{[0,8,2,11,12,3,9,6],[0,13,1,14,15,2,10,3],[1,9,5,12,15,7,10,14],[2,7,11,12,14,8,4,13],$
$[3,6,8,11,13,15,12,9],[4,12,7,14,15,10,13,5],[6,10,8,13,14,9,15,11]\}$
$D_2 = \{[0,8,2,11,12,3,9,7],[0,13,1,14,15,2,10,5],[1,9,5,12,15,6,13,4],[2,7,10,12,14,13,3,6],$
$[3,11,4,14,15,12,8,7],[6,8,13,10,11,14,9,15],[8,10,9,11,15,14,12,13]\}$
$D_3 = \{[0,8,2,11,12,3,9,13],[0,13,1,14,15,2,10,6],[1,9,5,12,15,7,14,4],[2,7,11,12,14,13,4,8],$
$[3,6,13,10,14,9,15,12],[3,11,8,13,15,14,10,5],[6,8,15,11,12,7,10,9]\}$
$D_4 = \{[0,8,2,11,12,3,9,14],[0,13,1,14,15,2,10,7],[1,9,5,12,15,6,13,11],[2,7,9,12,14,8,13,15],$
$[3,6,8,10,13,12,4,14],[3,11,6,14,15,9,10,5],[4,11,7,13,15,8,12,10]\}$
$D_5 = \{[0,8,2,11,12,3,9,15],[0,13,1,14,15,2,10,8],[1,9,5,12,15,6,10,7],[2,7,11,12,14,13,8,4],$
$[3,6,12,10,13,11,14,15],[3,11,9,14,15,4,13,5],[6,8,12,13,14,7,9,10]\}$
$D_6 = \{[0,8,2,11,12,3,10,6],[0,13,1,14,15,2,9,3],[1,10,5,12,15,7,9,11],[2,7,13,12,14,8,10,9],$
$[3,6,14,11,13,9,15,8],[4,8,6,13,14,12,15,5],[4,11,7,12,15,13,14,10]\}$
$D_7 = \{[0,8,2,11,12,3,10,7],[0,13,1,14,15,2,9,5],[1,10,5,12,15,6,13,4],[2,7,9,12,14,15,3,6],$
$[3,11,12,13,14,4,8,7],[6,8,13,9,11,14,10,15],[8,10,9,11,15,12,14,13]\}$
$D_8 = \{[0,8,2,11,12,3,10,9],[0,13,1,14,15,2,9,6],[1,10,6,12,15,13,3,14],[2,7,8,12,14,13,6,11],$
$[3,9,7,11,15,5,12,4],[4,8,11,13,14,10,15,9],[5,10,12,13,14,7,15,8]\}$
$D_9 = \{[0,8,2,11,12,3,10,13],[0,13,1,14,15,2,9,7],[3,9,6,11,13,5,15,4],[1,10,5,12,15,8,14,9],$
$[2,7,11,12,14,13,9,10],[3,6,12,14,15,13,8,11],[4,8,7,12,14,6,10,15]\}$
$D_{10} = \{[0,13,1,14,15,2,12,4],[6,8,13,11,14,7,9,15][3,6,12,11,13,10,8,4],[5,12,9,14,15,7,11,8],$
$[2,7,10,9,14,13,15,3],[1,9,5,10,15,6,13,12],[0,8,2,11,12,3,10,14],\}$
$D_{11} = \{[0,8,2,11,12,3,10,15],[0,13,1,14,15,2,12,5],[4,8,7,11,14,13,9,6],[1,9,5,10,15,3,13,6],$
$[2,7,12,9,14,10,13,15],[3,6,8,11,14,12,10,9],[4,12,11,13,15,8,14,7]\}$
$D_{12} = \{[0,8,2,11,12,3,13,4],[0,13,1,14,15,5,9,2],[1,10,2,12,15,8,7,9],[3,6,9,10,11,12,14,13],$
$[3,9,11,14,15,13,8,4],[5,10,15,12,14,6,11,7],[6,8,12,13,15,14,10,7]\}$

(5) $4 \in I_{12}(16)$. Let $I = \{[0,1,2,3,4,5,6,7],\ [0,2,4,6,7,5,3,1],\ [0,5,8,9,10,11,1,4],$
$[0,8,2,11,12,3,9,6]\}$.

