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Bifurcation set and index at infinity of polynomials

Gabriel Esteban Perico Monsalve

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Gabriel Esteban Perico Monsalve

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Conjunto de bifurcação e índice no infinito de polinômios

Tese apresentada ao Instituto de Ciências Matemáticas e de Computação – ICMC-USP, como parte dos requisitos para obtenção do título de Doutor em Ciências – Matemática. *VERSÃO REVISADA*

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quien me ha brindado su constante apoyo e inquebrantable amor desde los primeros pasos de mi educación.
Gracias por ser mi roca, mi guía y mi mayor defensora."

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RESUMO

MONSALVE, G. E. **Conjunto de bifurcação e índice no infinito de polinômios**. 2024. 85 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

No contexto de funções polinomiais $f : \mathbb{R}^2 \to \mathbb{R}$ de grau $d > 0$, enfrentamos três desafios fundamentais: a detecção eficaz de valores atípicos, o cálculo do índice no infinito e a estimativa do limite superior do índice em termos do grau d.

Demonstramos que a presença de fenômenos específicos no infinito das fibras, como o *desaparecimento e a divisão das componentes da fibra*, leva ao surgimento de *valores atípicos*, também conhecidos como *valores de bifurcação*. Para identificar esses fenômenos, utilizamos as componentes conexas do conjunto de Milnor do polinômio f fora de um compacto em $\mathbb{R}^2$, permitindo-nos descrever o comportamento topológico das fibras em proximidade do infinito.

Além disso, fornecemos uma caracterização detalhada de valores atípicos e aplicamos nossa abordagem para calculá-los em dois polinômios que exibem fenômenos intrigantes no infinito.

Em nosso estudo do *índice no infinito* $\mathrm{ind}_\infty f$ para funções polinomiais $f : \mathbb{R}^2 \to \mathbb{R}$ com singularidades isoladas, definimos este índice como o número de voltas do campo de vetores do gradiente $\mathrm{grad} f$ restrito a um círculo C que engloba todos os pontos singulares de f. Apresentamos uma fórmula que revela como o comportamento das fibras no infinito influencia este índice.

Por fim, investigamos os fenômenos que contribuem para a diferença entre $\mathrm{ind}_\infty f$ e o limite superior do índice, previamente estabelecido por Durfee.

Palavras-chave: Conjunto de Milnor, Conjunto de Bifurcação, Índice no Infinito.

ABSTRACT

MONSALVE, G. E. **Bifurcation set and index at infinity of polynomials**. 2024. 85 p. Tese (Doutorado em Ciências – Matemática) – Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos – SP, 2024.

In the context of polynomial functions $f:\mathbb{R}^2 \to \mathbb{R}$ of degree $d > 0$, we tackle three fundamental challenges: effective detection of atypical values, computation of the index at infinity, and estimation of the upper bound of the index in terms of the degree d.

We demonstrate that the presence of specific phenomena at infinity in the fibers, such as the *vanishing and splitting of fiber components*, leads to the emergence of *atypical values*, also known as *bifurcation values*. To identify these phenomena, we leverage the connected components of the Milnor set of the polynomial f outside a compact set in $\mathbb{R}^2$, allowing us to describe the topological behavior of fibers in proximity to infinity.

Furthermore, we provide a detailed characterization of atypical values and apply our approach to compute them for two polynomials exhibiting intriguing phenomena at infinity.

In our study of the *index at infinity* $\mathrm{ind}_\infty f$ for polynomial functions $f:\mathbb{R}^2 \to \mathbb{R}$ with isolated singularities, we define this index as the winding number of the gradient vector field $\mathrm{grad} f$ restricted to a circle C encompassing all singular points of f. We present a formula that unveils how the behavior of fibers at infinity influences this index.

Lastly, we investigate the phenomena contributing to the gap between $\mathrm{ind}_\infty f$ and the upper bound of the index, previously established by Durfee.

Keywords: Milnor set, Bifurcation set, index at infinity.

CONTENTS

CHAPTER

1

INTRODUCTION

In this thesis we give account of three problems concerning polynomial functions in two real variables, namely: *detection of their atypical values*, *computation of their index at infinity*, and *estimation of its upper bound*.

In the more general setting of a polynomial map $f : \mathbb{K}^n \to \mathbb{K}^k$ with $n > k$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the *submersion theorem* asserts that for each point $p \in \mathbb{K}^n$, outside the set $\mathrm{Sing} f \subset \mathbb{K}^n$ of points where f is not a submersion, there are two (small enough) neighborhoods $U_p \subset \mathbb{K}^n$ and $V_p \subset \mathbb{K}^k$ at p and $f(p)$, respectively, such that the restriction $f_| : U_p \cap f^{-1}(V_p) \to V_p$ is *differentialy isotopic*[1] to a trivial fibration. It is easy to see that the above conclusion does not hold globally, i.e. the restriction $f_| : f^{-1}(V) \to V$ is not necessarily differentialy isotopic to a trivial fibration for any neighborhood V of some points $p \in \mathbb{K}^n \backslash \mathrm{Sing} f$. The so called Broughton's example illustrates this phenomena:

Example 1.1. (BROUGHTON, 1983) The polynomial $f : \mathbb{K}^2 \to \mathbb{K}$ defined by $f(x,y) = x(xy+1)$ is a submersion over $\mathbb{K}^2$ since $\nabla f(x,y) = (2xy+1, x^2)$, and f is not a fibration over $f^{-1}(V)$ for any neighborhood $V \subset \mathbb{K}$ of 0, because $f^{-1}(0) \cong \mathbb{K} \sqcup \mathbb{K} \backslash \{0\}$ and $f^{-1}(\varepsilon) \cong \mathbb{K} \backslash \{0\}$ for any $\varepsilon \neq 0$ where the symbol "$\cong$" denotes the relation between topological spaces of being homeomorphic.

The above discussion raised up the problem of detect the values of f where the restriction $f_| : f^{-1}(V) \to V$ is not differentialy isotopic to a trivial fibration for any neighborhood V, or equivalently, the values where f is a locally trivial C^∞-fibration. After (TIBĂR, 2007) we define:

Definition 1.2. Let $f : \mathbb{K}^n \to \mathbb{K}^k$ with $n > k$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A fibre $f^{-1}(\lambda)$ is *typical* if and only if f is a C^∞-fibration over some neighborhood of $\lambda \in \mathbb{K}^k$ and its *bifurcation set*, or also *set of atypycal values*, is the minimal set $Atyp\, f \subset \mathbb{K}^k$ such that the restriction $f_| : \mathbb{K}^n \backslash f^{-1}(Atyp\, f) \to \mathbb{K}^k \backslash Atyp\, f$ is a locally trivial C^∞-fibration.

[1] We refer the reader to (MILNOR, 1997) for the precise definition.

The characterization of the set $Atyp\, f$ is a widely studied problem with interesting connections, for instance with: the Jacobian conjecture, optimization of polynomials $f : \mathbb{R}^n \to \mathbb{R}$, etc. We refer the reader to (DIAS; RUAS; TIBĂR, 2012) and the references on it, for further connections and different ways of approaching this characterization problem.

Only in the setting of polynomial functions $f : \mathbb{K}^2 \to \mathbb{K}$ we have characterizations of the set $Atyp\, f$. In the complex case the atypical values are detected by the change of the *Euler Characteristic* of fibres, see (SUZUKI, 1974; VUI; TRÁNG, 1984) and also §3 below. In the real setting, this problem is more subtle and a characterization only appears in (TIBĂR; ZAHARIA, 1999) with the study of the phenomena at infinity of fibres that may occur, namely *vanishing* and *splitting* at infinity.

The main result in this thesis, concerning the characterization of atypical values, is Theorem 3.5 and the new [2] effective detection of the vanishing and splitting (see §3.4 below) in terms of clusters of Milnor arcs (see Definitions 2.14 and 2.19). Our detection is inspired in the "clusters-technique" presented in (COSTE; DE LA PUENTE, 2001; VUI; THAO, 2011) with two main difference: we detect splitting intead of "cleaving", and we focus on the effectiveness.

Concerning the problem of computing the index at infinity, let us motivate the problem from the local case: let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. For each isolated singularity $p \in \mathbb{R}^2$ of f, *the index* $\mathrm{ind}_p f$ is defined as the winding number of the gradient vector field $\mathrm{grad} f$ restricted to a small enough circle C centred at p such that p is the unique singular point of f inside C. In (ARNOL'D, 1978, page 6) Arnold finds a formula for the index $\mathrm{ind}_p f$ depending only on the number of irreducible components of the curve $f^{-1}(f(p))$ at p. Let us state this result and, by completeness, we give an elementary proof.

Proposition 1.3. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be an analytic function germ. If the fiber $f^{-1}(0)$ is a curve with r irreducible components then $\mathrm{ind}_0 f = 1 - r$.

Proof. By (MILNOR, 1968, Corollary 2.9) there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the fiber $f^{-1}(0)$ intersects transversally the circle C_ε defined as the boundary ∂D_ε of the open disk D_ε centred at $0 \in \mathbb{R}^2$ with radius ε. By choosing ε_0 small enough we shall assume that p is the unique singular point of f inside the disk D_{ε_0}. Hence the number of points in $f^{-1}(0) \cap C_\varepsilon$ is constant equals $2r$ for all $0 < \varepsilon \leq \varepsilon_0$.

Fix $0 < \varepsilon \leq \varepsilon_0$. The set $C_\varepsilon \setminus f^{-1}(0)$ is the disjoint union $\cup_{i=1}^{2r} C_i$, where each C_i is a sector circle with exactly two boundary points a_i, a_{i+1} in $f^{-1}(0) \cap C_\varepsilon$ (note that $a_{2r+1} = a_1$). By our choice of ε, the sign of f change when it pass through one circle sector to another, since $\mathrm{grad} f$ at each boundary point a_i, $i = 1, \ldots, 2r$ is transverse to $f^{-1}(0)$. Let us assume, by multipliying by -1 if necessary, that $f > 0$ over C_i.

[2] we also refer the reader to (DIAS; TIBĂR, 2015; DIAS; TANABÉ; TIBĂR, 2017; DIAS; JOIŢA; TIBĂR, 2021) where the authors also address the problem of characterization and effectiveness

We shall find the variation β_i of the direction of grad$f$ in C_i: by $\overrightarrow{0a_i}$ and $\overrightarrow{0a_{i+1}}$ we denote the two radii of C_ε defined by the points a_i, a_{i+1}, respectively, by α_i we denote the angle spanned by $\overrightarrow{0a_i}$, $\overrightarrow{0a_{i+1}}$, and by θ_i we denote the angle between grad$f(a_i)$ and $\overrightarrow{0a_i}$. Therefore, $\beta_i, \theta_i, \theta_{i+1}, \alpha_i$ are the interior angles of a four-sides closed polygon satisfying $\beta_i = 2\pi - \alpha_i - \theta_i - \theta_{i+1}$, see Figure 1.

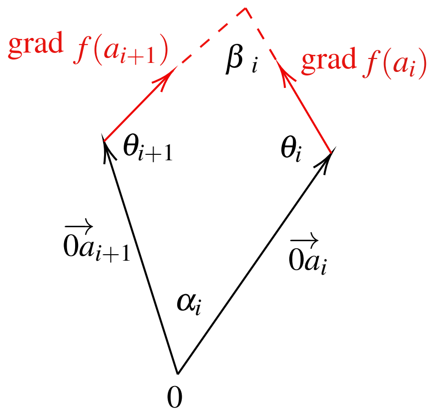

Figure 1

Now, we compute the variation β_{i+1}: since $f > 0$ in C_i, $f < 0$ in C_{i+1}, and thus, fixing notations as in the previous case, the angles $\beta_{i+1}, \theta_{i+1}, \theta_{i+2}, \alpha_{i+1}$ are the interior angles of a four-sides closed polygon, see Figure 2, satisfying, in this case, $\beta_{i+1} = -\alpha_{i+1} + \theta_{i+1} + \theta_{i+2}$.

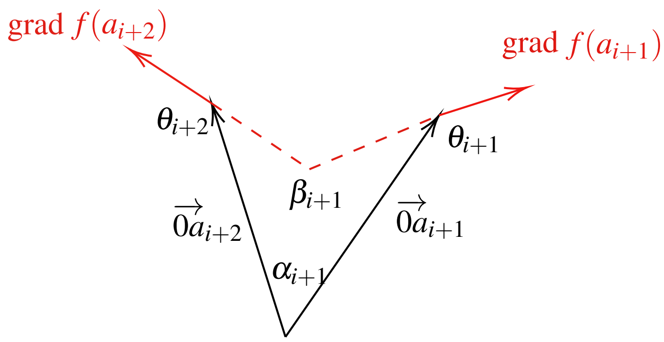

Figure 2

So far we computed the variation of direction of grad$f$ in two consecutive sector circles. Now, we use this variation to compute the total variation in C_ε:

$$\begin{aligned}\sum_{i=1}^{2r} \beta &= \sum_{k=1}^{r} (\beta_{2k-1} + \beta_{2k}) \\ &= 2\pi r - \sum_{i=1}^{2r} \alpha_i - \sum_{k=1}^{r} (\theta_{2k+1} - \theta_{2k-1}) \\ &= 2\pi(r-1),\end{aligned}$$

where the third equality follows from the observations: $\sum_{i=1}^{2r} \alpha_i = 2\pi$ and the terms of the sum $\sum_{k=1}^{r}(\theta_{2k+1} - \theta_{2k-1})$ cancel each other.

From the above proof the direction of grad$f$ moves clockwise when grad$f$ moves counterclockwise around C_ε. Therefore

$$\mathrm{ind}_0 f = -\frac{1}{2\pi}\sum_{i=1}^{2r}\beta_i = 1 - r.$$

□

Our computation problem for the index at infinity is a global counterpart of Proposition 1.3. For a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ with only isolated singularities, the index at infinity $\mathrm{ind}_\infty f$ is defined as the winding number of the gradient vector field grad$f$ restricted to a circle C such that all singular points of f are contained inside the circle C, see for instance (DURFEE, 1998; SEKALSKI, 2005) and also Definition 4.6 below. There are at least two families of polynomials where the local index coincide with the global index: *homogeneous* and *weighted homogeneous polynomials*.

Homogeneous polynomials: a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ is *homogeneous* if $f(x,y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$ where $d > 0$ is the degree of f. The equality for homogeneous polynomials follows from the fact that the unique singular point of a homogeneous polynomial f is the origin provided that f has only isolated singularities. Moreover, since f is homogeneous of two variables, it follows that f is product of irreducible homogeneous factors of degree 1 and 2. Consequently, the irreducible components of $f^{-1}(0)$ at the origin coincides with the linear real factors of f. If we denote by r the number of linear real factors of f then we conclude from Proposition 1.3 that $\mathrm{ind}_\infty f = \mathrm{ind}_0 f = 1 - r$.

Weighted-homogeneous polynomials: let $p, q \in \mathbb{N}^*$ be two relative prime positive integers. We say that the polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ is *weighted-homogeneous* of type (p,q) and degree $d > 0$ if $f(x,y) = \sum_{i=1}^{s} a_i x^{\tau_i} y^{\sigma_i}$ where $p\tau_i + q\sigma_i = d$ for all $i = 1, \ldots, s$. For weighted-homogeneous polynomials, a similar argument as in the homogeneous case shows that the origin is the unique singular point provided that f has only isolated singularities. Then $\mathrm{ind}_0 f = \mathrm{ind}_\infty f$.

As far as we known, the study of the index at infinity $\mathrm{ind}_\infty f$ originates in (DURFEE, 1998) and we refer the reader to the subsequent works (SEKALSKI, 2005; GWOŹDZIEWICZ, 2009). In Theorem 4.14 we prove a new formula for the index at infinity by describing the topological behaviour of fibres in a neighborhood at infinity.

Concerning the upper bounds for $\mathrm{ind}_\infty f$, we establish in Lemma 5.8 the origin of the "gaps" between the index realized by some examples and the theoretical upper bound $\mathrm{ind}_\infty f \leq \max\{1, d-3\}$, where d is the degree of f, see (DURFEE, 1998). We review some Durfee's results, clarify some uncleared points in his paper, and thus we present an slightly better upper bound in Theorem 5.11.

In §6 we present some examples for which we describe the behaviour of fibres in a neighborhood at infinity, we compute their index at infinity and also we show that the upper bound for the index, for some of these examples, based in our Lemma 5.8 gives the actual index.

CHAPTER

2

MILNOR SET AND FIBRE COMPONENTS AT INFINITY

Polynomial functions have an intriguing behavior in a neighborhood at infinity which, in the case of 2 real variables, is strongly related to regular atypical values, see (COSTE; DE LA PUENTE, 2001; TIBĂR; ZAHARIA, 1999; DIAS; JOIŢA; TIBĂR, 2021). For instance, in Example 1.1, the polynomial $f = x(xy+1)$ has empty singular set and 0 as its unique atypical value. A further study of its fibres when $\mathbb{K} = \mathbb{R}$ reveals that each fiber $f^{-1}(\pm\varepsilon)$ for $\varepsilon > 0$ close enough to $0 \in \mathbb{R}$, is composed by two fibre components, one of these containing a unique point, say $q_{\pm\varepsilon} \in f^{-1}(\pm\varepsilon)$, where the fibre is tangent to the circle centred at the origin of radius $\|q_{\pm\varepsilon}\|$, and $\lim_{\varepsilon\to 0}\|q_{\pm\varepsilon}\| = \infty$; moreover, when $\varepsilon \to 0$ such fibre component "splits" into two different components of the fibre $f^{-1}(0)$. After (TIBĂR; ZAHARIA, 1999), the splitting phenomenon described above is one among two phenomena[1] of fibres at infinity that produce atypical values. We refer the reader to the subsequent works (JOIŢA; TIBĂR, 2017; DIAS; JOIŢA; TIBĂR, 2021) on the phenomena at infinity of fibres, and also to (COSTE; DE LA PUENTE, 2001; VUI; THAO, 2011) for different point of view of these phenomena.

The above description of the fibres in Example 1.1, suggests that the subset in $\mathbb{R}^2$ where fibres are tangent to some circle detects the phenomena at infinity of fibres of f, such set of tangencies is known in the literature as *Milnor set* of f, see Definition 2.1 below[2]. In this chapter the description of the fibres, via unbounded components of the Milnor set (we name these components *Milnor arcs at infinity*, see Definition 2.14), is given for the class of *primitive* polynomials (see Definition 2.3). Let us remark that by definition the fibres of a *non-primitive* polynomial are all compact, thus, there is no phenomena at infinity to be detected. The main result of this chapter is Theorem 2.26 which states that there is an injective well-defined function

[1] namely, *splitting* and *vanishing* at infinity of fibre components

[2] In fact, the Milnor set has been widely used in the study of the global topology of fibres, i.e. (NÉMETHI; ZAHARIA, 1992; TIBĂR, 1998; TIBĂR, 1999; DIAS; TIBĂR, 2015; DIAS; JOIŢA; TIBĂR, 2021),etc.

between some clusters of Milnor arcs of f and fibre components of f outside a large enough disk.

This chapter is organized as follows: in §2.1 we define the *Milnor set* $M_a(f)$ *relative to* ρ_a of a real polynomial function f in two variables, and give necessary and sufficient conditions for $\dim M_a(f) = 1$, where "dim" denotes the semi-algebraic dimension as in (BOCHNAK; COSTE; ROY, 1998); in §2.2 we present an effective criteria for detecting bounded components of the Milnor set, see Corollary 2.12; in §2.3 we define Milnor arcs at infinity (Definition 2.14) and show that the restriction of f over each of these arcs, is either strictly monotone, or constant (Proposition 2.16); in §2.4 we describe the behavior of the fibers between two Milnor arcs at infinity (Lemmas 2.20 and 2.21), and define *μ-clusters of Milnor arcs at infinity* (Definition 2.19); finally, in §2.5 we prove there is an injective correspondence between μ-clusters and fibre components of f (Theorem 2.26).

2.1 Milnor set and primitive polynomials

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function, and let $\rho_a : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, $\rho_a(x,y) = (x-a_1)^2 + (y-a_2)^2$ be the square of the Euclidean distance to $a := (a_1,a_2) \in \mathbb{R}^2$. After (TIBĂR, 1998; TIBĂR, 1999; ARAÚJO DOS SANTOS; CHEN; TIBĂR, 2013; ARAÚJO DOS SANTOS; CHEN; TIBĂR, 2016), one defines:

Definition 2.1. The Milnor set of $f : \mathbb{R}^2 \to \mathbb{R}$ relative to ρ_a is the set of ρ_a-*nonregular* points of f, namely

$$M_a(f) := \{(x,y) \in \mathbb{R}^2 \mid \rho_a \not\pitchfork_{(x,y)} f\}.$$

Equivalently, the *Milnor set* $M_a(f)$ is the zero set of the determinant of the Jacobian matrix of the map $F_a := (f,\rho_a) : \mathbb{R}^2 \to \mathbb{R}^2$, namely $\mathrm{Jac} F_a(x,y) = 0$.

Notation 2.2. When a is the origin 0 of $\mathbb{R}^2$, we use the simpler notation without the lower index and then we write $M(f)$, ρ, and F instead of $M_0(f)$, ρ_0, and F_0, respectively.

The Milnor set was already used by other authors in the study of the topology of polynomial functions in relation to the phenomena around atypical points at infinity, cf (NÉMETHI; ZAHARIA, 1992; TIBĂR, 1999; DIAS; TIBĂR, 2015; DIAS; JOIŢA; TIBĂR, 2021) etc. In these studies, it is used that the Milnor set is a curve in $\mathbb{R}^2$; however, it does not hold that $M_a(f)$ is always a curve for all $a \in \mathbb{R}^2$, as one can see in $f = x^2 + y^2$, where $M(f) = \mathbb{R}^2$. Nevertheless, in the case of polynomial functions in any number of variables, it is well-known that there exists an open dense set Ω_f in $\mathbb{R}^2$ such that $M_a(f)$ is either empty or a curve for all $a \in \Omega(f)$, see for instance (TIBĂR, 1998; TIBĂR, 1999; ARAÚJO DOS SANTOS; CHEN; TIBĂR, 2013; ARAÚJO DOS SANTOS; CHEN; TIBĂR, 2016; DIAS; TANABÉ; TIBĂR, 2017). In the following we prove a more effective result in 2 variables, see Proposition 2.4 (b) and Lemma 2.5.

Definition 2.3. (DIAS; JOIŢA; TIBĂR, 2021, pag. 1548) We say that f is *primitive relative to* ρ_a, if $f \not\equiv P \circ \rho_a$ for any polynomial P in one variable.

The next result extends (DIAS; TIBĂR, 2015, Lemma 2.3) in our setting of 2 variables, and its proof relies on the characterization of primitive polynomials presented in Lemma 2.5 below.

Proposition 2.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a non-constant polynomial function. Then:

(a) for any $b \in \mathbb{R}^2$, $M_b(f)$ is unbounded and intersects all fibres of f;

(b) there exists at most one point $a \in \mathbb{R}^2$ such that f is not primitive relative to ρ_a;

(c) if f is primitive with respect to ρ_b, then $\dim(M_b(f) \setminus \mathrm{Sing} f) = 1$.

Proof. (a). If $\mathrm{Sing} f$ is unbounded, then $M_b(f)$ is so, since $\mathrm{Sing} f \subset M_b(f)$ by Definition 2.1. Otherwise, for all large enough circles C_b centred at b, the restriction of f to C_b has at least a maximum and a minimum point. These points are among the points where some fibre is tangent to the circle C_b, and thus these points belong to $M_b(f)$ by definition. This proves that $M_b(f)$ is unbounded.

Let us prove that any fibre $f^{-1}(t)$ intersects $M_b(f)$. If t is a critical value of f, then the statement holds since $\mathrm{Sing} f \subset M_b(f)$. If t is a regular value, then the Euclidean distance from the point b to the fibre $f^{-1}(t)$ has a minimum and thus $f^{-1}(t)$ intersects $M_b(f)$ along such points of minimum.

(b). If $\dim M_a(f) = 2$, then we have $f = P \circ \rho_a$ for some polynomial P of one variable by Lemma 2.5 below. For any $b = (b_1, b_2) \neq a = (a_1, a_2)$, the set $M_b(f)$ is defined by the equation:

$$\mathrm{Jac}(P(\rho_a), \rho_b) = 4P'(\rho_a)((a_2 - b_2)x + (b_1 - a_1)y + a_1 b_2 - a_2 b_1) = 0,$$

where P' denotes the derivative of P. Hence $M_b(f)$ is the union of a line with the set $\{P'(\rho_a) = 0\}$, where the later is a union of finitely many circles centred at a since $P'(t) = 0$ has finitely many solutions for $t > 0$. This shows that $\dim M_b(f) \leq 1$. Moreover, for any $b \neq a$, the set $\{(a_2 - b_2)x + (b_1 - a_1)y + a_1 b_2 - a_2 b_1 = 0\}$ is a nonempty line in $\mathbb{R}^2$ passing through b. This proves that $\dim M_b(f) = 1$ and $M_b(f)$ is unbounded, for any $b \neq a$.

(c). If f is primitive with respect to ρ_b, then $\dim M_b(f) < 2$ by Lemma 2.5. Since $M_b(f)$ intersects all fibres of f and since $\mathrm{Im} f$ is unbounded, it follows that $M_b(f)$ is unbounded. Moreover, since $\mathrm{Sing} f$ is contained in finitely many fibres, it also follows that $M_b(f) \setminus \mathrm{Sing} f$ is unbounded. Therefore there exists an unbounded component of $M_b(f)$ intersecting all fibres, and thus $\dim(M_b(f) \setminus \mathrm{Sing} f) = 1$. □

The following lemma is a characterization of primitive polynomials relative to ρ_a. When the center a is the origin of $\mathbb{R}^2$, Lemma 2.5 coincides with (DIAS; JOIŢA; TIBĂR, 2021, Remark 2.4).

Lemma 2.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function and let $a = (a_1, a_2) \in \mathbb{R}^2$. The following conditions are equivalent:

(a) $\dim M_a(f) = 2$,

(b) $\mathrm{Jac} F_a = 0$,

(c) f is not primitive with respect to ρ_a.

Proof. (a)$\Leftrightarrow$ (b) It follows from the fact that $M_a(f)$ is a proper variety of $\mathbb{R}^2$ if, and only if $\dim M_a(f) < 2$ (see for instance (MILNOR, 1968, page.10)).

