

Chapter 2

Toward a Fully Relativistic Theory of Quantum Information

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Information theory is a statistical theory dealing with the relative state of detectors and physical systems. Because of this physicality of information, the classical framework of Shannon needs to be extended to deal with quantum detectors, perhaps moving at relativistic speeds, or even within curved space-time. Considerable progress toward such a theory has been achieved in the last fifteen years, while much is still not understood. This review recapitulates some milestones along this road, and speculates about future ones.

1. Preface: From Nuclei to Quantum Information

I am sure I am one of the more junior contributors to this volume celebrating Gerry Brown's 85th birthday, and still I've known him for 25 years. I arrived as a young graduate student at Stony Brook University in 1986, and Gerry immediately introduced me to every member of his Nuclear Theory group, ending with his postdoc Ismail Zahed. He pointed to a chair in Ismail's office, said: "You guys talk", and left. I started to work with Ismail that day, and when he was promoted to Assistant Professor I became his first graduate student. Gerry and I only started to work together closely within the last two years of my Ph.D., and the collaboration intensified when he took me on his yearly Spring visits to the Kellogg Radiation Laboratory at the California Institute of Technology. There, I had the opportunity to meet Hans Bethe, who visited Caltech every Spring to work with Gerry. Over the following years, Hans and I became good

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friends and Hans's influence on my growth as a scientist would end up rivaling the influence that Gerry had on me [1]. In particular Hans was always very interested in my shifting interests from nuclear and high energy theory first towards quantum information theory and the foundations of quantum mechanics, and then to theoretical biology. At the same time, Gerry and Hans's collaboration on the physics of binary stars and in particular black holes continued to intrigue me. I ended up staying at Caltech for 12 years.

While I spend most of my time now working in biology, I still sometimes return to work in physics. People like Gerry and Hans have reinforced to me the fun that comes with attempting to understand the universe's basic principles, and when lucky enough, unravel a few of them. Perhaps it is not a coincidence that one of the striking applications of quantum relativistic information theory that I describe below is to the physics of black holes. Gerry and I discussed black holes and binary stars endlessly on walks in the mountains adjacent to Caltech, and on the phone (often on Sunday mornings) when he was back in New York. Why did I store away article after article on black holes in the 1990s when I wasn't nearly working on the subject? I am sure it was Gerry's influence, who taught me to go after your gut instinct, and not worry if you are called crazy. I've been called crazy in many a referee's review, and I've come to realize that this usually signals that I am on to something. Thus I dedicate this article to you Gerry: there are crazy things buried in here too.

2. Entropy and Information: Classical Theory

Since Shannon's historical pair of papers [2], information theory has changed from an engineering discipline to a full-fledged theory within physics [3]. While a considerable part of Shannon's theory deals with communication channels and codes [4], the concepts of entropy and information he introduced are crucial to our understanding of the physics of measurement, and turn out to be more general than thermodynamical entropy. Thus, information theory represents an important part of statistical physics both at equilibrium and away from it.

In the following, I present an overview of some crucial aspects of entropy and information in classical and quantum physics, with extensions to the special and general theory of relativity. While not exhaustive, the treatment is at an introductory level, with pointers to the technical literature where appropriate.

2.1. Entropy

The concepts of entropy and information are the central constructs of Shannon's theory. They quantify the ability of observers to make *predictions*, in particular how well an observer equipped with a specific measurement apparatus can make predictions about another physical system. Shannon entropies (also known as *uncertainties*) are defined for mathematical objects called *random variables*. A random variable X is a mathematical object that can take on a finite number of discrete states x_i , where $i = 1, \dots, N$ with probabilities p_i . Now, physical systems are not mathematical objects, nor are their states necessarily discrete. However, if we want to quantify our uncertainty about the state of a physical system, then in reality we only need to quantify our uncertainty about the *possible outcomes of a measurement* of that system. In other words, an observer's maximal uncertainty about a system is not a property of the system, but rather a property of the measurement device with which the observer is about to examine the system. For example, suppose I am armed with a measurement device that is simply a "presence-detector". Then the maximal uncertainty I have about the physical system under consideration is 1 bit, which is the amount of *potential information* I can obtain about that system, given this measurement device.

As a consequence, in information theory the entropy of a physical system is undefined if we do not specify the device that we are going to use to reduce that entropy. A standard example for a random variable (that is also a physical system) is the six-sided even die. Usually, the maximal entropy attributed to this system is $\log_2(6)$ bits. Is this all there is to know about this system? What if we are interested not only in the face of the die that is up, but also the angle that the die has made with respect to due North? Further, since the die is physical, it is made of molecules and these can be in different states depending on the temperature of the system. Are those knowable? What about the state of the atoms making up the molecules? All these could conceivably provide labels such that the number of states to describe the die is in reality much larger. What about the state of the nuclei? Or the quarks and gluons inside those?

This type of thinking makes it clear that we cannot speak about the entropy of an isolated system without reference to the coarse-graining of states that is implied by the choice of detector (but I will comment on the continuous variable limit of entropies below). So, even though detectors exist that record continuous variables (such as, say, a mercury thermometer), each detector has a finite resolution such that it is indeed appropriate to consider

only the *discrete* version of the Shannon entropy, which is given in terms of the probabilities p_i as^a

$$H(X) = - \sum_i^N p_i \log p_i . \quad (1)$$

For any physical system, how are those probabilities obtained? In principle, this can be done both by experiment and by theory. Once I have defined the N possible states of my system by choosing a detector for it, the *a priori* maximal entropy is defined as

$$H_{\max} = \log N . \quad (2)$$

Experiments using my detector can now sharpen my knowledge of the system. By tabulating the frequency with which each of the N states appears, we can estimate the probabilities p_i . Note, however, that this is a biased estimate that approaches the true entropy Eq. (1) only in the limit of an infinite number of trials [5]. On the other hand, some of the possible states of the system (or more precisely, possible states of my detector interacting with the system) can be eliminated by using some knowledge of the physics of the system. For example, we may know some initial data or averages that characterize the system. This becomes clear in particular if the degrees of freedom that we choose to characterize the system with are position, momentum, and energy, i.e. if we consider the *thermodynamical entropy* of the system (see below). In this respect it is instructive to consider for a moment the continuous variable equivalent of the Shannon entropy, also known as the *differential entropy*, defined as [4]

$$h(X) = - \int_S f(x) \log f(x) dx , \quad (3)$$

with a probability density function $f(x)$ with support S . It turns out that while $h(X)$ is invariant with respect to translations [$h(X+c) = h(X)$], it is not invariant under arbitrary coordinate transformations: the entropy is *renormalized* under such changes instead. For example,

$$h(cX) = h(X) + \log |c| . \quad (4)$$

In particular, this implies that if we introduce a discretization of continuous space (e.g. via $p_i = \int_{i\Delta}^{(i+1)\Delta} f(x) dx$) and consider the limit of the discretized

^aFrom now on, I shall not indicate the basis of the logarithm, which only serves to set the units of entropy and information (base 2, e.g. sets the unit to a “bit”).

version as $\Delta \rightarrow 0$, we find that

$$H_{\Delta}[p_i] = - \sum_i p_i \log p_i \xrightarrow{\Delta \rightarrow 0} h(X) - \log \Delta. \quad (5)$$

Thus, as the resolution of a measurement device is increased, the entropy is renormalized via an infinite term. Of course, we are used to such infinite renormalizations from quantum field theory, and just as in the field theory case, the “unphysical” renormalization is due to an unphysical assumption about the attainable precision of measurements. Just as in quantum field theory, differences of quantities do make sense: the shared (or mutual) differential entropy is finite in the limit $\Delta \rightarrow 0$ as the infinities cancel.

