The Torque on a Magnet

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The torque on a magnet

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The field of a 'soft' ellipsoidal magnet immersed in a permeable fluid, and the torque exerted on it by an externally applied uniform field, are obtained by simple direct methods.

Different ways of defining dipole moment are discussed and it is shown that, with suitable definitions, the moments $\boldsymbol{m}$ or $\boldsymbol{j}$ which specify the external field of the magnet also give the torque $\boldsymbol{T}$ with $\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B}_0 = \boldsymbol{j} \times \boldsymbol{H}_0$.

The permeability of the fluid does *not* enter simply as a factor, and neither of the conventional Kennelly or Sommerfeld approaches is correct.

1. Introduction

There is continuing controversy as to the 'correct' formulation for the torque on a magnet immersed in a permeable fluid in a magnetic field, even though the problem is essentially solved as part of the general discussion of the Coulomb Law Committee (1950) and the detailed discussions of Brown (1951, 1966, § 2). In the hope of ending the controversy the present paper gives a very simple and unambiguous calculation of the torque, and discusses why other calculations give incorrect answers.

As Brown has shown, arguments as to which of $\boldsymbol{B}$ and $\boldsymbol{H}$ is the 'effective' field are not relevant in this context; neither are the units (and dimensions) in which $\boldsymbol{B}$ and $\boldsymbol{H}$ are measured. In this paper (rationalized) SI units will be used. (For unrationalized c.g.s. e.m.u. put $\mu_0 = 1$, insert 4π in front of $\boldsymbol{j}$, $\boldsymbol{m}$, $\boldsymbol{J}$, $\boldsymbol{M}$, and insert $1/4\pi$ in front of N.)

In free space the vector fields $\boldsymbol{B}$ and $\boldsymbol{H}$ are related by $\boldsymbol{B} = \mu_0 \boldsymbol{H}$. In other media we have to introduce a third field, which specifies the local magnetization. We use either $\boldsymbol{M}$ or $\boldsymbol{J}$ for which

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0 \boldsymbol{H} + \boldsymbol{J}.$$

Clearly $\boldsymbol{M}$ and $\boldsymbol{J}$ differ only by the constant factor μ_0. Similarly we have two types of dipole moment, $\boldsymbol{m}$ and $\boldsymbol{j}$, which also differ dimensionally (but not necessarily numerically) by μ_0.

In § 2 of this paper various definitions of dipole moment are compared, and in § 3 the situation for which the torque is to be calculated is rigorously specified. § 4 gives the field distribution for a passive ellipsoid in a uniform field, and § 5 that from an ellipsoidal magnet. The results of § 5 are discussed in § 6 (and a more detailed discussion of one aspect given in appendix 1). The torque on any magnet in a uniform field is calculated in § 7 (another calculation of torque is summarized in appendix 2), and the defects of other calculations discussed in § 8. A concluding discussion is given in § 9.

2. Definition of dipole moment

A 'point dipole' in free space gives a field which can equally well be specified either by a scalar potential ϕ or a vector potential $\boldsymbol{A}$, for which

$$\boldsymbol{H} = -\operatorname{grad}\phi, \quad \phi = \frac{\boldsymbol{j}^*\cdot\hat{\boldsymbol{r}}}{\mu_0 4\pi r^2}, \tag{1a}$$

$$\boldsymbol{B} = \operatorname{curl}\boldsymbol{A}, \quad \boldsymbol{A} = \frac{\mu_0 \boldsymbol{m}^*\times\hat{\boldsymbol{r}}}{4\pi r^2}. \tag{1b}$$

As
$$-\operatorname{grad}\left(\frac{\boldsymbol{c}\cdot\hat{\boldsymbol{r}}}{4\pi r^2}\right) = \operatorname{curl}\left(\frac{\boldsymbol{c}\times\hat{\boldsymbol{r}}}{4\pi r^2}\right) = \frac{-\boldsymbol{c}+3(\boldsymbol{c}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}}}{4\pi r^3}$$

and $\boldsymbol{B} = \mu_0\boldsymbol{H}$ in this situation, we must have $\boldsymbol{j}^* = \mu_0\boldsymbol{m}^*$.

The (microscopic) field between atoms is obtained by integrating (1*a*) or (1*b*) over all such point sources. It follows that a finite-sized magnet giving a dipole field occupying a region V in otherwise free space has dipole† moments $\boldsymbol{m}^* = \boldsymbol{m}_{\mathrm{V}}$ and $\boldsymbol{j}^* = \boldsymbol{j}_{\mathrm{V}}$ where $\boldsymbol{m}_{\mathrm{V}}$ and $\boldsymbol{j}_{\mathrm{V}}$ are the volume moments defined by

$$\boldsymbol{m}_{\mathrm{V}} = \int_{\mathrm{V}} \boldsymbol{M}\,\mathrm{d}\tau, \quad \boldsymbol{j}_{\mathrm{V}} = \int_{\mathrm{V}} \boldsymbol{J}\,\mathrm{d}\tau, \tag{2}$$

where $\boldsymbol{M}$ and $\boldsymbol{J}$ are the dipole moments per unit volume. Clearly $\boldsymbol{j}_{\mathrm{V}} = \mu_0\boldsymbol{m}_{\mathrm{V}}$.

Now consider a magnet giving a dipole field in a permeable medium, where we are concerned now with the (differently) macroscopically averaged fields $\boldsymbol{H}$ or $\boldsymbol{B}$. The field originates partly in the magnetization of the magnet, and partly in the induced magnetization of the medium, but it is convenient to assign the total field to a single dipole moment; we are, in effect, replacing the real situation having distributed magnetization by a fictitious point dipole. This is an arbitrary procedure, and the definition of the moment of this fictitious dipole is itself arbitrary; several conventions have been used, and in particular the presence or absence of the factor μ_{e} in the definitions (3) below varies between authors. The conventions of (3), which are used throughout this paper, are self-consistent and are those which lead to the simplest results. The basic controversy cannot be resolved by any change of convention!