(c)$\Rightarrow$ (b) If $f = P \circ \rho_a$ for some polynomial in one variable P, then by the chain rule:

$$\mathrm{Jac} F_a(x,y) = P'(\rho_a(x,y))\Big((x-a_1)(y-a_2) - (x-a_1)(y-a_2)\Big) = 0$$

for every $(x,y) \in \mathbb{R}^2$, where P' denotes the derivative of P.

(b)$\Rightarrow$ (c) Let us assume that $\mathrm{Jac} F_a = 0$. Consider the translation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x - a_1, y - a_2)$ and the composition map $F_a \circ T^{-1} = (f \circ T^{-1}, \rho_a \circ T^{-1}) = (f \circ T^{-1}, \rho)$, where ρ is the square of the euclidean distance to the origin. By the chain rule $\mathrm{Jac}(F_a \circ T^{-1}) = \mathrm{Jac}(F_a)\mathrm{Jac}(T^{-1}) = \mathrm{Jac}(F_a) = 0$. Let us denote $g = f \circ T^{-1}$, and thus g is polynomial.

We use the polar change of coordinate $g(r,\theta) = g(r\cos\theta, r\sin\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$ and by the chain rule we obtain the derivatives:

$$\begin{cases} g_r = g_x \cos\theta + g_y \sin\theta, \\ g_\theta = g_x(-r\sin\theta) + g_y(r\cos\theta). \end{cases}$$

Since $\mathrm{Jac}(g,\rho) = 0$, it follows that $g_\theta = 0$. Then the function $g(r\cos\theta, r\sin\theta)$ is a polynomial depending only on r.

If $g(x,y) = \sum_{i,j} a_{i,j} x^i y^j$, then $g(r\cos\theta, r\sin\theta) = \sum_{i,j} a_{i,j} r^{i+j} \cos^i\theta \sin^j\theta$. Setting $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$ we obtain that $a_{i,j} = 0$ for $i+j$ odd, and that in $g(r\cos\theta, r\sin\theta) = \sum_k b_k(\theta) r^{2k}$ the coefficients b_k are independent of θ.

Then we get

$$g(x,y) = \sum_k b_k \left(\sqrt{x^2+y^2}\right)^{2k} = \sum_k b_k (x^2+y^2)^k.$$

Hence $g = P \circ \rho$ where $P(t) := \sum_k b_k t^k$. Therefore $f = P \circ \rho \circ T = P \circ \rho_a$ and this ends the proof. □

Next Corollary follows straightforward from Lemma 2.5 and Proposition 2.4.

Corollary 2.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. Then f is primitive with respect to ρ_a if and only if $\dim M_a(f) = 1$. Moreover, $M_a(f)$ is an unbounded real curve.

2.2 Bounded components of the Milnor set

As we have seen in §2.1 the Milnor set $M_a(f)$ of a primitive polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$, with respect to ρ_a is an unbounded real curve. In this section, for a primitive polynomial, we prove that there exists a compact set D_μ which contains all compact components of the Milnor set. Furthermore, the choice of such compact can be done effectively, see Proposition 2.9 and Remark 2.10. The set D_μ is a key element in our Definition 2.14 of Milnor arcs at infinity.

Let us fix a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$. It follows by Proposition 2.4(b) that the set $\mathbb{R}^2 \backslash \Omega(f)$ of centers a for which f is not primitive with respect to ρ_a is either empty or a singleton. After an adequate linear change of coordinates, if necessary, we assume without loss of generality that f is primitive with respect to ρ, and thus $M(f)$ is a non-empty and unbounded semi-algebraic set of dimension 1, see Definition 2.3 and Corollary 2.6.

Remark 2.7. It follows from Definition 2.1 that the set $M(f)$ is defined by the polynomial $h(x,y) = yf_x - xf_y$, where f_x, f_y denotes the partial derivatives of f.

We denote by $M(f)_{\text{red}}$ the reduced structure of the curve $M(f) = \{\text{Jac}(f,\rho) = 0\}$, i.e. $M(f)_{\text{red}}$ is the zero locus $\{g = g_1 \cdots g_s = 0\}$, where $g_1, \dots, g_s$ are all the polynomial irreducible factors in the decomposition of the polynomial $\text{Jac}(f,\rho)$ with $g_i \neq g_j$ when $i \neq j$.

Definition 2.8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. We define the set

$$\mu(M(f)) := \{p \in M(f) \mid \rho \not\pitchfork_p M(f)_{\text{red}}\}.$$

By definition, $\mu(M(f))$ is a real algebraic set, since it is defined by the equations $g(x,y) = 0$ and $yg_x(x,y) - xg_y(x,y) = 0$, where g is the reduced polynomial defining $M(f)_{\text{red}}$ and g_x, g_y denote the partial derivatives of g with respect to the variables x and y, respectively.

Proposition 2.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function. Then the set $\mu(M(f))$ is the union of finitely many points and circles centred at the origin. In particular, $\mu(M(f))$ is compact.

Proof. Let us consider a semialgebraic Whitney stratification on $M(f)_{\text{red}}$. By the Tarski-Seidenberg principle, all the levels of the distance function ρ, except for finitely many, are transversal to the strata of $M(f)_{\text{red}}$. Hence the restriction $\rho_{|M(f)_{\text{red}}}$ has finitely many critical values and thus $\mu(M(f)) \subset \cup_{i=1}^{s} C_{\lambda_i} \cup \{0\}$, where C_{λ_i} denotes a circle centred at the origin of radius $\lambda_i > 0$, and s is an integer greater or equal than zero. By Bézout Theorem one concludes that $\mu(M(f))$ is a union of finitely many points and circles centered at the origin. $\square$

Remark 2.10. Due to Proposition 2.9, the set $\mu(M(f))$ is contained in an open disk D. The restriction of ρ to a compact component of $M(f)$ has a point of maximum and, by definition, this point is in $\mu(M(f))$. On the other hand, any intersection point of the components of $M(f)$ is a singular point of the curve $M(f)_{\mathrm{red}}$ and by Definition 2.8 it belongs to $\mu(M(f))$. Altogether, this implies that $M(f)\backslash D$ is a finite union of one dimensional manifolds.

In the following, we present a criterion for detecting the finitely many circles in $\mu(M(f))$, cf Proposition 2.9.

Proposition 2.11. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a polynomial. If $\mu(M(f))$ contains a circle $C_r=\{x^2+y^2-r^2=0\}$ of radius $r>0$ centred at the origin, then:

(a) $C_r\subset M(f)$,

(b) C_r is contained in a single fibre of f,

(c) There exists $\lambda\in\mathbb{R}$ such that $f(x,y)-\lambda=(x^2+y^2-r^2)h(x,y)$ and thus $M(f)=C_r\cup M(h)$.

Proof. (a). Follows straightforward from the inclusion $\mu(M(f))\subset M(f)$.

(b). Let $\alpha_r:[0,2\pi]\to C_r$ be the parametric equation of C_r. By (a), the circle $C_r\subset M(f)$ and thus the gradient vector $\nabla f(\alpha_r(t))$ is a multiple scalar of $\alpha_r(t)$. Hence $\langle\nabla f(\alpha_r(t)),\alpha_r'(t)\rangle=0$ for every $t\in[0,2\pi]$. This proves that the restriction $f_{|C_r}$ is constant, equivalently, C_r is contained in a single fibre of f.

(c). It follows from (b) that there exists $\lambda\in\mathbb{R}$ such that the restriction $(f-\lambda)_{|C_r}=0$. Since $x^2+y^2-r^2$ is an irreducible real polynomial in two variables, and C_r is a one-dimensional algebraic set, it follows by (BOCHNAK; COSTE; ROY, 1998, Theorem 4.5.1) that $f-\lambda=(x^2+y^2-r^2)h(x,y)$, for some real polynomial h in two variables.

On the other hand, $M(f)$ is defined by the zero-set of the polynomial function

$$
\begin{aligned}
\mathrm{Jac}(f,\rho)=&x(2yh(x,y)+(x^2+y^2-r^2)h_y(x,y))\\
&-y(2xh(x,y)+(x^2+y^2-r^2)h_x(x,y))\\
=&(x^2+y^2-r^2)(xh_y(x,y)-yh_x(x,y)).
\end{aligned}
$$

This proves that $M(f)=C_r\cup M(h)$. □

From Proposition 2.11 we have:

Corollary 2.12. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a polynomial function and let $C_r=\{x^2+y^2-r^2=0\}$. The following conditions are equivalent:

(a) C_r is contained in $M(f)$,

(b) there exists $\lambda \in \mathbb{R}$ such that $f(x,y) = (x^2 + y^2 - r^2)h(x,y) + \lambda$,

(c) C_r is contained in a connected component of the fibre of f.

By Proposition 2.9 the set $\mu(M(f))$ is compact. Then there is a non-empty subset $\Lambda \subset \mathbb{R}_{>0}$ of positive real numbers $r > 0$ such that $\mu(M(f))$ is contained in the interior of the open disk $D(r)$ centred at the origin of radius r. We refer to the radius $R_\mu := \inf\{r \mid r \in \Lambda\}$ as the *Milnor radius at infinity of* f, and denote D_μ the disk centred at the origin with Milnor radius at infinity of f.

Remark 2.13. For a primitive polynomial f, the set $M(f)\backslash\overline{D_\mu}$ is a disjoint union of finitely many 1-dimensional manifolds.

2.3 Milnor arcs at infinity

This section aims to define *Milnor arcs at infinity* for a primitive polynomial function. As in §2.2, by a linear change of coordinates, if necessary, we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is primitive (with respect to ρ) and thus $M(f) \subset \mathbb{R}^2$ is an unbounded real curve, see Definition 2.3 and Proposition 2.4. Moreover, there exists a compact set $D_\mu \subset \mathbb{R}^2$ such that the semi-algebraic set $M(f)\backslash\overline{D_\mu}$ is a finite union of 1-dimensional unbounded manifolds.

Definition 2.14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function. A connected component γ of $M(f) \setminus \overline{D_\mu}$ will be called a *Milnor arc at infinity of f* or simply *Milnor arc*. We denote by $\mathfrak{M}_{\text{arc}}(f)$ the set of Milnor arcs at infinity of f.

Remark 2.15. The set of critical values $\text{Sing}f \subset \mathbb{R}^2$ of f may contain unbounded components. In this case, each unbounded component produce either one or two Milnor arcs that, by Remark 2.13, do not intersect any other Milnor arc. Consequently, only one of the following situations holds for a Milnor arc γ: either $\gamma \cap \text{Sing}f = \emptyset$ or $\gamma \subset \text{Sing}f$.

By definition, the semi-algebraic unbounded set $M(f)\backslash\overline{D_\mu} \subset \mathbb{R}^2$ of a primitive polynomial function f, is the union of finitely many one dimensional connected components $\gamma_1, \dots, \gamma_s$, see §2.2 to recall the choice of D_μ. One may endow each Milnor arc with a parametrization $\gamma_i :]R_\mu, +\infty[\to \mathbb{R}^2$, $i = 1, \dots, s$, where $R_\mu > 0$ denotes the Milnor radius at infinity of f as we have chose in §2.2. We refer the reader to (BOCHNAK; COSTE; ROY, 1998, Proposition 2.9.10) for further details on the parametrization of 1 dimensional semi-algebraic sets. As a convention, for each Milnor arc at infinity γ we denote by $\gamma(t)$ this parametrization and thus $\lim_{t\to+\infty} \|\gamma(t)\| \to +\infty$.

Proposition 2.16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function and let γ be a Milnor arc at infinity f. Then

(a) the restriction $\rho_{|\gamma(t)}$ is strictly monotonous and tends to $+\infty$ as the parameter t tends to $+\infty$.

(b) if $\gamma \not\subset \operatorname{Sing} f$, then the restriction $f_{|\gamma(t)}$ is strictly monotonous.

Proof. (a) If the composition $\rho(\gamma(t))$ is not monotonous at some parameter t, at the point $\gamma(t) \in \mathbb{R}^2$ the fibre of ρ is not transverse to $M(f)_{\text{red}}$, and thus $\gamma(t) \in \mu(M(f))$ by Definition 2.8. This is a contradiction with the choice of D_μ since $\gamma \subset \mathbb{R}^2 \setminus \overline{D_\mu}$ and $\mu(M(f)) \subset D_\mu$. Therefore the restriction $\rho_{|\gamma(t)}$ is strictly monotonous and tends to infinity.

(b) By contradiction, we assume that $f(\gamma(t))$ is not monotonous at some parameter t_0. By the chain rule

$$\langle \nabla f(\gamma(t_0)), \gamma'(t_0) \rangle = 0, \tag{2.1}$$

where γ' denotes the derivative of the parametrization of γ. On the other hand, our hypothesis $\gamma \not\subset \operatorname{Sing} f$ implies that $\gamma \cap \operatorname{Sing} f = \emptyset$ by Remark 2.15, and thus there exists a non-zero $\lambda \in \mathbb{R}$ such that $\nabla f(\gamma(t_0)) = \lambda \gamma(t_0)$ since $\gamma(t_0) \in M(f)$. Due to (2.1) one has $\langle \gamma(t_0), \gamma'(t_0) \rangle = 0$, which by Definition 2.8 implies that $\gamma(t_0) \in \mu(M(f))$. This is a contradiction since $M(f) \setminus \overline{D_\mu} \cap \mu(M(f)) = \emptyset$, by our choice of D_μ. $\square$

From Proposition 2.16 (b) the restriction of f to each Milnor arc is either strictly increasing or strictly decreasing, so we define:

Definition 2.17. Let f be a primitive polynomial function, and let γ be a Milnor arc at infinity of f such that $\lim_{t\to+\infty} f(\gamma(t)) = \lambda \in \mathbb{R} \cup \{\pm\infty\}$. We say that γ is:

(a) an *increasing Milnor arc to* $\lambda \in \mathbb{R} \cup \{+\infty\}$, if the restriction $f_{|\gamma(t)}$ is strictly increasing as $t \to +\infty$, and we denote this as $f \overset{\gamma}{\nearrow} \lambda$.

(b) a *decreasing Milnor arc to* $\lambda \in \mathbb{R} \cup \{-\infty\}$, if the restriction $f_{|\gamma(t)}$ is strictly decreasing as $t \to +\infty$, and we denote this as $f \overset{\gamma}{\searrow} \lambda$.

(c) a *constant Milnor arc to* $\lambda \in \mathbb{R}$, if the restriction $f_{|\gamma(t)}$ is a constant function, and we denote this as $f \overset{\gamma}{=} \lambda$.

By definition, the Milnor set $M(f)$ is the set of points where the fibres of f are not transverse to the level sets of the Euclidean distance function ρ, see Definition 2.14. For any point q of a Milnor arc γ, the fibre of f passing through q may be either

(a) locally inside the disk D, or

(b) locally outside D, or

(c) a local half-branch inside D and the other local half-branch outside D,

where $D = \{\rho(x,y) \leq \|q\|^2\}$ is the disk centred a the origin of radius $\|q\|$.

We say that the fiber of f at q has a *ρ-maximum*, *ρ-minimum*, or *ρ-inflectional* type of tangency if (a), (b), or (c) holds, respectively.

2.4 μ-Clusters of Milnor arcs

In this section we define *μ-clusters* of Milnor arcs at infinity for a primitive polynomial function f (Definition 2.19), and describe the behavior of the fibre components of f between two "consecutive" Milnor arcs at infinity (Lemmas 2.20 and 2.21). A similar definition of Milnor clusters was given in (VUI; THAO, 2011) in the setting of functions defined over unbounded surfaces in $\mathbb{R}^n$ instead of $\mathbb{R}^2$. Earlier, polar clusters have been defined in (COSTE; DE LA PUENTE, 2001). In this our study we write "μ-clusters" instead of only "clusters" to remark that Milnor clusters are different from polar clusters in (COSTE; DE LA PUENTE, 2001).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function with respect to ρ. By Definition 2.14 and Remark 2.13, any two Milnor arcs at infinity do not intersect mutually. So, if $C \subset \mathbb{R}^2$ is some large enough circle centred at the origin and such that intersects all Milnor arcs, then $M(f) \cap C$ is a finite set of points $\{p_1, \ldots, p_s\}$. We define the following counterclockwise relation between these points[3]: we say that "*p_j is the consecutive of p_k*", or that "*p_k is the antecedent of p_j*", if and only if starting from the point p_k and moving counterclockwise along the circle C one arrives at the point p_j without meeting any other point of the set $M(f) \cap C$.

We also say that $\{p_1, \ldots, p_k\}$ is a sequence of consecutive points of the set $M(f) \cap C$ if and only if p_{i+1} is the consecutive of p_i for all $i = 1, \ldots, k-1$. This relation between the points $M(f) \cap C$ on the circle C allows us to define a similar one among the Milnor arcs at infinity, as follows:

Definition 2.18. (Counterclockwise ordering of Milnor arcs at infinity)
We say that "*γ_j is the consecutive of γ_k*", or that "*γ_k is the antecedent of γ_j*", if and only if the point $p_j := \gamma_j \cap C$ is the consecutive of the point $p_k := \gamma_k \cap C$. This relation is independent on the size of the circle C, provided large enough. We also say that $\{\gamma_1, \ldots, \gamma_k\}$ is a sequence of consecutive Milnor arcs at infinity if and only if $\{p_1, \ldots, p_k\}$, where $p_i := \gamma_i \cap C$, is a sequence of consecutive points of the set $M(f) \cap C$.

With the above definition we present the notion of clusters of Milnor arcs. We refer the reader to Definition 2.17 to recall the notations $f \overset{\gamma_i}{\nearrow} \lambda$ and $f \overset{\gamma_i}{\searrow} \lambda$.

[3] Note that this is not an order relation.

Definition 2.19 (μ-clusters of Milnor arcs at infinity)**.** We call *increasing μ-cluster at $\lambda \in \mathbb{R}\cup\{+\infty\}$* a sequence of consecutive Milnor arcs at infinity $\gamma_k,\ldots,\gamma_{k+l}$, $l \geq 0$, such that the condition $f \overset{\gamma_i}{\nearrow} \lambda$ holds precisely for all $i = k,\ldots,k+l$ and does not hold for the antecedent of γ_k nor for the consecutive of γ_{k+l}.

Similarly, we define a *decreasing μ-cluster at $\lambda \in \mathbb{R}\cup\{-\infty\}$* by replacing $\searrow$ instead of $\nearrow$ in the above definition.

Let C_R be the circle centred at the origin with radius $R > R_\mu$, where R_μ denotes the Milnor radius at infinity of the primitive polynomial f, see §2.2. Let $\gamma_i,\gamma_{i+1} \in \mathfrak{M}_{\mathrm{arc}}(f)$ such that γ_{i+1} is the consecutive of γ_i and let $p_i = \gamma_i \cap C_R$ and $p_{i+1} = \gamma_{i+1}\cap C_R$. We denote by $\Gamma_R^{i,j}$ the set of all points in C_R that one meets when moving counterclockwise along C from p_i to p_{i+1}. Finally, one defines *the band* between γ_i and γ_{i+1} (see also (VUI; THAO, 2011, Definition 2.4)) as:

$$]\gamma_i,\gamma_j[:= \bigcup_{R>R_\mu} \Gamma_R^{i,j}, \tag{2.2}$$

and we denote its topological closure in $\mathbb{R}^2$ by $\left[\gamma_i,\gamma_j\right]$.

The following lemma is adapted from (VUI; THAO, 2011, Lemma 3.1), and it gives an effective criterion to determine the type of tangency of the fibres in each Milnor arc.

Lemma 2.20. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a primitive polynomial function, let $\gamma_i \in \mathfrak{M}_{\mathrm{arc}}(f)$ be increasing to $\lambda\in\mathbb{R}$. Then there exists $\varepsilon > 0$ such that for every $t \in \,]\lambda-\varepsilon,\lambda[$ there exists a unique $s_t > R_\mu$ such that $f(\gamma_i(s_t)) = t$. Moreover:

(a) If $\mathrm{Jac}(f,\rho) > 0$ in $]\gamma_{i-1},\gamma_i[$ and $\mathrm{Jac}(f,\rho) < 0$ in $]\gamma_i,\gamma_{i+1}[$, then there exist δ_- and δ_+ in $\left[R_\mu, s_t\right[$ such that the intersection of $f^{-1}(t)$ with the band $]\gamma_{i-1},\gamma_i[$ $\left(\text{resp. }]\gamma_i,\gamma_{i+1}[\,\right)$ is a continuous curve $\widetilde{h}_{i-1}:]\delta_-,s_t[\to\mathbb{R}^2$ $\left(\text{resp. } \widetilde{h}_{i+1}:]\delta_+,s_t[\to\mathbb{R}^2\right)$ with $\|\gamma_i(s)\| = \|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]\delta_-,s_t[$ $\left(\text{resp. } s\in\,]\delta_+,s_t[\right)$.

(b) If $\mathrm{Jac}(f,\rho) < 0$ in $]\gamma_{i-1},\gamma_i[$ and $\mathrm{Jac}(f,\rho) > 0$ in $]\gamma_i,\gamma_{i+1}[$, then there exist δ_- and δ_+ in $]s_t,+\infty[$ such that the intersection of $f^{-1}(t)$ with the band $]\gamma_{i-1},\gamma_i[$ (resp. $]\gamma_i,\gamma_{i+1}[$) is a continuous curve $\widetilde{h}_{i-1}:]s_t,\delta_-[\to\mathbb{R}^2$ $\left(\text{resp. } \widetilde{h}_{i+1}:]s_t,\delta_+[\to\mathbb{R}^2\right)$ with $\|\gamma_i(s)\| = \|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]s_t,\delta_-[$ $\left(\text{resp. } s\in\,]s_t,\delta_+[\right)$.

(c) If $\mathrm{Jac}(f,\rho)$ has the same sign in $]\gamma_{i-1},\gamma_i[$ and in $]\gamma_i,\gamma_{i+1}[$, then there exist δ_- and δ_+ in $\left]R_\mu,\infty\right[$ such that the intersection of $f^{-1}(t)$ with the band $]\gamma_{i-1},\gamma_i[$ $\left(\text{resp. }]\gamma_i,\gamma_{i+1}[\right)$ is a continuous curve $\widetilde{h}_{i-1}:]\delta_-,s_t[\to\mathbb{R}^2$ $\left(\text{resp. } \widetilde{h}_{i+1}:]s_t,\delta_+[\to\mathbb{R}^2\right)$ with $\|\gamma_i(s)\| = \|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]\delta_-,s_t[$ $\left(\text{resp. } s\in\,]s_t,\delta_+[\right)$, or a continuous curve $\widetilde{h}_{i-1}:]s_t,\delta_-[\to\mathbb{R}^2$ $\big(\text{resp. } \widetilde{h}_{i+1}:]\delta_+,s_t[\to\mathbb{R}^2\big)$ with $\|\gamma_i(s)\| = \|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]s_t,\delta_-[$ $\left(\text{resp. } s\in\,]\delta_+,s_t[\right)$.

Proof. Let us remark that by Definition 2.17 (a), $f_{|\gamma_i(t)}$ takes values $< \lambda$. This, together with the continuity of the same function prove the first part of the statement.

As our convention in this chapter, for $r > 0$ we denote by C_r the circle centred at the origin of radius r. For $r \geq R_\mu$, let us denote by C_r^- the circle sector $C_r \cap [\gamma_{i-1}, \gamma_i]$ endowed with a counterclockwise parametrization denoted by α_r^-. Also the circle sector C_r^+ endowed with a counterclockwise parametrization α_r^+ is defined as above by writing $[\gamma_i, \gamma_{i+1}]$ instead of $[\gamma_{i-1}, \gamma_i]$. The chain rule, applied to $f \circ \alpha_r^\pm$ leads to the equality

$$\frac{d}{d\theta} f(\alpha_r^\pm(\theta)) = -\frac{1}{2} \mathrm{Jac}(f(\alpha_r^\pm(\theta)), \rho(\alpha_r^\pm(\theta))), \tag{2.3}$$

where $\pm$ is replaced by $+$ or $-$ in all its occurrences. It follows then, by (2.3), that $f \circ \alpha_r^\pm$ is strictly monotonous, and whether if $f \circ \alpha_r^\pm$ is increasing or decreasing, it does not depend on r provided $r > R_\mu$, since $\mathrm{Jac}(f, \rho)$ has constant sign in any band between consecutive Milnor arcs. The following proof shows that the sign of $\mathrm{Jac}(f, \rho)$ in each band determines the topological behavior of the fibers within the band, so the arguments of (a), (b), and (c) are analogous. We present the proof only for (a), in order to avoid repetition.

If $\mathrm{Jac}(f, \rho) > 0$ in $]\gamma_{i-1}, \gamma_i[$ it follows by (2.3) that $f \circ \alpha_r^-$ is decreasing, and thus the fibre component F_t of $f^{-1}(t)$ containing $\gamma_i(s_t)$ does not intersect the circle sector $C_r^- \setminus \gamma_i$, where $\gamma_i(s_t)$ denotes the unique point in $C_r^- \cap \gamma_i$. The component F_t is one-dimensional, and thus in $]\gamma_{i-1}, \gamma_i[$, it is contained either in the closed disk $\overline{D_r}$ or in $\mathbb{R}^2 \setminus \overline{D_r}$. For all $r' > r$ the function $f \circ \alpha_{r'}^-$ is also decreasing, and thus it takes values greater than t (recall that $f(\gamma(s_t) = t)$ since γ_i is increasing. Hence $F_t \cap]\gamma_{i-1}, \gamma_i[\subset \overline{D}_r$. Take $R_\mu \leq \delta_- < s_t$ such that the interval $]\delta_-, s_t[$ is the maximal interval contained in $]R_\mu, s_t[$ such that all sector circles C_s^- with $s \in]\delta_-, s_t[$ intersect F_t in a unique point. Furthermore: if $\delta_- = R_\mu$ then F_t intersects C_μ^-, and if $\delta_- > R_\mu$ then $C_{\delta_-}^- \cap F_t \in \gamma_i$ in $]\gamma_{i-1}, \gamma_i]$ since γ_{i-1}, γ_i are consecutive. From the choice of δ_- the correspondence

$$\begin{aligned} \widetilde{h}_{i-1} :]\delta_-, s_t[&\to \mathbb{R}^2 \cap [\gamma_{i-1}, \gamma_i] \\ s &\mapsto \widetilde{h}_{i-1}(s), \end{aligned}$$

where $\widetilde{h}_{i-1}(s)$ is the unique point in $F_t \cap C_s^-$, defines a continuous curve with $\|\gamma_i(s)\| = \|\widetilde{h}_{i-1}(s)\|$.