2.2. Conditional entropy

Let us look at the basic process that reduces uncertainty: a measurement. When measuring the state of system X , I need to bring it into contact with a system Y . If Y is my measurement device, then usually I can consider it to be completely known (at least, it is completely known with respect to the degrees of freedom I *care* about). In other words, my device is in a particular state y_0 with certainty. After interacting with X , this is not the case anymore. Let us imagine an interaction between the systems X and Y that is such that

$$x_i y_0 \rightarrow x_i y_i \quad i = 1, \dots, N, \quad (6)$$

that is, the states of the measurement device y_i end up reflecting the states of X . This is a perfect measurement, since no state of X remains unresolved. More generally, let X have N states while Y has M states, and let us suppose that $M < N$. Then we can imagine that each state of Y reflects an *average* of a number of X 's states, so that the probability to find Y in state y_j is given by q_j , where $q_j = \sum_i p_{ij}$, and p_{ij} is the joint probability to find X in state x_i and Y in state y_j . The measurement process then proceeds as

$$x_i y_0 \rightarrow \langle x \rangle_j y_j \quad (7)$$

where

$$\langle x \rangle_j = \sum_i p_{i|j} x_i. \quad (8)$$

In Eq. (8) above, I introduced the *conditional probability*

$$p_{i|j} = \frac{p_{ij}}{q_j} \quad (9)$$

that X is in state i given that Y is in state j . In the perfect measurement above, this probability was 1 if $i = j$ and 0 otherwise (i.e. $p_{i|j} = \delta_{ij}$), but in the imperfect measurement, X is distributed across some of its states i with a probability distribution $p_{i|j}$, for each j .

We can then calculate the *conditional entropy* (or remaining entropy) of the system X given we found Y in a particular state y_j after the measurement:

$$H(X|Y = y_j) = - \sum_i^N p_{i|j} \log p_{i|j}. \quad (10)$$

This remaining entropy is guaranteed to be smaller than or equal to the unconditional entropy $H(X)$, because the worst case scenario is that Y doesn't resolve *any* states of X , in which case $p_{i|j} = p_i$. But since we didn't know anything about X to begin with, $p_i = 1/N$, and thus $H(X|Y = y_j) \leq \log N$.

Let us imagine that we did learn something from the measurement of X using Y , and let us imagine furthermore that this knowledge is permanent. Then we can express our new-found knowledge about X by saying that we know the probability distribution of X , p_i , and this distribution is *not* the uniform distribution $p_i = 1/N$. Of course, in principle we should say that this is a conditional probability $p_{i|j}$, but if the knowledge we have obtained is permanent, there is no need to constantly remind ourselves that the probability distribution is conditional on our knowledge of certain other variables connected with X . We simply say that X is distributed according to p_i , and the entropy of X is

$$H_{\text{actual}}(X) = - \sum_i \log p_i \log p_i. \quad (11)$$

According to this strict view, all Shannon entropies of the form (11) are conditional if they are not maximal. And we can quantify our knowledge about X simply by subtracting this uncertainty from the maximal one:

$$I = H_{\text{max}}(X) - H_{\text{actual}}(X). \quad (12)$$

This knowledge, of course, is *information*. We can see from this expression that the entropy H_{max} can be seen as *potential information*: it quantifies how much is knowable about this system. If my actual entropy vanishes, then all of the potential information is realized.

2.3. Information

In Eq. (12), we quantified our knowledge about the states of X by the difference between the maximal and the actual entropy of the system. This was a special case because we assumed that after the measurement, Y was in state y_j with certainty, i.e. there is no remaining uncertainty associated with the measurement device Y (of course, this is appropriate for a measurement device). In a more general scenario where two random variables are correlated with each other, we can imagine that Y (after the interaction with X) instead is in state y_j with probability q_j (in other words, we have reduced our uncertainty about Y somewhat, but we don't know everything about it, just as for X). We can then define the *average conditional entropy* of X simply as

$$H(X|Y) = \sum_j q_j H(X|Y = y_j) \quad (13)$$

and the information that Y has about X is then the difference between the unconditional entropy $H(X)$ and Eq. (13) above,

$$H(X : Y) = H(X) - H(X|Y). \quad (14)$$

The colon between X and Y in the expression for the information $H(X : Y)$ is conventional, and indicates that it stands for an entropy *shared* between X and Y . According to our strict definition of unconditional entropies given above, $H(X) = \log N$, but in the standard literature $H(X)$ refers to the actual uncertainty of X given whatever knowledge allowed me to obtain the probability distribution p_i , that is, Eq. (11). In the case nothing is known *a priori* about X , Eq. (14) equals Eq. (12).

Equation (14) can be rewritten to display the symmetry between the observing system and the observed:

$$H(X : Y) = H(X) + H(Y) - H(XY), \quad (15)$$

where $H(XY)$ is just the joint entropy of both X and Y combined. This joint entropy would equal the sum of each of X 's and Y 's entropy only in the case that there are *no correlations* between X 's and Y 's states. If that would be the case, we could not make any predictions about X just from knowing something about Y . The information (15), therefore, would vanish.

2.3.1. *Example: thermodynamics*

We can view Thermodynamics as a particular case of Shannon theory. First, if we agree that the degrees of freedom of interest are position and momentum, then the maximal entropy of any system is defined by its volume in phase space:

$$H_{\max} = \log \Delta\Gamma, \quad (16)$$

where $\Delta\Gamma = \frac{\Delta p \Delta q}{k}$ is the number of states within the phase space volume $\Delta p \Delta q$. Now the normalization factor k introduced in (16) clearly serves to coarse grain the number of states, and should be related to the resolution of our measurement device. In quantum mechanics, of course, this factor is given by the amount of phase space volume occupied by each quantum state, $k = (2\pi\hbar)^n$ where n is the number of degrees of freedom of the system. Does this mean that in this case it is not my type of detector that sets the maximum entropy of the system? Actually, this is still true, only that here we assume a quantum mechanical perfect detector, while still averaging over certain internal states of the system inaccessible to this detector.

Suppose I am contemplating a system whose maximum entropy I have determined to be Eq. (16), but I have some additional information. For example, I know that this system has been undisturbed for a long time, and I know its total energy E , and perhaps even the temperature T . Of course, this kind of knowledge can be obtained by a number of different ways. It could be obtained by experiment, or it could be obtained by inference, or theory. How does this knowledge reduce my uncertainty? In this case, we use our *knowledge of physics* to predict that the probabilities $\rho(p, q)$ going into our entropy

$$H(p, q) = - \sum_{\Delta p \Delta q} \rho(p, q) \log \rho(p, q) \quad (17)$$

is given by the canonical distribution^b

$$\rho(p, q) = \frac{1}{Z} e^{-E(p, q)/T}, \quad (18)$$

^bWe set Boltzmann's constant equal to 1 throughout. This constant, of course, sets the scale of thermodynamical entropy, and would end up multiplying the Shannon entropy just like any particular choice of base for the logarithm would.

where Z is the usual normalization constant, and the sum in (17) goes over all positions and momenta in the phase space volume $\Delta p \Delta q$. The amount of knowledge we have about the system according to Eq. (12) is then just the difference between the maximal and actual uncertainties:

$$I = \log \frac{\Delta \Gamma}{Z} - \frac{E}{T}, \quad (19)$$

where $E = \sum_{\Delta p \Delta q} \rho(p, q) E(p, q)$ is the energy of the system.