Let the medium have relative permeability μ_{e} so that $\boldsymbol{B} = \mu_{\mathrm{e}}\mu_0\boldsymbol{H}$. Again we can equally well express $\boldsymbol{H}$ in terms of ϕ, or $\boldsymbol{B}$ in terms of $\boldsymbol{A}$.

These favouring the magnetostatic, or Kennelly, approach (thinking of a magnetic dipole producing a constant external flux of $\boldsymbol{B}$ regardless of μ_{e}, by analogy with electrostatic charges) would define an (effective) moment $\boldsymbol{j}_{\mathrm{e}}$ such that

$$\phi = \frac{\boldsymbol{j}_{\mathrm{e}}\cdot\hat{\boldsymbol{r}}}{4\pi\mu_{\mathrm{e}}\mu_0 r^2}. \tag{3a}$$

† In the absence of any net magnetic pole strength, the dipole (or first-order) moment is in fact an invariant, independent of origin: it, and its field, can be considered independently of any higher moments. All these moments (dipole and higher) are relevant only outside the region of integration.

The $\boldsymbol{B}$ flux is then proportional to j_e. It is shown later that the torque exerted on this dipole by a uniform field $\boldsymbol{H}_0$ is $\boldsymbol{j}_e \times \boldsymbol{H}_0$, and it follows that the mutual torque between two dipoles $\boldsymbol{j}_{e1}, \boldsymbol{j}_{e2}$ is proportional to $j_{e1} j_{e2}/\mu_e$.

Those favouring the electromagnetic, or Sommerfeld, approach (thinking of a current loop producing a constant flux of $\boldsymbol{H}$) would define an effective moment $\boldsymbol{m}_e$ such that

$$\boldsymbol{A} = \frac{\mu_e \mu_0 \boldsymbol{m}_e \times \hat{\boldsymbol{r}}}{4\pi r^2}, \tag{3b}$$

with the $\boldsymbol{H}$ flux proportional to m_e. The torque in a uniform field $\boldsymbol{B}_0$ is shown to be $\boldsymbol{m}_e \times \boldsymbol{B}_0$, and the mutual torque between $\boldsymbol{m}_{e1}$ and $\boldsymbol{m}_{e2}$ is then proportional to $\mu_e m_{e1} m_{e2}$.

These definitions of effective moment are quite consistent provided $\boldsymbol{j}_e = \mu_e \mu_0 \boldsymbol{m}_e$. However it has usually been assumed (see e.g. Whitworth & Stopes-Roe 1971) that the magnet can be specified by its volume moments (2); clearly $\boldsymbol{j}_e$ and $\boldsymbol{m}_e$ cannot *both* be equal to the volume moments $\boldsymbol{j}_V$ and $\boldsymbol{m}_V$, for which $\boldsymbol{j}_V = \mu_0 \boldsymbol{m}_V$. Although often expressed as an argument as to whether the torque is given by $\boldsymbol{j} \times \boldsymbol{H}_0$ or $\boldsymbol{m} \times \boldsymbol{B}_0$, essentially the controversy has been as to which of $\boldsymbol{j}_e$ or $\boldsymbol{m}_e$ equals $\boldsymbol{j}_V$ or $\boldsymbol{m}_V$ (and is invariant to change of the external medium).

To resolve the controversy it is necessary to determine unambiguously the field distribution of, and torque on, a reasonably unsymmetrical finite-sized magnet immersed in a permeable medium. This is now done, and for completeness the magnet is allowed to be 'soft'. It turns out that in general *none* of the four moments are invariant or equal, all of them depending on the shape of the magnet and the permeability of the outside medium as well as the properties of the magnet material.

3. Specification of the problem

The external medium is assumed to be an incompressible fluid; it is homogeneous, isotropic and linear, and has relative permeability μ_e (subscript e for external).

Before the magnet is inserted there exists a uniform field $\boldsymbol{B}_0 = \mu_e \mu_0 \boldsymbol{H}_0$. The medium is of sufficient extent and the field sources sufficiently distant, that the insertion of the magnet does not significantly alter the reluctance seen by the sources. Nor are the sources affected by the field of the magnet.

The magnet *material* is also assumed to be homogeneous, isotropic, and linear, with relative permeability μ_i (subscript i for internal), but in addition to any induced magnetization there is a constant, uniform, 'permanent' magnetization $\boldsymbol{J}_0 = \mu_0 \boldsymbol{M}_0$. Thus

$$\boldsymbol{B} = \mu_0(\boldsymbol{H}+\boldsymbol{M}) = \mu_0(\boldsymbol{H}+\boldsymbol{M}_{\text{ind}}) + \mu_0 \boldsymbol{M}_0 = \mu_i \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}_0, \tag{4}$$

where $\boldsymbol{H}$ is the *total* $\boldsymbol{H}$, due to both the external sources and the magnet itself. (This is a reasonable approximation to the behaviour of a well stabilized magnet. $\boldsymbol{M}_0$ is the magnetization at zero $\boldsymbol{H}$, and μ_i the 'reversible' or 'recoil' permeability;

for a 'perfectly hard' magnet $\mu_i = 1$.) To enable the problem to be solved algebraically the magnet is assumed to be of ellipsoidal shape (so that the internal fields are uniform) with semi-axes a, b, c.

As $\operatorname{div} \boldsymbol{B} = 0$ and $\operatorname{curl} \boldsymbol{H} = 0$ everywhere, both inside and outside the magnet (except at current sources of $\boldsymbol{H}_0$), we can specify the local fields equally well either by $\boldsymbol{H}$ and ϕ or by $\boldsymbol{B}$ and $\boldsymbol{A}$. For algebraic simplicity $\boldsymbol{H}$ and ϕ will be used. Dipole moments are as defined in (3).

As both media are effectively linear, the field distribution when the magnet is in the external field can be obtained by adding that of the magnet in the medium ($\boldsymbol{H}_0 = 0$) to that of the passive (magnet) material in the external field ($\boldsymbol{M}_0 = 0$).

In solving the problem no representation of the magnet in terms of poles or currents is made; the field solutions given are simply those which satisfy the boundary conditions and the constitutive equations $\boldsymbol{B} = f(\boldsymbol{H})$, and in calculating the torque only these fields are used.