If $\mathrm{Jac}(f, \rho) < 0$ in the band $]\gamma_i, \gamma_{i+1}[$, the function $f \circ \alpha_r^+$ is strictly increasing. Proceeding as in the previous case, one finds that $F_t \cap [\gamma_i, \gamma_{i+1}]$ is contained in $\overline{D_r}$, and thus for all s in the maximal interval $]\delta_+, s_t[$ such that C_s^+ intersects F_t in a unique point, one defines the continuous curve $\widetilde{h}_{i+1} :]\delta_+, s_t[\to \mathbb{R}^2 \cap [\gamma_i, \gamma_{i+1}]$ with $\|\gamma_i(s)\| = \|\widetilde{h}_{i+1}(s)\|$. This proves (a).

Let us remark that by changing the sign of $\mathrm{Jac}(f, \rho)$ in any of the considered above bands, lets say for instance that $\mathrm{Jac}(f, \rho) < 0$ in $]\gamma_{i-1}, \gamma_i[$, it leads to the situation where $F_t \cap [\gamma_{i-1}, \gamma_i]$ is contained outside D_r, and thus one finds for all s in the maximal interval of the form $]s_t, \delta_-[$, C_s^- intersect in a unique point the fibre component F_t. Therefore one constructs the continuous curve $\widetilde{h}_{i-1} :]s_t, \delta_-[\to \mathbb{R}^2 \cap [\gamma_{i-1}, \gamma_i]$ with $\|\gamma_i(r)\| = \|\widetilde{h}_{i+1}(r)\|$. In this case, if $\delta_- = +\infty$, $F_t \cap]\gamma_{i-1}, \gamma_i[$ goes to infinity in this band; on the other hand, if $\delta_- < +\infty$, then $F_t \cap C_{\delta_-}^-$ is a unique point in γ_i.

After applying these considerations in all cases (b) and (c) we finish the proof. □

One also have Lemma 2.20 for decreasing Milnor arcs. Let us state it by completeness. Its proof follows as in Lemma 2.20.

Lemma 2.21. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a primitive polynomial function, let $\gamma_i\in\mathfrak{M}_{\mathrm{arc}}(f)$ be decreasing to $\lambda\in\mathbb{R}$. Then there exists $\varepsilon>0$ such that for every $t\in\,]\lambda,\lambda+\varepsilon[$ there exists a unique $s_t>R_\mu$ such that $f(\gamma_i(s_t))=t$. Moreover:

(a) If $\mathrm{Jac}(f,\rho)>0$ in $]\gamma_{i-1},\gamma_i[$ and $\mathrm{Jac}(f,\rho)<0$ in $]\gamma_i,\gamma_{i+1}[$, then there exist δ_- and δ_+ in $]s_t,+\infty[$ such that the intersection of $f^{-1}(t)$ with the band $]\gamma_{i-1},\gamma_i[$ (resp. $]\gamma_i,\gamma_{i+1}[$) is a continuous curve $\widetilde{h}_{i-1}:\,]s_t,\delta_-[\,\to\mathbb{R}^2$ $\big($resp. $\widetilde{h}_{i+1}:\,]s_t,\delta_+[\,\to\mathbb{R}^2\big)$ with $\|\gamma_i(s)\|=\|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]s_t,\delta_-[$ $\big($resp. $s\in\,]s_t,\delta_+[\big)$.

(b) If $\mathrm{Jac}(f,\rho)<0$ in $]\gamma_{i-1},\gamma_i[$ and $\mathrm{Jac}(f,\rho)>0$ in $]\gamma_i,\gamma_{i+1}[$, then there exist δ_- and δ_+ in $\big[R_\mu,s_t\big[$ such that the intersection of $f^{-1}(t)$ with the band $]\gamma_{i-1},\gamma_i[$ $\big($resp. $]\gamma_i,\gamma_{i+1}[\,\big)$ is a continuous curve $\widetilde{h}_{i-1}:\,]\delta_-,s_t[\,\to\mathbb{R}^2$ $\big($resp. $\widetilde{h}_{i+1}:\,]\delta_+,s_t[\,\to\mathbb{R}^2\big)$ with $\|\gamma_i(s)\|=\|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]\delta_-,s_t[$ $\big($resp. $s\in\,]\delta_+,s_t[\big)$.

(c) If $\mathrm{Jac}(f,\rho)$ has the same sign in $]\gamma_{i-1},\gamma_i[$ and in $]\gamma_i,\gamma_{i+1}[$, then there exist δ_- and δ_+ in $\big]R_\mu,\infty\big[$ such that the intersection of $f^{-1}(t)$ with the band $]\gamma_{i-1},\gamma_i[$ $\big($resp. $]\gamma_i,\gamma_{i+1}[\big)$ is a continuous curve $\widetilde{h}_{i-1}:\,]\delta_-,s_t[\,\to\mathbb{R}^2$ $\big($resp. $\widetilde{h}_{i+1}:\,]s_t,\delta_+[\,\to\mathbb{R}^2\big)$ with $\|\gamma_i(s)\|=\|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]\delta_-,s_t[$ $\big($resp. $s\in\,]s_t,\delta_+[\big)$, or a continuous curve $\widetilde{h}_{i-1}:\,]s_t,\delta_-[\,\to\mathbb{R}^2$ $\big($resp. $\widetilde{h}_{i+1}:\,]\delta_+,s_t[\,\to\mathbb{R}^2\big)$ with $\|\gamma_i(s)\|=\|\widetilde{h}_{i\pm1}(s)\|$ for all $s\in\,]s_t,\delta_-[$ $\big($resp. $s\in\,]\delta_+,s_t[\big)$.

□

Let us recall that a fibre component intersecting a Milnor arc γ has a type of tangency at its intersection point q with γ, i.e. either ρ-maximum, or ρ-minimum, or ρ-inflectional, see §2.3.

Corollary 2.22. Let γ_i be a Milnor arc at infinity of f, at every point in γ the type of tangency between the fibre component and the circle passing trough is the same. Moreover, this type of tangency is a:

(a) ρ-maximum if, and only if, γ_i is increasing and (a) of Lemma 2.20 holds, or γ_i is decreasing and (b) of Lemma 2.21 holds;

(b) ρ-minimum if and only if γ_i is increasing and (b) of Lemma 2.20 holds, or γ_i is decreasing and (a) of Lemma 2.21 holds.

(c) ρ-inflectional if and only if (c) of Lemma 2.20.

Proof. It follows from Lemma 2.20 and Lemma 2.21 that the behavior of any fibre in the bands $]\gamma_{i-1},\gamma_i[$, $]\gamma_i,\gamma_{i+1}[$ depends only on the constant non-zero sing of $\mathrm{Jac}(f,\rho)$ in each band. This implies that the first part of the corollary. The second part follows form item (a), (b), (c) in Lemmas 2.20 and 2.21. □

Corollary 2.23. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function, let $\mathscr{C}$ be a μ-cluster of f at $\lambda \in \mathbb{R}\cup\{\pm\infty\}$.

(a) if $\mathscr{C}$ is increasing (or decreasing) at $\lambda \in \mathbb{R}$, then there exists $\varepsilon > 0$ such that for all $t \in]\lambda-\varepsilon,\lambda[$ (or $t \in]\lambda,\lambda+\varepsilon[$, respectively) the fibre $f^{-1}(t)$ intersects each Milnor arc in $\mathscr{C}$ in a unique point. Moreover, all these points belong to the same fibre component of $f^{-1}(t)$;

(b) if $\mathscr{C}$ is increasing (or decreasing) at $\lambda = +\infty$ (or $\lambda = -\infty$, respectively), then there exists $M > 0$ large enough such that for all $t > M$ (or $t < -M$, respectively) the fibre $f^{-1}(t)$ intersects each Milnor arc in $\mathscr{C}$ in a unique point. Moreover, all these points belong to the same fibre component of $f^{-1}(t)$.

Proof. By Definition 2.19 all the Milnor arcs in $\mathscr{C}$ are increasing (or decreasing), if $\mathscr{C}$ is increasing (or decreasing, respectively) and tend to $\lambda \in \mathbb{R}\cup\{\pm\infty\}$ in the sense of Definition 2.17. First we prove (a) so we start assuming $\lambda \in \mathbb{R}$. It follows from Lemmas 2.20 and 2.21 that there exists $\varepsilon > 0$ such that for all $t \in]\lambda-\varepsilon,\lambda[$ (or $t \in]\lambda,\lambda+\varepsilon[$, respectively) the fibre $f^{-1}(t)$ intersects each Milnor arc in $\mathscr{C}$ in a unique point; moreover, for any two consecutive Milnor arcs γ_i,γ_{i+1} in $\mathscr{C}$, the fibre $f^{-1}(t)$ defines a unique continuous curve in the band $]\gamma_i,\gamma_{i+1}[$, by (a), (b), and (c) of Lemmas 2.20 and 2.21, and therefore the same connected component $F_{t,i}$ of $f^{-1}(t)$ intersects γ_i and γ_{i+1}. The same argument applied to all (finitely many) pairs of consecutive Milnor arcs in $\mathscr{C}$ leads to the conclusion that $F_{t,i}$ intersects all the Milnor arcs in $\mathscr{C}$. Notice that $F_{t,i}$ is the unique fibre component of $f^{-1}(t)$ intersecting all the arcs in $\mathscr{C}$, since the restriction of f to each Milnor arc is strictly monotonous by Proposition 2.16. This ends the proof of (a).

Let us prove (b), so let $\mathscr{C}$ be increasing (or decreasing) at $+\infty$ (or at $-\infty$, respectively). Since there are finitely many μ-clusters, the amount of values $\beta \in \mathbb{R}$ for which there are clusters at β is finite and thus bounded in $\mathbb{R}$. Let $M > 0$ such that for all values $t \in]M,+\infty[$ (or $t \in]-\infty,-M[$, respectively) there are no clusters at t. As in the proof of (a), Lemmas 2.20 and 2.21 lead us to the conclusion that there is a unique fibre component $F_{t,i}$ of $f^{-1}(t)$ intersecting all the Milnor arcs in $\mathscr{C}$ for all $t > M$ (or $t < -M$, respectively). This ends the proof of (b). □

It follows from Corollary 2.22 that two consecutive Milnor arcs in the same μ-cluster can not have the same type of tangency being ρ-maximum or ρ-minimum. Moreover:

Corollary 2.24. Let $\mathscr{C}$ be a μ-cluster and let $\gamma_i,\gamma_{i+l} \in \mathscr{C}$ with $l \geq 1$, such that

(a) the type of tangency of γ_i is ρ-maximum (or, ρ-minimum), and

(b) the type of tangency of γ_{i+k} is ρ-inflectional for all $k = 1,\dots,l-1$.

Then the type of tangency of γ_{i+l} is a ρ-minimum (or ρ-maximal, respectively).

2.5 μ-Clusters and fibre components

As the general assumption in this chapter, we fix a primitive polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ and let $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ such that there is at least one increasing (or decreasing) μ-cluster at λ. In Corollary 2.23 above we proved that for values t close enough to λ, there exists a unique connected component of the fibre $f^{-1}(t)$ intersecting each Milnor arc in $\mathscr{C}$. One may read this as: for $t < \lambda$ (or $t > \lambda$,respectively) close enough to λ the correspondence between increasing (decreasing, respectively) μ-clusters at λ and fibre components defined as:

$$\begin{aligned} \alpha_t : \{\mathscr{C}_i\}_{i=1,\dots,r} &\to \{\text{fibre components of } f^{-1}(t)\} \\ \mathscr{C}_i &\mapsto F_{t,i}, \end{aligned} \tag{2.4}$$

is a well defined function, where $F_{t,i}$ denotes the fibre component of $f^{-1}(t)$ intersecting all Milnor arcs of f in $\mathscr{C}_i$.

It is not true that (2.4) above is injective. For instance, $f(x,y) = (x-y^2)((x-y^2)(y^2+1)-1)$ in Example 6.3, has non-empty compact fibers for all values in $]-\frac{1}{4},0[$ which intersect two different increasing μ-clusters, namely, $\{\gamma_7\},\{\gamma_3\}$ at the value 0. In this section we prove that by considering the fibre components of the restriction of f outside some compact set in $\mathbb{R}^2$, (2.4) is injective, see Theorem 2.26.

Remark 2.25. For all $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ there are finitely many increasing (or decreasing) Milnor arcs at infinity of f to λ, and therefore there is a finite amount of increasing (or decreasing, respectively) μ-clusters associated to λ that we denote by $\mathscr{C}_1,\dots,\mathscr{C}_k$. By Corollary 2.23 there are positive real numbers $\varepsilon_1,\dots\varepsilon_k$ small enough such that for all $t \in]\lambda-\varepsilon_i,\lambda[$ (or $t \in]\lambda,\lambda+\varepsilon_i[$, respectively) there is a unique fibre component $F_{t,i}$ of the fibre $f^{-1}(t)$ intersecting all the Milnor arcs in C_i for all $i = 1\dots,k$. By taking the minimum, one may assume that there exists $\varepsilon > 0$ such that for all $t \in]\lambda-\varepsilon,\lambda[$ (or $]\lambda,\lambda+\varepsilon[$, respectively) $\alpha_t(\mathscr{C}_i)$ intersects $\mathscr{C}_i$, $i = 1,\dots,k$

Let us fix $\lambda \in \mathbb{R}$ such that $\mathscr{C}_1,\dots,\mathscr{C}_r$ are increasing (or decreasing) μ-clusters associated to λ and let us assume that the fibre $f^{-1}(\lambda)$ has a compact critical set. It follows from (DIAS; JOIŢA; TIBĂR, 2021, Lemma 2.9) that there exists $R_\lambda > 0$ large enough and depending on λ such that $f^{-1}(\lambda) \cap M(f) \setminus \overline{D_{R_\lambda}} = \emptyset$. Let $f_{R_\lambda} : \mathbb{R}^2 \setminus \overline{D_{R_\lambda}} \to \mathbb{R}$ denote the restriction of f outside $\overline{D_{R_\lambda}}$. With this we have the following Theorem:

Theorem 2.26. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function. Let $\mathscr{C}_1,\dots,\mathscr{C}_r$ be all the increasing (or decreasing) μ-clusters associated to $\lambda \in \mathbb{R}$. Then there exists $\eta > 0$ such that for every $t \in]\lambda-\eta, \lambda[$ (or $t \in]\lambda, \lambda+\eta[$, respectively) the function is injective:

$$\begin{aligned} \alpha_t : \{\mathscr{C}_i\}_{i=1,\dots,r} &\to \{\text{Connected components of } f_{R_\lambda}^{-1}(t)\} \\ \mathscr{C}_i &\mapsto F_{t,i}, \end{aligned} \tag{2.5}$$

where $F_{t,i}$ is a connected component of $f_{R_\lambda}^{-1}(t)$ intersecting all Milnor arcs at infinity of f in $\mathscr{C}_i$.

Proof. Let us agree that for two Milnor arcs γ, β, by "$\gamma < \beta$" we mean that β is the consecutive of γ, and equivalently γ is the antecedent of β, as in Definition 2.18. With this convention, let $\mathscr{C}_i = \{\gamma_1^i < \ldots < \gamma_s^i\}$ and $\mathscr{C}_j = \{\gamma_1^j < \ldots < \gamma_l^j\}$ be two different increasing (decreasing, respectively) μ-clusters associated to λ. We remark that none of the Milnor arcs in $\mathscr{C}_i$ is consecutive nor antecedent to some Milnor arc in $\mathscr{C}_j$, otherwise the considered μ-clusters are not different, see Definition 2.18. There exists a sequence of consecutive Milnor arcs $S = \{\xi_0 < \xi_1 < \ldots < \xi_{k-1} < \xi_k\}$[4] with $k > 1$, such that $\xi_0 = \gamma_s^i, \xi_k = \gamma_1^j$ or $\xi_0 = \gamma_l^j, \xi_k = \gamma_1^i$. Without loss of generality we assume that $\gamma_s^i = \xi_0, \gamma_1^j = \xi_k$, and thus there is at least one Milnor arc in S such that it is not increasing (decreasing) to λ, since S is not a μ-cluster. We denote by ξ such Milnor arc in S.

It follows by Corollary 2.23 and Remark 2.25 that there is $\eta > 0$ small enough such that $\alpha_t(\mathscr{C}_i)$ and $\alpha_t(\mathscr{C}_i)$ intersect $\mathscr{C}_i$ and $\mathscr{C}_j$[5], respectively for $t \in]\lambda-\eta, \lambda[$ ($t \in]\lambda, \lambda+\eta[$) and all these intersections occurs outside $\overline{D_{R_\lambda}}$. This implies that there are components of $f_{R_\lambda}^{-1}(t)$ intersecting these two clusters. The injectivity of (2.5) follows after proving that the component $F_{i,t}$ of $f_{R_\lambda}^{-1}(t)$ intersecting $\mathscr{C}_i$ does not intersect $\mathscr{C}_j$. By contradiction, let us suppose that $F_{i,t}$ also intersects $\mathscr{C}_j$, and thus it intersects γ_s^i and γ_1^j. It follows from the connectedness that $F_{i,t}$ intersects ξ outside $\overline{D_{R_\lambda}}$, and thus ξ is an increasing μ-cluster, since γ_s^i is increasing. Moreover, it follows from our construction that ξ is a Milnor arc at some $\beta > \lambda$ ($\beta < \lambda$). All this together and since the restriction $f(\xi(s))$ is increasing (decreasing) (see Proposition 2.16 (b)) implies that $f^{-1}(\lambda) \cap M(f) \setminus \overline{D_{R_\lambda}} \neq \emptyset$. This is a contradiction with the choice of R_λ. □

4 Notice that, by construction S is not a μ-cluster.
5 defined by (2.4)

CHAPTER

3

CHARACTERIZATION OF ATYPICAL VALUES

Let $f : \mathbb{K}^2 \to \mathbb{K}$ be a polynomial function where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The set of *Atypical values* or *Bifurcation set of* f (Definition 1.2) is, by definition, the set of values λ where f is not locally trivial at λ. It was proved in (THOM, 1969) that for $\mathbb{K} = \mathbb{C}$ the set $Atyp\, f$ is finite. In the real setting i.e. $\mathbb{K} = \mathbb{R}$, Verdier proved in (VERDIER, 1976) that $Atyp\, f$ is also finite. It is known that $f(\mathrm{Sing} f) \subset Atyp\, f$, where $\mathrm{Sing} f$ denotes the singular set of f, see for instance (TIBĂR, 2007). However, $Atyp\, f$ may contain regular values as Example 1.1 shows since $\mathrm{Sing} f = \emptyset$ and $Atyp\, f = \{0\}$. Therefore, the set $Atyp\, f$ is the union of the finite set of critical values of f with the set of atypical values which are regular values. The problem of finding $Atyp\, f$ relies on the detection of regular atypical values.

In the complex setting ($\mathbb{K} = \mathbb{C}$), and motivated by the Jacobian conjecture, Suzuki proved in (SUZUKI, 1974)[1] that: a regular value λ is atypical if and only if $\chi(f^{-1}(\lambda)) \neq \chi(f^{-1}(t))$, where $f^{-1}(t)$ is a general fibre and χ denotes the Euler characteristic. In 1995 this characterization was extended in (PARUSIŃSKI, 1995) and (SIERSMA; TIBĂR, 1995) to polynomials $f : \mathbb{C}^n \to \mathbb{C}$ with *isolated singularities at infinity*.

In the real setting, the variation of the Euler characteristic of fibres is not enough to conclude the existence of atypical values. In (TIBĂR; ZAHARIA, 1999), it is proved that for a function $f : X \to \mathbb{R}$ which is the restriction of a polynomial function $F : \mathbb{R}^n \to \mathbb{R}$ to a smooth non-compact algebraic surface $X \subset \mathbb{R}^n$, the atypical values are produced by some phenomena of fibres that may occur at infinity, namely, *vanishing* and *splitting* of fibres at infinity, see Definition 3.2 below. After this characterization, the next aim was to give a method to detect the vanishing and splitting at infinity. In (DIAS; JOIŢA; TIBĂR, 2021) the authors present an effective algorithm that detects these fibre phenomena at infinity of a polynomial in two real variables based on the localization of the behavior of fibres at some points at infinity, see also

[1] See also the subsequent work (VUI; TRÁNG, 1984)

§3.2. In what concerns the detection of atypical values, the authors in (COSTE; DE LA PUENTE, 2001) use certain truncated parametrizations of semialgebraic affine curves without treating the effectivity aspect.

In this chapter we present a new method of detection of atypical values inspired by (COSTE; DE LA PUENTE, 2001; VUI; THAO, 2011) and we focused in the effective aspects. The main result of this chapter is Theorem 3.5 where we use the notion of *μ-clusters* (Definition 2.19) of a primitive polynomial function in order to detect the phenomena of vanishing and splitting as defined in (DIAS; JOIŢA; TIBĂR, 2021), see also Definition 3.2 below.

The detection of regular atypical values in this chapter is based in the detailed study of the Milnor set $M(f)$ presented in §2, and it has mainly two new aspects:

(i) we give an effective way of detecting a compact set in $\mathbb{R}^2$ such that on its complement the clusters are well-defined (cf. the compact set K in Definition 3.1). In (COSTE; DE LA PUENTE, 2001; VUI; THAO, 2011) the existence of such compact set is theoretically proved with semi-algebraic properties but the effectivity is not treated. We also remark that in (COSTE; DE LA PUENTE, 2001) and (VUI; THAO, 2011) one uses the notion of clusters to detect *cleaving* instead of splitting,

(ii) Theorem 2.26 proves an injective correspondence between clusters and connected components of fibres of f outside an effectively determined compact set contained in the compact set in (i).

The chapter is organized as follows: in §3.1 after (DIAS; JOIŢA; TIBĂR, 2021), we present the definition of vanishing and splitting. In §3.2 we present the main result, Theorem 3.5, which characterize atypical values in terms of *odd* clusters, see Definition 3.4. In §3.3 we recall some results of (DIAS; JOIŢA; TIBĂR, 2021) revealing the behavior of the fibre components of values close to an atypical fiber in a neighborhood of infinity. Finally, in §3.4 we describe the algorithmic aspect of our effective detection based in the results presented in §2 and the previous sections of this Chapter. This process has several steps: finding the Milnor radius, finding the values associated to the Milnor arcs, finding the μ-clusters, and finally finding the atypical values. The algorithm is applied in two examples selected from (TIBĂR; ZAHARIA, 1999), see §A. These examples show intriguing phenomena: Example A.1 shows that both vanishing and splitting at infinity may occurs in a single atypical value. Example A.2 shows that clusters may exist at $\lambda \in \mathbb{R}$ without λ being an atypical value.

3.1 Vanishing and Splitting at infinity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. In this section we define the phenomena at infinity that produce the atypical regular values of f, i.e. *vanishing* and *splitting at infinity* (Definition 3.2). We present a characterization of the atypical values, proved in (DIAS; JOIŢA; TIBĂR, 2021) (see Proposition 3.3).

In the setting of a differentiable map $g : M \to \mathbb{R}^p$ defined over a Riemann compact surface M, Ehresmann theorem says that g is a locally trivial fibration over a small neighborhood of a regular value $v \in \mathbb{R}^p$ of g. In our context of polynomial function defined in $\mathbb{R}^2$, one cannot apply this result because we cannot control the trivialization in a "neighborhood at infinity", see for instance Example 1.1 above. In order to focus in the behavior at infinity of the fibers and separate it from its behavior in a compact set, one may replace "atypical values" by the notion of "*atypical values at infinity*". We consider the following definition:

Definition 3.1. (DIAS; JOIŢA; TIBĂR, 2021, Definition 1.2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. We say that λ is a *typical value at infinity* of f if there exists an interval $I :=]\lambda - \eta, \lambda + \eta[\subset \mathbb{R}$ centred at λ and a compact set $K \subset \mathbb{R}^2$ such that the restriction $f_| : f^{-1}(I) \setminus K \to I$ is a C^∞ trivial fibration. If the contrary occurs, then we say that λ is an *atypical value at infinity* of f and then its corresponding fibre $f^{-1}(\lambda)$ is also called *atypical at infinity*.

After (DIAS; JOIŢA; TIBĂR, 2021) we define the phenomena of *vanishing* and *splitting at infinity* of fibre components. First we introduce the notion of limit of sets: let $\{M_t\}_{t\in\mathbb{R}}$ be a family of sets in $\mathbb{R}^2$, the limit set of the family $\{M_t\}_{t\in\mathbb{R}}$ when $t \to \lambda$ denoted by $\lim_{t\to\lambda} M_t$ is the set of points $x \in \mathbb{R}^2$ such that there exists a sequence $t_k \in \mathbb{R}$ with $t_k \to \lambda$ and a sequence of points $x_k \in M_{t_k}$ such that $x_k \to x$.

A non-empty fibre $f^{-1}(t)$ is the union of finitely many non-empty fibre components $F_{t,i}$, i.e. $f^{-1}(t) = \sqcup F_{t,i}$. With the definition of limit set and the notation of fibre components we define:

Definition 3.2. (DIAS; JOIŢA; TIBĂR, 2021) Let $\lambda \in \mathbb{R}$ such that $\mathrm{Sing} f^{-1}(\lambda)$ is a compact set.

(i) One says that f *has a vanishing at infinity at* λ, if either $\lim_{t\nearrow\lambda} \max_j \inf_{q\in F_{t,j}} \|q\| = \infty$ or $\lim_{t\searrow\lambda} \max_j \inf_{q\in F_{t,j}} \|q\| = \infty$. Otherwise, we say that f *has no vanishing at infinity at* λ and we denote it shortly by $\mathrm{NV}(\lambda)$.

(ii) One says that f *has a splitting at infinity at* λ if there exists $\eta > 0$ and a continuous family of analytic paths $\phi_t : [0,1] \to f^{-1}(t)$ for $t \in]\lambda - \eta, \lambda[$ or for $t \in]\lambda, \lambda + \eta[$, such that:

(1) $\mathrm{Im}\phi_t \cap M(f) \neq \emptyset$, with $\lim_{t\nearrow\lambda} \|q_t\| = \infty$ (or $\lim_{t\searrow\lambda} \|q_t\| = \infty$, resp.) for any $q_t \in \mathrm{Im}\phi_t \cap M(f)$, and

(2) the limit set $\lim_{t\nearrow\lambda} \mathrm{Im}\phi_t$ (or $\lim_{t\searrow\lambda} \mathrm{Im}\phi_t$, resp.) is not connected.