3. Quantum Theory

In quantum mechanics, the concept of entropy translates very easily, but the concept of information is thorny. John von Neumann introduced his eponymous quantum mechanical entropy as early as 1927 [6], a full 21 years before Shannon introduced its classical limit! In fact, it was von Neumann who suggested to Shannon to call his formula (1) “entropy”, simply because, as he said, “your uncertainty function has been used in statistical mechanics under that name” [7].

3.1. Measurement

In quantum mechanics, measurement plays a prominent role, and is still considered somewhat mysterious in many respects. The proper theory to describe measurement dynamics in quantum physics, not surprisingly, is quantum information theory. As in the classical theory, the uncertainty about a quantum system can only be defined in terms of the detector states, which in quantum mechanics is a discrete set of *eigenstates* of a measurement operator. The quantum system itself is described by a wave function, given in terms of the quantum system’s eigenbasis, which may or may not be the same as the measurement device’s basis.

For example, say we would like to “measure an electron”. In this case, we may mean that we would like to measure the position of an electron, whose wave function is given by $\Psi(q)$, where q is the coordinate of the electron. Further, let the measurement device be characterized initially by its eigenfunction $\phi_0(\xi)$, where ξ may summarize the coordinates of the device. Before measurement, i.e. before the electron interacts with the measurement device, the system is described by the wave function

$$\Psi(q)\phi_0(\xi). \quad (20)$$

After the interaction, the wave function is a superposition of the eigenfunctions of electron and measurement device

$$\sum_n \psi_n(q)\phi_n(\xi). \quad (21)$$

Following orthodox measurement theory [8], the classical nature of the measurement apparatus implies that after measurement the “pointer” variable ξ takes on a well-defined value at each point in time; the wave function, as it turns out, is thus *not* given by the entire sum in (21) but rather by the single term

$$\psi_n(q)\phi_n(\xi). \quad (22)$$

The wave function (21) is said to have collapsed to (22).

Let us now study what actually happens in such a measurement in detail. For ease of notation, let us recast this problem into the language of state vectors instead. The first stage of the measurement involves the interaction of the quantum system Q with the measurement device (or “ancilla”) A . Both the quantum system and the ancilla are fully determined by their state vector, yet, let us assume that the state of Q (described by state vector $|x\rangle$) is unknown whereas the state of the ancilla is prepared in a special state $|0\rangle$, say. The state vector of the combined system $|QA\rangle$ before measurement then is

$$|\Psi_{t=0}\rangle = |x\rangle|0\rangle \equiv |x, 0\rangle. \quad (23)$$

The von Neumann measurement [9] is described by the unitary evolution of QA via the interaction Hamiltonian

$$\hat{H} = -\hat{X}_Q\hat{P}_A, \quad (24)$$

operating on the product space of Q and A . Here, \hat{X}_Q is the observable to be measured, and \hat{P}_A the operator *conjugate* to the degree of freedom of A that will reflect the result of the measurement. We now obtain for the state vector $|QA\rangle$ after measurement (e.g. at $t = 1$, putting $\hbar = 1$)

$$|\Psi_{t=1}\rangle = e^{i\hat{X}_Q\hat{P}_A}|x, 0\rangle = e^{ix\hat{P}_A}|x, 0\rangle = |x, x\rangle. \quad (25)$$

Thus, the pointer variable in A that previously pointed to zero now also points to the position x that Q is in. This operation appears to be very much like the classical measurement process Eq. (6), but it

turns out to be quite different. In general, the unitary operation (25) introduces quantum *entanglement* between the system being measured and the measurement apparatus, a concept that is beyond the classical idea of correlations.

That entanglement is very different from correlations becomes evident if we apply the unitary operation described above to an initial quantum state which is in a quantum *superposition* of two states:

$$|\Psi_{t=0}\rangle = |x + y, 0\rangle. \quad (26)$$

Then, the linearity of quantum mechanics implies that

$$|\Psi_{t=1}\rangle = e^{i\hat{X}_Q \hat{P}_M} (|x, 0\rangle + |y, 0\rangle) = |x, x\rangle + |y, y\rangle. \quad (27)$$

This state is very different from what we would expect in classical physics, because Q and A are not just correlated (like, e.g. the state $|x + y, x + y\rangle$ would be) but rather they are *quantum entangled*. They now form *one* system that cannot be thought of as composite.

This nonseparability of a quantum system and the device measuring it is at the heart of all quantum mysteries. Indeed, it is at the heart of *quantum randomness*, the puzzling emergence of unpredictability in a theory that is unitary, i.e. where all probabilities are conserved. What is being asked here of the measurement device, namely to describe the system Q , is logically impossible because after entanglement the system has grown to QA . Thus, the detector is being asked to describe a system that is *larger* (with respect to the possible number of states) than the detector, because it includes the detector *itself*. This is precisely the same predicament that befalls a computer program that is asked to determine its own halting probability, in the famous *Halting Problem* [10] analogue of Gödel's famous Incompleteness Theorem [11]. Chaitin [12] showed that the self-referential nature of the question that is posed to a computer program written to solve the Halting Problem gives rise to randomness in pure Mathematics: the halting probability $\Omega = \sum_{p \text{ halts}} 2^{-|p|}$, where the sum goes over all the programs p that halt and $|p|$ is the size of those programs, is random in every way that we measure randomness [13]. A quantum measurement is self-referential in the same manner, since the detector is asked to describe its own state, which is logically impossible. Thus we see that quantum randomness has mathematical, or rather logical, randomness at its very heart.

3.2. von Neumann entropy

Because of the uncertainty inherent in standard projective measurements, measurements of a quantum system Q are described as *expectation values*, which are averages of an observable over the system's *density matrix*, so that

$$\langle \hat{O} \rangle = \text{Tr}(\rho_Q \hat{O}), \quad (28)$$

where \hat{O} is an operator associated with the observable we would like to measure, and

$$\rho_Q = \text{Tr}_A |\Psi_{QA}\rangle \langle \Psi_{QA}|, \quad (29)$$

is obtained from the quantum wave function Ψ_{QA} (for the combined system QA , since neither Q nor the measurement device A separately have a wave function after the entanglement occurred) by *tracing out* the measurement device. However, technically, we are observing the states of the detector, not the states of the quantum system, so instead we need to obtain

$$\rho_A = \text{Tr}_Q |\Psi_{QA}\rangle \langle \Psi_{QA}|, \quad (30)$$

by averaging over the states of the quantum system (which strictly speaking is *not* being observed) and the expectation value of the measurement is instead

$$\langle \hat{O} \rangle = \text{Tr}(\rho_A \hat{O}). \quad (31)$$

The uncertainty about the quantum system is then assumed to be given by the uncertainty in the measurement device A , and can be calculated simply using the von Neumann entropy [6,9]:

$$S(\rho_A) = -\text{Tr} \rho_A \log \rho_A. \quad (32)$$

If Q has been measured in A 's eigenbasis, then the density matrix ρ_A is diagonal, and von Neumann entropy turns into Shannon entropy, as we expect. Indeed, measuring with respect to the system's eigenbasis is precisely the classical limit: entanglement does not happen under these conditions.

