A Fan-Type Condition for Hamiltonian Graphs*

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Abstract

For two non-adjacent vertices u, v in a graph G, we use $\alpha(u, v)$ to denote the maximum cardinality of an independent vertex set of G containing both u and v. In this paper, we prove that if G is a 2-connected graph of order n and $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices u, v of G with $1 \leq |N(u) \cap N(v)| < \alpha(u, v) - 1$, then either G is Hamiltonian or else G belongs to a family of exceptional graphs.

1. Introduction

We use [2] for notation and terminology not defined here and consider simple graphs only.

Let G be a graph and u, v be two vertices of G. The distance between u and v in G is denoted by $d(u, v)$, the maximum cardinality of an independent vertex set of G containing both u and v by $\alpha(u, v)$, and the number of vertices in a maximum independent set of G by $\alpha(G)$. We denote the neighborhood and the degree of vertex v in G by $N(v)$ and $d(v)$, respectively. For two subgraphs H and F of G, we define

$$N(H) = \cup_{v \in V(H)} N(v)$$

and

$$N_F(H) = N(H) \cap V(F).$$

If C is a cycle of G, we denote by $\vec{C}$ the cycle with a given orientation. If $u, v \in V(C)$, then $u\,\vec{C}\,v$ denotes the subpath of C on $\vec{C}$ from u to v. The same vertices, in reverse order, are given by $v\,\overleftarrow{C}\,u$. For $S \subseteq V(X)$, we use S^+ (resp. S^-) to denote the successors (resp. predecessors) of vertices of S on $\vec{X}$. Let uHv denote a u–v path in which all internal vertices belong to H.

The development of the theory of Hamiltonian graphs has seen a series of results based on controlling the degrees of the vertices of G. A classical result is due to Ore [8].

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Theorem 1 (Ore [8]). *Let G be a graph of order n in which $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices u, v. Then G is Hamiltonian.*

Many generalizations of Ore's Theorem have been found. The strongest known result of this type is the Closure Theorem of Bondy and Chvátal [1]. Define the $k-closure$ of G to be the graph obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least k, until no such pair remains. Their main result for Hamiltonian graphs is the following:

Theorem 2 (Bondy and Chvátal [1]). *A graph G of order n is Hamiltonian if and only if its $n-closure$ is Hamiltonian.*

Another very interesting approach was introduced by Fan [3] in 1984.

Theorem 3 (Fan [3]). *Let G be a 2-connected graph of order n in which $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of vertices u, v with $d(u, v) = 2$ in G. Then G is Hamiltonian.*

Fan's Theorem implies that we need not consider all pairs of nonadjacent vertices, but only a particular subset of these pairs. This idea was used were to provide some generalizations of Fan's Theorem.

Theorem 4 (Chen [5], Song [9]). *Let G be a 2-connected graph of order n in which $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices u, v with $1 \leq |N(u) \cap N(v)| < \alpha$. Then G is Hamiltonian.*

Theorem 5 (Wang and Chu [11]). *Let G be a 2-connected graph of order n in which $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices u, v with $1 \leq |N(u) \cap N(v)| < \alpha(u, v)$. Then G is Hamiltonian.*

In this paper, we shall prove the following theorem.

Theorem 6. *Let G be a 2-connected graph of order n in which $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for every pair of nonadjacent vertices u, v with $1 \leq |N(u) \cap N(v)| < \alpha(u, v) - 1$. Then either G is Hamiltonian or G is a spanning subgraph of the nonhamiltonian graph $(\bigcup_{i=0}^{k} K_{n_i}) \vee K_k$.*

Remarks. Since $d(u, v) = 2$ if and only if $|N(u) \cap N(v)| \geq 1$ for a pair of nonadjacent vertices u, v of G, and $\alpha(G) = max\{\alpha(u, v) | u, v \in V(G) \text{ and } uv \notin E(G)\}$, Theorems 1 and 3-5 are immediate consequences of Theorem 6. In fact, there are many Hamiltonian graphs showing that Theorem 6 is stronger than Theorems 1 and 3-5. One of these is constructed as follows: For two integers m and n with $m \geq 3$ and $n \geq 4(m+3)$, define the graph $G_{m,n}$ obtained from six vertex disjoint graphs $H_1, H_2, \cdots, H_6$ with $H_1 \cong \bar{K}_m, H_2 \cong K_{m+2}, H_3 \cong K_3, H_4 \cong \bar{K}_2, H_5 \cong K_{m+2}$ and $H_6 \cong K_{n-3m-9}$ such that every vertex in H_i is joined to all vertices of H_{i+1} for $i = 1, 2, \cdots, 5$. It is easy to check that $G_{m,n}$ satisfies the conditions of Theorem 6, but not that of Theorems 1 and 3-5. Moreover, $G_{m,n}$ does not satisfy also the following sufficient conditions for a 2-connected graph to be Hamiltonian.