$D_1 = \{[0,13,1,14,15,2,10,3],[4,12,7,14,15,10,13,5],[2,7,11,12,14,8,4,13],$
$[3,6,8,11,13,15,12,9],[1,9,5,12,15,7,10,14],[6,10,8,13,14,9,15,11]\}$
$D_2 = \{[0,13,1,14,15,2,10,5],[1,9,5,12,15,7,13,8],[2,7,11,12,14,10,13,15],$
$[3,6,13,14,15,10,9,11],[3,10,8,11,13,12,14,4],[4,8,7,12,15,6,14,9]\}$
$D_3 = \{[0,13,1,14,15,2,10,6],[1,9,5,12,15,7,10,8],[2,7,13,12,14,11,9,10],$
$[3,6,13,10,14,8,11,15],[3,11,4,13,15,12,14,5],[4,8,7,12,15,13,14,9]\}$
$D_4 = \{[0,13,1,14,15,2,10,7],[1,9,5,12,15,10,13,8],[2,7,13,12,14,9,15,4],$
$[3,6,8,11,15,14,10,12],[3,10,6,13,14,5,15,11],[4,8,7,11,13,14,12,9]\}$
$D_5 = \{[0,13,1,14,15,2,10,8],[1,9,10,12,15,14,3,13],[2,7,9,12,14,10,13,4],$
$[3,6,14,11,15,10,5,9],[4,8,13,11,15,12,14,7],[5,12,7,13,15,11,8,6]\}$
$D_6 = \{[0,13,1,14,15,2,10,9],[1,9,5,12,15,13,14,7],[2,7,11,12,14,10,8,4],$
$[3,6,14,11,13,10,15,4],[3,10,12,14,15,5,13,8],[6,8,7,13,15,12,9,11]\}$
$D_7 = \{[0,13,1,14,15,2,10,12],[1,9,5,12,15,7,14,4],[2,7,11,12,14,13,4,8],$
$[3,6,8,11,15,13,10,14],[3,10,5,13,14,6,15,9],[7,8,15,10,12,13,11,9]\}$

$D_8 = \{[0,13,1,14,15,2,12,4],[1,9,5,10,15,11,12,13],[2,7,9,10,14,12,15,6],$
$[3,6,8,10,14,13,11,7],[3,11,4,13,15,14,8,7],[5,13,9,14,15,8,12,10]\}$
$D_9 = \{[0,13,1,14,15,2,12,5],[1,9,5,10,15,7,13,8],[2,7,12,10,14,8,11,3],$
$[3,6,10,13,15,14,7,11],[4,8,12,14,15,6,13,9],[4,11,9,12,13,14,10,15]\}$
$D_{10} = \{[0,13,1,14,15,2,12,7],[1,9,5,10,15,11,13,6],[2,7,9,10,14,13,15,8],$
$[3,6,8,13,15,14,11,12],[3,10,7,11,14,12,8,4],[4,12,9,13,15,5,14,10]\}$
$D_{11} = \{[0,13,1,14,15,2,12,8],[1,9,5,10,15,7,12,13],[2,7,8,10,14,11,4,15],$
$[3,6,14,10,11,8,13,9],[3,13,4,14,15,11,12,9],[5,13,7,14,15,6,10,12]\}$
$D_{12} = \{[0,13,1,14,15,2,12,9],[1,9,5,10,15,7,13,12],[2,7,8,10,14,12,4,13],$
$[3,6,8,11,15,10,14,4],[3,10,9,13,14,7,11,15],[5,12,11,14,15,8,13,6]\}$

(6) $5 \in I_{12}(16)$. Let $I = \{[0,13,3,14,15,6,10,7]$, $[1,13,8,14,15,11,4,12]$, $[2,7,11,14,15,8,12,13]$, $[3,6,8,11,15,14,10,9]$, $[4,13,9,14,15,10,12,5]\}$