Quantum Information Theory, of course, needs concepts such as conditional entropies and mutual entropies besides the von Neumann entropy. They can be defined in a straightforward manner [14,15], but their

interpretation needs care. For example, we can define a conditional entropy in analogy to Shannon theory as

$$\begin{aligned} S(A|B) &= S(AB) - S(B) \\ &= -\text{Tr}_{AB}(\rho_{AB} \log \rho_{AB}) + \text{Tr}_B(\rho_B \log \rho_B), \end{aligned} \quad (33)$$

where $S(AB)$ is the joint entropy of two systems A and B . But can we write this entropy in terms of a conditional density matrix, just as we were able to write the conditional Shannon entropy in terms of a conditional probability? The answer is yes and no: a definition of conditional von Neumann entropy in terms of a conditional density *operator* $\rho_{A|B}$ exists [15, 16], but this operator is technically not a density matrix (its trace is not equal to one), and the eigenvalues of this matrix are very peculiar: they can *exceed* one (this is of course not possible for probabilities). Indeed, the eigenvalues can exceed one only when the system is entangled. As a consequence, quantum conditional entropies can be *negative* [15]. This negative quantum entropy has an operational meaning in quantum information theory: it quantifies how much additional information must be conveyed in order to transport a quantum state if part of a distributed quantum system is known [17]. If this “partial information” is negative, the sender and receiver can use the states for future communication. In Fig. 1(a), we can see a quantum communication process known as “quantum teleportation” [18], in which the quantum wavefunction of a *qubit* (the quantum analogue of the usual bit, which is a quantum particle that can exist in superpositions of zero and one) is transported from the sender “A” (often termed “Alice”) to the receiver “B” (conventionally known as “Bob”). This can be achieved using an entangled pair of particles $e\bar{e}$ (an ebit–anti-ebit pair), where “ebit” stands for entangled bit [19]. This pair carries no information, but each element of the pair carries partial information: in this case the ebit carries one bit, while the anti-ebit carries minus one bit. Bob sends the ebit over to Alice, who performs a joint measurement M of the pair and sends the two classical bits of information back to Bob (see Fig. 1(a)). Armed with the two classical bits, Bob in turn can now perform a unitary operation U on the anti-ebit he has been carrying around, and transform it into the original qubit that Alice had intended to convey. In this manner, Bob has used the negative “partial information” in his anti-ebit to recover the full quantum state, using only classical information. Note that the anti-ebit with negative partial information traveling forwards in time can be seen as an ebit with positive partial information traveling backwards in time [15]. The process of

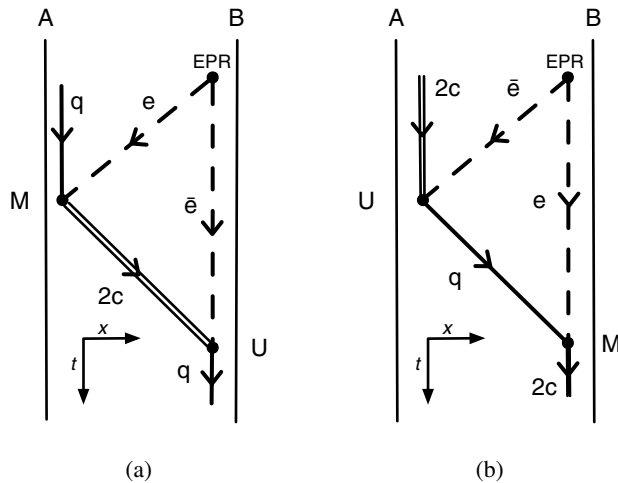


Fig. 1. Using negative partial information for quantum communication. (a) In these diagrams, time runs from top to bottom, and space is horizontal. The line marked “A” is Alice’s space-time trajectory, while the line marked “B” is Bob’s. Bob creates an $e\bar{e}$ pair (an Einstein-Podolski-Rosen pair) close to him, and sends the ebit over to Alice. Alice, armed with an arbitrary quantum state q , performs a joint measurement M on both e and q , and sends the two classical bits $2c$ she obtains from this measurement back to Bob (over a classical channel). When Bob receives these two cbits, he performs one out of four unitary transformations U on the anti-ebit he is still carrying, conditionally on the classical information he received. Having done this, he recovers the original quantum state q , which was “teleported” over to him. The partial information in e is one bit, while it is minus one for the anti-ebit. (b) In superdense coding, Alice sends two classical bits of information $2c$ over to Bob, but using only a single qubit in the quantum channel. This process is in a way the “dual” to the teleportation process, as Alice encodes the two classical bits by performing a conditional unitary operation U on the anti-ebit, while it is Bob that performs the measurement M on the ebit he kept and the qubit Alice sent. Figure adapted from [15].

super-dense coding [20] can be explained in a similar manner (see Fig. 1(b)), except here Alice manages to send 2 classical bits by encoding them on the single anti-ebit she received from Bob.

Quantum mutual entropy is perhaps even harder to understand. We can again define it simply in analogy to (15) as [14, 15, 21]

$$S(A : B) = S(A) + S(B) - S(AB), \tag{34}$$

but what does it mean? For starters, this quantum mutual entropy can be twice as large as the entropy of any of the subsystems, so A and B can share more quantum entropy than they even have by themselves! Of

course, this is due to the fact, again, that “selves” do not exist anymore after entanglement. Also, in the classical theory, information, that is, shared entropy, could be used to make predictions, and therefore to reduce the uncertainty we have about the system that we share entropy with. But that’s not possible in quantum mechanics. If, for example, I measure the spin of a quantum particle that is in an even superposition of its spin-up and spin-down state, my measurement device will show me spin-up half the time, and spin-down half the time, that is, my measurement device has an entropy of one bit. It can also be shown that the shared entropy is *two bits* [15]. But this shared entropy cannot be used to make predictions about the actual spin. Indeed, for the case of the even superposition, I still do not know anything about it [22]! On the other hand, it is possible, armed with my measurement result, to make predictions about the state of *other* detectors measuring the same spin. And even though all these detectors will agree about their result, technically they agree about a random variable (the state of the measurement device), not the actual state of the spin they believe their measurement device to reflect [23]. Indeed, what else could they agree on, since the spin does not have a state? Only the combined system with all the measurement devices that have ever interacted with it, does [24].

Still, the quantum mutual entropy plays a central role in quantum information theory, because it plays a similar role as the classical mutual entropy in the construction of the capacity of an entanglement-assisted channel [25, 26]. In this respect, it is unsurprising that the mutual entropy between two qubits can be as large as 2, as this is the capacity of the superdense coding channel described in Fig. 1(b) [25].