4. Passive ellipsoid in uniform field in permeable medium

The ellipsoid is placed with its a axis parallel to the previously uniform field $\boldsymbol{H}_0 = H_0\hat{\boldsymbol{x}}$. Inside the ellipsoid, $\boldsymbol{B}_\mathrm{f} = \mu_\mathrm{i}\mu_0 \boldsymbol{H}_\mathrm{f}$, and outside $\boldsymbol{B}_\mathrm{f} = \mu_\mathrm{e}\mu_0 \boldsymbol{H}_\mathrm{f}$ (subscript f for field). The boundary conditions are that $\boldsymbol{H}_\mathrm{f} \times \hat{\boldsymbol{n}}$ and $\boldsymbol{B}_\mathrm{f} \cdot \hat{\boldsymbol{n}}$ must be continuous on the surface of the ellipsoid, and that $\boldsymbol{H}_\mathrm{f}$ should be finite everywhere and tend to $\boldsymbol{H}_0$ at large distance.

The solution is (see, for example, Stratton 1941, § 3.27) inside:

$$\boldsymbol{H}_\mathrm{f} = \boldsymbol{H}_0 \mu_\mathrm{e}/D_a \tag{5}$$

outside:

$$\boldsymbol{H}_\mathrm{f} = \boldsymbol{H}_0 - \operatorname{grad} \phi_\mathrm{f},$$

where

$$\phi_\mathrm{f} = H_0 x F_a(\xi)\,(\mu_\mathrm{i} - \mu_\mathrm{e})/D_a. \tag{6}$$

The quantity ξ is the ellipsoidal coordinate specifying the size of the ellipsoidal surfaces (the magnet surface is $\xi = 0$),

$$F_a(\xi) = \tfrac{1}{2}abc \int_\xi^\infty \frac{\mathrm{d}s}{(s+a^2)\,[(s+a^2)\,(s+b^2)\,(s+c^2)]^{\frac{1}{2}}},$$

and

$$D_a \equiv N_a \mu_\mathrm{i} + (1 - N_a)\,\mu_\mathrm{e} = \mu_\mathrm{e} + N_a(\mu_\mathrm{i} - \mu_\mathrm{e}),$$

where

$$N_a \equiv F_a(0)$$

is the demagnetizing factor appropriate to the a axis.

At large distances $F(\xi)$ tends to $abc/3r^2$, so that ϕ_f becomes the potential of a dipole of moment

$$\boldsymbol{j}_\mathrm{f}/\mu_\mathrm{e}\mu_0 = \boldsymbol{m}_\mathrm{f} = \tfrac{4}{3}\pi abc \boldsymbol{H}_0(\mu_\mathrm{i} - \mu_\mathrm{e})/D_a; \tag{7}$$

the ellipsoid produces this dipole field, together with the fields of higher multipoles.

If the field $\boldsymbol{H}_0$ is not along one of the principal axes of the ellipsoid it can be resolved along these axes and the three solutions added; the total internal field, and the total dipole moment, are still alined, but are then *not* parallel to $\boldsymbol{H}_0$.

For a sphere, $N = \frac{1}{3}$ for all axes, and the external field is exactly dipolar.

5. Field of ellipsoidal magnet in permeable medium

The magnet has $\boldsymbol{M}_0 = M_0\hat{\boldsymbol{z}}$ along the c axis.

$$\left.\begin{array}{ll}\text{Inside the magnet} & \boldsymbol{B}_{\mathrm{m}} = \mu_{\mathrm{i}}\mu_0\boldsymbol{H}_{\mathrm{m}} + \mu_0\boldsymbol{M}_0, \\ \text{and outside} & \boldsymbol{B}_{\mathrm{m}} = \mu_{\mathrm{e}}\mu_0\boldsymbol{H}_{\mathrm{m}},\end{array}\right\} \tag{8}$$

where the subscript m indicates fields produced by the magnet. The boundary conditions are that $\boldsymbol{H}_{\mathrm{m}} \times \hat{\boldsymbol{n}}$ and $\boldsymbol{B}_{\mathrm{m}} \cdot \hat{\boldsymbol{n}}$ must be continuous on the magnet surface, and that the fields should be finite everywhere and decrease at least as $1/r^3$ at large distance.

Many texts give the solution for a hard spherical magnet in a vacuum. For a permeable medium, Wilberforce (1933) and Smythe (1939, § 12.10) give the solution for a hard spherical magnet, and Page (1933), Preston (1950), and Sommerfeld & Ramberg (1950) for a hard spheroidal magnet. I have not been able to find in the literature any full solution for the 'soft' ellipsoidal magnet, though partial solutions are given by Brown (1951), Diesselhorst (1948), Knapp (1953) and Timotkin & Ciric (1971).

The solution is in fact

inside:
$$\boldsymbol{H}_{\mathrm{m}} = -\boldsymbol{M}_0 N_c/D_c, \tag{9}$$

$$\boldsymbol{B}_{\mathrm{m}} = \mu_0\boldsymbol{M}_0\mu_{\mathrm{e}}(1-N_c)/D_c, \tag{10}$$

$$\boldsymbol{M}_{\mathrm{m}} = \boldsymbol{M}_0[N_c + (1-N_c)\mu_{\mathrm{e}}]/D_c, \tag{11}$$

outside:
$$\boldsymbol{H}_{\mathrm{m}} = -\operatorname{grad}\phi_{\mathrm{m}}, \tag{12}$$

where
$$\phi_{\mathrm{m}} = M_0 z F_c(\xi)/D_c.$$

$$F_c(\xi) = \tfrac{1}{2}abc\int_{\xi}^{\infty}\frac{\mathrm{d}s}{(s+c^2)\,[(s+a^2)\,(s+b^2)\,(s+c^2)]^{\frac{1}{2}}},$$

and
$$D_c \equiv N_c\mu_{\mathrm{i}} + (1-N_c)\,\mu_{\mathrm{e}} = \mu_{\mathrm{e}} + N_c(\mu_{\mathrm{i}} - \mu_{\mathrm{e}}),$$

where $N_c \equiv F_c(0)$ is the demagnetizing factor appropriate to the c axis.