Otherwise we say that f *has no splitting at infinity at* λ and we denote it by $\mathrm{NS}(\lambda)$.

The notion of $\mathrm{NS}(\lambda)$ in Definition 3.2 (ii) is precisely the "strong non-splitting" in (JOIŢA; TIBĂR, 2017, Definintion 4.3), and $\mathrm{NV}(\lambda)$ coincides with "non-vanishing" in (JOIŢA; TIBĂR, 2017, Definition 3.1). The proof of Theorem 3.5 relies on the equivalence between the existence of atypical values and the vanishing and splitting phenomena. This is proved in (JOIŢA; TIBĂR, 2017, Corollary 4.7) and subsequently extended in (DIAS; JOIŢA; TIBĂR, 2021, Theorem 2.8) as follows:

Proposition 3.3. (DIAS; JOIŢA; TIBĂR, 2021, Theorem 2.8) Let $f^{-1}(\lambda)$ be a fibre with a compact set. Then $NV(\lambda)$ and $\mathrm{NS}(\lambda)$ if and only if λ is a typical value of f.

3.2 Atypical values and Clusters

In this section we prove our main Theorem 3.5 of characterization of atypical values, which follows the spirit of the results given in (COSTE; DE LA PUENTE, 2001; VUI; THAO, 2011). Different from the statements in the two mentioned papers, our main theorem involves the number of Milnor arcs in a μ-cluster (see Definition 3.4), and its proof is based in the effective detection of clusters presented in §2.

Let us recall that a polynomial f which is not primitive with respect to $\rho := x^2 + y^2$ is the composition $f = P \circ \rho$ for some polynomial P in one variable (see Definition 2.3), and thus its fibres are circles centred at the origin. It follows from Proposition 3.3 that $Atyp\, f$ coincides with the set of critical values of f, since $\mathrm{NV}(\lambda)$ and $\mathrm{NS}(\lambda)$ for all $\lambda \in \mathbb{R}$, see Definition 3.2. Therefore, we focus our study to the case of a primitive polynomial function f, and so $\dim M(f) = 1$, see Lemma 2.5 and Proposition 2.4.

By Proposition 3.3 the atypical fibres of a primitive polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ are exactly the ones for which the vanishing or splitting at infinity occurs, and by Definition 3.2, for each of these phenomena at infinity there exists at least a Milnor arc at infinity at the same atypical value. On the other hand the presence of Milnor arcs at some finite value $\lambda \in \mathbb{R}$, is not an indicator of an atypical fibre, as shown by Example A.2, which has 2 μ-clusters at 0, each of them composed by two Milnor arcs, and so 4 Milnor arcs at the value 0, but $Atyp\, f = \emptyset$. [2]

The following combinatorial property on μ-clusters differentiate the clusters detecting atypical fibres.

Definition 3.4. Let $\mathscr{C}$ be a μ-cluster associated to λ. We say that $\mathscr{C}$ is odd (or even) if the number of Milnor arcs at infinity with ρ-maximum type in $\mathscr{C}$ added by the number of Milnor arcs at infinity with ρ-minimum type in $\mathscr{C}$ is an odd (resp. even) number.

We are ready to prove the main theorem of this section.

Theorem 3.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function. A regular value $\lambda \in \mathbb{R}$ is an atypical value of f if and only if there exists an odd μ-cluster associated to λ.

[2] Let us remark that in the setting of complex polynomials of 2 variables where the existence of the Milnor set at a point at infinity is a precise indicator of the existence of an atypical fibre, see e.g. (SIERSMA; TIBĂR, 1995; TIBĂR, 2007)

Proof. The proof for increasing and decreasing μ-clusters is analogous. Without loss of generality, we assume that $\mathscr{C}$ is an increasing μ-cluster associated to λ. Since the vanishing and splitting at infinity of fibres are phenomena that occur outside some compact set, we focus in the restriction $f_{|R}$ of f outside the compact disk $\overline{D}_R$ with $R > 0$, where the later is the radius given by Theorem 2.26 an also let us consider $\eta > 0$ given by the same Theorem. With this, for each μ-cluster $\mathscr{C}$ at λ it corresponds a unique fibre component $\alpha_t(\mathscr{C})$ of $f_{|R}^{-1}(t)$ for all $t \in]\lambda - \eta, \lambda[$, and by taking a smaller $\eta > 0$ if necessary, we assume that all values in $]\lambda - \eta, \lambda[$ are regular values of f.

We study all possible behaviors of $\alpha_t(\mathscr{C})$, depending on the parity of $\mathscr{C}$, see Definition 3.4. Notice that among the finitely many Milnor arcs in $\mathscr{C}$, there may be arcs with ρ-inflectional type of tangency as defined at the end of §2.3. By dismissing such “inflectional” arcs we obtained a sequence of arcs $\gamma_{s_1}, \cdots, \gamma_{s_l}$ in $\mathscr{C}$ (not necessarily consecutive) such that for each $i = 1, \ldots l-1$ either $\gamma_{s_{i+1}}$ is the consecutive of γ_{s_i} or the sequence of consecutive arcs between γ_{s_i} and $\gamma_{s_{i+1}}$ in the order of arcs are all “inflectional”. Furthermore, the type of tangency of each arc in the sequence $\gamma_{s_1}, \cdots, \gamma_{s_l}$ is either ρ-maximum or ρ-minimum.

By Corollary 2.24 any two arcs $\gamma_{s_i}, \gamma_{s_{i+1}}$ cannot have the same type of tangency and so the following situations do not occur: $\mathscr{C}$ is odd and $\gamma_{s_1}, \gamma_{s_l}$ have different type of tangency; $\mathscr{C}$ is even and $\gamma_{s_1}, \gamma_{s_l}$ have the same type of tangency. This lead us to consider only the four other situations, namely:

(1) $\mathscr{C}$ is odd, γ_{s_1} and γ_{s_l} have ρ-minimum type of tangency. In this case, the restriction $\rho_{|\alpha_t(\mathscr{C})}$ has a global minimum at some $p \in \alpha_t(\mathscr{C})$ that we denote by $\inf_{p\in\alpha_t(\mathscr{C})} \|p\|$. Since $\mathscr{C}$ is increasing and associated to λ, one has that $\lim_{t\nearrow\lambda} \inf_{x\in\alpha_t(\mathscr{C})} \|x\| = \infty$. This proves that f has a vanishing at infinity at λ by Definition 3.2 (i).

(2) $\mathscr{C}$ is odd, γ_{s_1} and γ_{s_l} have ρ-maximum type of tangency. In this case, we construct a family of continuous analytic paths $\phi_t : [0,1] \to \alpha_t(\mathscr{C})$ such that $\mathrm{Im}\phi_t \cap \gamma_{s_1} = q_t$ and $\lim_{t\nearrow\lambda} \|q_t\| = \infty$, by applying the Implicit Function Theorem to each regular point $q_t \in \gamma_{s_1} \cap \alpha_t(\mathscr{C})$. It follows from Lemma 2.20 that for all $t \in]\lambda - \eta, \lambda[$, $\alpha_t(\mathscr{C}) \cap]\gamma_{s_1-1}, \gamma_{s_1}[\neq \emptyset$ and $\alpha_t(\mathscr{C}) \cap]\gamma_{s_{2k+1}}, \gamma_{s_{2k+1}+1}[\neq \emptyset$, where γ_{s_i-1} denotes the antecedent of γ_{s_1} and $\gamma_{s_{2k+1}+1}$ denotes the consequent of $\gamma_{s_{2k+1}} = \gamma_{s_l}$, see Definition 2.18. Since $\lim_{t\nearrow\lambda} \|q_t\| = \infty$ we conclude that $\lim_{t\nearrow\lambda} \mathrm{Im}\phi_t$ has at least two connected components: one in $]\gamma_{s_1-1}, \gamma_{s_1}[$ and another one in $]\gamma_{s_{2k+1}}, \gamma_{s_{2k+1}+1}[$. This proves that $\alpha_t(\mathscr{C})$ has a splitting at infinity by Definition 3.2 (ii).

(3) $\mathscr{C}$ is even, γ_{s_1} is a ρ-maximum and γ_{s_l} is a ρ-minimum. In this case, the intersection $\alpha_t(\mathscr{C}) \cap]\gamma_{s_1-1}, \gamma_{s_1}[$ is not empty and it is contained in the disk $D_{\|q_t^1\|}$, where $q_t^1 = \gamma_{s_1} \cap \alpha_t(\mathscr{C})$ for all $t \in]\lambda - \eta, \lambda[$. This implies that $\lim_{t\nearrow\lambda} \alpha_t(\mathscr{C}) \neq \emptyset$, and thus $\alpha_t(\mathscr{C})$ has no vanishing at infinity by Definition 3.2 (i). On the other hand, the set $\alpha_t(\mathscr{C}) \cap]\gamma_{s_l}, \gamma_{s_l+1}[$ is contained in the exterior of the disk $D_{\|q_t^l\|}$, where $q_t^l = \alpha_t(\mathscr{C}) \cap \gamma_l$, and thus $\lim_{t\nearrow\lambda} \alpha_t(\mathscr{C}) \cap]\gamma_{s_l}, \gamma_{s_l+1}[= \emptyset$, since $\lim_{t\nearrow\lambda} \|q_t^l\| = \infty$. Therefore for every family of continuous paths $\phi_t : [0,1] \to \alpha_t(\mathscr{C})$ as in

Definition 3.2 (ii) the limit set $\lim_{t\nearrow\lambda} \mathrm{Im}\phi_t$ is contained in $\lim_{t\nearrow\lambda} \alpha_t(\mathscr{C}) \subset]\gamma_{s_1-1}, \gamma_{s_1}[$, and thus $\mathrm{Im}\phi_t$ is connected. This implies that $\alpha_t(\mathscr{C})$ has no splitting at infinity.

(4) $\mathscr{C}$ is even, γ_{s_1} is a ρ-minimum and γ_{s_l} is a ρ-maximum. This case is analogous as in (3) and we conclude that $\alpha_t(\mathscr{C})$ has no vanishing and no splitting at infinity.

Altogether this implies that: $\mathrm{NV}(\lambda)$ and $\mathrm{NS}(\lambda)$ if and only if the μ-clusters associated to λ are all even. After applying Proposition 3.3 we finish the proof. □

3.3 Points at infinity of μ-clusters

In the previous section we detected the phenomena of fibres at infinity that produce the atypical values of a polynomial f (Theorem 3.5) with the affine properties of its Milnor set $M(f)$. For this section we present a different approach based on the effective detection of vanishing and splitting at infinity in (DIAS; JOIŢA; TIBĂR, 2021), namely, the localization of the behavior of fibres at certain points at infinity (see Definition 3.7). We prove that the Milnor arcs in a μ-cluster at some finite value have all the same point at infinity (Corollary 3.11 and Lemma 3.12), and we also prove that the tangent line of a splitting and vanishing component is shared by the tangent line of the Milnor arcs in the μ-cluster intersecting the component, see Propositions 3.13 and 3.14. These results are key for the estimation of the *index at infinity* in §5

Let f be a polynomial function of degree $d > 1$, and let $\widetilde{f}(x,y,z)$ be the homogenization of degree d with respect to the variable z of f.

Let $X := \{\tilde{f}(x,y,z) - tz^d = 0\} \subset \mathbb{P}^2 \times \mathbb{R}$, let $\tau : X \to \mathbb{R}$ be the projection on the second factor, and let $X_t := \tau^{-1}(t)$ be the fibre of τ over t. Let $L^\infty := \{z = 0\} \simeq \mathbb{P}^1$ be the line at infinity of $\mathbb{P}^2$. The part at infinity $X^\infty := X \cap L^\infty \times \mathbb{R} = \{f_d = 0\} \times \mathbb{R}$ consists of finitely many lines. The algebraic space X may be endowed with a Whitney stratification such that X^∞ is a union of strata. This consists of the open stratum $\mathbb{R}^2 \subset X$ of dimension 2, and the finitely many strata of X^∞ which are either of dimension 1 or of dimension 0. Let us denote by $\mathscr{S}_0$ the finite set which is the union of these strata of dimension 0.

Then $X_t \subset \mathbb{P}^2$ is a closed set and contains the closure $\overline{F_t}$ of the fibre $F_t := f^{-1}(t)$. The part at infinity $X_t \cap L^\infty = \{f_d = 0\} \subset \mathbb{P}^1$ is a finite set and it is independent of $t \in \mathbb{R}$. We have the inclusion $\{f_d = 0\} \supset \overline{F_t} \cap L^\infty$, which may be strict, like in the example $f = x^2 + y^4$ where we have $\overline{F_t} \cap L^\infty = \emptyset$, but $X_t \cap L^\infty = [1:0]$ for all $t \in \mathbb{R}$.

Definition 3.6. Let $\mathscr{L}_f := \{\overline{F_t} \cap L^\infty \mid t \in \mathbb{R}\}$, which is a finite set of points included in the part at infinity $\{f_d = 0\}$, and let $\ell_f := \#\mathscr{L}_f$.

After a linear change of coordinates, let us assume without loss of generality that $p = [0;1;0] \in L^\infty$, and let $U \simeq \mathbb{R}^2 \subset \mathbb{P}^2$ be a chart with origin at p and local coordinates (x,z).

The family of polynomial functions $g_t := \widetilde{f}(x,1,z) - tz^d$ with parameter t define a family of algebraic curve germs $\{g_t = 0\}$ at p for t close enough to some $\lambda \in \mathbb{R}$.

Definition 3.7. (DIAS; JOIŢA; TIBĂR, 2021, Definition 2.1) Let $p \in L^\infty$. We say that f has a *splitting at* $(p,\lambda) \in L^\infty \times \mathbb{R}$, shortly $(\mathrm{S}_{(p,\lambda)})$, if there is a small disk D_ε at $p \in U \simeq \mathbb{R}^2$ such that the representative of the curve C_t in D_ε has a connected component C_t^i such that $C_t^i \cap \partial D_\varepsilon \neq \emptyset$ for all $t > \lambda$ (or $t < \lambda$) close enough to λ and the local Euclidean distance $\mathrm{dist}(C_t^i, p) \neq 0$ and tends to 0 when $t \to 0$.

We say that f has a *vanishing loop at* $(p,\lambda) \in L^\infty \times \mathbb{R}$, shortly $(\mathrm{V}_{(p,\lambda)})$, if there is a small disk D_ε at $p \in \mathbb{R}^2$ such that $C_t \cap D_\varepsilon \setminus \{p\}$ has a non-empty connected component $C_t^i \setminus \{p\}$ with $C_t^i \cap D_\varepsilon = \emptyset$ for all $t < \lambda$ (or for all $t > \lambda$) close enough to λ, such that $\lim_{t\to\lambda} C_t^i \cap D_\varepsilon = \{p\}$.

Definition 3.8. (DIAS; JOIŢA; TIBĂR, 2021, Definition 2.2) (Atypical points at infinity) We say that $(p,\lambda) \in L^\infty \times \mathbb{R}$ is an *atypical point at infinity* of f if there is $(\mathrm{S}_{(p,\lambda)})$ or $(\mathrm{V}_{(p,\lambda)})$.

It turns out that the existence of atypical points at infinity is equivalent to the existence of an atypical fiber component. In order to state this result we follow (DIAS; JOIŢA; TIBĂR, 2021), and thus by (V_λ) and (S_λ) we mean that the value $\lambda \in \mathbb{R}$ has a vanishing and splitting fibre component at infinity, respectively, see Definition 3.2.

Proposition 3.9. (DIAS; JOIŢA; TIBĂR, 2021, Theorem 2.1) Let $f^{-1}(\lambda)$ be a fiber with compact singular set. Then

(a) $(\mathrm{V}_\lambda) \Leftrightarrow \exists p \in L^\infty$ such that $(\mathrm{V}_{(p,\lambda)})$,

(b) $(\mathrm{S}_\lambda) \Leftrightarrow \exists p \in L^\infty$ such that $(\mathrm{S}_{(p,\lambda)})$.

Theorem 3.10. (DIAS; JOIŢA; TIBĂR, 2021, Theorem 1.1) A value $\lambda \in \mathbb{R}$ is an atypical value at infinity of f (cf Definition 3.1) if and only if there exists $p \in \mathscr{L}_f$ such that (p,λ) is an atypical point at infinity. And more precisely, if $\mathscr{C}$ is a splitting cluster at λ, then after splitting, the two fibre components have the same point p at infinity, and if $\mathscr{C}$ is a vanishing cluster at λ, then before vanishing, the fibre component has the unique point p at infinity.

Proof. The proof follows from the results of (DIAS; JOIŢA; TIBĂR, 2021). The first part of the statement is (DIAS; JOIŢA; TIBĂR, 2021, Theorem 1.1) and we refer the reader to its proof. Let us denote by $\overline{\alpha_t(\mathscr{C})}_p$ the representative, in the chart $\mathbb{R}^2$ of $\mathbb{P}^2$ at p, of the unique fibre component associated to $\mathscr{C}$ via the injective function in Theorem 2.26. If $\mathscr{C}$ is a splitting cluster, the fibre component $\alpha_t(\mathscr{C})$ is splitting in the sense of Definition 3.2 and by Proposition 3.9 f has a splitting at $(p,\lambda) \in L^\infty \times \mathbb{R}$. This is $\mathrm{dist}(\overline{\alpha_t(\mathscr{C})}_p, p) \neq 0$ and tends to 0 as $t \to \lambda$, where "dist" denotes the local Euclidean distance at p. Therefore p is a point at infinity of the fibre components after splitting on $\mathscr{C}$.

On the other hand, if $\mathscr{C}$ is a vanishing cluster, the representatives $\overline{\alpha_t(\mathscr{C})}_p$ is a vanishing loop at the atypical point at infinity $(p,\lambda) \in \mathscr{L}_f \times \mathbb{R}$ by Proposition 3.9. Therefore p is the unique point at infinity of $\alpha_t(\mathscr{C})$. This ends the proof for all t close enough to λ. $\square$

Corollary 3.11. To an odd cluster $\mathscr{C}$ at λ there corresponds a unique atypical point $(p,\lambda) \in \mathscr{L}_f \times \mathbb{R}$, such that all the Milnor arcs in the cluster $\mathscr{C}$ have the same point p at infinity.

Proof. The components $\alpha_t(\mathscr{C})$ associated to a splitting cluster $\mathscr{C}$ split into two fibre components of the atypical value λ with the same point at infinity by Theorem 3.10 and are intersected by all the Milnor arcs that compose the cluster $\mathscr{C}$. This implies that these Milnor arcs have the same point at infinity; otherwise, the fibre components after splitting have not the same point at infinity.

If $\alpha_t(\mathscr{C})$ is a fibre component of a vanishing cluster, it follows by Theorem 3.10 that $\alpha_t(\mathscr{C})$ produce a vanishing loop at the point $p \in \mathscr{L}_f$ for which its interior is filled by fibre components all of them intersected by all the Milnor arcs in $\mathscr{C}$. This implies that these Milnor arcs have the same point at infinity p; otherwise, the fibre component cannot be a vanishing loop at p. $\square$

Let us note that Corollary 3.11 is not anymore true for clusters corresponding to connected components of fibres of f which tend to the values $\pm\infty$, see Example 6.4.

Lemma 3.12. All the Milnor arcs in an even cluster $\mathscr{C}$ at $\lambda \in \mathbb{R}$ have the same point at infinity.

Proof. By contradiction, let us assume that two Milnor arcs γ and δ in an even cluster $\mathscr{C}$ have different points at infinity, and that they are consecutive in the sense of Definition 2.18. Since $\mathscr{C}$ is even, the corresponding components $\alpha_t(\mathscr{C})$, for t close enough to λ, are not vanishing, otherwise $\mathscr{C}$ must be an odd cluster. This implies that there is a fibre component of $f^{-1}(\lambda)$ contained in the region outside a big enough disk and between the two consecutive Milnor arcs γ, δ. This component has two points at infinity, namely the same points at infinity of γ and δ. This implies that the restriction of ρ to such fibre component has a critical point, that by definition belongs to $M(f)$ and thus it produce a Milnor arc passing through such point. It follows by Definition 2.14 that the Milnor arcs do not intersect each other, and thus the produced Milnor arc is between γ and β. This contradicts our hypothesis that γ and β are consecutive. Therefore all the Milnor arcs in $\mathscr{C}$ have the same point at infinity. $\square$

We recall that the notation $\mathscr{L}_f$ stands for the points at infinity of all the fibres of a non-constant polynomial $f : \mathbb{R}^2 \to \mathbb{R}$, and $\ell_f = \#\mathscr{L}_f$. We will use in this section the well defined injective function α_t between the clusters associated to λ and fibre components of the restriction $f_{|R}$ as in Theorem 2.26, for t close enough to λ. Via this function, when $\mathscr{C} \mapsto \alpha_t(\mathscr{C})$, it also implies that $\alpha_t(\mathscr{C})$ intersects all the Milnor arcs of the cluster $\mathscr{C}$

Also recall that a vanishing or a splitting cluster at some value λ contains an odd number of Milnor arcs, by Theorem 3.5, and that all these arcs contain the point p in their closure at infinity, by Corollary 3.11.

Proposition 3.13. Let $\mathscr{C}$ be an odd cluster and let $(p,\lambda) \in \mathscr{L}_f \times \mathbb{R}$ be its atypical point at infinity. Then:

(a) If $\mathscr{C}$ is a splitting cluster then, after splitting, the resulting two germs at p of fibre components, $C_{p,1}$ and $C_{p,2}$, are tangent to some semi-line, call it T, and all the Milnor arcs of the cluster $\mathscr{C}$ are also tangent to the same semi-line T at p.

(b) If $\mathscr{C}$ is a vanishing cluster, let C_1 and C_2 be the two local arcs at p of the component $\alpha_t(\mathscr{C})$ which vanishes at p when $t \to \lambda$. Then C_1 and C_2 are tangent at p to the same semi-line T, and all the Milnor arcs of this cluster are tangent to the same semi-line T at p.

Proof. We give the proof for (a) only, as the one for (b) is analogous.

The component $\alpha_t(\mathscr{C})$ contained in the fibre $f^{-1}(t)$ which splits at p tends to an atypical fibre; so let $\lambda \in \mathbb{R}$ denote this corresponding atypical value. By contradiction, suppose that one of the two resulting local components, $C_{p,1}$ or $C_{p,2}$, is not tangent to T, say for instance $T_pC_{p,1} = R \neq T$, where R is a semi-line with origin at p. In local coordinates, by intersecting with a small enough disk neighbourhood D_ε at p, we may also assume that the component $C_{p,1} \cap D_\varepsilon$ is in the interior of the angle δ at p spanned by the semi-lines T and R, and of measure less than π.

We consider some semi-line L between T and R in the angle δ, different from both. For t close enough to λ, the component $\alpha_t(\mathscr{C})$ must intersect L, otherwise $C_{p,1}$ cannot be tangent to R (if not, then $C_{p,1}$ is in the interior of the angle delimited by L and T, which contradicts its tangency to R since R is not in this angle). Therefore the restriction $f_{|L}$ of f to the line L is not constant, and if so, then it must be unbounded, as $f_{|L}$ is a non-constant polynomial of one variable. This implies that, for any t close to λ, there is a point of intersection $q(t) \in \alpha_t(\mathscr{C}) \cap L$ which tends to p when $t \to \lambda$, and the value $f_{|L}(q(t))$ tends to infinity as $t \to \lambda$.

On the other hand, the same fibre component $\alpha_t(\mathscr{C})$ intersects a fixed Milnor arc $\gamma \in \mathscr{C}$ at a point $r(t)$ tending to p as $t \to \lambda$, and $f_{|\gamma}(r(t)$ tends to the atypical value λ by our hypotheses.

But by our construction, the values $f_{|L}(q(t))$ and $f_{|\gamma}(r(t)$ coincide. This yields a contradiction to our initial assumption $R \neq T$. □

Proposition 3.14. Let γ and δ be two consecutive Milnor arcs (in the order of arcs, cf Definition 2.18) such that they have either different points at infinity, or the same point at infinity $p \in \mathscr{L}_f$ but different tangent semi-lines at p. Let $\mathscr{C}_\gamma$ and $\mathscr{C}_\delta$ be their respective clusters. Then the corresponding fibre components $\alpha_t(\mathscr{C}_\gamma)$ and $\alpha_t(\mathscr{C}_\delta)$ cannot both split.

Proof. Suppose, by contradiction, that the components $\alpha_t(\mathscr{C}_\gamma)$ and $\alpha_t(\mathscr{C}_\delta)$ both split at p. Note that, the hypothesis implies, via Lemma 3.12 or Proposition 3.13 , respectively that $\alpha_t(\mathscr{C}_\gamma) \neq \alpha_t(\mathscr{C}_\delta)$. First assume that γ and δ have the same point $p \in \mathscr{L}_f$ at infinity but different tangent semi-lines at p. The splitting can happen only at atypical values of f, so let $\lambda_\gamma, \lambda_\delta \in \mathbb{R}$ be the atypical values where $\alpha_t(\mathscr{C}_\gamma)$ and $\alpha_t(\mathscr{C}_\delta)$ split at p, respectively.

Let T be the semi-line tangent to δ at p, and let L be the semi-line tangent to γ at p. By our hypothesis, $L \neq T$ and they belong to different lines.

The component $\alpha_t(\mathscr{C}_\gamma)$ splits as $t \to \lambda_\gamma$ into two branches C^1_γ and C^2_γ, and the component $\alpha_t(\mathscr{C}_\delta)$ splits as $t \to \lambda_\delta$ into two branches C^1_δ and C^2_δ. Since there is no other Milnor arc between γ and δ in the consecutive ordering, there must be a family of fibre components between C^i_γ and C^j_δ, for appropriate $i, j \in \{1,2\}$, which is a topologically trivial family at infinity (in the sense employed in Definition 3.1). But there cannot be a trivial fibration at infinity since all the fibres in such a trivial fibration must have the same tangent at p. This implies that there exist an atypical point at infinity (p,λ), with λ in the open interval between a Milnor arc at p between λ_γ and λ_δ, thus there exists a Milnor arc "between" γ and δ in the consecutive ordering, which is a contradiction to our assumption.