The extension of Shannon’s theory into the quantum regime not only throws new light on the measurement problem, but it also helps in navigating the boundary between classical and quantum physics. According to standard lore, quantum systems (meaning systems described by a quantum wave function) “become” classical in the macroscopic limit, that is, if the action unit associated with that system is much larger than \hbar . Quantum information theory has thoroughly refuted this notion, since we now know that macroscopic bodies can be entangled just as microscopic ones can [27]. Instead, we realize that quantum systems appear to follow the rules of classical mechanics if parts of their wave function are averaged over [such as in Eq. (29)], that is, if the experimenter is not in total control of all the degrees of freedom that make up the quantum system. Because entanglement, once achieved, is not undone by the distance between

entangled parts, almost all systems will seem classical unless expressly prepared, and then protected from interaction with uncontrollable quantum systems. Unprotected quantum systems spread their state over many variables very quickly: a process known as *decoherence* of the quantum state [28].

3.2.1. Quantum thermodynamics

A simple example that illustrates the use of information theory in (quantum) thermal physics is the Heisenberg dimer model, defined by the Hamiltonian

$$H = J\vec{s}_1 \otimes \vec{s}_2 = \frac{J}{4}\vec{\sigma} \otimes \vec{\sigma}, \quad (35)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The system has three degenerate excited states with energy $J/4$, and a (singlet) ground state with energy $-3J/4$. The thermal density matrix of the two-spin system can be written in the product basis as (here, $\beta = 1/T$ is the inverse temperature)

$$\rho_{12} = \frac{e^{-\beta H}}{Z} = \frac{e^{\frac{\beta J}{4}}}{Z} \begin{pmatrix} e^{-\frac{\beta J}{2}} & 0 & 0 & 0 \\ 0 & \cosh \frac{\beta J}{2} & -\sinh \frac{\beta J}{2} & 0 \\ 0 & -\sinh \frac{\beta J}{2} & \cosh \frac{\beta J}{2} & 0 \\ 0 & 0 & 0 & e^{-\frac{\beta J}{2}} \end{pmatrix}$$

where $Z = \text{Tr}e^{-\beta H} = e^{\frac{3\beta J}{4}} + 3e^{-\frac{\beta J}{4}}$. We can calculate the von Neumann entropy of the joint system as

$$S(\rho_{12}) = -\text{Tr}\rho_{12} \log \rho_{12} = \log Z + \beta E, \quad (36)$$

where E is the energy

$$E = \text{Tr}\rho_{12}H = \frac{3J}{4} \frac{1 - e^{\beta J}}{3 + e^{\beta J}}. \quad (37)$$

The marginal density matrices for each of the spin subsystems turn out to be

$$\rho_1 = \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as can easily be seen from inspecting ρ_{12} above, so that $S(\rho_1) = S(\rho_2) = 1$. Using (34) we can calculate the mutual entropy between the quantum subsystems to find

$$S(1 : 2) = 2 - \log Z - \beta E, \tag{38}$$

which is formally analogous to the classical result (19), but has very peculiar quantum properties instead. In the infinite temperature limit $\beta \rightarrow 0$ we see that $Z \rightarrow 4$ while $E \rightarrow 0$, so the shared entropy vanishes in that limit as it should: no interactions can be maintained. But it is clear that at any finite temperature, the quantum interaction between the spins creates correlations that can be quantified by the mutual von Neumann entropy between the spins. In particular, in the limit of zero temperature we find

$$\log Z + \beta E \xrightarrow{T \rightarrow 0} 0, \tag{39}$$

that is, the joint entropy of the spins $S(\rho_{12})$ vanishes and $S(1 : 2) \rightarrow 2$. In that case, the mutual von Neumann entropy is that of a pure Einstein-Podolski-Rosen pair: the singlet solution

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \tag{40}$$

and exceeds by a factor of two the entropy of any of the spins it is composed of. We recognize the wavefunction of the ground state of the Heisenberg dimer at zero temperature as the entangled $e\bar{e}$ pair that we encountered earlier, and that was so useful in quantum teleportation and superdense coding. We will study its behavior under Lorentz transformations below.

4. Relativistic Theory

Once convinced that information theory is a statistical theory about the relative states of detectors in a physical world, it is clear that we must worry not only about quantum detectors, but about moving ones as well. Einstein's special relativity established an upper limit for the speed at which information can be transmitted, without the need to cast this problem in an information-theoretic language. But in hindsight, it is clear that the impossibility of superluminal signaling could just as well have been the result of an analysis of the information transmission capacity of a communication channel involving detectors moving at constant speed with respect to each other. As a matter of fact, Jarett and Cover calculated

the capacity of an “additive white noise Gaussian” (AWNG) channel [4] for information transmission for the case of moving observers, and found [29]

$$C = W \log(1 + \alpha \text{SNR}), \quad (41)$$

where W is the bandwidth of the channel, SNR is the signal-to-noise ratio, and $\alpha = \nu'/\nu$ is the Doppler shift. As the relative velocity $v/c \rightarrow 1$, $\alpha \rightarrow 0$ and the communication capacity vanishes. In the limit $\alpha = 1$, the common capacity formula for the common Gaussian channel with limited bandwidth [4] is recovered. Note that in the limit of an infinite bandwidth channel, Eq. (41) becomes

$$C = \alpha \text{SNR} \log_2(e) \text{ bits per second.} \quad (42)$$

Historically, this calculation seems to have been an anomaly: no-one else seems to have worried about an “information theory of moving bodies”, not the least because such a theory had, or indeed has, little immediate relevance. Interestingly, the problem that Jarett and Cover addressed with their calculation [29] was the famous “twin-paradox”: a thought experiment in special relativity that involves a twin journeying into space at high-speed, only to turn around to find that his identical twin that stayed behind has aged faster. Relativistic information theory gives a nice illustration of the resolution of the paradox, where the U-turn that the traveling twin must undergo creates a switch in reference frames that affects the information transmission capacities between the twins, and accounts for the differential aging.

A standard scenario that would require relativistic information theory thus involves two random variables moving with respect to each other. The question we may ask is whether relative motion is going to affect any shared entropy between the variables. First, it is important to point out that Shannon entropy is a *scalar*, and we therefore do not expect it to transform under Lorentz transformations. This is also intuitively clear if we adopt the “strict” interpretation of entropy as being *unconditional* (and therefore equal to the logarithm of the number of degrees of freedom). On the other hand, probability distributions (and the associated conditional entropies) could conceivably change under Lorentz transformations. How is this possible given the earlier statement that entropy is a scalar?

We can investigate this with a *gedankenexperiment* where the system under consideration is an ideal gas, with particle velocities distributed

according to the Maxwell distribution. In order to define entropies, we have to agree about which degrees of freedom we are interested in. Let us say that we only care about the two components of the velocity of particles confined in the $x - y$ -plane. Even at rest, the mutual entropy between the particle velocity components $H(v_x : v_y)$ is non-vanishing, due to the finiteness of the magnitude of v . A detailed calculation^c using continuous variable entropies of the Maxwell distribution shows that, at rest

$$H(v_x : v_y) = \log(\pi/e). \quad (43)$$

The Maxwell velocity distribution, on the other hand, will surely change under Lorentz transformations in, say, the x -direction, because clearly the components are affected differently by the boost. In particular, it can be shown that the mutual entropy between v_x and v_y will *rise* monotonically from $\log(\pi/e)$, and tend to a constant value as the boost-velocity $v/c \rightarrow 1$. But of course, β is just another variable characterizing the moving system, and if this is known precisely, then we ought to be able to recover Eq. (43), and the apparent change in information is due entirely to a reduction in the uncertainty $H(v_x)$. This example shows that in information theory, even if the entire system's entropy does not change under Lorentz transformations, the entropies of subsystems, and therefore also information, can.