On the external surface of the magnet it can be shown that (12) gives

$$\boldsymbol{H}_{\mathrm{m}}(\xi = 0)_{\mathrm{ext}} = [-N_c\boldsymbol{M}_0 + (\boldsymbol{M}_0\cdot\hat{\boldsymbol{n}})\,\hat{\boldsymbol{n}}]/D_c, \tag{13}$$

the result obtained by Diesselhorst (equation (21)) by a different method.

The external field has a dipole component corresponding to a moment

$$\boldsymbol{j}_{\mathrm{m}}/\mu_{\mathrm{e}}\mu_0 = \boldsymbol{m}_{\mathrm{m}} = \tfrac{4}{3}\pi abc\boldsymbol{M}_0/D_c, \tag{14}$$

a result obtained indirectly by Brown (1951).

If $\boldsymbol{M}_0$ is not along one of the principal axes of the ellipsoid it can be resolved along these axes; the internal fields $\boldsymbol{H}_{\mathrm{m}}$, $\boldsymbol{B}_{\mathrm{m}}$ and $\boldsymbol{M}_{\mathrm{m}}$, and the effective dipole moment, will then all be in different directions, none of them parallel to $\boldsymbol{M}_0$.

From (13) and (9) we see that the external $\boldsymbol{H}$ flux is $\pi abM_0(1-N_c)/D_c$, and the internal $\boldsymbol{H}$ flux $\pi ab M_0 N_c/D_c$; the demagnetizing factor N gives the fraction of the total $\boldsymbol{H}$ flux from the surface which returns internally, regardless of the permeability contrast. (The usually quoted expression $\boldsymbol{H}_{\mathrm{m}} = -N\boldsymbol{M}_{\mathrm{m}}$ is true only if $\mu_{\mathrm{e}} = 1$.)

6. Discussion of field result

Equation (14) shows that, even for a hard magnet, neither effective moment (and hence neither flux) is independent of the outside medium; a comparison with (11) shows also that neither is equal to the integrated moment – for skew magnetization they are not even parallel. That a volume integral over the magnet cannot in general give the correct field is clear from the following consideration.

The field and potentials at any point can certainly be expressed as volume integrals of magnetization, but the integral must include *all* regions of magnetization, induced as well as permanent; only for a magnet in a vacuum can the integral be taken only over the magnet. When we represent the field of a magnet in a permeable medium by a single moment, this moment includes a contribution from the magnetization of the medium. The nature of this contribution is discussed in appendix I.

Also discussed in appendix I is the exact representation of the situation in terms of 'fictitious' or 'bound' surface current or pole distributions. In our situation it turns out that two quite simple models of the magnet, in which the permanent magnetization $\boldsymbol{M}_0$ is replaced by (constant) surface sources, also give the correct external field. Outside, $\boldsymbol{B} = \mu_{\mathrm{e}}\mu_0 \boldsymbol{H}$ as before, and the magnet material is replaced by a passive medium having $\boldsymbol{B} = \mu_{\mathrm{i}}\mu_0 \boldsymbol{H}$. A 'real' or 'free' fixed surface pole distribution $\boldsymbol{M}_0 \cdot \hat{\boldsymbol{n}}$ (for which $\operatorname{div} \boldsymbol{B} \neq 0$) gives the correct $\boldsymbol{H}$ everywhere (and the correct $\boldsymbol{B}$ outside), while a 'real' or 'free' fixed current distribution $\boldsymbol{M}_0 \times \hat{\boldsymbol{n}}/\mu_{\mathrm{i}}$ (for which $\operatorname{curl} \boldsymbol{H} \neq 0$) gives the correct $\boldsymbol{B}$ everywhere (and the correct $\boldsymbol{H}$ outside).† It must be emphasized that these models are *not* exact. However, they do characterize the Kennelly and Sommerfeld approaches, and show the difficulties of trying to use these approaches for a finite sized magnet in a permeable medium rather than the classical point dipole in free space. In particular, the $\boldsymbol{B}$ flux is independent of μ_{e} only if the outside medium is allowed to penetrate the pole or current distribution ($\mu_{\mathrm{i}} = \mu_{\mathrm{e}}$), or for a long thin magnet ($N \to 0$), while the $\boldsymbol{H}$ flux is independent only for a short fat magnet ($N \to 1$).

The way in which Φ_{m}, the total external (and internal) $\boldsymbol{B}$ flux (and hence the external $\boldsymbol{B}$ and $\boldsymbol{H}$) varies with μ_{e} can be seen by using the surface current model and the concept of magnetic reluctance. For a closed magnetic circuit the reluctance R_{m} relates the (constant) flux Φ to the magnetomotive force F_{m} by $F_{\mathrm{m}} = \Phi R_{\mathrm{m}}$. With our magnet most of the flux 'leaks' sideways, but we can define an effective R_{m} by using the total flux Φ_{m}. The (real) surface current density $\boldsymbol{M}_0 \times \hat{\boldsymbol{n}}/\mu_{\mathrm{i}}$ gives

† The factor $1/\mu_{\mathrm{i}}$ is at first surprising, but there is an essential difference between these two models: a pole distribution is not changed, but a resistanceless current is reduced by a factor μ_{r}, as space is filled with a medium of relative permeability μ_{r}.

a m.m.f. $2cM_0/\mu_i$. From § 5 we have $\Phi_m = \mu_e\mu_0\pi abM_0(1-N_c)/[N_c\mu_i+(1-N_c)\mu_e]$, so we obtain for the effective reluctance

$$R_m = \frac{2c}{\mu_0\pi ab}\left(\frac{1}{\mu_i}+\frac{N_c}{1-N_c}\frac{1}{\mu_e}\right).$$

(The same expression can be obtained from the real situation by using

$$H_i = [(\Phi_m/\pi ab)-\mu_0 M_0]/\mu_i\mu_0,$$

and equating the line integral of $\boldsymbol{H}$ along the axis to zero.) In this expression the two terms obviously give the internal and external reluctances respectively. For constant N_c, increasing μ_e decreases only the external part of R_m, so Φ_m and B increase less rapidly than μ_e, and H decreases. For long thin magnets ($N_c \to 0$) the internal reluctance dominates, so Φ_m becomes independent of μ_e; for short fat magnets (magnetic shells; $c \to 0$, $N_c \to 1$) the internal reluctance is negligible, and the whole (constant) m.m.f. acts directly on the fluid, giving Φ_m and B proportional to μ_e, with H constant.