Let us now assume that the consecutive Milnor arcs γ and δ do not have the same point at infinity, and that the corresponding clusters are splitting like described above. Then the region $\mathscr{R}_{\rangle|}$ outside a large enough disk D and between the two corresponding consecutive fibre components C^i_γ and C^j_δ is either filled with a trivial fibration (defined by the appropriate restriction of f), or there is no such trivial fibration containing C^i_γ and C^j_δ. Since the connected components C^i_γ and C^j_δ have different points at infinity, they cannot live in a trivial family of connected fibres. But if there is no fibration between C^i_γ and C^j_δ, then there exists an atypical fibre at infinity in that region $\mathscr{R}_{\rangle|}$, and thus there exists another Milnor arc "between" γ and δ, which is a contradiction to our assumption that γ and δ are consecutive Milnor arcs. □

Remark 3.15. Proposition 3.14 is illustrated by Example 6.2. It shows that one can have two consecutive Milnor arcs γ and δ belonging to different clusters, such that both fibre components $\alpha_t(\mathscr{C}_\gamma)$ and $\alpha_t(\mathscr{C}_\beta)$ split. However γ and δ have a common tangent at p.

3.4 Algorithmic aspects

In the previous sections we presented theoretical results which led us to the characterization of atypical values in Theorem 3.5. In this section we aim the algorithmic aspects of a new effective detection of atypical values. Remark that for a polynomial function which is non-primitive with respect to ρ_a for some $a \in \mathbb{R}^2$, its fibres are all circles centred at a, and thus the vanishing and splitting does not occur, since the appearance of at least one of these phenomena implies the existence of non-compact fibres by Definition 3.2. In the sight of Proposition 3.3, a non-primitive polynomial function f satisfies $Atyp\, f = f(\mathrm{Sing} f)$, and we remark that the singular locus of f is contained in finitely many fibres. So one can detect $\mathrm{Sing} f$ as the common zeroes of the two polynomial components of the gradient vector field of f. Therefore we consider, without loss of generality, a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ which is primitive with respect to ρ_a for all $a \in \mathbb{R}$, so it is primitive with respect to ρ after an appropriate translation of coordinates.

This algorithm involves several steps: find the Milnor radius at infinity μ, where outside of it, the Milnor arcs are defined; compute the values $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ for which f tends along each Milnor arc; cast the Milnor arcs in μ-clusters; detect the odd and even μ-clusters.

3.4.1 Milnor radius at infinity

Let us find the Milnor radius at infinity, see Remark 2.13. Let d be the degree of the primitive polynomial f, and thus its Milnor set $M(f)$ is an unbounded real curve by Lemma 2.5 and Proposition 2.16 (a). Consider $h(x,y)$ the reduced polynomial defining $M(f)_{\mathrm{red}}$ as in Definition 2.8. The set $\mu(M(f))$ is, by Definition 2.8, the solutions of the system

$$\begin{cases} h(x,y) = 0, \\ xh_y(x,y) - yh_x(x,y) = 0, \end{cases} \tag{3.1}$$

where h_x, h_y denote the partial derivatives of h. By Proposition 2.16, it is enough to choose $R > \max_{q \in \mu(M(f))}\{\|q\|\}$ and this choice can be done effectively since by Proposition 2.11 the set $\mu(M(f))$ is a union of finitely many points and circles centred at the origin. Moreover, the circles $\{x^2+y^2-r^2=0\}$ contained in $M(f)$ are identified with the factors $x^2+y^2-r^2$ occurring in the irreducible decomposition of $h(x,y)$. By choosing all such factors $x^2+y^2-r_i^2$, $i = 1,\ldots,l_1$ and the isolated solutions q_i, $i = 1,\ldots,l_2$ of the system (3.1), we take

$$R > \max\{r_1,\ldots,r_{l_1}, \|q_1\| \ldots, \|q_{l_2}\|\}.$$

3.4.2 Regular values associated to Milnor arcs.

Consider $\mathbb{P}^2 = \mathbb{R}^2 \cup \mathbb{P}^1$ the real projective compactification of $\mathbb{R}^2$ where $\mathbb{P}^1 = L^\infty$ denotes the real line at infinity. Since f is primitive of degree d, the intersection $\overline{M(f)} \cap L^\infty$ is a non-empty finite set with at most d points by Bézout, where $\overline{M(f)}$ denotes the projective closure of $M(f)$ in $\mathbb{P}^2$ and it is defined by the homogenization $\widetilde{h}(x,y,z)$ of degree d with respect to the new variable z of the polynomial $h(x,y)$ that defines $M(f)$, see Remark 2.7. By Theorem 3.5 a regular value $\lambda \in \mathbb{R}$ is atypical if and only if there is an odd cluster associated to λ. This implies that the atypical values of f are among the values in $\cup_{p \in \overline{M(f)} \cap L^\infty} S_p(f)$ where

$$S_p(f) := \{\lambda \in \mathbb{R} \ : \ \text{there exists a Milnor arc at } p \text{ tending to } \lambda\}$$

We will compute the values in $S_p(f)$ for each point p in the finite set $\overline{M(f)} \cap L^\infty$. Let $p \in \overline{M(f)} \cap L^\infty$, and modulo an appropriate change of coordinates we assume that $p = [0;1;0]$.

In the chart $\{y \neq 0\}$ of $\mathbb{P}^2$ at p with local coordinates (x,z), the germ of curve $\overline{M(f)}$ at p is defined by the polynomial $\hat{h}(x,z) := \widetilde{h}(x,1,z)$. The set $\{\hat{h}(x,z)=0\}$ have finitely many branches at p, each of them define two half branches which by our construction coincide the Milnor arcs at p in $\{y \neq 0\}$. Let us consider the real germ of curve at p defined by the polynomial $\hat{f}(x,z) := \widetilde{f}(x,1,z)$, where $\widetilde{f}(x,y,z)$ denotes the homogenization of degree d of $f(x,y)$ with respect to the variable z. The problem of finding the values in $S_p(f)$ is equivalent to the problem:

$$(*) \text{ find all limits } \lim_{(x,z)\to p} \frac{\hat{f}(x,z)}{z^d} \text{ and } \hat{h}(x,z)=0.$$

This problem is addressed in (DIAS; JOIŢA; TIBĂR, 2021) by using a truncated Newton-Puiseux algorithm in a finite number of steps. In the following, we briefly describe their algorithm applied to it for the effective determination of our problem $(*)$. For further details we refer the reader to (DIAS; JOIŢA; TIBĂR, 2021).

Passing to complex variables, the Newton-Puiseux algorithm, shows all the complex roots of $\hat{h}(x,z)$, by starting from each edges of the Newton polygon and following all the choices of constants in each step. A Puiseux parametrization of a complex root of $\hat{h}(x,z)$ has the form $z=T^n, x=\sum_{1\leq j} a_j T^j$. As remarked in (DIAS; JOIŢA; TIBĂR, 2021) the series may be infinite even in the polynomial case; however, one can truncate the Newton Puiseux series in the dn first steps due to the equality:

$$\lim_{T\to 0} \frac{\widehat{f}(\sum_{1\leq j} a_j T^j, T^n)}{T^{dn}} = \lim_{T\to 0} \frac{\widehat{f}(\sum_{1\leq j\leq dn} a_j T^j, T^n)}{T^{dn}}.$$

Furthermore $n \leq \min\{k!, d\}$ where k is the lowest point in the x-axis of the Newton polygon of $\hat{h}$. After finding all limits with the effective truncation of each Puiseux parametrization, one might chose only the real limits.

3.4.3 Detecting μ-clusters.

In this step we give an effective criteria that determine all the μ-clusters. In the previous step we have found the values λ for which f tends to it along the Milnor arcs. It may happen that λ is a critical value, in this case $\lambda \in Atyp\, f$ since $f(\mathrm{Sing} f) \subset Atyp\, f$. Without loss of generality, in this step we assume that λ as found in the previous step is regular. As we have seen in §3.4.2, $Atyp\, f \subset \cup_{p\in \overline{M(f)}\cap L^\infty} S_p(f)$, and this union in the left is effectively determined. It follows from Theorem 3.5 that a value $\lambda \in \cup_{p\in \overline{M(f)}\cap L^\infty} S_p(f)$ is atypical if and only if there is an odd cluster associated to λ. We identify the Milnor arcs tending to λ by applying the change of coordinates $x=\frac{x}{z}, y=\frac{1}{z}$ to each truncated real Puiseux parametrization $z=T^n, x=\sum_{1\leq j\leq nd} a_j T^j$ such that $\lim_{(x,z)\to p} \frac{\hat{f}(x,z)}{z^d} = \lambda$ as found in §3.4.2. It follows from Definition 2.19 that each μ-cluster is formed by a consecutive sequence[3], so one has to determine which are the increasing and

[3] In the sense of Definition 2.18

decreasing Milnor arcs, see Definition 2.17. From Definition 2.14 at every point in a Milnor arc at infinity γ the gradient vector grad$f$ of f is parallel to the position vector and from Proposition 2.16 (b) the restriction $f_{|\gamma}$ is strictly monotonous. This implies that at all points $q \in \gamma$ one has that the sign of the inner product $\langle \mathrm{grad} f(q), q \rangle$ in $\mathbb{R}^2$ is constant. This remark gives us the following criteria:

(i) if $\mathrm{sgn}\langle \mathrm{grad} f(q), q \rangle = +1$, then γ is an increasing Milnor arc at infinity,

(ii) if $\mathrm{sgn}\langle \mathrm{grad} f(q), q \rangle = -1$, then γ is an decreasing Milnor arc at infinity,

(iii) if $\langle \mathrm{grad} f(q), q \rangle = 0$, then by Proposition 2.16 and Definition 2.14, $\gamma \subset \mathrm{Sing} f$. In such case we have that λ is an atypical value by Definition 3.1.

Denote by $\mathscr{I}_\lambda$ and $\mathscr{D}_\lambda$ the sets of Milnor arcs at infinity tending to λ which are increasing and decreasing, respectively. The increasing (or decreasing) μ-clusters associated to λ are by Definition 2.19 the sequences of consecutive Milnor arcs in $\mathscr{I}_\lambda$ (or in $\mathscr{D}_\lambda$, respectively).

3.4.4 Detecting atypical values.

In §3.4.3 we have identified the increasing and decreasing μ-clusters associated to each $\lambda \in \cup_{p \in \overline{M(f)} \cap L^\infty} S_p(f)$. Following Theorem 3.5 the odd μ-clusters are the ones which produce an atypical value. Hence we apply Corollary 2.22 to each Milnor arc in a μ-cluster associate to some finite value λ in order to know the type of tangency of each Milnor arc i.e. either ρ-maximum, or ρ-minimum, or ρ-inflectional type of tangency as defined in §2.3. If the number of Milnor arcs in a μ-cluster $\mathscr{C} := \{\gamma_k, \dots, \gamma_{k+l}\}$ associated to $\lambda \in \mathbb{R}$ with type of tangecy different than ρ-inflectional is odd (or even) then $\mathscr{C}$ is odd (or even,respectively) by Definition 3.4. Moreover, it follows from the proof of Theorem 3.5 the following criteria for odd μ-clusters:

(i) If γ_k is has ρ-maximum type of tangency, then $\alpha_t(\mathscr{C})$ is splitting at infinity.

(ii) If γ_k is has ρ-minimum type of tangency, then $\alpha_t(\mathscr{C})$ is vanishing at infinity.

In both cases λ is a regular atypical value. On the other hand, if all μ-cluster associated λ are even, then λ is a typical value.

CHAPTER

4

INDEX OF THE GRADIENT AT INFINITY

The index of a vector field with isolated singularities enters in the celebrated Poincaré-Hopf Theorem. Without entering into details, for a smooth vector field with isolated zeroes defined on a compact manifold M, the Poincaré-Hopf Theorem asserts that the sum of the indices at the zeroes of such a vector field is equal to the *Euler characteristic* of M. In particular, the sum of indices is a topological invariant of M (it does not depend on the chosen vector field), we refer the reader to (MILNOR, 1997, §6) for further details. The index of a gradient vector field of a polynomial function on $\mathbb{R}^n$ is far from satisfying the above theorem due to the non-compact nature of $\mathbb{R}^n$.

In this chapter we prove in Theorem 4.14 a formula for the *index at infinity* of a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$, see Definition 4.6 below, which is a global counterpart of the local index theorem in (ARNOL'D, 1978). Arnold theorem asserts that the index at an isolated singular point $p \in C$ of a real plane curve $C := \{g = 0\}$ with r branches at p satisfies the equality $\mathrm{ind}_p(\mathrm{grad}(g)) = 1 - r$, where $\mathrm{grad}(g)$ denotes the gradient vector field of g, see §1 where we present an alternative proof of this local theorem.

This chapter is organized as follows: in §4.1 we recall the the *Brouwer degree* of a mapping defined between two one-dimensional manifolds. In the sequence we define the *index at infinity* of a polynomial function f as the Brouwer degree of its gradient vector field restricted to a large enough circle, see Definition 4.6. In §4.2 we assign to each Milnor arc at infinity an index in $\{\pm\frac{1}{2}, 0\}$, and we relate this index with the type of tangency between fibres and level sets of the Euclidean distance ρ along the Milnor arcs, see Lemma 4.10. We present in Theorem 4.9 a new proof of a formula for the index at infinity already proved by Durfee in (DURFEE, 1998).

By summing up the indices of Milnor arcs in each μ-cluster, one obtain an index for each μ-cluster, so in §4.3, we compute the index of even and odd μ-cluster, see Lemma 4.12 and Proposition 4.13. With these cluster indices we prove a new formula in Theorem 4.14 which also reveals how the phenomena at infinity of fibres influence in the index at infinity.

4.1 Brouwer degree and index at infinity

In this section we introduce the definition of *index at infinity*, see Definition 4.6. We start by recalling the definition of *Brouwer degree* in the context of a mapping between two circles, which are one-dimensional oriented manifolds. We remark that this degree is defined over a more general class of maps, and we refer the reader to (MILNOR, 1997, pag. 27) for further details. In the sequence we prove that one can chose a pair of points diametrically opposed in the unite circle which are both regular. We finish this section with the definition of index at infinity for a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ with isolated singularities.

Let $D_R \subset \mathbb{R}^2$ be the open disk centred at the origin with radius $R > 0$, and let C_R denote its boundary ∂D_R. We endow C_R and C_1 with their positive orientations, where C_1 denotes the boundary of the unite circle centred at the origin. Let $\Psi : C_R \to C_1$ be a $\mathscr{C}^\infty$-map, at each regular point $p \in C_R$ of Ψ. The derivative $D_p\Psi$ of Ψ at p defines a tangent line $T_p\Psi$ that we endow with a fixed orientation $\mathrm{or}(T_p\Psi) = +1$ or $\mathrm{or}(T_p\Psi) = -1$, if and only if the determinant of $D_p\Psi$ is positive or negative, respectively. After (MILNOR, 1997, Pag. 27), the Brouwer degree of Ψ is defined by

$$\deg \Psi := \sum_{p \in \Psi^{-1}(q)} \mathrm{or}(T_p\Psi),$$

for some regular value $q \in C_1$ of Ψ. The definition of $\deg \Psi$ does not depend on the choice of the regular value q, see for instance (MILNOR, 1997, Pag. 28. Theorem A).

Lemma 4.1. Let $\Psi : C_R \to C_1$ be a $\mathscr{C}^1$-map. Let $\pi : C_1 \to \mathbb{P}^1$ be the canonical quotient map defined by $\pi(q) := [q] \in \mathbb{P}^1$. If $[q] \in \mathbb{P}^1$ is a regular value of the mapping $\pi \circ \Psi : C_R \to \mathbb{P}^1$ then q and $-q$ are regular values of Ψ.

Proof. If $\Psi^{-1}(q) = \emptyset$ or $\Psi^{-1}(-q) = \emptyset$, then q or $-q$, respectively, is regular value of Ψ by definition. Let us assume that $\Psi^{-1}(q) \cup \Psi^{-1}(-q) \neq \emptyset$, let $p \in (\pi \circ \Psi)^{-1}([q])$, and let U be a local system of coordinates at the point $\Psi(p) \in C_1$ such that $\pi_{|U} : U \to \mathbb{P}^1$ is a diffeomorphism over its image (see for instance (LEE, 2003, Example 1.33)). The point p is a regular point of the map $\pi \circ \Psi$ if and only if p is a regular point for Ψ since $D(\pi \circ \Psi)(p) = D\pi_{|U}(\Psi(p))D\Psi(p)$ where $D\pi_{|U}(\Psi(p))$ is nonzero because $\pi_{|U}$ is a diffeomorphism at p. Since $[q]$ is a regular value of $\pi \circ \Psi$ and due to the obvious equality $(\pi \circ \Psi)^{-1}([q]) = \Psi^{-1}(q) \cup \Psi^{-1}(-q)$, one concludes that q and $-q$ are regular values of Ψ. $\square$

Remark 4.2. By Sard's Theorem (see for instance (MILNOR, 1997, Chapter 3)) the critical values of $\pi \circ \Psi$ has measure zero in $\mathbb{P}^1$. This together with Lemma 4.1 imply that there are infinitely many values $q \in C_1$ of Ψ such that q and $-q$ are regular values of Ψ.

Lemma 4.3. Let $\Psi : C_R \to C_1$ be a $\mathscr{C}^1$-mapping. Then there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $1, -1 \in C_1$ are regular values of the mapping $T \circ \Psi : C_R \to C_1$. Moreover $\deg T \circ \Psi = \deg \Psi$.

Proof. By Lemma 4.1 and Remark 4.2 we choose a pair of regular values $q, -q \in C_1$ of Ψ out of the infinitely many such pairs in C_1. If $q = \pm 1 \in C_1$, the result follows by taking T as the identity in $\mathbb{R}^2$. Otherwise, we consider the linear transformation T as the rotation multiplied by an adequate scalar such that $T(q) = 1 \in C_1$, and hence $T(-q) = -1 \in C_1$ by the linearity of T. Note that $\det T \neq 0$ and thus, after applying the chain rule, the equality $\det(D(T \circ \Psi)(p)) = \det(D\Psi(p))$ holds for every $p \in (T \circ \Psi)^{-1}(1) \cup (T \circ \Psi)^{-1}(-1)$, where $\det(D\Psi(p)) \neq 0$ since q and $-q$ are both regular values of Ψ. This proves that $1, -1 \in C_1$ are regular values of the mapping $T \circ \Psi$.

It remains to prove that $\deg T \circ \Psi = \deg \Psi$: consider the smooth homotopy between $T \circ \Psi$ and Ψ defined by

$$\begin{aligned} H : C_R \times [0,1] &\to C_1 \\ (q,t) &\mapsto H(q,t) := \frac{t\Psi(q) - (1-t)(T \circ \Psi)(q)}{\|t\Psi(q) - (1-t)(T \circ \Psi)(q)\|}, \end{aligned}$$

This ends the proof since the Brouwer degree of two smooth homotopic maps is the same, see for instance (MILNOR, 1997, §5, Theorem B). □

Proposition 4.4. Let $\Psi : C_R \to C_1$ be a $\mathscr{C}^1$-mapping, and let q and $-q$ in C_1 be regular values of Ψ. Then

$$\deg \Psi = \frac{1}{2} \sum_{p \in \Psi^{-1}(\{q,-q\})} \operatorname{or}(T_p\Psi). \tag{4.1}$$

Proof. Since $\deg \Psi$ does not depend on the choice of the regular value (see (MILNOR, 1997, Pag. 28. Theorem A)) we obtain:

$$2\deg \Psi = \sum_{p \in \Psi^{-1}(q)} \operatorname{or}(T_p\Psi) + \sum_{p \in \Psi^{-1}(-q)} \operatorname{or}(T_p\Psi) = \sum_{p \in \Psi^{-1}(q) \cup \Psi^{-1}(-q)} \operatorname{or}(T_p\Psi).$$

Due to the above equality and from the obvious equality $\Psi^{-1}(q) \cup \Psi^{-1}(-q) = \Psi^{-1}(\{-q,q\})$ we conclude (4.1). □

We finish this section with the definition of index at infinity. In order to do it, let us fix $f : \mathbb{R}^2 \to \mathbb{R}$ a polynomial function with isolated singularities, we denote by $\operatorname{grad} f$ the gradient vector field of f. Fix a circle $C_R \subset \mathbb{R}^2$ centred at the origin with radius $R > 0$ large enough such that $\operatorname{Sing} f$ is contained in the interior of the open disk D_R with border C_R.

Notation 4.5. By $\varphi := \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|} : \mathbb{R}^2 \backslash \operatorname{Sing} f \to C_1$ we denote the Gauss map of $\operatorname{grad} f$, and $\varphi_{C_R} : C_R \to C_1$ denote its restriction to the fixed circle $C_R \subset \mathbb{R}^2 \backslash \operatorname{Sing} f$.

Under this consideration we define the index at infinity of f, see also (DURFEE, 1998) and the subsequent work (SEKALSKI, 2005).

Definition 4.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function with isolated singularities. The *index at infinity of f* is defined as

$$\operatorname{ind}_\infty f := \deg \varphi_{C_R}. \tag{4.2}$$

This definition does not depend on the circle C_R large enough.

4.2 Index along Milnor arcs and Durfee's index formula.

In (DURFEE, 1998), Durfee presents, as far as we know, the first formula of the index at infinity of a polynomial function with isolated singularities. This formula involves the sum of some well-defined indices assigned to each Milnor arcs at infinity (see Definition 2.14), which in Durfee's work are originally called unbounded components of the curve of tangencies. This assignation of indices to each Milnor arc depends on the type of tangency of the fibres with some circle at each point of the Milnor arcs, see the possible types of tangencies (a), (b), (c) at the end of §2.3. This section aims to justify Durfee's index assignation (Lemma 4.10). We also present Durfee's formula for the index at infinity in Theorem 4.9 and give a proof for the shake of completeness.

Let us recall that the Milnor arcs at infinity of a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ are the unbounded components of the Milnor set $M(f)$ (see Definition 2.1), which, turns out to be finitely many one-dimensional manifolds (see Definition 2.14 and Proposition 2.16) provided f primitive with respect to $\rho(x,y) := x^2 + y^2$ (see Definition 2.3) . If f is not primitive with respect to ρ, one has $M(f) = \mathbb{R}^2$ by Lemma 2.5, and thus one cannot define Milnor arcs at infinity. In this section we assume that f has only isolated singularities and it is primitive with respect to ρ. Hence the Milnor arcs of f are defined as the components of $M(f)\backslash\overline{D}_\mu$ where D_μ denotes the open disk centred at the origin with Milnor radius at infinity, see Remark 2.13. We refer the reader to §2.1 above for further details on the Milnor arcs.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function with respect to ρ with only isolated singularities. Let us endow the unitary circle C_1 centered at the origin with the complex multiplication denoted by "$\cdot$". For any circle $C_R \subset \mathbb{R}^2$ centred at the origin with radius $R > 0$ such that the singular set $\operatorname{Sing} f$ is contained in the open disk defined by C_R[1], we consider the mapping $\psi_{C_R} : C_R \to C_1$, $\psi_{C_R}(z) := \varphi_{C_R}(z) \cdot (z/\|z\|)^{-1}$ which is well defined since $C_R \subset \mathbb{R}^2\backslash\operatorname{Sing} f$, and we assume that $1, -1 \in C_1$ are regular values of ψ_{C_R}, see Lemmas 4.1 and 4.3.

Lemma 4.7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function with only isolated singularities, and let γ be a Milnor arc at infinity of f. Then the orientation $\operatorname{or}(T_p\psi_{C_{\|p\|}})$, for every $p \in \gamma$ does not depend on the choice of the point $p \in \gamma$.

[1] In particular we may choose $R > R_\mu$ where R_μ denotes the Milnor radius at infinity of f, see Remark 2.13

Proof. Let us prove that $\mathrm{or}(T_p\psi_{C_{\|p\|}}) = +1$ and $\mathrm{or}(T_p\psi_{C_{\|p\|}}) = -1$ are both open conditions in the Milnor arcs of f. With that, the result follows from the connectedness of the Milnor arcs, see Definition 2.14. In order to prepare the proof, we make the following considerations: by regarding $\psi_{C_{\|p\|}}(p) \in C_1$ as a complex number one has $\psi_{C_{\|p\|}}(p) = e^{i\arg(\psi_{C_{\|p\|}}(p))}$, where "arg" denotes the argument function which is locally well defined in C_1. Hence the orientation $\mathrm{or}(T_p\psi_{C_{\|p\|}})$ coincides with the sign of the derivative $\frac{d}{d\theta}\arg\psi_{C_{\|p\|}}(\alpha_{\|p\|}(0))$, where $\alpha_{\|p\|} :]-\varepsilon,\varepsilon[\to C_{\|p\|}$ is a counterclockwise local parametrization at $p \in \gamma\cap C_{\|p\|}$. On the other hand, since all the Milnor arcs are one-dimensional manifolds, after a local linear change of coordinates one may assume, that in a small neighborhood of the form $U :=]-\varepsilon,\varepsilon[\times]\|p\|-\delta,\|p\|+\delta[$, the arc γ coincides with the segment $(0,t)\cap U$ and the horizontal lines are the circles C_t for $t \in]\|p\|-\delta,\|p\|+\delta[$. First let us assume that $p \in \psi_{C_{\|p\|}}^{-1}(1)$, hence for every $(\theta,t) \in]-\varepsilon,\varepsilon[\times]\|p\|-\delta,\|p\|+\delta[$

$$\arg(\psi_{C_t}(\alpha_t(\theta))) = 0 \text{ if and only if } \theta = 0, \tag{4.3}$$

and by taking $\varepsilon > 0$ small enough, we assume that $\arg(\psi_{C_t}\circ\alpha_t)$ has non-zero constant sign in the intervals $]-\varepsilon,0[$ and $]0,\varepsilon[$. Altogether this implies:

(i) $\mathrm{or}(T_{(0,t)}\psi_{C_t}) = +1$ if, and only if $\arg(\psi_{C_t}\circ\alpha_t)$ is increasing, and

(ii) $\mathrm{or}(T_{(0,t)}\psi_{C_t}) = -1$ if, and only if $\arg(\psi_{C_t}\circ\alpha_t)$ is decreasing.