(I) (Chen [6]). $uv \notin E(G) \Rightarrow 2|N(u) \cup N(v)| + d(u) + d(v) \geq 2n - 1$

(II)(Flandrin, Jung and Li [4]). $uv \notin E(G) \Rightarrow |N(u) \cup N(v)| + max\{d(u), d(v)\} \geq n$

(III)(Flandrin, Jung and Li [4]). For every independent set $\{u, v, w\}$ of G, $d(u) + d(v) + d(w) \geq n + |N(u) \cap N(v) \cap N(w)|$

(IV)(Jackson [7]). $uv \notin E(G) \Rightarrow |N(u) \cup N(v)| \geq \frac{n}{2}$ and G does not belong to one of three families of exceptional graphs.

2. The proof of Theorem 6

Suppose that $G = (V, E)$ is a nonhamiltonian graph with maximal number of edges, which satisfies the hypothesis of Theorem 6. Set $B = \{v \in V(G) \mid d(v) \geq \frac{n}{2}\}$. By Theorem 2, the induced subgraph G[B] is clique. Thus we may let C be a longest cycle of G containing all vertices of B and let H be a component of $G - V(C)$ with largest number on $N(H) \cap C$. Let $v_1, v_2, \cdots, v_k$ be the elements of $N_C(H)$ occurring on $\overrightarrow{C}$ in consecutive order and let $x_i \in N(v_i) \cap V(H)$ for $i = 1, 2, \cdots, k$. Since G is 2-connected, $k \geq 2$. Note that, for any $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$, the path

$$v_i^+ \overrightarrow{C} v_j H v_i \overleftarrow{C} v_j^+$$

contains all vertices of C and contains at least one vertex of H. By the maximality of C, we conclude that $v_i^+ v_j^+ \notin E(G)$. Thus it follows from the definition of $N_C^+(H)$ that

(2.1) For any i with $1 \leq i \leq k$, $\{x_i\} \cup N_C^+(H)$ is an independent set.

Since $G[B]$ is a clique, $|N_C^+(H) \cap B| \leq 1$. Without loss of generality, we assume that $d(v_i^+) < \frac{n}{2}$ for $i = 1, 2, \cdots, k-1$. Note that $v_i \in N(v_i^+) \cap N(x_i) \subseteq \{v_1, v_2, \cdots, v_k\}$ and $\max\{d(x_i), d(v_i^+)\} < \frac{n}{2}$ for $i = 1, 2, \cdots, k-1$. Hence, the following two statements hold by the assumption of the theorem and (2.1).

(2.2) $\alpha(x_i, v_i^+) = k + 1$ *and* $N(x_i) \cap N(v_i^+) = \{v_1, v_2, \cdots, v_k\}$, $i = 1, 2, \cdots, k-1$.

(2.3) *If* $d(v_k^+) < \dfrac{n}{2}$, *then* $\alpha(x_k, v_k^+) = k + 1$ *and* $N(x_k) \cap N(v_k^+) = \{v_1, v_2, \cdots, v_k\}$.

For $i \neq j$, set $R = v_i^+ \overrightarrow{C} v_j^+$ and $S = V(C) - R$. Then we conclude that

(2.4) $$N_R^-(v_i^+) \cap N_R(v_j^+) = \emptyset.$$

To prove (2.4) suppose $v \in N_R^-(v_i^+) \cap N_R(v_j^+)$. By (2.1), $v \neq v_i^+$ and $v \neq v_j$. Hence we see that

$$v_i H v_j \overleftarrow{C} v^+ v_i^+ \overrightarrow{C} v v_j^+ \overrightarrow{C} v_i$$

is a cycle longer than C. This contradiction shows (2.4).

An analogous argument proves that the following statement also holds.

(2.5) $$N_S^+(v_i^+) \cap N_S(v_j^+) = \emptyset.$$

Now, we shall give a characterization of G by showing the following four statements (2.6)-(2.9). Set $V_0 = V(H)$ and $V_i = v_i^+ \overrightarrow{C} v_{i+1}^-$ for $i = 1, 2, \cdots, k$, (indices taken modulo k).

(2.6) For each $i = 1, 2, \cdots, k$, $V_i \setminus \{v_i^+\} \subset N(v_i^+)$.

If $|V_i| \leq 2$, then we are done. So assume that $|V_i| \geq 3$. If $i \neq k$, then by (2.2), $v_i^+ v_{i+1} \in E(G)$ and hence (2.4) implies $v_{i+1}^- \notin N(v_j^+)$ for every $j \neq i$ with $1 \leq j \leq k$. Thus, by using (2.1) and (2.2), we obtain $v_i^+ v_{i+1}^- \in E(G)$ for otherwise $\{x_i, v_{i+1}^-\} \cup N_C^+(H)$ would be an independent set containing $\{x_i, v_i^+\}$ of cardinality $k+2$ since $v_i^{++} \neq v_{i+1}^-$. This contradicts (2.2). By continuing the process if $|V_i| > 3$, the conclusion follows. Now, the proof only for the case $i = k$ remains. If $d(v_k^+) < \frac{n}{2}$, then, by an argument analogous to one above, we have finished. So assume that $d(v_k^+) \geq \frac{n}{2}$. Then we conclude that $v_1^- v_k^+ \in E(G)$. When $d(v_1^-) \geq \frac{n}{2}$ this follows since $\{v_1^-, v_k^+\} \in B$. If $d(v_1^-) < \frac{n}{2}$, then by symmetry of C we also obtain that $V_k \setminus \{v_1^-\} \subset N(v_1^-)$. Hence, using (2.4) we have