7. Calculation of torque

Given the field distribution, the torque on the magnet can be calculated either directly or by calculating the appropriate energy as a function of orientation.

Energy calculations can be tricky, but Timotkin & Ciric (1971) have used one based on $\delta U = \delta(i\Phi)$ which appears to be valid and which gives the torque in terms of a volume integral over the magnet. (The magnet can have arbitrary shape and be inhomogeneous, but the resultant internal field distribution must be known.) Their calculation is reproduced in appendix II.

The torque can be determined directly using the Maxwell stress concept, and this approach leads to a very simple interpretation. The total torque $\boldsymbol{T}$, from both body and surface forces, is obtained by performing the integral

$$T = \mu_e\mu_0\int_S \boldsymbol{r}\times[(\boldsymbol{H}\cdot\hat{\boldsymbol{n}})\,\boldsymbol{H}-\tfrac{1}{2}H^2\hat{\boldsymbol{n}}]\,\mathrm{d}S \tag{15}$$

over any surface in the fluid medium surrounding the magnet; $\boldsymbol{r}$ is the radius vector from any origin, and $\boldsymbol{H}$ is the total field from all sources. (The analogous electrostatic expression is a standard result, and (15) is derived similarly (Brown 1951, § 2.4).) Previous authors have integrated over the surface of spheroidal magnets. Diesselhorst (1948), by using (13), was able to calculate the torque when both the magnetization $\boldsymbol{M}_0$ and the applied field $\boldsymbol{H}_0$ were parallel to the central plane. Preston (1950) calculated the torque for axial magnetization, as did Sommerfeld & Ramburg (1950) in the limit of a long thin magnet.

A more general, and simpler, calculation is now given.

If in (15) we choose the integration surface to be a sphere about the origin of $\boldsymbol{r}$, the second integrand vanishes. Now expand the $\boldsymbol{H}$ of internal origin in terms of fields derived from spherical harmonic potentials $\phi_n(\theta,\lambda)/r^{n+1}$. The torque must be independent of the radius of the integration surface, so it is clear that the only non-zero contribution comes from the interaction of the external uniform field $\boldsymbol{H}_0$ with the field of the central dipole. (If $\boldsymbol{H}_0$, is not uniform, then higher multipoles will also contribute to the torque; there will also be a net force.)

Now at radius $\boldsymbol{r}$ a dipole of moment $\boldsymbol{j}_\mathrm{e} = \mu_\mathrm{e}\mu_0 \boldsymbol{m}_\mathrm{e}$ gives a field

$$\left.\begin{aligned} \boldsymbol{H} &= 3(\boldsymbol{F}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{F}, \\ \text{where}\qquad \boldsymbol{F} &= \boldsymbol{j}_\mathrm{e}/\mu_\mathrm{e}\mu_0\, 4\pi r^3 = \boldsymbol{m}_\mathrm{e}/4\pi r^3. \end{aligned}\right\} \tag{16}$$

Adding this dipole field to $\boldsymbol{H}_0$ we find

$$\boldsymbol{T} = \mu_\mathrm{e}\mu_0 \int \boldsymbol{r}\times(\boldsymbol{H}_0 - \boldsymbol{F})\,(\boldsymbol{H}_0\cdot\hat{\boldsymbol{n}} + 2\boldsymbol{F}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S.$$

This consists of four terms of the form $\int \boldsymbol{r}\times \boldsymbol{A}(\boldsymbol{B}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant vectors. Diesselhorst (1948) showed that in this case

$$\int_\mathrm{S} \boldsymbol{r}\times\boldsymbol{A}(\boldsymbol{B}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S = \int_\mathrm{V} \boldsymbol{B}\times\boldsymbol{A}\,\mathrm{d}\tau, \tag{17}$$

a result obtained more easily from the tensor generalization of the divergence theorem (Milne 1948, § 101)

$$\int_\mathrm{S} K_{ij\ldots}\, n_l\,\mathrm{d}S = \int_\mathrm{V} \frac{\partial K_{ij\ldots}}{\partial x_l}\,\mathrm{d}\tau.$$

Hence

$$\begin{aligned} \boldsymbol{T} &= \tfrac{4}{3}\pi r^3 \mu_\mathrm{e}\mu_0\, 3\boldsymbol{F}\times\boldsymbol{H}_0 \\ &= \boldsymbol{j}_\mathrm{e}\times\boldsymbol{H}_0 = \boldsymbol{m}_\mathrm{e}\times\boldsymbol{B}_0, \end{aligned} \tag{18}$$

from (16) and (14).

Thus we have rigorously confirmed that the torque is given by the vector product of the appropriate dipole moment and the uniform applied field, where the moment is that which specifies the external field of the magnet.

In (18) $\boldsymbol{j}_\mathrm{e}$ is the total moment producing dipole field. In our situation it is the sum of the magnetic moment $\boldsymbol{j}_\mathrm{m}$ and that induced by the applied field $\boldsymbol{j}_\mathrm{f}$. The torque on $\boldsymbol{j}_\mathrm{f}$ will be ignored in the following discussion. (If $\boldsymbol{H}_0$ is along one of the principal axes of the ellipsoid then $\boldsymbol{j}_\mathrm{f}\times\boldsymbol{H}_0 = 0$. However, if $\boldsymbol{H}_0$ is skew there is a torque; see, for example, Stratton (1941, § 3.29).)