While a full theory of relativistic information does not exist, pieces of such a theory can be found when digging through the literature. For example, relativistic thermodynamics is a limiting case of relativistic information theory, simply because as we have seen above, thermodynamical entropy is a limiting case of Shannon entropy. But unlike in the case constructed above, we do not have the freedom to choose our variables in thermodynamics. Hence, the invariance of entropy under Lorentz transformations is assured via Liouville's theorem, because the latter guarantees that the phase-space volume occupied by a system is invariant. Yet, relativistic thermodynamics is an odd theory, not the least because it is intrinsically inconsistent: the concept of equilibrium becomes dubious. In thermodynamics, equilibrium is defined as a state where all relative motion between the subsystems of an ensemble have ceased. Therefore, a joint system where one part moves with a constant velocity with respect to the other cannot be at equilibrium, and relativistic information theory has to be used instead.

^cR.M. Gingrich, unpublished.

One of the few questions of immediate relevance that relativistic thermodynamics has been able to answer is how the temperature of an isolated system will appear from a moving observer. Of course, temperature itself is an equilibrium concept and therefore care must be taken in framing this question [30]. Indeed, both Einstein [31] and Planck [32] tackled the question of how to Lorentz-transform temperature, with different results. The controversy [33, 34] can be resolved by realizing that no such transformation law can in fact exist [35], as the usual temperature (the parameter associated with the Planckian blackbody spectrum) becomes direction-dependent if measured with a detector moving with velocity $\beta = v/c$ and oriented at an angle θ' with respect to the radiation [36, 37]

$$T' = \frac{T\sqrt{1-\beta^2}}{1-\beta\cos\theta'}. \quad (44)$$

In other words, an ensemble that is thermal in the rest frame is non-thermal in a moving frame, and in particular cannot represent a standard heat bath because it will be *non-isotropic*.

5. Relativistic Quantum Theory

While macroscopic quantities like temperature lose their meaning in relativity, microscopic descriptions in terms of probability distributions clearly still make sense. But in a quantum theory, these probability distributions are obtained from quantum measurements specified by local operators, and the space-time relationship between the detectors implementing these operators becomes important. For example, certain measurements on a joint (i.e. composite) system may require communication between parties, while certain others are impossible *even though* they do not require communication [38]. In general, a relativistic theory of quantum information needs to pay close attention to the behavior of the von Neumann entropy under Lorentz transformation, and how such entropies are being reduced by measurement. In this section, I discuss the effect of a Lorentz transformation on the entropy of a single particle, or a pair of entangled particles. For the latter case, I study how quantum entanglement between particles is affected by global Lorentz boosts. This formalism has later been used to study the effect of *local* Lorentz transformations on the von Neumann entropy of a single particle or a pair of entangled particles, and I will summarize those results too.

5.1. Boosting quantum entropy

The entropy of a qubit (which we take here for simplicity to be a spin-1/2 particle) with wave function

$$|\Psi\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}}(a|\uparrow\rangle + b|\downarrow\rangle), \quad (45)$$

(a and b are complex numbers), can be written in terms of its density matrix $\rho = |\Psi\rangle\langle\Psi|$ as

$$S(\rho) = -\text{Tr}(\rho \log \rho). \quad (46)$$

A wave function is by definition a completely known state (called a “pure state”), because the wave function is a complete description of a quantum system. As a consequence, (46) vanishes: we have no uncertainty about this quantum system. As we have seen earlier, it is when that wave function interacts with uncontrolled degrees of freedom that mixed states arise. And indeed, just by Lorentz-boosting a qubit, such mixing will arise [39]. The reason is not difficult to understand. The wave function (45), even though I have just stated that it completely describes the system, in fact only completely describes the *spin* degree of freedom! Just as we saw in the earlier discussion about the classical theory of information, there may always be other degrees of freedom that our measurement device (here, a spin-polarization detector) cannot resolve. Because we are dealing with particles, ultimately we have to consider their *momenta*. A more complete description of the qubit state then is

$$|\Psi\rangle = |\sigma\rangle \times |\vec{p}\rangle, \quad (47)$$

where σ stands for the spin-variable, and \vec{p} is the particle’s momentum. Note that the momentum wave function $|\vec{p}\rangle$ is in a product state with the spin wave function $|\sigma\rangle$. This means that both spin and momentum have their own state, they are *unmixed*. But as is taught in every first-year quantum mechanics course, such momentum wave functions (plane waves with perfectly well-defined momentum \vec{p}) do not actually exist; in reality, they are wave packets with a momentum *distribution* $f(\vec{p})$, which we may take to be Gaussian. If the system is at rest, the momentum wave function does not affect the entropy of (47), because it is a product.

What happens if the particle is boosted? The spin and momentum degrees *do* mix, which we should have expected because Lorentz transformations always imply frame rotations as well as changes in linear velocity.

The product wave function (47) then turns into

$$|\Psi\rangle \longrightarrow \sum_{\sigma} \int f(\vec{p}) |\sigma, \vec{p}\rangle d\vec{p}, \quad (48)$$

which is a state where spin-degrees of freedom and momentum degrees of freedom are entangled. But our spin-polarization detector is insensitive to momentum! Then we have no choice but to average over the momentum, which gives rise to a spin density matrix that is mixed

$$\rho_{\sigma} = \text{Tr}_{\vec{p}}(|\Psi\rangle\langle\Psi|), \quad (49)$$

and that consequently has positive entropy. Note, however, that the entropy of the joint spin-momentum density matrix remains unchanged, at zero. Note also that if the momentum of the particle was truly perfectly known from the outset, i.e. a plane wave $|\vec{p}\rangle$, mixing would also not take place [40].

While the preceding analysis clearly shows what happens to the quantum entropy of a spin-1/2 particle under Lorentz transformations (a similar analysis can be done for photons [41]), what is most interesting in quantum information theory is the entanglement *between* systems. While some aspects of entanglement are captured by quantum entropies [42] and the spectrum of the conditional density operator [16], quantifying entanglement is a surprisingly hard problem, currently without a perfect solution. However, some good measures exist, in particular for the entanglement between two-level systems (qubits) and three-or-fewer level systems [43].

5.2. Boosting quantum entanglement

If we wish to understand what happens to the entanglement between two massive spin-1/2 particles, say, we have to keep track of four variables, the spin states $|\sigma\rangle$ and $|\lambda\rangle$ and the momentum states $|\vec{p}\rangle$ and $|\vec{q}\rangle$. A Lorentz transformation on the joint state of this two-particle system will mix spins and momenta just as in the previous example. Let us try to find out how this affects entanglement.

A good measure for the entanglement of mixed states, i.e. states that are not pure such as (47), is the so-called *concurrence*, introduced by Wootters [44]. This concurrence $C(\rho_{AB})$ can be calculated for a density matrix ρ_{AB} that describes two subsystems A and B of a larger system, and quantifies the entanglement *between* A and B . For our purposes, we will be interested in the entanglement between the spins σ and λ of our pair. The

concurrence is one if two degrees of freedom are perfectly entangled, and vanishes if no entanglement is present.