$$N(v_1^{--}) \cap \{x_i, v_1^+, v_2^+, \cdots, v_{k-1}^+\} = \emptyset$$

and thus $v_1^{--} v_k^+ \in E(G)$ if $|V_k| > 3$ since otherwise $\{x_1, v_1^{--}\} \cup N_C^+(H)$ would be an independent set containing $\{x_1, v_1^+\}$ of cardinality $k+2$, which contradicts (2.2). By continuing the same process, it follows that $V_k \setminus \{v_k^+\} \subset N(v_k^+)$. Thus, (2.6) is verified.

By symmetry, it follows immediately that

(2.7) For each $i = 1, 2, \cdots, k$, $V_i \setminus \{v_{i+1}^-\} \subset N(v_{i+1}^-)$.

(2.8) For any $i, j \in \{0, 1, \cdots, k\}$ with $i \neq j$, $N(V_i) \cap V_j = \emptyset$.

Otherwise, assume there exists two vertices $u \in V_i$ and $v \in V_j$ with $i \neq j$ such that $uv \in E(G)$. By the assumption, we have $i, j \neq 0$. Without loss of generality, we suppose that $i \neq k$. It is easy to see that

$$C' = \begin{cases} uv \overleftarrow{C} v_j^+ v^+ \overrightarrow{C} v_i H v_j \overleftarrow{C} u^+ v_i^+ \overrightarrow{C} u & \text{if } v \neq v_{j+1}^- \\ uv \overleftarrow{C} v_{i+1} H v_{j+1} \overrightarrow{C} u^- v_{i+1}^- \overleftarrow{C} u & \text{if } v = v_{j+1}^- \end{cases}$$

is a cycle longer than C. This contradiction shows (2.8).

(2.9). If $H' \neq H$ is another component of $G - V(C)$ and $N(H') \cap V_i \neq \emptyset$ for $i \in \{1, 2, \cdots, k\}$, then $N(H') \subset V_i \cup N(H)$.

Suppose the contrary. We first conclude that

$$|N_C(H') \cap N_C^+(H)| = 1 \text{ and } |N_C(H') \cap N_C^-(H)| = 1.$$

If $N(H') \cap N_C^+(H) = \emptyset$ or $N(H') \cap N_C^-(H) = \emptyset$, then $\{x_1, y\} \cup N_C^+(H)$ or $\{x_1, y\} \cup N_C^-(H), y \in V(H')$ would be an independent set containing $\{x_1, v_1^+\}$ or $\{x_1, v_1^-\}$ of cardinality $k+2$ contradicting (2.2). If there exist two vertices $v_i^+, v_j^+ \in N_C(H') \cap N_C^+(H)$ with $i \neq j$, then the cycle

$$v_i^+ H' v_j^+ \overrightarrow{C} v_i H v_j \overleftarrow{C} v_i^+$$

is longer than C, a contradiction. Similarly, $|N_C(H') \cap N_C^-(H)| = 1$.

Without loss of generality, assume that $v_i^+, v_j^- \in N_C(H')$ with $1 \leq i, j \leq k$. Then we claim $j = i+1$. Otherwise we see that the paths

$$v_{i+1}^+ \overrightarrow{C} v_j^- H' v_i^+ \overrightarrow{C} v_{i+1} H v_i \overleftarrow{C} v_j$$

and

$$v_{i+1}^- \overleftarrow{C} v_i^+ H' v_j^- \overleftarrow{C} v_{i+1} H v_i \overleftarrow{C} v_j$$

contain all vertices of C and contain at least one vertex of H. Hence it follows from the maximality of C that $v_{i+1}^+, v_{i+1}^- \notin N(v_j)$ and therefore we have $d(v_{i+1}^+) \geq \frac{n}{2}$ and $d(v_{i+1}^-) \geq \frac{n}{2}$, which imply $v_{i+1}^+ v_{i+1}^- \in E(G)$ contradicting (2.8).

If there exists some $s \neq i$ such that $v \in N(H') \cap V_s \neq \emptyset$, then we see easily that

$$v_i^+ H' v \overleftarrow{C} v_s^+ v^+ \overrightarrow{C} v_i H v_s \overleftarrow{C} v_i^+$$

is longer than C, a contradiction. Thus (2.9) is proved.

By combining the statements (2.6) through (2.9), it is easily seen that G is a spanning subgraph of the nonhamiltonian graph $(\bigcup_{i=0}^{k} K_{n_i}) \vee K_k$. The proof of Theorem 6 is complete. ■

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