8. Discussion of torque result

For our ellipsoidal magnet (18) gives

$$\begin{aligned} \boldsymbol{T} &= \tfrac{4}{3}\pi abc \frac{\mu_\mathrm{e}}{N_c\mu_\mathrm{i} + (1-N_c)\,\mu_\mathrm{e}} \boldsymbol{J}_0\times\boldsymbol{H}_0 \\ &= \tfrac{4}{3}\pi abc \frac{1}{N_c\mu_\mathrm{i} + (1-N_c)\,\mu_\mathrm{e}} \boldsymbol{M}_0\times\boldsymbol{B}_0. \end{aligned} \tag{19}$$

We see that the torque is proportional to H_0 and independent of μ_e only for a long thin magnet ($N_c \to 0$), and proportional to B_0 and independent of μ_e only for a short fat magnet ($N_c \to 1$).

As the torque can be evaluated entirely in terms of the fields outside the magnet, it might be thought that any replacement of the magnet by a model which gives the same external fields could be used to determine the torque by other methods. This is in fact so, but care must be taken to consider *all* the relevant terms.

For example, if $\boldsymbol{M}_0$ is replaced by a surface current density $\boldsymbol{M}_0 \times \hat{\boldsymbol{n}}/\mu_i$ then calculation of the interaction energy by considering flux linkage with the current producing $\boldsymbol{H}_0$ leads to the correct torque. So also does calculating the energy required to produce a surface pole density $\boldsymbol{M}_0 \cdot \hat{\boldsymbol{n}}$ by bringing poles either from the origin, or from infinity, in the presence of $\boldsymbol{H}_0$, assumed due to permanent magnetization.

However, direct calculations of torque, by considering the forces on the surface currents or poles, give a different torque $\boldsymbol{T}'$ because they involve the demagnetizing factor appropriate to the direction of $\boldsymbol{H}_0$ and not that of $\boldsymbol{M}_0$. (To obtain consistent results we have to assume that the currents or poles are just *inside* the surface, so that the relevant fields are the internal fields.) In fact, $\boldsymbol{T}'$ is the body torque, and these calculations omit the torque exerted on the surface by the pressure of the magnetized fluid. For an ellipsoidal magnet $\boldsymbol{T}' = \boldsymbol{T}$ only when $\mu_i = \mu_e$ (in practice a hard magnet in a vacuum), when there is no surface torque, and N does not occur. $\boldsymbol{T}' = \boldsymbol{T}$ also for a spherical magnet, for which three is no surface pressure torque and for which N is independent of direction, giving simplifications which can be misleading; Smythe's (1939, § 12.10) correct result for the torque on a spherical hard magnet was obtained this way.

Scott (1959, § 8.3) showed that for a hard magnet in a vacuum the (body) torque due to $\boldsymbol{H}_0$ is given by the volume integral over the magnet of

$$\boldsymbol{M}_0 \times \boldsymbol{B}_f = \boldsymbol{J}_0 \times \boldsymbol{H}_f = \boldsymbol{J}_0 \times \boldsymbol{H}_0,$$

where $\boldsymbol{B}_f$ and $\boldsymbol{H}_f$ are the fields inside the magnet due to the external sources. (If these fields are non-uniform there will also be a contribution to the torque from the resultant body forces.) For $\mu_e \neq 1$, and for a hard magnet (not explicitly stated), Brown (1951, § 1.3) also used $\boldsymbol{M}_0 \times \boldsymbol{B}_f$. It is likely that in general the body torque contribution is

$$\begin{aligned}\boldsymbol{M}_{\text{total}} \times \boldsymbol{B}_{\text{total}} &= \boldsymbol{J}_{\text{total}} \times \boldsymbol{H}_{\text{total}} \\ &= (\boldsymbol{J}_0 \times \boldsymbol{H}_f) + (\boldsymbol{J}_0 \times \boldsymbol{H}_m)\end{aligned}$$

in our situation. (Note that in general $\boldsymbol{J}_0 \times \boldsymbol{H} \neq \boldsymbol{M}_0 \times \boldsymbol{B}$.) The first term again gives $\boldsymbol{T}'$, the body torque from $\boldsymbol{H}_0$. (That it is not the total torque is not stated by either author.) The second term is non-zero for skew $\boldsymbol{J}_0 = \mu_0 \boldsymbol{M}_0$, but is presumably cancelled by the appropriate surface torque.

9. Discussion

Papers on (theoretical implications of) the Kennelly/Sommerfeld controversy are too numerous to list. A welcome innovation was the paper by Whitworth & Stopes-Roe (1971), who showed *experimentally* that j_e was in fact constant for a long thin magnet. Unfortunately they interpreted their result as support for Kennelly (though still expected the torque to depend on the shape of the magnet). McCaig (1973) discussed their result and, in effect, preferred a different factorization of equation (14) for moment, to give the (constant) moment in vacuum together with a (different) shape factor, to 'defining the moment of a magnet in a medium by $\boldsymbol{T} = \boldsymbol{j} \times \boldsymbol{H} = \boldsymbol{m} \times \boldsymbol{B}$'. Knapp (1953) discusses this and other factorizations, some of which use the total moment $(\boldsymbol{j}_m + \boldsymbol{j}_f)$. (In some deflexion experiments in the Earth's field it is this total moment which is relevant.)

In conclusion it must be emphasized that *all* external effects of a (dipolar) magnet can be specified by *one* moment (though there are several ways of defining it). This moment is *not* in general invariant or equal to the volume integral of magnetization. *Both* the Kennelly and Sommerfeld notations are usable, (in the sense that $\boldsymbol{J}$ and $\boldsymbol{M}$ are interchangeable, and that the torque can be expressed as *either* $\boldsymbol{j} \times \boldsymbol{H}_0$ or $\boldsymbol{m} \times \boldsymbol{B}_0$), but the usual formulations of both (which equate j or $\boldsymbol{m}$ to the volume moment) are *not* valid.

I must thank the very large number of colleagues without whose helpful discussion over two years this paper would not have been written.