In order to do this calculation we first have to specify our initial state. We take this to be a state with spin and momentum wave function in a product, but where the spin-degrees of freedom are perfectly entangled in a so-called Bell state:

$$|\sigma, \lambda\rangle = \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle). \tag{50}$$

The concurrence of this state can be calculated to be maximal: $C(\rho_{\sigma\lambda}) = 1$. We now apply a Lorentz boost to this joint state, i.e. we move our spin-polarization detector with speed $\beta = v/c$ with respect to this pair (or, equivalently, we move the pair with respect to the detector). If the momentum degrees of freedom of the particles at the outset are Gaussian distributions unentangled with each other and the spins, the Lorentz boost will entangle them, and the concurrence will drop [45]. How much it drops depends on the ratio between the spread of the momentum distribution σ_r (not to be confused with the spin σ) and the particle's mass m . In Fig. 2 below, the concurrence is displayed for two different such ratios, as a function of the *rapidity* ξ . The rapidity ξ here is just a transformed velocity: $\xi = \sinh \beta$, such that $\xi \rightarrow \infty$ as $\beta \rightarrow 1$. We can see that if the ratio σ_r/m is not too large, the concurrence will drop but not disappear altogether. But

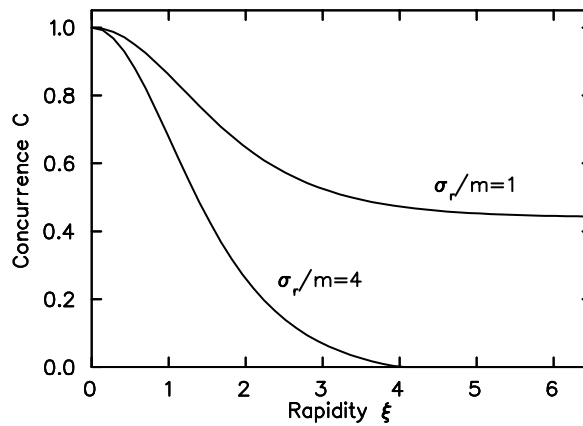


Fig. 2. Spin-concurrence as a function of rapidity, for an initial Bell state with momenta in a product Gaussian. Data is shown for $\sigma_r/m = 1$ and $\sigma_r/m = 4$ (from Ref. [45]).

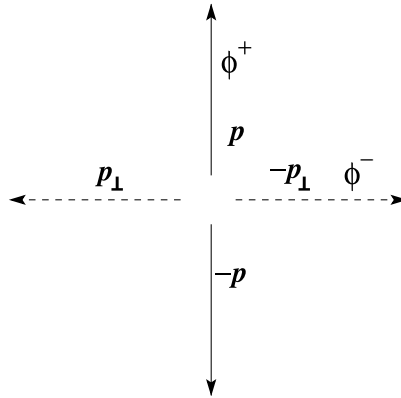


Fig. 3. Superposition of Bell-states Φ^+ and Φ^- at right angles, with the particle pair moving in opposite directions.

if the momentum spread is large compare to the mass, all entanglement can be lost.

Let us consider instead a state that is initially unentangled in spins, but fully entangled in momenta. I depict such a wave function in Fig. 3, where a pair is in a superposition of two states, one moving in opposite directions with momentum \vec{p}_{\perp} in a relative spin state Φ^- (this is one of the four Bell spin-entangled states, Eq. (50)), and one moving in a plane in opposite orthogonal directions with momentum p , in a relative spin-state Φ^+ (which is Eq. (50) but with a plus sign between the superpositions). It can be shown that if observed at rest, the spins are indeed unentangled. But when boosted to rapidity ξ , the concurrence actually *increases* [45], as for this state (choosing $m = 1$)

$$C(\rho_{AB}) = \frac{p^2(\cosh^2(\xi) - 1)}{(\sqrt{1 + p^2 \cosh(\xi)} + 1)^2}. \quad (51)$$

Thus, Lorentz boosts can, under the right circumstances, create entanglement where there had been none before.

A similar analysis can be performed for pairs of entangled photons, even though the physics is quite different [46]. First of all, photons are massless and their quantum degree of freedom is the photon polarization. The masslessness of the photons makes the analysis a bit tricky, because issues of gauge invariance enter into the picture, and as all particles move with constant velocity (light speed), there cannot be a spread in momentum

as in the massive case. Nevertheless, Lorentz transformation laws acting on polarization vectors can be identified, and an analysis similar to the one described above can be carried through. The difference is that the entangling effect of the Lorentz boost is now entirely due to the spread in momentum *direction* between the two entangled photon beams. This implies first of all that fully-entangled photon polarizations cannot exist, even at rest, and second that existing entanglement can either be decreased or increased, depending on the angle with which the pair is boosted (with respect to the angle set by the entangled pair), and the rapidity [46].

5.3. Entanglement in accelerated frames

The logical extension of the work just described is to allow for *local* Lorentz transformations on quantum particles, that is, to study particles on relativistic (accelerating) orbits or in classical gravitational fields. Alsing and Milburn, for example, studied the quantum teleportation channel I discussed earlier [47]. Because quantum teleportation relies on an entangled pair of particles, the fidelity of quantum teleportation (how well Bob's version of Alice's quantum state agrees with the original) would suffer if acceleration of either Bob or Alice leads to a deterioration of entanglement. This is precisely what happens, but the origin of the deterioration of entanglement is different here: it is not due to the mixing of spin and momentum degrees of freedom, but rather due to the emergence of "Unruh radiation" in the rest frame of the accelerated observer [47, 48]. Unruh radiation is a peculiar phenomenon that is due to the appearance of a sort of "event horizon" for accelerated observers: there are regions of spacetime that are causally disconnected from an accelerated observer, and this disconnected region affects the vacuum fluctuations that occur anywhere in space [49–51]. In this sense, the Unruh radiation is analogous to Hawking radiation, which I will discuss in more detail in the following section. Unruh radiation produces thermal noise in the communication channel, which leads to the breakdown of the fidelity of quantum teleportation. Because this reasoning applies to all quantum communication that relies on the assistance of entanglement, we can conclude that generally the capacity of entanglement-assisted channels would be reduced between accelerated observers [52]. A similar conclusion holds for entangled particles near strong gravitational fields [53]. In that case, it is indeed the Hawking radiation that leads to the deterioration of entanglement between Einstein-Podolski-Rosen pairs.

6. Information in Curved Space Time

While there are clearly many other questions that can conceivably be posed (and hopefully answered) within the relatively new field of relativistic quantum information theory [54], I would like to close this review with some speculations about quantum information theory in curved space time.