Let us consider the function

$$\Gamma :]-\varepsilon,\varepsilon[\times]\|p\|-\delta,\|p\|+\delta[\to\mathbb{R},\;\; \Gamma(\theta,t) := \arg(\psi_{C_t}(\alpha_t(\theta))).$$

After these considerations let us assume by contradiction that for two different values $t_1,t_2 \in]\|p\|-\delta,\|p\|+\delta[$, one has $\mathrm{or}(T_{(0,t_1)}\psi_{C_{t_1}}) = +1$ and $\mathrm{or}(T_{(0,t_2)}\psi_{C_{t_2}}) = -1$. From (i) and (ii) $\arg(\psi_{C_{t_1}}(\alpha_{t_1}(\theta_0))) > 0$ and $\arg(\psi_{C_{t_2}}(\alpha_{t_2}(\theta_0))) < 0$, for any $\theta_0 \in]-\varepsilon,0[$. Therefore the function in one variable defined by restricting Γ to some open segment L in U passing through $\alpha_{t_1}(\theta_0)$ and $\alpha_{t_2}(\theta_0)$ satisfies $\Gamma_{|L}(\theta_0,t_3) = 0$ for some $(\theta_0,t_3) \in L$. This leads to a contradiction with (4.3) since $\theta_0 \neq 0$ by our construction, and thus $\mathrm{or}(T_p\psi_{C_{\|p\|}}) = +1$ and $\mathrm{or}(T_p\psi_{C_{\|p\|}}) = -1$ are both open conditions in γ.

Notice that the proof for $p \in \psi_{C_{\|p\|}}^{-1}(-1)$ is analogous to the one presented above, with the difference that equivalence (4.3) becomes:

$$\arg(\psi_{C_t}(\alpha_t(\theta))) = \pi \text{ if and only if } \theta = \pi.$$

In order to avoid repetitions, we let the details to the reader. This ends the proof.

□

Definition 4.8. For each Milnor arc at infinity γ of f we define *the index of f along the Milnor arc at infinity γ* as the number $i(\gamma) := \frac{1}{2}\mathrm{or}(T_p\psi_{C_R})$ for some $p \in \gamma\cap C_R$. This definition does not depends on the choice of the point $p \in \gamma$ by Lemma 4.7.

We prove Durfee's index formula (DURFEE, 1998, pag. 1344, Equation (1)):

Theorem 4.9. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a primitive polynomial function with only isolated singularities such that $1,-1\in C_1$ are regular values of ψ_{C_R} for some $R>R_\mu$ where R_μ denotes the Milnor radius at infinity of f(cf. Remark 2.13). Then

$$\mathrm{ind}_\infty f = 1+\sum_\gamma i(\gamma), \tag{4.4}$$

where γ runs over all Milnor arcs at infinity of f.

Proof. Notice that the circle C_R intersects all Milnor arcs of f since $R>R_\mu$. Since $1,-1\in C_1$ are regular values of ψ_{C_R}, it follows by Proposition 4.4:

$$\deg\psi_{C_R} = \sum_{p\in\psi^{-1}(\{1,-1\})}\frac{1}{2}\mathrm{or}(T_p\psi_{C_R}) = \sum_{p\in M(f)\cap C_R}\frac{1}{2}\mathrm{or}(T_p\psi_{C_R}) = \sum_\gamma \mathrm{i}_\gamma, \tag{4.5}$$

where the second equality follows from the obvious equality of sets $M(f)\cap C_R=\psi_{C_R}^{-1}(\{1,-1\})$; the third equality follows from Definition 4.8, and the sum in the right side runs over all Milnor arcs at infinity of f. On the other hand $\deg\psi_{C_R}=\deg\varphi_{C_R}-1=\mathrm{ind}_\infty(f)=-1$, by the properties of the topological degree and Definition 4.6. After putting together last equality and (4.5) one obtain (4.4). □

We finish this section with a result that shows the relation between the type of tangency of fibres with some circles along the Milnor arcs and the index along each Milnor arc in Definition 4.8

Lemma 4.10. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a primitive polynomial function with only isolated singularities. Let γ be a Milnor arc at infinity of f and let $1,-1\in C_1$ be regular values of the mapping ψ_{C_R} with $R>R_\mu$, where R_μ denotes the Milnor radius at infinity of f. Then:

(a) if γ has ρ-maximum type then $i(\gamma)=+\frac{1}{2}$.

(b) if γ has ρ-minimum type then $i(\gamma)=-\frac{1}{2}$.

Proof. By Corollary 2.22 and Lemma 4.7 the type of tangency between the fibers and concentric circles at the origin passing through some point $p\in\gamma$, and the orientation $\mathrm{or}(T_p\psi_{C_{\|p\|}})$ do not depend on the point $p\in\gamma$. Hence it is enough to study the relation between this type of tangency and this orientation at a fixed point $p\in\gamma$. Within the proof of Lemma 4.7 we proved that: $\mathrm{or}(T_p\psi_{C_R})=+1$ if, and only if $\frac{d}{d\theta}\arg(\psi_{C_R}\circ\alpha_R(0))>0$; and $\mathrm{or}(T_p\psi_{C_R})=-1$ if, and only if $\frac{d}{d\theta}\arg(\psi_{C_R}\circ\alpha_R(0))<0$, where $\alpha_R:]-\varepsilon,\varepsilon[\to C_R$ is a counterclockwise local parametrization of C_R at p with $\varepsilon>0$ small enough such that $\arg(\psi_{C_R}\circ\alpha_{C_R})$ is well defined in $]-\varepsilon,\varepsilon[$. Note that $\mathrm{grad} f(\alpha_{C_R}(\theta))$ is perpendicular to the fibre $f^{-1}(f(\alpha_R(\theta)))$ at the point $\alpha_R(\theta)\in\mathbb{R}^2$ for every $\theta\in]-\varepsilon,\varepsilon[$; moreover, all such fibers intersect γ with the same type of tangency. Altogether, this implies: if γ has a ρ-maximum type of tangency at p then $\arg(\psi_{C_R} \circ \alpha_R)$ is increasing, and thus $i(\gamma) = \frac{1}{2}\mathrm{or}(T_p\psi_{C_R}) = +\frac{1}{2}$; and if γ has a ρ-minimum type of tangency at p then $\arg(\psi_{C_R} \circ \alpha_R)$ is decreasing, and thus $i(\gamma) = \frac{1}{2}\mathrm{or}(T_p\psi_{C_R}) = -\frac{1}{2}$. This ends the proof. $\square$

4.3 Index at infinity and atypical values.

Let us recall that Theorem 4.9 gives a formula to compute the index at infinity of a primitive polynomial function in terms of the indices associated to each Milnor arcs. These indices turn out to be well-defined, see Lemma 4.7. In this section we want a formula involving the phenomena of fibres at infinity. For primitive polynomial functions we compute the sum of the indices of Milnor arcs in each μ-cluster (Lemma 4.12 and Proposition 4.13) and this will lead us to formula in Theorem 4.14 in terms of vanishing and splitting at infinity of fibres, see §3.2 above where we study these phenomena at infinity of fibres. We also treat the non-primitive case where we compute its index at infinity as the a winding number of its gradient vector field, see Remark 4.15.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function of degree $\deg f = d$ with isolated singularities. Let $\overline{M(f)} \subset \mathbb{P}^2$ be the projective closure of the Milnor set, i.e. if $h(x,y)$ denotes the polynomial of degree d defining $M(f) \subset \mathbb{R}^2$ and $\widetilde{h}(x,y,z)$ its homogenization of degree d with respect to the new variable z, then $\overline{M(f)} = \{\widetilde{h}(x,y,z) = 0\} \subset \mathbb{P}^2$. Let $L^\infty := \{z = 0\} \subset \mathbb{P}^2$ the line at infinity. Since f is primitive with respect to ρ, $M(f)$ is an unbounded curve, and thus $\overline{M(f)} \cap L^\infty$ is composed by finitely many points in L^∞ by Bézout Theorem. Among these points are the points in the projective closure $\overline{\gamma}$ of each Milnor arc γ

Definition 4.11. Any Milnor arc γ has a unique point at infinity $p \in L^\infty \cap \overline{\gamma}$; we shall say that "the Milnor arc γ has the point p at infinity".

Recall that the Milnor set $M(f)$ is the set of points where the fibres of f are tangent to some level set of the Euclidean distance function ρ, and that, by Definition 2.14, the Milnor arcs do not intersect $\mathrm{Sing} f$. For any point q of a Milnor arc γ, the fibre of f passing through q has only one of the following three behaviors:

(a) the fibre is locally inside the disk D, and then one defines the index $i(\gamma) := +\frac{1}{2}$

(b) the fibre is locally outside D, and then one defines the index $i(\gamma) := -\frac{1}{2}$,

(c) the fibre has a local half-branch inside D and the other local half-branch outside D, in which case one defines the index $i(\gamma) := 0$.

where in this case, D denotes the closed disk of radius $\|q\|$ centred at the origin. Remark that the definition of the indices above relies on Lemmas 4.7 and 4.10, and so each Milnor arcs has well defined index depending on the type of tangency of the fibres with level sets of ρ along the Milnor arc.

Let us set the following notations:

(i) $i_{p,c} := \sum_\gamma i(\gamma)$, and $i^{\text{abs}}_{p,c} := \sum_\gamma |i(\gamma)|$, where both sums run over all Milnor arcs γ such that $p \in \overline{\gamma} \cap L^\infty$ and $\lim f_{|\gamma} = c$.

(ii) $i_p := \sum_{c \in \mathbb{R}\cup\{\pm\infty\}} i_{p,c}$.

By gathering the indices of the Milnor arcs with the same point at infinity p and tending to the same value c i.e. $\lim f_{|\gamma} = c$, one recast (4.4) as

$$\operatorname{ind}_\infty f = 1 + \sum_{\substack{p\in L^\infty \\ c\in\mathbb{R}}} i_{p,c} + \sum_{p\in L^\infty} i_{p,\infty}, \tag{4.6}$$

which is the index formula in (DURFEE, 1998, Proposition 3.3).

In Lemma 4.12 and Proposition 4.13 below, we show that the behavior of the fibres intersecting a μ-cluster defines the index of the cluster. Let us recall that to each μ-cluster $\mathscr{C}$ one associates a unique fibre component $\alpha_t(\mathscr{C})$ as in Theorem 2.26.

Lemma 4.12. Let $\mathscr{C}$ be a μ-cluster associate to some $\lambda \in \mathbb{R}\cup\{\pm\infty\}$, and let $\alpha_t(\mathscr{C})$ its associated fibre component (cf Theorem 2.26). Then

(a) $\sum_{\gamma\in\mathscr{C}} i(\gamma) = 0$, if $\mathscr{C}$ is even (cf Definition 3.4);

(b) $\sum_{\gamma\in\mathscr{C}} i(\gamma) = -\frac{1}{2}$, if $\alpha_t(\mathscr{C})$ is vanishing at infinity at λ;

(c) $\sum_{\gamma\in\mathscr{C}} i(\gamma) = \frac{1}{2}$ if $\alpha_t(\mathscr{C})$ is splitting at infinity at λ.

Proof. Notice that each Milnor arc in $\mathscr{C}$ with zero index does not contribute to any of the sums in (a), (b), or (c), and these arcs are exactly the arcs in $\mathscr{C}$ with ρ-inflectional type of tangency by Lemma 4.10. If necessary, we take the subsequence $S_{\mathscr{C}} = \{\gamma_{j_1}, \dots, \gamma_{j_k}\}$ of all arcs in $\mathscr{C}$ with type of tangency different than ρ-inflectional, and so $\sum_{\gamma\in\mathscr{C}} i(\gamma) = \sum_{l=1}^k i(\gamma_{j_l})$. The consecutive relation of arcs in $\mathscr{C}$ induces on $S_{\mathscr{C}}$ a "$<$" relation i.e. $\gamma_{j_s} < \gamma_{j_r}$ if and only if γ_{j_r} is the consecutive, or γ_{j_s} or for the sequence of consecutive arcs $\{\gamma_{t_1}, \dots, \gamma_{t_l}\}$ such that γ_{t_1} is the consecutive of γ_{j_s} and γ_{t_l} is the antecedent of γ_{j_r}, the index $i(\gamma_{t_q}) = 0$ for $q = 1, \dots, l$. By our construction $\sum_{l=1}^k i(\gamma_{j_l})$ is an alternating sum of $\pm\frac{1}{2}$, by Corollary 2.24.

If $\mathscr{C}$ is even then $\sum_{l=1}^k i(\gamma_{j_l}) = 0$ since the amount of positive and negative arc indices are the same. This proves (a).

If $\mathscr{C}$ is odd, then $\sum_{\gamma\in\mathscr{C}} i(\gamma) = \sum_{l=1}^{2p+1} i(\gamma_{j_l}) = i(\gamma_{j_1})$, for some $p \geq 0$ since the set of indices $\{i(\gamma_2), \dots, i(\gamma_{2k+1})\}$ has an even number of altering indices. Moreover, as we have seen in the proof of Theorem 3.5: if $\alpha_t(\mathscr{C})$ is vanishing (splitting) then γ_1 has a ρ-minimum (ρ-maximum, respectively) type of tangency. By Lemma 4.10 one finish the proof of (b) and (c), respectively. □

Proposition 4.13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function with isolated singularities and such that at least one fibre component is non-compact. If $\mathscr{C}$ is a μ-cluster associated to $\pm\infty$ (cf Definition 2.19), then $\sum_{\gamma\in\mathscr{C}} i(\gamma) = -\frac{1}{2}$.

Proof. It follows by Lemma 5.1 below that if the level sets $|f| = A$ contain a compact fibre component for some $A > 0$ large enough such that A and $-A$ are regular values of f, then all fibres of f are compact. Our hypothesis implies that all components $\alpha_t(\mathscr{C})$ associated to each μ-cluster $\mathscr{C}$ at $\pm\infty$[2] has two ends approaching to the line at infinity $L_\infty \subset \mathbb{P}^2$. The two Milnor arcs in $\mathscr{C}$ that are the first and last arc in $\mathscr{C}$, respectively, (in the order of consecutive arcs) with type of tangency different than ρ-inflectional have both ρ-minimum type of tangency, otherwise the two ends of $\alpha_t(\mathscr{C})$ do not approach to L_∞. Then $\sum_{\gamma\in\mathscr{C}} i(\gamma) = -\frac{1}{2}$ by the alternation of tangencies in Corollary 2.24 and the associated index of each arc as in Lemma 4.10. □

Let us recall some notions from §3: the vanishing and splitting of components are phenomena that occur at some point $p \in L^\infty$ and some value $\lambda \in \mathbb{R}$. A point $(p,\lambda) \in L^\infty \times \mathbb{R}$ where one of these phenomena occurs is called atypical point at infinity. Let $\mathrm{Sp}(p,\lambda)$ and $\mathrm{Va}(p,\lambda)$ denote the numbers of connected components of fibres of f which are splitting or vanishing, respectively, at the atypical point $(p,\lambda) \in L^\infty \times \mathbb{R}$. By F_+ and F_- we denote a fibre of a regular value in the right and left unbounded interval of the set $\mathbb{R}\backslash Atyp\, f$, respectively. Let $\mathrm{Va}(\pm\infty)$ denote the number of components of $F_+ \cup F_-$. Note that there are two type of components which are counted in $\mathrm{Va}(\pm\infty)$: those which tend to a nontrivial segment of the line at infinity as the value of f tends to infinity, and those which tend to a point $p \in L^\infty$ as the value of f tends to infinity. These different types are illustrated in Example 5.5; see also (DURFEE, 1998, p. 1351, Fig. 6).

Theorem 4.14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a primitive polynomial function with isolated singularities and containing non-compact fibres. Then:

$$\mathrm{ind}_\infty f = 1 + \frac{1}{2}\sum_{p\in\mathscr{L}_f,\lambda\in\mathbb{R}} \mathrm{Sp}(p,\lambda) - \frac{1}{2}\sum_{p\in\mathscr{L}_f,\lambda\in\mathbb{R}} \mathrm{Va}(p,\lambda) - \mathrm{Va}(\pm\infty), \tag{4.7}$$

where $\mathscr{L}_f$ denotes the points at infinity of fibres as in Definition 3.6.

Proof. By gathering the indices of the Milnor arcs of the same cluster, one recasts (4.4) as:

$$\mathrm{ind}_\infty(f) = 1 + \sum_{\mathscr{C}} \sum_{\gamma\in\mathscr{C}} i(\gamma), \tag{4.8}$$

[2] via the well defined injective function defined in Theorem 2.26.

where the sum runs over all Milnor clusters.

Let us compute the total index $\sum_{\gamma\in\mathscr{C}} i(\gamma)$ for each cluster $\mathscr{C}$. From Theorem 2.26 to each cluster $\mathscr{C}$ it corresponds a fibre component $\alpha_t(\mathscr{C})$ and this is an injective correspondence. Moreover, for each vanishing and splitting components at $\lambda\in\mathbb{R}$ it corresponds an odd μ-cluster $\mathscr{C}$ with total index $-\frac{1}{2}$ and $+\frac{1}{2}$, respectively by Lemma 4.12 (b), (c). This implies that the number of odd μ-clusters of total index $-\frac{1}{2}$ or $+\frac{1}{2}$ at $\lambda\in\mathbb{R}$ is equal the number $\sum_{p\in\mathscr{L}_f,\lambda\in\mathbb{R}}\mathrm{Va}(p,\lambda)$ or $\sum_{p\in\mathscr{L}_f,\lambda\in\mathbb{R}}\mathrm{Sp}(p,\lambda)$, respectively. On the other hand the number of clusters at $\pm\infty$ is equal to the number $\mathrm{Va}(\pm\infty)$.

Our formula (4.7) follows by plugging in all these data in (4.8). $\square$

Remark 4.15. By definition, the fibres of a non-primitive polynomial with respect to ρ coincide with the level sets of ρ and thus are compact. One may compute the index at infinity (4.2) as the winding number of its gauss map restricted to a large enough circle C such that the singular locus of f is contained in the bounded region defined by C. In the case of a non-primitive polynomial, the circle C is contained in a single fibre of f, and thus $\mathrm{ind}_\infty(f)=1$, since along C its gradient vector field points either outwards or inwards.

CHAPTER

5

ESTIMATION OF THE INDEX AT INFINITY

In the 80's the known mathematician V. I. Arnold proposed, to a summer research group leaded by A. Durfee, the problem of relating the index at infinity of a polynomial function f in two real variables with its degree. In (DURFEE *et al.*, 1993, Proposition 2.5) Durfee's research group applied Bézout Theorem to find the number of intersection points (counted with multiplicities) between some big enough circle and the curve $\{f_y = 0\}$. Their reasoning lead them to the bound:

$$|\mathrm{ind}_\infty f| \leq d-1 \tag{5.1}$$

where d denotes the degree of f. For any degree d, one can realize the lower bound $1-d$ with a polynomial such that its zero fibre is a line arrangement of d generic lines. We refer the reader to (DURFEE *et al.*, 1993, §4) where the authors present this construction. The subsequent work (DURFEE, 1998) shows that the upper bound $d-1$ in (5.1) is not realized. Based on his inequality (DURFEE, 1998, Proposition 7.4) (see also Corollary 5.10 below) and a further study in the case $\ell_f = 1$, Durfee proves:

$$\mathrm{ind}_\infty f \leq \max\{1, d-3\}. \tag{5.2}$$

and raised the problem of estimating better upper bounds, since many examples show a gap between the theoretical bound (5.2) and the actual realized indices. This chapter aims to present a slightly improvement of Durfee's upper bound (5.2) (Theorem 5.11). We review some of Durfee's results in (DURFEE, 1998) as well clarify a couple of shadow points in his paper, see Remark 5.2 and Corollary 5.3.

5.1 Properties of fibres

Recall that the set $\mathscr{L}_f$ of "points at infinity" is the finite set of points in the real line at infinity $L^\infty \subset \mathbb{P}^2$ contained in the projective closure of some fibres of the real polynomial f, see Definition 3.6. Unlike the complex setting where the image of $f_\mathbb{C}$ is $\mathbb{C}$ and all fibres have the same points at infinity, in our real setting some fibres may be empty (Example 6.4), some fibres may be compact and some others not (Example 6.3), or those fibres which contain non-compact components may not have the same points at infinity (Example 5.5). In this section we give necessary conditions for $\mathscr{L}_f$ be empty or, equivalently, the fibres of f are either compact or empty (Lemma 5.1). For $p \in \mathscr{L}_f$, we prove in Corollary 5.3, that p belongs to the closure of fibres for values either close to $+\infty$ or $-\infty$. Finally, we recall a lower bound for the number of connected components in the level sets $|f| = A$ for some $A > 0$ large enough, see Proposition 5.4 and (DURFEE, 1998).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial with isolated singularities. As we have seen in §3.3 the map $\tau : X \to \mathbb{R}$, where $X := \{\tilde{f}(x,y,z) - tz^d = 0\} \subset \mathbb{P}^2 \times \mathbb{R}$, has finitely many stratified critical points, namely $\operatorname{Sing}\tau = \mathscr{S}_0 \cup \operatorname{Sing} f$. In particular the set $\tau(\operatorname{Sing}\tau)$ of critical values of τ is finite, and it is well-known, see e.g. (TIBĂR, 1999), (TIBĂR; ZAHARIA, 1999), (TIBĂR, 2007), that we have the inclusion $Atyp\, f \subset \tau(\operatorname{Sing}\tau)$.

The set $\mathbb{R}\backslash Atyp\, f$ is therefore a union of intervals, two of which are unbounded (and they coincide with $\mathbb{R}$ in case $Atyp\, f = \emptyset$). Let us denote the infinite intervals by I_+ (the one towards $+\infty$), and by I_- (the one towards $-\infty$). Consequently $f_| : f^{-1}(I) \to I$ is a trivial fibration, where I is either I_+ or I_-.

Let then F_- and F_+ denote the fibre of f over some point of I_-, and of I_+, respectively.

Lemma 5.1. Assume that the fibre F_+ contains a compact component C. Then $F_+ = C$ and F_- is empty, and moreover, all the non-empty fibres of f are compact.

The same statement holds if we switch the roles of F_- and F_+.

Remark 5.2. In (DURFEE, 1998, Prop 4.3, point 2 of the conclusion) it is stated that if all fibres of f are compact or empty, then its homogeneous part of highest degree f_d has no linear factors. A simple polynomial $f = x^4 + y^2$ shows that this conclusion is not true: we have $d = 4$; for $t < 0, f^{-1}(t) = \emptyset$; the fibre of each $t \geq 0$ is connected and bounded by the square $]-(t+1), t+1[\times]-(t+1), t+1[\subset \mathbb{R}^2$, so it is compact; and $f_4 = x^4$ has four linear factors x.

Proof of Lemma 5.1. Let $F_+ = f^{-1}(a)$ for some $a \in I_+$. Consider the surface $S_+ := f^{-1}(I_+) \subset \mathbb{R}^2$.

Let us denote by C_+ the connected component of S_+ which contains C. We may refer to (JOIŢA; TIBĂR, 2017) where this family plays a role in certain proofs. Then C_+ must be equal to the exterior of the oval C. Indeed, along each direction outside the circle, the values of the function f tend to infinity, and therefore the fibres of f must fill in the region of $\mathbb{R}^2$ outside C, which is connected. In particular this implies the equality $S_+ = C_+$.

Moreover, this also shows that f takes values less than a inside the oval C, and since f is bounded inside the oval C, it follows that the fibres $f = t$ must be empty for all t in some interval $]-\infty, b[$. $\square$

Corollary 5.3. (DURFEE, 1998, Prop 4.4(1) and Prop 4.3) If $p \in \mathscr{L}_f$ then $p \in \overline{F_+} \cap L^\infty$ or $p \in \overline{F_-} \cap L^\infty$.

Proof. We consider $D := X \backslash (\{p\} \times \mathbb{R})$, which is a connected set, and we apply the proof of Lemma 5.1 to D instead of $\mathbb{R}^2$. Namely, by contradiction, if $p \notin \overline{F_+} \cap L^\infty$ then $\overline{F_+}$ is compact in D and so, by Lemma 5.1, all the fibres $X_t \cap D$ are compact in D or empty. This shows that $p \notin X_t$ for any $t \in \mathbb{R}$, which contradicts our hypothesis. $\square$

We remark that Corollary 5.3 is not true any more if we write "and" instead of "or". Indeed, the polynomial presented in Example 6.4 satisfies that all its positive fibres have $p = [0;1;0] \in \mathscr{L}_f$ as point at infinity (so $p \in \overline{F_+} \cap L^\infty$), but its negative fibres are all empty (so $p \notin \overline{F_-} \cap L^\infty$).

We give an account of Durfee's proof of the following result.

Proposition 5.4. (DURFEE, 1998, Cor 4.2 and Prop. 4.4(2)) Let $A \in \mathbb{R}_+$ such that $Atyp\,(f) \cap (\mathbb{R} \backslash]-A, A[) = \emptyset$. Then the number of connected components of $F_+ \cup F_-$ is at least $2\ell_f$, where $\ell_f := \#\mathscr{L}_f$.

Proof. For the reader's convenience, we recall here Durfee's proof. One considers a resolution at infinity of f[1]. This produces a space X and a proper holomorphic $\hat{f} : X \to \mathbb{C}$ which extends f. One may regard the space X as the disjoint union of $\mathbb{C}^2$ and a finite number of divisors at infinity. Some of them are "horizontal", i.e. the restriction of $\hat{f}_{|E} : E \to \mathbb{C}$ is a non-constant polynomial of one variable, and the others are "vertical", i.e. the restriction of $\hat{f}_{|E} : E \to \mathbb{C}$ is constant.

For each point of $\mathscr{L}_f$ there is at least one horizontal divisor. Considering a horizontal divisor E, if $\hat{f}_{|E}$ is a polynomial of odd degree, then it is injective outside a compact interval $[-A, A]$, and therefore $(\hat{f})^{-1}(a)$ has one solution for every a such that $|a| > A > 0$. If the degree is even, then one has two solutions towards $-\infty$ and no solution towards $+\infty$, or the other way around. To each such solution corresponds a local branch of $(\hat{f})^{-1}(a)$, and for each half-branch we count one connected component of the fibre $f^{-1}(a)$. In this way each component of $f^{-1}(a)$ is counted twice, whether or not its two intersection points with the horizontal divisors at infinity coincide or not. We therefore get our inequality $\#F_+ \cup F_- \geq 2\ell_f$. $\square$

[1] we refer the reader to (TRÁNG; WEBER, 1995; DURFEE, 1998; FOURRIER, 1996) for resolution at infinity of polynomials in 2 variables.