Appendix 1

Integral expressions for the fields, potentials, and moments

By considering the moment $\boldsymbol{M}\,\mathrm{d}\tau$ of a small volume element of magnetization to be an elementary dipole, we see that the resultant field at an observation point is

$$\boldsymbol{H} = \frac{1}{4\pi}\int \frac{-\boldsymbol{M} + 3(\boldsymbol{M}\cdot\hat{\boldsymbol{R}})\,\hat{\boldsymbol{R}}}{R^2}\,\mathrm{d}\tau, \qquad \text{(A 1)}$$

where the integration includes *all* regions of magnetization, induced as well as permanent; $\boldsymbol{R}$ is the vector from the integration point to the observation point. Similarly

$$\phi = \frac{1}{4\pi}\int \frac{\boldsymbol{M}\cdot\hat{\boldsymbol{R}}}{R^2}\,\mathrm{d}\tau, \qquad \text{(A 2)}$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{M}\times\hat{\boldsymbol{R}}}{R^2}\,\mathrm{d}\tau. \qquad \text{(A 3)}$$

These integrals are all convergent when the observation point is in free space (and distant from any magnetization by a large number of lattice spacings). However, we would also like to use them to give the macroscopically averaged fields inside a region of magnetization. The $1/R^2$ variation now gives difficulty in performing the integrations at the observation point; we must define the integrals as the limiting values as a small volume δV excluding the observation point is shrunk to zero. It turns out that the integral (A 1) is only semi-convergent; it gives values ranging from $\boldsymbol{H}$ to $\boldsymbol{B}/\mu_0$ depending on the shape of δV. The integrals for ϕ and $\boldsymbol{A}$ are themselves convergent, but if differentiated algebraically to give the fields of course give (A 1) again.

However, by using appropriate vector transformations, integrals can be obtained which are fully convergent. One such transformation gives

$$\phi = \frac{1}{4\pi}\Sigma\left\{\int_{\mathrm{V}}(-\operatorname{div}\boldsymbol{M})\frac{1}{R}\mathrm{d}\tau + \int_{\mathrm{S}}(\boldsymbol{M}\cdot\hat{\boldsymbol{n}})\frac{1}{R}\mathrm{d}S\right\} \tag{A 4}$$

and

$$\boldsymbol{H} = \frac{1}{4\pi}\Sigma\left\{\int_{\mathrm{V}}(-\operatorname{div}\boldsymbol{M})\frac{\hat{\boldsymbol{R}}}{R^2}\mathrm{d}\tau + \int_{\mathrm{S}}(\boldsymbol{M}\cdot\hat{\boldsymbol{n}})\frac{\hat{\boldsymbol{R}}}{R^2}\mathrm{d}S\right\}, \tag{A 5}$$

each term in the sum being the contribution from one region V of magnetization bounded by the surface S, $\hat{\boldsymbol{n}}$ being the unit outward normal on S. Equations (A 4) and (A 5) are in fact mathematically identical to those we would obtain if $(-\operatorname{div}\boldsymbol{M})$ and $(\boldsymbol{M}\cdot\hat{\boldsymbol{n}})$ were the volume and surface densities of a magnetic pole distribution in free space. If we wish we can interpret them in terms of such a distribution of 'fictitious' or 'bound' magnetic poles. (It is perhaps worth reminding the reader that μ_e, the relative permeability of the medium, is simply one way of representing the presence of induced magnetization; it does not enter into equations, such as (A 4) or (A 5), which directly consider such induced magnetization.) Equation (A 5) gives the correct $\boldsymbol{H}$ everywhere (and hence, by using the constitutive equation $\boldsymbol{B} = f(\boldsymbol{H})$ appropriate to the observation point, the correct $\boldsymbol{B}$ everywhere).

Similarly, another transformation gives

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\Sigma\left\{\int_{\mathrm{V}}(\operatorname{curl}\boldsymbol{M})\frac{1}{R}\mathrm{d}\tau + \int_{\mathrm{S}}(\boldsymbol{M}\times\hat{\boldsymbol{n}})\frac{1}{R}\mathrm{d}S\right\} \tag{A 6}$$

and

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\Sigma\left\{\int_{\mathrm{V}}(\operatorname{curl}\boldsymbol{M})\times\frac{\hat{\boldsymbol{R}}}{R^2}\mathrm{d}\tau + \int_{\mathrm{S}}(\boldsymbol{M}\times\hat{\boldsymbol{n}})\times\frac{\hat{\boldsymbol{R}}}{R^2}\mathrm{d}S\right\}. \tag{A 7}$$

Again, if we wish, we can interpret $(\operatorname{curl}\boldsymbol{M})$ and $(\boldsymbol{M}\times\hat{\boldsymbol{n}})$ as volume and surface densities of a 'fictitious' or 'bound' electric current distribution. (A 7) gives the correct $\boldsymbol{B}$ everywhere.

By expanding the surface integrand in (A 5) and (A 7) and by using the divergence theorem it can be confirmed that these equations do in fact give values of (macroscopic) $\boldsymbol{H}$ and $\boldsymbol{B}/\mu_0$ which differ by the local value of $\boldsymbol{M}$.

It must be emphasized that these transformations are purely mathematical, and are both valid regardless of the physical plausibility of any interpretation we may give to the transformed 'sources'.

Now consider a magnet in a homogeneous permeable linear medium. As (A 5) and (A 7) give the correct $\boldsymbol{H}$ and $\boldsymbol{B}$ in the medium, the corresponding integrated dipole moments must be equivalent to $\boldsymbol{j}_e$ and $\boldsymbol{m}_e$. In this effectively free space situation we must use the moment definitions of (1), i.e. those of (3) without the factor μ_e; such moments are equal to, and will be denoted by, $[\boldsymbol{j}/\mu_e]$ and $[\mu_e\boldsymbol{m}]$. If the medium is linear, in it $\operatorname{div} \boldsymbol{M}_e = \operatorname{curl} \boldsymbol{M}_e = 0$; using the conventional expressions for the dipole moment of pole and current distributions we then have