That something interesting might happen to entropies in curved space time has been suspected ever since the discovery of Hawking radiation [55] that gave rise to the black hole information paradox [56]. The paradox has two parts and can be summarized as follows: According to standard theory, a non-rotating and uncharged black hole can be described by an entropy that is determined entirely in terms of its mass M (in units where $\hbar = G = 1$):

$$S_{BH} = 4\pi M^2. \quad (52)$$

Presumably, a state that is fully known (that is, one that is correlated with another system that an observer has in its possession) can be absorbed by the black hole. Once that state disappears behind the event horizon, the correlation between that state and its description in the observer's hands seems to disappear: the information cannot be retrieved any longer. Even worse, after a long time, the black hole will have evaporated entirely into thermal (Hawking) radiation, and the information is not only irretrievable, it must have been destroyed. A more technical discussion would argue that black holes appear to have the capability to turn pure states into mixed states *without* disregarding (tracing over) parts of the wave function. Such a state of affairs is not only paradoxical, but it is in fact completely incompatible not only with the standard model of physics, but with the basic principle of probability conservation.

The second part of the problem has to do with the entropy balance between the black hole and the radiation it emits. When the black hole evaporates via Hawking radiation, the emitted radiation is thermal, and carries entropy

$$S_{\text{rad}} \sim T_H^3 \quad (53)$$

with black hole temperature $T_H = (8\pi M)^{-1}$. But the black hole's entropy must also change at the same time, and this is determined by the amount of energy that had to be spent in order to create the virtual particle pairs that gave rise to the radiation. Because mass and temperature of the black hole are inversely related, the entropy decrease of the black hole and the entropy

of the emitted radiation *cannot match*. Indeed, we roughly find that

$$dS_{\text{rad}} \approx 4/3 dS_{\text{BH}}. \quad (54)$$

Now it should be pointed out that the preceding results were obtained within equilibrium thermodynamics in curved space time. But since black holes have negative heat capacity [57], they can *never* be at equilibrium, and the assumptions of that theory are strongly violated. As the concept of information itself is a non-equilibrium one, we should not be surprised if paradoxical results are obtained if equilibrium concepts are used to describe such a case. Still, a resolution of the microscopic dynamics in black hole evaporation is needed. One possible approach is to use quantum information theory to characterize the relative states of the black hole, the stimulated radiation emitted during the formation of the black hole, and the Hawking radiation (spontaneous emission of radiation) created in the subsequent evaporation [23, 58]. As we have lost track of the stimulated radiation, we must always average over it (“trace” it out), which (along with tracing out the causally disconnected region that lies beyond the Schwarzschild radius) creates the positive black hole entropy. In the flat space-time treatment of Ref. [23], the entropy balance between the black hole and the Hawking radiation can be maintained because entanglement is spread between the stimulated radiation, Hawking radiation, and the black hole. While all three are strongly entangled, tracing over the stimulated radiation produces a state of *no* correlations between Hawking radiation and black hole, implying that the Hawking radiation appears purely thermal. But of course, the joint system is still highly entangled, but in order to discover this entanglement we would have to have access to the lost radiation emitted during the formation process. Still, this treatment is unsatisfying because it does not resolve the ultimate paradox: the unitary description only works up until the black hole has shrunk to a particular small size. At that point it appears to break down.

One reason for this breakdown might lie in the inappropriate treatment of quantum entropy in curved space time (the preceding formalism ignored curvature). A more thorough analysis must take into account the causal structure of space time. For example, not all quantum measurements are realizable [38], because only those variables can be simultaneously measured whose separation is space-like. In physics, we do have a theory that correctly describes how different observables interact in a manner compatible with the causal structure of space-time, namely quantum field theory. In order to consistently define quantum entropies then, we must define them within quantum field theory in curved space-time.

The first steps toward such a theory involve defining quantum fields over a manifold separated into an accessible and an inaccessible region. This division will occur along a world-line, and we shall say that the “inside” variables are accessible to me as an observer, while the outside ones are not. Note that the inaccessibility can be due either to causality, or due to an event horizon. Both cases can be treated within the same formalism. States in the inaccessible region have to be averaged over, since states that differ only in the outside region are unresolvable. Let me denote the inside region by R , while the entire state is defined on E . We can now define a set of commuting variables X that can be divided into X_{in} and X_{out} . By taking matrix elements of this density matrix of the entire system

$$\rho = |E\rangle\langle E| \tag{55}$$

with the complete set of variables $(X_{\text{in}}, X_{\text{out}})$, we can construct the inside density matrix (defined on R) as

$$\rho_{\text{in}} = \text{Tr}_{X_{\text{out}}}(\rho_{X_{\text{in}}X_{\text{out}}}). \tag{56}$$

which allows me to define the *geometric entropy* [59] of a state E for an observer restricted to R

$$S_{\text{geom}} = -\text{Tr}(\rho_{\text{in}} \log \rho_{\text{in}}). \tag{57}$$

Here, the trace is performed using the inside variables only.

This, however, is just the beginning. As with most quantities in quantum field theory, this expression is divergent and needs to be renormalized. Rather than being an inconvenience, this is precisely what we should have expected: after all, we began this review by insisting that entropies only make sense when discussed in terms of the possible measurements that can be made to this system. This is, of course, precisely the role of renormalization in quantum field theory. Quantum entropies can be renormalized via a number of methods, either using Hawking’s zeta function regularization procedure [60] or by the “replica trick”, writing

$$S_{\text{geom}} = - \left[\frac{d}{dn} \text{Tr}(\rho_{\text{in}}^n) \right]_{n=1}, \tag{58}$$

and then writing $dS(n)$ in terms of the expectation value of the stress tensor. A thorough application of this program should reveal components of the entropy due entirely to the curvature of space-time, and which vanish in the flat-space limit. Furthermore, the geometric entropy can be used to

write equations relating the entropy of the inside and the outside space-time regions, as

$$S(E) = S(\rho_{\text{in,out}}) = S(\rho_{\text{in}}) + S(\rho_{\text{out}}|\rho_{\text{in}}). \quad (59)$$

If $S(\rho_{\text{in}})$ is the entropy of the black hole radiation (together with the stimulated radiation), then $S(\rho_{\text{out}}|\rho_{\text{in}})$ is the conditional black hole entropy *given* the radiation field, a most interesting quantity in black hole physics.

7. Summary

Entropy and information are statistical quantities describing an observer's capability to predict the outcome of the measurement of a physical system. Once couched in those terms, information theory can be examined in all physically relevant limits, such as quantum, relativistic, and gravitational. Information theory is a non-equilibrium theory of statistical processes, and should be used under those circumstances (such as measurement, non-equilibrium phase transitions, etc.) where an equilibrium approach is meaningless. Because an observer's capability to make predictions (quantified by entropy) is not a characteristic of the object the predictions apply to, it does not have to follow the same physical laws (such as reversibility) as that befitting the objects. Thus, the arrow of time implied by the loss of information under standard time-evolution is even less mysterious than the second law of thermodynamics, which is just a consequence of the former.

In time, a fully relativistic theory of quantum information, defined on curved space-time, should allow us to tackle a number of problems in cosmology and other areas that have as yet resisted a consistent treatment. These developments, I have no doubt, would make Shannon proud.

Acknowledgments

I am grateful to N.J. Cerf for years of very fruitful collaboration in quantum information theory, as well as to R.M. Gingrich and A.J. Bergou for their joint efforts in the relativistic theory. I would also like to acknowledge crucial discussions on entropy, information, and black holes, with P. Cheeseman, J.P. Dowling, and U. Yurtsever. This work was carried out in part with support from the Army Research Office. I would like to acknowledge support from Michigan State University's BEACON Center for the Study of Evolution in Action, where part of this review was written.

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