$D_1 = \{[0,1,2,3,4,5,6,7],[0,2,4,6,7,5,3,1],[0,5,8,9,10,11,1,4],$
$[0,8,2,11,12,3,9,6],[1,9,5,10,12,7,13,2]\}$
$D_2 = \{[0,1,2,3,4,5,8,9],[0,2,5,6,10,4,7,1],[0,5,9,7,11,10,2,6],$
$[0,8,1,9,12,3,4,6],[1,3,5,11,12,7,13,2]\}$
$D_3 = \{[0,1,2,3,4,5,13,7],[0,2,9,6,7,5,3,1],[0,5,8,9,10,11,2,4],$
$[0,8,1,11,12,3,4,6],[1,9,5,10,12,7,6,2]\}$
$D_4 = \{[0,1,2,3,4,6,12,7],[0,2,5,7,8,9,4,1],[0,5,6,11,12,3,9,1],$
$[0,6,7,9,10,2,13,5],[1,3,8,5,10,4,2,11]\}$
$D_5 = \{[0,1,2,3,4,7,13,5],[0,2,12,7,10,11,6,5],[0,5,1,6,8,2,4,9],$
$[0,9,5,11,12,3,8,1],[1,3,7,9,10,4,6,2]\}$
$D_6 = \{[0,1,2,3,5,4,6,7],[0,2,5,6,7,4,3,1],[0,4,9,8,11,10,1,5],$
$[0,9,5,10,12,7,13,2],[1,8,2,11,12,3,9,6]\}$
$D_7 = \{[0,1,2,3,5,7,9,8],[0,2,5,6,7,12,1,4],[0,4,10,11,12,3,1,6],$
$[0,8,1,9,10,2,11,5],[2,4,9,6,13,5,3,7]\}$
$D_8 = \{[0,1,2,3,5,8,9,4],[0,2,5,6,10,11,1,4],[0,4,2,8,9,7,12,3],$
$[0,7,5,11,12,1,9,6],[1,3,7,6,10,5,13,2]\}$
$D_9 = \{[0,1,2,3,6,5,8,9],[0,2,9,5,11,6,1,10],[0,4,6,7,8,3,12,1],$
$[0,9,4,10,12,7,5,2],[1,3,7,4,11,5,13,2]\}$
$D_{10} = \{[0,1,2,3,6,5,9,4],[0,2,13,7,10,11,5,4],[0,4,2,5,9,1,12,7],$
$[0,8,1,11,12,3,7,6],[1,3,9,6,10,5,8,2]\}$
$D_{11} = \{[0,1,2,3,6,5,13,7],[0,2,4,5,9,6,1,3],[0,4,9,7,12,3,8,1],$
$[0,8,2,10,11,5,9,1],[2,5,10,11,12,7,4,6]\}$
$D_{12} = \{[0,1,2,3,6,7,5,4],[0,2,13,7,9,6,5,3],[0,4,2,10,12,7,9,1],$
$[0,5,9,8,11,10,4,1],[1,3,8,5,6,12,2,11]\}$

(7) $6 \in I_{12}(16)$. Let $I = \{[0,13,1,14,15,2,10,3]$, $[1,9,5,12,15,7,10,14]$, $[2,7,11,12,14,8,4,13]$, $[3,6,8,11,13,15,12,9]$, $[4,12,7,14,15,10,13,5]$, $[6,10,8,13,14,9,15,11]\}$

$D_1 = \{[0,1,2,3,4,5,6,7],[0,2,4,6,7,5,3,1],[0,5,8,9,10,11,1,4],[0,8,2,11,12,3,9,6]\}$
$D_2 = \{[0,1,2,3,7,5,6,4],[0,2,5,4,6,9,3,1],[0,5,8,9,10,11,2,4],[0,8,1,11,12,3,7,6]\}$
$D_3 = \{[0,1,2,4,5,3,9,6],[0,2,8,9,10,11,1,4],[0,3,2,6,7,4,5,1],[0,8,5,11,12,3,7,6]\}$
$D_4 = \{[0,1,3,5,9,2,4,6],[0,2,5,8,12,6,7,3],[0,3,9,4,11,2,8,1],[0,6,1,7,10,11,5,4]\}$
$D_5 = \{[0,1,3,5,9,6,4,2],[0,2,1,8,12,6,7,3],[0,3,9,4,11,2,8,5],[0,6,5,7,10,11,1,4]\}$
$D_6 = \{[0,1,3,9,10,11,5,4],[0,2,1,7,12,6,4,3],[0,3,8,5,11,2,9,6],[0,4,7,6,8,2,5,1]\}$
$D_7 = \{[0,1,3,9,11,5,7,6],[0,2,8,5,10,11,1,4],[0,3,5,6,7,4,2,1],[0,4,9,8,12,6,2,3]\}$
$D_8 = \{[0,1,4,3,6,5,2,9],[0,2,8,9,10,11,5,4],[0,4,7,5,12,6,1,3],[0,7,3,8,11,6,2,1]\}$
$D_9 = \{[0,1,5,4,6,2,3,7],[0,2,9,3,7,5,8,1],[0,5,6,9,10,11,1,4],[0,8,2,11,12,3,4,6]\}$
$D_{10} = \{[0,1,5,11,12,3,4,6],[0,2,6,7,10,11,1,4],[0,3,7,5,9,8,1,2],[0,4,9,6,8,2,3,5]\}$
$D_{11} = \{[0,1,6,4,9,3,5,2],[0,2,8,3,10,11,5,4],[0,5,7,6,8,1,4,9],[0,7,1,11,12,3,2,6]\}$
$D_{12} = \{[0,1,7,3,9,2,5,8],[0,2,6,7,10,11,5,4],[0,4,9,6,8,2,3,1],[0,5,1,11,12,3,4,6]\}$