The following example shows that the inequality of Proposition 5.4 may be very far from an equality.

Example 5.5. (DURFEE, 1998, p. 1347) Let $f = x(y+1)\cdots(y+k)$, $k \geq 2$. The fibre $f^{-1}(0)$ produces a partition of the plane into $2(k+1)$ horizontal strips between parallel lines delimited by the vertical axis. Each horizontal strip contains a connected component of $F_+ \cup F_-$, whereas $\ell_f = 2$.

5.2 The index gaps at infinity

We continue to consider a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ of degree $d \geq 2$. Recall from §4.3 that we endowed each Milnor arc at infinity of f with an index in $\{\pm\frac{1}{2}, 0\}$ which indicates the type of tangency along Milnor arcs between the fibres of f with level sets of the function $\rho(x,y) = x^2 + y^2$ i.e. ρ-maximum or ρ-minimum or ρ-inflectional, see (a), (b) and (c) after Definition 4.11. Here we relate the sum of Milnor arc indices at some points in the real line at infinity $L^\infty \subset \mathbb{P}^2$, with the intersection number between the complex Milnor set of f and the complex line at infinity $L^\infty_{\mathbb{C}}$, see Lemma 5.8. This is a key result that will be used for finding an upper bound of the index at infinity.

Definition 5.6. Let d_{Re} denote the number of real solutions of the equation $f_d = 0$ counted with multiplicity. We call it the *real degree* of f_d.

We denote by d_p the order of f_d at the point $p \in \{f_d = 0\} \subset \mathbb{P}^1$. This is equal to the multiplicity of the linear factor of f_d corresponding to p.

Remark 5.7. The inequality $d_p > 0$ does not imply that $p \in \mathscr{L}_f$, like in the example $f = xy^2 + x$. Indeed, $p := [1;0;0] \in \{f_d = 0\}$ with $d_p = 2$, but $p \notin \mathscr{L}_f$ since each fibre $f^{-1}(t) = (\frac{t}{y^2+1}, y)$ is contained in the vertical strip $]-(t+1), t+1[\times\mathbb{R}$.

By this reason, out of the obvious inequalities:

$$d_{Re} \geq \sum_{p \in \{f_d=0\} \cap L^\infty} d_p \geq \sum_{p \in \mathscr{L}_f} d_p, \tag{5.3}$$

the second may be strict, for instance in the example $f = x^4y + y^3$ where $d_{Re} = 5$, but one has $\sum_{p \in \mathscr{L}_f} d_p = 1$ because $\mathscr{L}_f = \{[1;0;0]\}$.

We consider the projective closure $\overline{M_{\mathbb{C}}(f)} \subset \mathbb{P}^2_{\mathbb{C}}$ of the complex Milnor set $M_{\mathbb{C}}(f)$. Let $p \in \{f_d = 0\} \cap L^\infty$. The following equality is displayed in (DURFEE, 1998, Lemma 7.3):

$$\mathrm{mult}_p(\overline{M_{\mathbb{C}}(f)}, L^\infty_{\mathbb{C}}) = d_p - 1. \tag{5.4}$$

We provide here an explicit proof of (5.4).

Proof of (5.4). One may assume (by an adequate linear change of coordinates) that $p = [1;0;0]$, and thus we have:

$$f(x,y) = y^{d_p} r(x,y) + \text{l.o.t}$$

where r is a homogeneous polynomial of degree $d - d_p$ and not divisible by y. In the chart $\{x \neq 0\}$, the Milnor set $\overline{M_{\mathbb{C}}(f)}$ has equation:

$$\hat{h}(y,z) = -d_p y^{d_p - 1} r(1,y) - y^{d_p} r_y(1,y) + y^{d_p+1} r_x(1,y) + zq(1,y,z) = 0 \tag{5.5}$$

where r_x and r_y denote the partial derivatives of r, and $q(x,y,z)$ is a homogeneous polynomial of degree $d-1$. By our assumption, we also have $r(1,y) = c_0 + \cdots + c_k y^k$, where $c_0 \neq 0$, and $k \leq d = d_p$. One has by definition:

$$\text{mult}_p(\overline{M_{\mathbb{C}}(f)}, L^\infty_{\mathbb{C}}) = \text{ord}_y \left(\hat{h}(y,z)_{|L^\infty_{\mathbb{C}}} \right)$$

and due to (5.5), the later is precisely $d_p - 1$. This ends the proof of equation (5.4).

□

Let $p \in L^\infty \cap \{f_d = 0\}$ and let $\overline{M(f)} \subset \mathbb{P}^2$ denote the real projective closure of the Milnor set $M(f)$ in $\mathbb{P}^2$. Each local branch β at p of the real germ of curve defined by $\overline{M(f)}$ is contained in its complexified branch which is a local branch of the complex germ at p of $\overline{M_{\mathbb{C}}(f)}$, and we denote by $\beta_{\mathbb{C}}$ the complexified branch of β.

Lemma 5.8. Let $p \in L^\infty \cap \{f_d = 0\}$. Then:

$$\sum_{c \in \mathbb{R}} i_{p,c} \leq d_p - 1. \tag{5.6}$$

The following phenomena are producing the difference between the two terms in the inequality (5.6), to which we shall refer as "**index gap**":

(a) A Milnor arc γ at p of index $i(\gamma) = -\frac{1}{2}$, or 0, yields a gap of at least 1, or $\frac{1}{2}$, respectively.

(b) A Milnor arc γ at p such that $\lim f_{|\gamma} = \pm\infty$, of any index, yields a gap of at least $\frac{1}{2}$.

(c) A Milnor branch β at p such that $\beta_{\mathbb{C}}$ is tangent to $L^\infty_{\mathbb{C}}$, or that $\beta_{\mathbb{C}}$ is singular at p, yields a gap of at least 1.

(d) A complex Milnor branch at p that is not the complexified of a real Milnor branch produces a gap of at least 1. If moreover this branch verifies the hypotheses of (c), then the gap increases to at least 2.

Moreover, the "sign gaps" (a), as well as the gaps (b), cumulate with the "singularity gaps" (c).

Remark 5.9. Situation (a) is illustrated by Example 6.4. Situation (b) can be seen in Example 6.3 by the cluster $\{\gamma_8, \gamma_1, \gamma_2\}$. In the same Example 6.3, the Milnor arcs γ_3 and γ_7 have index $+\frac{1}{2}$ and are tangent to L^∞, which means case (c). For case (d), one may consider the polynomial $f(x,y) = -5/3y^3 + y^2 - 2y + 4x^2 + x$. The closure of the complex Milnor set contains the point $p = [1;0;0]$, which means that there are complex Milnor branches at p. Nevertheless, there are no real branches by the simple reason that the initial part (of the lowest order) of $\overline{M}(f)$, which is $2z^2 + 6zy + 5y^2$, is irreducible over the reals.

Proof of Lemma 5.8. For every local complex branch β of the germ at p of $\overline{M_{\mathbb{C}}(f)}$, the intersection number $\mathrm{mult}_p(\beta, L^\infty_{\mathbb{C}})$ is a well defined positive integer, and $\mathrm{mult}_p(\overline{M_{\mathbb{C}}(f)}, L^\infty_{\mathbb{C}}) = \sum_\beta \mathrm{mult}_p(\beta, L^\infty_{\mathbb{C}})$.

Now (5.6) follows from the obvious inequalities:

$$\sum_{c\in\mathbb{R}} i_{p,c} \leq \sum_{c\in\mathbb{R}} i_{p,c} + \sum_{c\in\{\pm\infty\}} |i_{p,c}| \leq \sum_{c\in\mathbb{R}\cup\{\pm\infty\}} |i_{p,c}| \leq \mathrm{mult}_p(\overline{M_{\mathbb{C}}(f)}, L^\infty_{\mathbb{C}}) = d_p - 1 \tag{5.7}$$

all of which may be strict, and where the last equality is (5.4).

Each real Milnor branch at p has two Milnor arcs, of indice $-\frac{1}{2}$ or $+\frac{1}{2}$. The inequality (5.6) compares the indices of *real Milnor arcs* with intersection multiplicities of the complex Milnor branches. The proof of the gap situations goes as follows.

(a). A Milnor arc γ at p with $i(\gamma) = -\frac{1}{2}$ becomes $|i(\gamma)| = \frac{1}{2}$ on the right side of (5.7), which produces a gap of 1 in (5.6).

Any real Milnor arc γ of index 0 yields a null contribution on the left hand side of (5.6), whereas its corresponding complex branch gives a contribution of at least 1 on the right hand side. Therefore the arc γ yields a gap of at least $\frac{1}{2}$.

(b). If γ is a Milnor arc at p such that $f_{|\gamma} \to \pm\infty$ then it does not exist in the sum of the left side of (5.6), whereas it contributes to the right side of (5.7) by $|i(\gamma)| = \frac{1}{2}$.

(c). A Milnor branch β at p such that its complexification $\beta_{\mathbb{C}}$ is tangent to $L^\infty_{\mathbb{C}}$ or singular at p contributes by at most 1 in the left hand side of (5.7), whereas the multiplicity $\mathrm{mult}_p(\beta_{\mathbb{C}}, L^\infty_{\mathbb{C}})$ contributes by at least 2 in the right hand side of (5.7).

(d). Any real Milnor branch has a unique complexification. However, not all local complex branches are complexifications of real branches: there may be some purely complex Milnor branches, and all these count in the intersection index $\mathrm{mult}_p(\overline{M_{\mathbb{C}}(f)}, L^\infty_{\mathbb{C}})$, thus they give positive integer contributions in the right hand side of (5.7) whereas they do not exist in the left hand side of (5.7) □

From Equation (4.6), Lemma 5.8 via (5.3), and Proposition 5.4 one can deduce Durfee's result:

Corollary 5.10. (DURFEE, 1998, Prop 7.4)

$$\operatorname{ind}_\infty(f) \leq 1 + d_{Re} - 2\ell_f.$$

Proof. From (4.6):

$$\operatorname{ind}_\infty(f) = 1 + \sum_{p\in\mathscr{L}_f, c\in\mathbb{R}} i_{p,c} + \sum_{q\in L^\infty} i_{q,\infty}.$$

By Lemma 5.8, for each $p \in \mathscr{L}_f$ we have:

$$\sum_{c\in\mathbb{R}} i_{p,c} \leq \sum_{c\in\mathbb{R}} i^{\text{abs}}_{p,c} \leq \sum_{c\in\mathbb{R}\cup\{\pm\infty\}} i^{\text{abs}}_{p,c} \leq \operatorname{mult}_p(\bar{M}_{\mathbb{C}}, L^\infty_{\mathbb{C}}) = d_p - 1.$$

To each component counted in $\operatorname{Va}(\pm\infty)$ corresponds injectively (as shown in Theorem 2.26) a cluster $\mathscr{C}$, and we have seen in Proposition 4.13 that for this cluster we have the sum of indices $\sum_{\gamma\in\mathscr{C}} i(\gamma) = -1/2$. By Proposition 5.4, we have $\operatorname{Va}(\pm\infty) \geq 2\ell_f$. We thus obtain:

$$\sum_{q\in L^\infty} i_{q,\infty} = \sum_{\mathscr{C}} \sum_{\gamma\in\mathscr{C}} i(\gamma) \leq -\frac{1}{2} 2\ell_f = -\ell_f$$

where the sum is taken over all clusters $\mathscr{C}$ associated to $\pm\infty$. Altogether we get:

$$\operatorname{ind}_\infty(f) = 1 + \sum_{p\in\mathscr{L}_f, c\in\mathbb{R}} i_{p,c} + \sum_{q\in L^\infty} i_{q,\infty} \leq 1 + \sum_{p\in\mathscr{L}_f} (d_p - 1) - \ell_f \leq 1 + d_{Re} - 2\ell_f,$$

where in the last inequality one uses (5.3). □

5.3 Reviewing Durfee's upper bound

Durfee has bonded the index at infinity only in terms of the degree d, namely $\operatorname{ind}_\infty f \leq \max\{1, d-3\}$. Revising and completing his proof, we show here the following slight improvement:

Theorem 5.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of degree $d \geq 2$ with isolated singularities. Then:

(a) If $\ell_f \geq 2$, then $\operatorname{ind}_\infty(f) \leq d - 3$.

(b) If $\ell_f = 1$, then $\operatorname{ind}_\infty(f) \leq d - 3$ for $d \geq 4$, and $\operatorname{ind}_\infty(f) \leq 0$ for $d \leq 3$.

(c) If $\ell_f = 0$ then $\operatorname{ind}_\infty(f) = 1$.

Proof. (a). Follows immediately from Corollary 5.10.

(c). By Lemma 5.1, the set $\{|f(x,y)| = R\}$ for $R \gg 1$ is diffeomorphic to a circle, and the winding number over a circle is $+1$, thus $\operatorname{ind}_\infty(f) = 1$. This trivial fact was also observed in (DURFEE, 1998, Theorem 7.8).

(b). Let $\mathscr{L}_f = \{p\}$. Then, by a linear change of coordinates, we may assume that $p = [1;0;0]$. For $\ell_f = 1$, Theorem 4.14 and Corollary 5.10 yield:

$$\mathrm{ind}_\infty(f) = 1 + \frac{1}{2}\Big(\sum_{\lambda\in\mathbb{R}} \mathrm{Sp}(p,\lambda) - \sum_{\lambda\in\mathbb{R}\cup\{\pm\infty\}} \mathrm{Va}(p,\lambda)\Big) \le d_p - 1 \tag{5.8}$$

If $d_p \le d-2$ then Theorem 5.11 follows directly from (5.8). We next consider in the following the two other cases: $d_p = d-1$ and $d_p = d$.

<u>Case 1.</u> $d_p = d-1$.

Remark 5.12. Durfee claims in (DURFEE, 1998, pag. 1359), this case is not possible. His argument is the following: the roots of f_d other than p are complex, and hence occur in conjugate pairs, thus $d_p \le d-2$". This seems to be grounded on the same belief which made the object of Remark 5.2: the assertion $\mathscr{L}_f = \emptyset \implies d_{Re} = 0$ is false, as shown by the simple example $f = x^4 + y^2$, whereas its converse is obviously true.

A more particular assertion could be nevertheless true, and we formulate it as a conjecture:

$$d_p = d_{Re} - 1 > 0 \implies \mathscr{L}_f \neq \emptyset, \text{ and if moreover } p \in \mathscr{L}_f \text{ then } \ell_f = 2.$$

Instead, we can prove in the next lemma its version with d_{Re} replaced by d. This confirms Durfee's claim.

Lemma 5.13. If $d_p = d-1 > 0$ then $\mathscr{L}_f \neq \emptyset$, and if moreover $p \in \mathscr{L}_f$ then $\ell_f = 2$.

Proof. After an appropriate linear change of coordinates, we assume that $p = [1;0;0]$. Condition $d_p = d-1$ implies that the highest order homogeneous term f_d has the form xy^{d-1}, and if so, there exist a polynomial $h(x,y)$ of degree less than $d-2$ and a polynomial in one variable $u(y)$ such that

$$f(x,y) = xy^{d-1} + xh(x,y) + u(y).$$

We prove that the fibre of $t = 0$ has $[0;1;0]$ as point at infinity: the restriction of f to the line $\{x = 0\}$ coincides with the polynomial in one variable $u(y)$ with lead coefficient $b \in \mathbb{R}$. If $u = 0$ then $\{x = 0\} \subset f^{-1}(0)$, and if so the point $[0;1;0] \in \mathscr{L}_f \neq \emptyset$. Let us assume then that $b \neq 0$ and we choose a non-zero $x_0 \in \mathbb{R}$ such that the lead coefficient $x_0 + r$ of the polynomial in one variable $f(x_0,y)$ has opposite sign than b.[2] For large enough positive values of the variable y the sign of the polynomials $u(y)$ and $f(x_0,y)$ in the variable y, are constant equals to the sign of their respective lead coefficients. It follows from the choose of x_0 that $f(0,y)f(x_0,y) < 0$, for large enough positive values of y, and therefore the fibre of 0 intersects all this lines parallel to the x-axis given by large enough values of y. This proves that $[0;1;0]$ is a point at infinity of f.

[2] Note that r is zero when the degree of u is less than $d-1$. Otherwise, it is equal to b.

On the other hand, we have seen that the number of points at infinity of f is bounded above by the number of real linear factors in f_d. Altogether this implies that if, in addition $p \in \mathscr{L}_f$, then $\ell_f = 2$ □

This shows that the case $d = d_p - 1$, where $f = xy^{d-1}$+ l.o.t. having the unique point $[1;0;0]$ at infinity, does not exist.

Case 2. $d_p = d$.

We may assume as above that $p = [1;0;0]$ and since $d_p = d$, one may also assume that, modulo some appropriate linear change of coordinates, one has $f_d = y^d$.

Lemma 5.14. Let $p = [1;0;0] \in \mathscr{L}_f$ and let $f_d = y^d$. Then either there exists a Milnor branch at p which is tangent to L^∞, or $f(x,y) = y^d + v(x) + u(y)$, where $\deg v \le 2 < d$ and $\deg u \le d-1$.

Proof. Let us assume that f contains mixed terms, namely let $f = y^d + xyq(x,y) + v(x) + u(y)$, where $q \not\equiv 0$ is a polynomial of degree $\le d-3$, and $v(x)$ and $u(y)$ are some polynomials of degrees $\le d-1$. We write explicitly the equation $\hat{h}(1,y,z) = 0$ of the closure $\overline{M(f)}$ of the Milnor set in the chart $\{x = 1\}$. This is a polynomial of degree at least $d-1$ because it contains the term dy^{d-1}. Its order at $(0,0)$ is ord $\hat{h}(1,y,z) \le$ ord $z\hat{q}(1,y,z) + zy\hat{q}(1,y,z) \le d-2$, where $\hat{q}(x,y,z)$ denotes the homogenization of $q(x,y)$ of degree $d-3$ by the variable z. Moreover, all terms of $\hat{h}(1,y,z)$ different from dy^{d-1} are multiple of z. Altogether this implies that $\overline{M(f)}_p$, i.e. the germ at p of the Milnor set, contains a branch which is tangent to the line at infinity $L^\infty = \{z = 0\}$.

Let us now treat the complementary case whenever f contains no mixed terms, i.e. $f = y^d + v(x) + u(y)$, where $v(x)$ and $u(y)$ are some polynomials of degrees $\le d-1$. Then the Milnor set germ $\overline{M(f)}_p$ in the chart $\{x = 1\}$ is defined by the equation:

$$\hat{h}(1,y,z) = -dy^{d-1} + zy\hat{v}_x(1,z) - z\hat{u}_y(y,z), \tag{5.9}$$

where $\hat{v}_x(x,z)$ and $\hat{u}_y(y,z)$ denote the homogenization of degree $d-2$ of the derivatives v_x and u_y. The 1st and the 3rd terms of (5.9) are homogeneous of degree $d-1$, while the term in the middle is of order $\le d-2$ if and only if $\deg v \ge 3$. We deduce that the tangent cone $\mathrm{Cone}\overline{M(f)}_p$ contains the line $\{z = 0\}$ if and only if $\deg v \ge 3$. □

We compute the index in the special case of Lemma 5.14.

Lemma 5.15. Let $f = v(x) + u(y)$, where $\deg u = \deg f = d$, and $\deg v \le 2 < d$ such that $v_x \not\equiv 0$. Then $\mathrm{ind}_\infty(f) = 0$ for $d = 3$ and $|\mathrm{ind}_\infty(f)| = 1$ for $d > 3$.

Proof. If $\deg v \le 1$ then $\mathrm{Sing} f = \emptyset$ and therefore $\mathrm{ind}_\infty(f) = 0$. If $\deg v = 2$, then the derivative $v_x = ax + b$, where $a \ne 0$ by our assumption, changes sign one time, precisely at $x = -b/a$. Consider a large enough circle $C \subset \mathbb{R}^2$ centred at the origin such that we compute the index at infinity of f as the winding number over C. Consider the two points $N, S = C \cap \{x = -b/a\}$.

If $d = 3$ then u_y is a polynomial of degree 2 and therefore has a constant sign outside a compact subset of $\mathbb{R}$. This implies that on each half circle cut out of C by the vertical line $\{x = -b/a\}$, the variation of the vector field grad$f$ over the circle C between the two points N and S is zero. In case $d > 3$ it also follows that this variation can be either zero, or of π or $-\pi$ on each half circle. □

We continue the proof of Case 2. By Lemma 5.14 and by Lemma 5.15, if there is no Milnor branch tangent to L^∞ at p, then we obtain $\text{ind}_\infty(f) = 0$ when $d = 3$, and $\text{ind}_\infty(f) \le d - 3$ when $d > 3$, hence Theorem 5.11 is proved in this situation.

In the following we therefore focus on the remaining case established by Lemma 5.14: there exists at least a Milnor branch β tangent to L^∞ at p. Then the study falls into the following 4 situations:

(i). There are at least two Milnor branches at p which are tangent to L^∞, then by Lemma 5.8 (c) we get a gap of at least 1 for each of these branches. Therefore $\text{ind}_\infty(f) \le d_p - 1 - 2 = d_p - 3$.

(ii). There is a single Milnor branch β tangent to L^∞ at p, and such that $\text{mult}_p(\beta, L^\infty_{\mathbb{C}}) \ge 2$. By (5.6), this implies $d_p \ge 3$.

If $\text{mult}_p(\beta, L^\infty_{\mathbb{C}}) > 2$ then we have a gap of at least 2, and thus $\text{ind}_\infty(f) \le d_p - 3$. If $\text{mult}_p(\beta, L^\infty_{\mathbb{C}}) = 2$ and not all indices of the arcs of β are $+\frac{1}{2}$ then we get a gap of at least $3/2$. Since the index is an integer, the gap is of at least 2, and therefore $\text{ind}_\infty(f) \le d_p - 3$ again.

Last case, let $\text{mult}_p(\beta, L^\infty_{\mathbb{C}}) = 2$ and the index is $+\frac{1}{2}$ on each of the two arcs of β. Since β is also tangent to L^∞, it follows that β is nonsingular at p, more precisely the germ of β at p is equivalent, after some linear change of coordinates, with the curve $z = y^2$. Thus the two Milnor arcs are in the same half-plane of the chart $\mathbb{R}^2$ cut by the line $z = 0$. According to Proposition 3.13, after each of the two splittings we obtain two components of the fibres of f which are tangent to the the line $L := \{z = 0\}$, and moreover, along the two Milnor arcs the tangency is to different semi-lines, say L_+ and L_-. This implies that, in the absence of other splittings or vanishings at p, there should exist a trivial fibration connecting two components tangent to different semi-lines, which is treated by Proposition 3.14, and which tells that this situation is impossible.

(iii). There is a single Milnor branch tangent to L^∞, and there are also non-tangent Milnor branches at p, such that at least one of the Milnor arcs has index $< \frac{1}{2}$. Then by Lemma 5.8 (a),(c) we obtain a gap of at least 1.5, thus of at least 2 since both sides of (5.8) are integers. Therefore we get $\text{ind}_\infty(f) \le d_p - 3$.

(iv). There is single tangent Milnor branch at p, there are one or more transversal Milnor branches, and such that all the Milnor arcs at p have index $+\frac{1}{2}$. This means that all arcs are of splitting type, and in particular each arc is a cluster. Our Lemma 3.14 shows that this situation is impossible. $\square$

CHAPTER

6

EXAMPLES

We consider here four examples. For three of them we will use pictures to encode information, and in order to draw the frame we will use the following construction employed in (DURFEE, 1998). Let $\mathbb{R}^2 \hookrightarrow \mathbb{P}^2 \simeq \mathbb{R}^3 \setminus \{0\}/\mathbb{R}^*$ be the embedding defined by $(x,y) \mapsto [x;y;1]$, and let

$$S := \{(a,b,0) \in \mathbb{R}^3 \setminus \{0\}\}/\mathbb{R}_+,$$

be the circle which is a double covering of the line at infinity $L^\infty \subset \mathbb{P}^2$. The compactification $\mathbb{R}^2 \sqcup S$ of $\mathbb{R}^2$ may be represented as a 2-disk D with boundary S.

The dashed circle is the boundary ∂D_R of the disk D_R centred at the origin of radius $R \gg 1$ as in Remark 2.13. The Milnor arcs live in the annulus between ∂D_R and S. By enumerating the Milnor arcs as $\gamma_1, \ldots, \gamma_k$ we mean that they are consecutive in the counterclockwise ordering (Definition 2.18).

The limit $\lambda \in \mathbb{R} \cup \{\pm\infty\}$ to which $f_{|\gamma}$ tends along some Milnor arc γ is written near the Milnor arc γ close to S (see Proposition 2.16). The index $i(\gamma)$ is attached to each Milnor arc γ at the intersection with the doted circle in the middle of the annulus, and the respective little arrow indicates the direction of the gradient of the Milnor arc. We write "Sp" or "Va" next to a cluster when the corresponding fibre component is splitting or vanishing, respectively. Whenever a cluster contains more than one Milnor arc, we connect all its arcs by a thicker curve (e.g. like in Figure 4).

Example 6.1. Let $f(x,y) = x^2y + x$. This polynomial has two clusters at the atypical value $\lambda = 0$, each of them composed by a single Milnor arc of positive index. Both clusters have the point $p = [0;1;0]$ at infinity, and no other Milnor arcs abut to this point. Thus $i_p = 1, d_p = 2$, and (5.6) is an equality.

The polynomial f has two clusters having the point $q = [1;0;0]$ at infinity, the corresponding fibre components of which tend to the value $+\infty$. And there are two more clusters with the corresponding fibre components tending to the value $-\infty$. These 4 clusters being vanishing clusters at $\lambda = \pm\infty$, all of them have index $-\frac{1}{2}$ by Proposition 4.13.

It then follows from Theorem 4.14 that $\mathrm{ind}_\infty f = 1 + 2\cdot\frac{1}{2} - 4\cdot\frac{1}{2} = 0$. Comparing to Lemma 5.8, there are no index gaps of any kind, and this example realizes the maximal index at infinity that a polynomial of degree 3 may have, cf Theorem 5.11(a).