(8) $7 \in I_{12}(16)$. Let $I = \{[0,5,8,9,10,11,1,4], [0,8,2,12,13,4,14,1], [0,11,3,14,15,4,12,5], [1,9,5,10,15,7,13,6], [2,7,12,13,15,14,6,3], [8,10,9,11,12,14,13,15], [8,13,11,14,15,10,12,9]\}$

$D_1 = \{[0,1,2,3,4,5,6,7],[0,2,4,6,7,5,3,8],[1,3,9,6,7,10,2,11]\}$
$D_2 = \{[0,1,2,3,4,6,11,7],[0,2,9,6,7,5,3,8],[1,3,4,5,7,10,2,6]\}$
$D_3 = \{[0,1,2,3,6,7,4,5],[0,2,6,4,7,5,1,3],[2,9,3,10,11,6,8,7]\}$
$D_4 = \{[0,1,2,3,7,5,6,8],[0,2,5,4,6,9,3,7],[1,3,4,6,7,10,2,11]\}$
$D_5 = \{[0,1,2,4,7,3,9,6],[0,2,10,3,6,11,7,5],[1,5,2,6,7,4,3,8]\}$
$D_6 = \{[0,1,3,2,4,5,7,6],[0,3,5,6,7,4,2,11],[1,2,9,6,7,10,3,8]\}$
$D_7 = \{[0,1,3,2,4,7,8,6],[0,3,5,6,7,10,2,11],[1,2,4,5,6,9,3,7]\}$
$D_8 = \{[0,1,3,2,6,5,7,11],[0,3,5,4,7,10,2,6],[1,2,9,6,7,4,3,8]\}$
$D_9 = \{[0,1,3,4,6,2,10,7],[0,2,9,3,7,5,6,8],[1,5,4,6,7,3,2,11]\}$
$D_{10} = \{[0,1,5,2,3,7,6,9],[0,4,2,6,7,5,3,8],[1,2,10,3,6,11,7,4]\}$
$D_{11} = \{[0,1,5,3,7,6,2,10],[0,2,3,4,6,11,7,5],[1,2,9,3,7,4,6,8]\}$
$D_{12} = \{[0,1,5,4,6,2,3,7],[0,2,4,3,7,11,6,1],[2,5,6,9,10,7,8,3]\}$

(9) $8 \in I_{12}(16)$. Let $I = \{[0,5,8,9,10,11,1,4], [0,8,2,11,12,3,9,6], [0,13,1,14,15,2,10,3], [1,9,5,12,15,7,10,14], [2,7,11,12,14,8,4,13], [3,6,8,11,13,15,12,9], [4,12,7,14,15,10,13,5], [6,10,8,13,14,9,15,11]\}$.

$D_1 = \{[0,1,2,3,4,5,6,7],[0,2,4,6,7,5,3,1]\}$ $D_2 = \{[0,1,2,3,6,7,4,5],[0,2,6,4,7,5,1,3]\}$
$D_3 = \{[0,1,3,2,4,5,7,6],[0,3,5,6,7,4,2,1]\}$ $D_4 = \{[0,1,3,2,7,6,4,5],[0,3,7,4,6,5,1,2]\}$
$D_5 = \{[0,1,5,4,6,2,3,7],[0,2,4,3,7,5,6,1]\}$ $D_6 = \{[0,1,5,4,6,7,3,2],[0,2,1,3,7,5,6,4]\}$
$D_7 = \{[0,1,5,4,7,3,2,6],[0,2,4,3,6,1,7,5]\}$ $D_8 = \{[0,1,5,4,7,6,2,3],[0,2,1,3,6,4,7,5]\}$
$D_9 = \{[0,1,6,2,4,5,7,3],[0,3,5,6,7,1,2,4]\}$ $D_{10} = \{[0,1,6,2,7,3,4,5],[0,3,2,4,6,5,1,7]\}$
$D_{11} = \{[0,1,7,3,4,5,6,2],[0,2,1,6,7,5,3,4]\}$ $D_{12} = \{[0,1,7,3,6,2,4,5],[0,2,3,4,7,5,1,6]\}$