Example 6.2. Let $f(x,y) = y^5 + x^2y^3 - y$. The Milnor set $M(f)$ is defined by the equation $x(-1+3x^2y^2+3y^4) = 0$.

We have $d = 5$, $\mathscr{L}_f = \{p\}$ with $p := [1;0;0]$, and there are two other complex non-real points at infinity due to the factor y^2+x^2 of the top homogeneous part f_5. In the chart $\{x=1\}$ of $\mathbb{P}^2$, the germ at p of the Milnor set $\overline{M(f)}$ is defined by the equation $\hat{h}(y,z) = -z^4 + 3y^2 + 3y^4 = 0$, thus $\{y=0\}$ is the only line at p tangent to the Milnor branches. There are 4 clusters having the point $p \in \mathscr{L}_f$ at infinity, each containing a single Milnor arc, all being splitting clusters at the value 0, and one pair of clusters is tangent to a semi-line, and the second pair of clusters is tangent to the other semi-line. This example shows in particular that in Proposition 3.14, the hypothesis "two consecutive Milnor arcs at p tangent to different tangent semi-lines" cannot be removed.

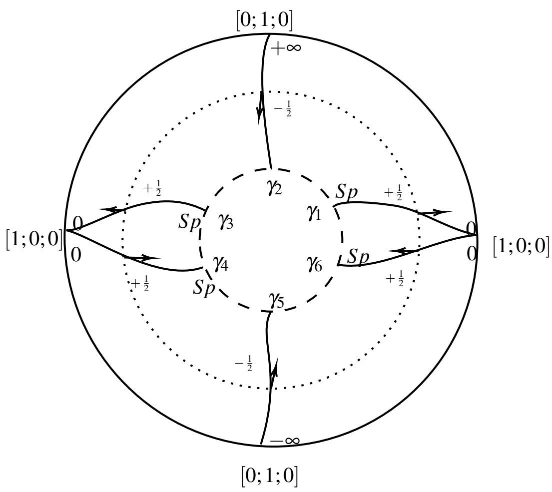

Figure 3 – Milnor arcs of $f = y^5 + x^2y^3 - y$

The fibres at infinity $F_+ \cup F_-$ (see §5.1 for the notation) consist of two components; one corresponds to the cluster $\{\gamma_2\}$, see Figure 3, and covers the upper semi-circle of S, and the other corresponds to the cluster $\{\gamma_5\}$ and covers the lower semi-circle of S. By Theorem 4.14, one then has $\mathrm{ind}_\infty(f) = 2$, which is the highest possible index at infinity of a polynomial of degree $d = 5$ with $\ell_f = 1$, according to Theorem 5.11(b).

Example 6.3. Let $f(x,y) = (x-y^2)\Big((x-y^2)(y^2+1) - 1\Big)$.

Its Milnor set is defined by the equation: $y(1 - 4x^2 + 2x^3 + 2y^2 + 2xy^2 - 8x^2y^2 + 2y^4 + 6xy^4) = 0$.

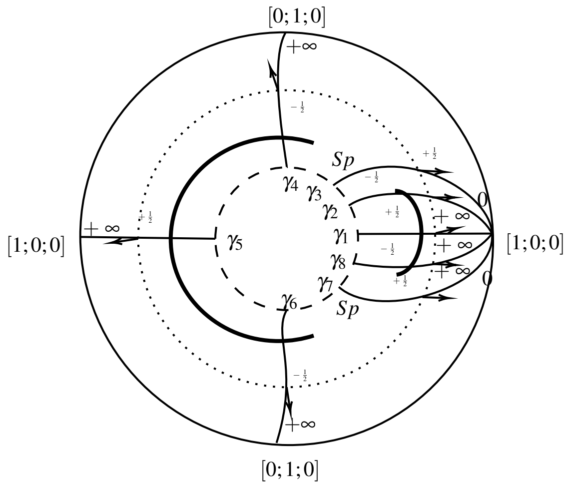

Figure 4 – Milnor arcs of $f = (x-y^2)((x-y^2)(y^2+1)-1)$

One has $d = d_{Re} = 6$, and $\mathscr{L}_f = \{p\}$ where $p := [1;0;0]$, with $d_p = 6$ (cf Definition 5.6). In the chart $\{x = 1\}$ of $\mathbb{P}^2$, the germ at p of $\overline{M(f)}$ is defined by the equation:

$$\hat{h}(y,z) = y(6y^4 - 8y^2z + 2y^4z + 2z^2 + 2y^2z^2 - 4z^3 + 2y^2z^3 + z^5) = 0,$$

and therefore the tangent cone at p of $\overline{M_{\mathbb{C}}(f)}$ is the union $L^\infty \cup \{y = 0\}$.

The polynomial f has a global minimum at the point $(\frac{1}{2}, 0) \in \mathbb{R}^2$, with critical value $-\frac{1}{4}$. The fibre of f is empty over the interval $]-\infty, -\frac{1}{4}[$. Over $[-\frac{1}{4}, 0[$, the fibre of f is compact and connected, having outside the disk D_R two arcs, one of which is splitting along the cluster $\{\gamma_7\}$, and the other is splitting along the cluster $\{\gamma_3\}$; both Milnor clusters are tangent to L^∞ at the point p. Over the interval $]0, +\infty[$, the fibre of f has two connected components: one of them corresponds to the cluster $\{\gamma_8, \gamma_1, \gamma_2\}$, and is vanishing[1] at the point p with the value of f tending to $+\infty$. The other component corresponds to the vanishing cluster $\{\gamma_4, \gamma_5, \gamma_6\}$ and covers the entire line at infinity L^∞ as $t \to +\infty$. By Theorem 4.14 we get $\mathrm{ind}_\infty(f) = 1 + 2\frac{1}{2} - 2\frac{1}{2} = 1$.

By direct computations we see that the germ $\overline{M_{\mathbb{C}}(f)_p}$ has 3 non-singular branches: one is $\{y = 0\}$ and contains the Milnor arcs γ_1 and γ_5. The other two branches are tangent to the line at infinity[2] and have both their two arcs on the same side of it, namely γ_8 with γ_2, and γ_7 with γ_3, respectively. By Lemma 5.8(c) and (b), 2 tangent branches to $L^\infty_{\mathbb{C}}$ and 4 Milnor arcs at p at $\pm\infty$, produce in (5.6) a gap of at least 2 and 2 respectively. Proceeding as in Corollary 5.10 one

[1] The cluster $\{\gamma_8, \gamma_1, \gamma_2\}$ may be contrasted to Proposition 3.13 in which such a situation cannot happen for a cluster associated to a finite limit value of f.

[2] We get $\mathrm{mult}(\overline{M_{\mathbb{C}}(f)_p}, L^\infty) = 1 + 2 + 2 = 5$, each tangency producing multiplicity 2 in (5.4).

obtains the upper bound

$$\begin{aligned}\operatorname{ind}_{\infty} f &\leq 1 + \sum_{c\in\mathbb{R}} i_{p,c} + \sum_{q\in L^{\infty}} i_{q,\infty} \\ &\leq 1 + (d_p - \ell_f - 2 - 4\cdot\frac{1}{2}) - \ell_f \\ &= 1\end{aligned}$$

which coincide with the actual index of f.

Example 6.4. Let $f(x,y) = x^2 + (xy-1)^2$, see Figure 5. One has $d = d_{Re} = 4$, and $\mathscr{L}_f = \{p, q\}$ where $p := [1;0;0]$ and $q := [0;1;0]$, with $d_p = d_q = 2$. Note that f has empty fibres over $]-\infty, 0[$.

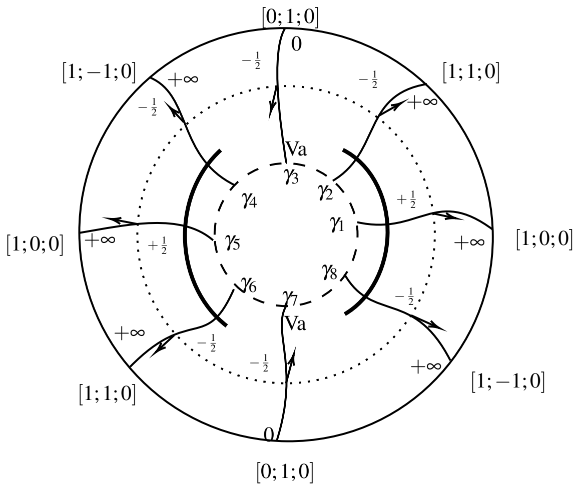

Figure 5 – Milnor arcs of $f = x^2 + (xy-1)^2$

The Milnor set $M(f)$ is defined by the equation $x^2 + xy - x^3y - y^2 + xy^3 = 0$. At $q \in L^{\infty}$ there are two clusters with a single Milnor arc, namely $\{\gamma_3\}$ and $\{\gamma_7\}$, and the corresponding fibre components are both vanishing at the value 0. There are two more clusters, namely $\{\gamma_8, \gamma_1, \gamma_2\}$, and $\{\gamma_4, \gamma_5, \gamma_6\}$, the fibre components of which are both tending to $+\infty$ and cover half the circle S each of them.

By Theorem 4.14 we get:

$$\operatorname{ind}_{\infty}(f) = 1 + \frac{1}{2}\sum \operatorname{Sp}(p,\lambda) - \frac{1}{2}\sum \operatorname{Va}(p,\lambda) - \frac{1}{2}\operatorname{Va}(\pm\infty) = 1 + \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 2 - \frac{1}{2}\cdot 2 = -1,$$

where the sums are over $\{p \in \mathscr{L}_f, \lambda \in \mathbb{R}\}$.

The tangent cones of $\overline{M_{\mathbb{C}}(f)}$ at q and p are the lines $\{x = 0\}$ and $\{y = 0\}$, respectively, both with multiplicity 1. By considering the sign gaps as in Lemma 5.8(a), one actually obtains gap of 2 at the point $q \in \mathscr{L}_f$ due to the two Milnor arcs with index $-\frac{1}{2}$ at the value 0 of f. We then get

$$ind_{\infty}(f) \leq 1 + 4 - 4 - 2 = -1,$$

which coincides with the actual index at infinity of f computed above.

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APPENDIX

A

COMPUTING ATYPICAL VALUES

Recall that Chapter §2 presents a detailed study of the Milnor set (Definition 2.1), in particular we proved in Theorem 2.26 that for each μ-cluster (Definition 2.19) and for certain values t there exists a unique fibre component of f in a neighborhood at infinity. The correspondence between μ-clusters and fibre components is key in the characterization of atypical values Theorem 3.5. Furthermore, one can effectively detect the atypical values as we have seen in Section 3.4.

In this appendix we apply the algorithm decribed in Section 3.4 to compute the set of atypical values at infinity for two examples extracted from (TIBĂR; ZAHARIA, 1999). We use the software Mathematica to find solution of equations and the adequate truncation of the Newton-Puiseux expansion of the points at infinity in $\overline{M(f)} \cap L^{\infty}$, see Section 3.4.2.

We use the following notations: $\mathrm{Jac}(f,\rho)$ denotes the polynomial defining the Milnor set i.e. $yf_x - xf_y$, by D_r we denote a disk centred at the origin with radius $r > 0$ and C_r denotes its border in $\mathbb{R}^2$.

A.1 Example 1

The polynomial

$$f(x,y) = x^2y^3(y^2-25)^2 + 2xy(y^2-25)(y+25) - (y^4+y^3-50y^2-51y+575).$$

has four singular points and the regular value $\lambda = 0$ has two vanishing components and two splitting components. Following 3.4 we compute the Milnor set $M(f)$ which is defined by the polynomial:

$$\begin{aligned}\mathrm{Jac}(f,\rho) =& 51x - 1250x^2 + 100xy - 100x^2y + 1250y^2 - 3xy^2 + 150x^2y^2 + 1875x^3y^2 \\ &+ 50y^3 - 4xy^3 + 8x^2y^3 - 50y^4 - 1250xy^4 - 250x^3y^4 - 2y^5 + 100xy^6 \\ &+ 7x^3y^6 - 2xy^8.\end{aligned}$$

$\mathrm{Jac}(f,\rho)$ is an irreducible polynomial, Then $\mathrm{Jac}(f,\rho) = h'(x,y)$ and has no irreducible components of the form $x^2+y^2-r^2$ by Proposition 2.11. Then $\mu(M(f))$ has no circles and it is the union of finitely many points which are solution of the system $h' = 0, xh'_y - yh'_x = 0$ by Definition 2.8. Let us list the approximations of thepoints in $\mu(M(f))$:

$$(-5.93546, 7.5999), \quad (2.53818, -5.32629), \quad (-3.48373, 3.85988),$$

$$(-1.06481, 0.16992), \quad (-0.243122, -1.08353), \quad (0,0).$$

After computing the norms of each point, we have that D_{10} contains all points in $\mu(M(f))$. By Proposition 2.16, the circle C_{10} intersects transversaly each Milnor arc at infinity. In the following we list the solutions of the system of equations $\mathrm{Jac}(f,\rho) = 0$ and $x^2+y^2-10^2 = 0$ and ordered them in the counterclockwise direction as in Definition 2.18

$$\begin{array}{ccc} p_1 \approx (9.99661, 0.260218), & p_2 \approx (9.47272, 3.2043), & p_3 \approx (8.66828, 4.98606), \\ p_4 \approx (4.30914, 9.02393), & p_5 \approx (-0.00466653, 10), & p_6 \approx (-4.30884, 9.02407). \\ p_7 \approx (-8.65228, 5.01378), & p_8 \approx (-9.47605, 3.19444), & p_9 \approx (-9.47674, -3.19239) \\ p_{10} \approx (-8.65494, -5.0092), & p_{11} \approx (-4.3085, -9.02423), & p_{12} \approx (-0.00200005, -10) \\ p_{13} \approx (4.30916, -9.02392), & p_{14} \approx (8.6656, -4.99072), & p_{15} \approx (9.47244, -3.20512) \\ & p_{16} \approx (9.99674, -0.255396). & \end{array}$$

Hence the polynomial f has sixteen Milnor arcs at infinity and we can order them as $\gamma_1, \dots, \gamma_{16}$ with the order induced by the point $p_1, \dots, p_{16}$, see Definition 2.18.

Now we will find the points in $\overline{M(f)} \cap L^\infty$. As in Section 3.4.2 we need to homogenize $\mathrm{Jac}(f,\rho)$ with respect to the new variable z and find the points in L^∞. The homogenization of $\mathrm{Jac}(f,\rho)$ is

$$\begin{aligned}\widetilde{h}(x,y,z) =& 7x^3y^6 - 2xy^8 - 250x^3y^4z^2 + 100xy^6z^2 + 1875x^3y^2z^4 + 8x^2y^3z^4 - 1250xy^4z^4 \\ &- 2y^5z^4 + 150x^2y^2z^5 - 4xy^3z^5 - 50y^4z^5 - 100x^2yz^6 - 3xy^2z^6 + 50y^3z^6 \\ &- 1250x^2z^7 + 100xyz^7 + 1250y^2z^7 + 51xz^8.\end{aligned}$$

Hence by finding the points in $\mathbb{P}^2$ such that $\widetilde{h}(x,y,z) = 0$ and $z = 0$ we have that $\overline{M(f)} \cap L^\infty = \{[0:1:0], [1:0:0], [\sqrt{2}:\sqrt{7}:0], [-\sqrt{2}:\sqrt{7}:0]\}$.

By the truncated Newton-Puiseux process (see Section 3.4.2 and (DIAS; JOIŢA; TIBĂR, 2021)) applied to each point in $\overline{M(f)} \cap L^\infty$ we list the limits $\lim_{t\to\infty} f(\gamma_i(t))$:

$$0 = \lim_{t\to\infty} f(\gamma_3(t)) = \lim_{t\to\infty} f(\gamma_7(t)) = \lim_{t\to\infty} f(\gamma_{10}(t)) = \lim_{t\to\infty} f(\gamma_{14}(t)),$$

$$\infty = \lim_{t\to\infty} f(\gamma_2(t)) = \lim_{t\to\infty} f(\gamma_4(t)) = \lim_{t\to\infty} f(\gamma_6(t)) = \lim_{t\to\infty} f(\gamma_8(t)) = \lim_{t\to\infty} f(\gamma_{16}(t)),$$

$$-\infty = \lim_{t\to\infty} f(\gamma_1(t)) = \lim_{t\to\infty} f(\gamma_5(t)) = \lim_{t\to\infty} f(\gamma_9(t)) = \lim_{t\to\infty} f(\gamma_{11}(t)) = \lim_{t\to\infty} f(\gamma_{12}(t)),$$

$$-\infty = \lim_{t\to\infty} f(\gamma_{13}(t)) = \lim_{t\to\infty} f(\gamma_{15}(t))$$

From Definition 3.2 the vanishing and splitting at infinity are detected by Milnor arcs at infinity γ such that the limits $\lim_{t\to\infty} f(\gamma(t))$ are finite. Hence we will concentrate our study on $\gamma_3, \gamma_7, \gamma_{10}, \gamma_{14}$. The gradient vector of f evaluated at each point p_3, p_7, p_{10}, p_{14} is:

$$\begin{aligned} \operatorname{grad} f(p_3) &\approx (0.00538847, 0.00310546), & \operatorname{grad} f(p_7) &\approx (-0.00525811, 0.00305082), \\ \operatorname{grad} f(p_{10}) &\approx (-0.00156177, -0.000898007), & \operatorname{grad} f(p_{14}) &\approx (108.077, -2.76115). \end{aligned}$$

Consequently

$$\begin{aligned} \operatorname{Sgn}\langle p_3, \operatorname{grad} f(p_3)\rangle &= +1, & \operatorname{Sgn}\langle p_7, \operatorname{grad} f(p_7)\rangle &= +1, \\ \operatorname{Sgn}\langle p_{10}, \operatorname{grad} f(p_{10})\rangle &= +1, & \operatorname{Sgn}\langle p_{14}, \operatorname{grad} f(p_{14})\rangle &= +1. \end{aligned}$$

From the criteria presented in Section 3.4.3 one has $\mathscr{I}_0 = \{\gamma_3, \gamma_7, \gamma_{10}, \gamma_{16}\}$ and $\mathscr{D}_0 = \emptyset$. From Definition 2.19 we need to choose the maximal sequences on $\mathscr{I}_0$ of consecutive Milnor arcs at infinity. then we conclude that $\mathscr{C}_1 := \{\gamma_3\}, \mathscr{C}_2 := \{\gamma_7\}, \mathscr{C}_3 := \{\gamma_{10}\}, \mathscr{C}_4 := \{\gamma_{14}\}$ are the increasing μ-clusters associated to $\lambda = 0$ and there are no decreasing μ-clusters associated to $\lambda = 0$.

Now let us identify the ρ-type of tangency of each Milnor arc at infinity associated to $\lambda = 0$, for that we compute the sing of $\operatorname{Jac}(f,\rho)$ in each band between consecutive Milnor arcs, see (2.2):

$$\begin{aligned} \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_2, \gamma_3[& \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_3, \gamma_4[\\ \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_6, \gamma_7[& \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_7, \gamma_8[\\ \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_9, \gamma_{10}[& \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_{10}, \gamma_{11}[\\ \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_{13}, \gamma_{14}[& \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_{14}, \gamma_{15}[. \end{aligned}$$

By Lemma 2.20 one concludes that γ_3, γ_7 have ρ-maximum type of tangency and γ_{10}, γ_{14} have ρ-minimum type of tangency. Hence, by Theorem 3.5 $\lambda = 0$ is an atypical value at infinity of f since it has four odd μ-clusters. Moreover, $\mathscr{C}_1, \mathscr{C}_2$ have splitting components and $\mathscr{C}_3, \mathscr{C}_4$ have vanishing components by Definition 3.2.

A.2 Example 2

In (TIBĂR; ZAHARIA, 1999) it is proved that

$$f(x,y) = 2x^2y^3 - 9xy^2 + 12y,$$

has empty singular set and no atypical values. Here we show that there are exactly 4 Milnor arcs to finite value (see Definition 2.17), forming 2 even μ-clusters at 0 cf Theorem 3.5.

Its Milnor set is defined by the polynomial:

$$\mathrm{Jac}(f,\rho) = 12x - 18x^2y + 6x^3y^2 + 9y^3 - 4xy^4,$$

and then $\mathrm{Jac}(f,\rho)$ is an irreducible polynomial with no components of the form $x^2+y^2-r^2$. By Proposition 2.11 $\mu(M(f))$ is the union of finitely many points which are solution of the system $h' = 0, xh'_y - yh'_x = 0$, where $h(x,y) = \mathrm{Jac}(f,\rho)$ (see Definition 2.8).

The approximations of the points in $\mu(M(f))$ are:

$$\begin{array}{ccc} (-1.31838, -1.85956), & (1.31838, 1.85956), & (1.61395, 0.787331), \\ (-1.61395, -0.787331), & (0,0). & \end{array}$$

After computing the norms of each point above, we have that D_3 contains all point in $\mu(M(f))$. Then from Proposition 2.16 the circle C_3 intersects transversaly each Milnor arc at infinity. In the following we list the solutions of the system of equations $\mathrm{Jac}(f,\rho) = 0$ and $x^2+y^2-3^2 = 0$ and ordered them in the counterclockwise direction as in Definition 2.18

$$\begin{array}{ccc} p_1 \approx (2.98077, 0.339151), & p_2 \approx (2.92246, 0.677678), & p_3 \approx (1.98013, 2.25368), \\ p_4 \approx (0.782521, 2.89615), & p_5 \approx (-1.80028, 2.39979), & p_6 \approx (-2.98077, -0.339151), \\ p_7 \approx (-2.92246, -0.677678), & p_8 \approx (-1.98013, -2.25368), & p_9 \approx (-0.782521, -2.89615) \\ & p_{10} \approx (1.80028, -2.39979) & \end{array}$$

Hence the polynomial f has sixteen Milnor arcs at infinity and we can order them as $\gamma_1, \ldots, \gamma_{10}$ with the order induced by the point $p_1, \ldots, p_{10}$.

Now we will find the points in $\overline{M(f)} \cap L^\infty$. As in section 3.4.2 we need to homogenize equation $\mathrm{Jac}(f,\rho)$ with respect to the variable z and find the points in L^∞. The homogenization of $\mathrm{Jac}(f,\rho)$ is

$$\widetilde{h}(x,y,z) = 6x^3y^2 - 4xy^4 - 18x^2yz^2 + 9y^3z^2 + 12xz^4.$$

Hence by finding the points in $\mathbb{P}^2$ such that $\widetilde{h}(x,y,z) = 0$ and $z = 0$ we have that $\overline{M(f)} \cap L^\infty = \{[0:1:0],[1:0:0],[\sqrt{2}:\sqrt{3}:0],[-\sqrt{2},\sqrt{3}:0]\}$.

By the truncated Newton-Puiseux process (see section 3.4.2 and (DIAS; JOIŢA; TIBĂR, 2021)) applied to each point in $\overline{M(f)} \cap L^\infty$ we list the limits $\lim_{t\to\infty} f(\gamma_i(t))$:

$$0 = \lim_{t\to\infty} f(\gamma_1(t)) = \lim_{t\to\infty} f(\gamma_2(t)) = \lim_{t\to\infty} f(\gamma_6(t)) = \lim_{t\to\infty} f(\gamma_7(t))$$

$$\infty = \lim_{t\to\infty} f(\gamma_3(t)) = \lim_{t\to\infty} f(\gamma_4(t))$$

$$-\infty = \lim_{t\to\infty} f(\gamma_5(t)) = \lim_{t\to\infty} f(\gamma_8(t)) = \lim_{t\to\infty} f(\gamma_9(t)) = \lim_{t\to\infty} f(\gamma_{10}(t))$$

From Definition 3.2 the vanishing and splitting at infinity are detected by Milnor arcs at infinity γ such that the limits $\lim_{t\to\infty} f(\gamma(t))$ are finite. Hence we will concentrate our study on $\gamma_1,\gamma_2,\gamma_6,\gamma_7$. With the approximation of the points p_1,p_2,p_6,p_7 listed above, let us find approximations of the gradient vector of f evaluated at each point.

$$\begin{aligned} \operatorname{grad} f(p_1) &\approx (-0.570088,-0.0648644), & \operatorname{grad} f(p_2) &\approx (-0.495097,-0.114806),\\ \operatorname{grad} f(p_6) &\approx (-0.570088,-0.0648644), & \operatorname{grad} f(p_7) &\approx (-0.495097,-0.114806). \end{aligned}$$

then we obtain that

$$\begin{aligned} \operatorname{Sgn}\langle p_1, \operatorname{grad} f(p_1)\rangle &= -1, & \operatorname{Sgn}\langle p_2, \operatorname{grad} f(p_2)\rangle &= -1,\\ \operatorname{Sgn}\langle p_6, \operatorname{grad} f(p_6)\rangle &= +1, & \operatorname{Sgn}\langle p_7, \operatorname{grad} f(p_7)\rangle &= +1. \end{aligned}$$

Then we conclude that $\mathscr{I}_0 = \{\gamma_6,\gamma_7\}$ and $\mathscr{D}_0 = \{\gamma_1,\gamma_2\}$. From Definition 2.19 we need to choose the maximal sequences on $\mathscr{I}_0$ of consecutive Milnor arcs at infinity, then we conclude that $\mathscr{C}_1 := \{\gamma_1,\gamma_2\}$ is an increasing μ-cluster associated to $\lambda = 0$ and $\mathscr{C}_2 := \{\gamma_6,\gamma_7\}$ is a decreasing μ-cluster associated to $\lambda = 0$.

Now let us identify the ρ-type of tangency of each Milnor arc at infinity associated to $\lambda = 0$:

$$\begin{aligned} \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_{10},\gamma_1[& \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_1,\gamma_2[\\ \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_1,\gamma_2[& \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_2,\gamma_3[\\ \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_5,\gamma_6[& \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_6,\gamma_7[\\ \operatorname{Jac}(f,\rho) &< 0 \text{ in }]\gamma_6,\gamma_7[& \operatorname{Jac}(f,\rho) &> 0 \text{ in }]\gamma_7,\gamma_8[. \end{aligned}$$

By Lemma 2.20 one concludes that γ_1,γ_6 have ρ-maximum type of tangency and γ_2,γ_3 have ρ-minimum type of tangency. Let us observe that both $\mathscr{C}_1$ and $\mathscr{C}_2$ are even μ-clusters. Then by Theorem 3.5 $\lambda = 0$ is a typical value at infinity of f.

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