$\boxed{K_{13} \setminus K_4}$ Let the vertex set of $K_{13} \setminus K_4$ be $\{0,1,2,3\} \cup \{4,5,\ldots,12\}$, with the hole on the vertices $\{0,1,2,3\}$. Then $0 \in I_{12}(K_{13} \setminus K_4)$.

$D_1 = \{[0,4,1,5,6,2,7,3],[0,7,4,8,9,5,6,10],[1,8,11,9,10,12,4,3],[0,10,2,11,12,5,8,6],$
$[1,6,7,11,12,9,8,3],[2,5,4,9,12,11,10,7]\}$
$D_2 = \{[0,4,1,5,6,2,7,8],[0,7,3,8,9,4,5,10],[0,10,6,11,12,7,5,2],$
$[1,6,7,9,12,3,11,5],[1,8,4,10,11,9,6,12],[2,8,12,9,10,11,4,3]\}$
$D_3 = \{[0,4,1,5,6,2,7,9],[0,7,3,8,9,4,5,11],[0,10,6,11,12,2,5,7],$
$[1,6,7,8,12,4,10,5],[1,9,8,10,11,3,6,12],[2,8,12,9,11,4,3,10]\}$
$D_4 = \{[0,4,1,5,6,2,7,10],[0,7,3,8,9,4,11,5],[0,10,8,11,12,3,6,7],$
$[1,6,12,10,11,9,8,2],[1,8,7,9,12,4,5,2],[3,4,6,5,9,10,11,12]\}$
$D_5 = \{[0,4,1,5,6,2,7,11],[0,7,3,8,9,4,10,5],[0,10,9,11,12,7,6,4],$
$[1,6,8,10,11,12,9,2],[1,8,7,9,12,2,5,3],[3,4,5,6,11,8,12,10]\}$
$D_6 = \{[0,4,1,5,6,2,7,12],[0,7,3,8,9,4,10,6],[0,10,7,11,12,8,5,3],$
$[1,6,7,9,11,4,8,2],[1,8,9,10,12,11,5,2],[3,4,5,6,9,12,10,11]\}$
$D_7 = \{[0,4,1,5,6,2,8,3],[0,7,1,9,10,4,6,5],[0,8,4,11,12,7,3,10],$
$[1,10,7,11,12,9,2,5],[2,10,8,11,12,6,9,3],[4,5,7,9,12,8,6,11]\}$
$D_8 = \{[0,4,1,5,6,2,8,7],[0,7,1,9,10,3,6,4],[0,8,4,11,12,5,3,9],$
$[1,10,5,11,12,7,9,8],[2,5,4,7,10,6,12,11],[2,9,6,11,12,10,8,3]\}$
$D_9 = \{[0,4,1,5,6,2,8,9],[0,7,1,9,10,3,6,11],[0,8,4,11,12,5,10,1],$
$[2,5,3,9,12,6,8,7],[2,7,6,10,11,5,4,12],[3,4,7,11,12,9,10,8]\}$
$D_{10} = \{[0,4,1,5,6,2,8,10],[0,7,1,9,10,3,6,12],[0,8,7,11,12,4,5,3],$
$[1,10,9,11,12,7,6,5],[2,5,9,7,11,8,3,4],[2,9,4,10,12,8,6,11]\}$
$D_{11} = \{[0,4,1,5,6,2,8,11],[0,7,1,9,10,3,6,5],[0,8,4,11,12,6,7,2],$
$[1,10,7,11,12,9,8,3],[2,5,3,9,10,12,4,6],[4,5,7,9,10,8,12,11]\}$
$D_{12} = \{[0,4,1,5,6,2,8,12],[0,7,1,9,10,3,6,8],[0,8,5,11,12,4,6,9],$
$[1,10,4,11,12,7,5,2],[2,7,4,9,10,11,3,5],[3,8,7,9,12,11,6,10]\}$

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