$$[\boldsymbol{j}/\mu_e] = \mu_0 \left\{ \int_V \boldsymbol{r}(-\operatorname{div} \boldsymbol{M}_i)\,d\tau + \int_S \boldsymbol{r}(\boldsymbol{M}_i \cdot \hat{\boldsymbol{n}})\,dS - \int_S \boldsymbol{r}(\boldsymbol{M}_e \cdot \hat{\boldsymbol{n}})\,dS \right\}, \quad (A\ 8)$$

$$\mu_0[\mu_e \boldsymbol{m}_e] = \mu_0 \left\{ \frac{1}{2} \int_V \boldsymbol{r} \times \operatorname{curl} \boldsymbol{M}_i\,d\tau + \frac{1}{2} \int_S \boldsymbol{r} \times (\boldsymbol{M}_i \times \hat{\boldsymbol{n}})\,dS - \frac{1}{2} \int_S \boldsymbol{r} \times (\boldsymbol{M}_e \times \hat{\boldsymbol{n}})\,dS \right\}. \quad (A\ 9)$$

Here V is the region occupied by the magnet, and S its surface; $\hat{\boldsymbol{n}}$ has the same direction in both surface integrals. (The corresponding terms for the outer boundary of the medium are omitted. This boundary would give a small uniform field at the observation point, and it is assumed that the boundary is sufficiently distant that this field can be neglected.)

In (A 8) and in (A 9) the first two terms together give the contribution of the magnet material. By using the divergence theorem it can be shown that these 'internal' contributions are equal and are in fact the 'volume' moments of § 2, producing field contributions for which $\boldsymbol{B} = \mu_0 \boldsymbol{H}$:

$$[\boldsymbol{j}_i/\mu_e] = \mu_0[\mu_e \boldsymbol{m}_i] = \mu_0 \int_V \boldsymbol{M}_i\,d\tau = \int_V \boldsymbol{J}_i\,d\tau$$
$$= \mu_0 \boldsymbol{m}_V = \boldsymbol{j}_V. \quad (A\ 10)$$

(It must be remembered that $\boldsymbol{j}_i$ and $\boldsymbol{m}_i$ are not observable quantities, and that the magnetization $\boldsymbol{M}_i$ of the magnet may well have been altered by the presence of the external medium.) It is the third term, which gives the contribution of the surrounding medium, which makes $[\boldsymbol{j}_e/\mu_e]$ and $[\mu_e \boldsymbol{m}_e]$ differ from $\boldsymbol{j}_V$ and $\boldsymbol{m}_V$; it can be shown that it does in fact give total moments such that

$$\mu_e[\boldsymbol{j}_e/\mu_e] = \mu_0[\mu_e \boldsymbol{m}_e] \quad \text{or} \quad \boldsymbol{j}_e = \mu_e \mu_0 \boldsymbol{m}_e. \quad (A\ 11)$$

In the situation considered in § 5 we have the further simplification that

$$\operatorname{div} \boldsymbol{M}_i = \operatorname{curl} \boldsymbol{M}_i = 0,$$

so that the only effective 'sources' are the surface pole and current distributions, $(\boldsymbol{M}_i - \boldsymbol{M}_e) \cdot \hat{\boldsymbol{n}}$ and $(\boldsymbol{M}_i - \boldsymbol{M}_e) \times \hat{\boldsymbol{n}}$, which turn out to be $\boldsymbol{M}_0 \cdot \hat{\boldsymbol{n}}/D_c$ and $\mu_e \boldsymbol{M}_0 \times \hat{\boldsymbol{n}}/D_c$. Putting these values into (A 5) and (A 7) gives the correct $\boldsymbol{H}$ and $\boldsymbol{B}$ respectively everywhere, both inside and outside the magnet. However, such a 'free space' representation cannot easily be extended to the externally applied field $\boldsymbol{B}_0 = \mu_e \mu_0 \boldsymbol{H}_0$, as the details of its sources are not specified.

Appendix 2

Summary of a calculation given by Timotkin & Ciric (1971)

A circuit carrying current i_a produces the applied field $\boldsymbol{B}_a = \mu_e \mu_0 \boldsymbol{H}_a$. With $i_a = 0$ a region V is replaced with magnetized material which gives $\boldsymbol{B}_m = \mu_i \mu_0 \boldsymbol{H}_m + \mu_0 \boldsymbol{M}_0$ inside V, and $\boldsymbol{B}_m = \mu_e \mu_0 \boldsymbol{H}_m$ outside (region W). (Here μ_i, μ_e, $\boldsymbol{H}_a$ and $\boldsymbol{M}_0$ can be arbitrary functions of position.) The field $\boldsymbol{B}_{\mathrm{m}}$ has vector potential $\boldsymbol{A}_{\mathrm{m}}$ and produces a flux linkage Φ with the current circuit.

The interaction energy is (Stratton 1941, §2.14)

$$U_{\mathrm{am}} = i_{\mathrm{a}}\,\Phi = \int_{\mathrm{V+W}} \boldsymbol{I}_{\mathrm{a}}\cdot\boldsymbol{A}_{\mathrm{m}}\,\mathrm{d}\tau = \int(\operatorname{curl}\boldsymbol{H}_{\mathrm{a}})\cdot\boldsymbol{A}_{\mathrm{m}}\,\mathrm{d}\tau$$
$$= \int \operatorname{div}(\boldsymbol{H}_{\mathrm{a}}\times\boldsymbol{A}_{\mathrm{m}})\,\mathrm{d}\tau + \int \boldsymbol{H}_{\mathrm{a}}\cdot\operatorname{curl}\boldsymbol{A}_{\mathrm{m}}\,\mathrm{d}\tau.$$

By Gauss's theorem the first integral is zero, so

$$U_{\mathrm{am}} = \int_{\mathrm{V+W}} \boldsymbol{H}_{\mathrm{a}}\cdot\boldsymbol{B}_{\mathrm{m}}\,\mathrm{d}\tau = \int_{\mathrm{V}} \boldsymbol{H}_{\mathrm{a}}\cdot(\mu_{\mathrm{i}}\mu_0\boldsymbol{H}_{\mathrm{m}}+\mu_0\boldsymbol{M}_0)\,\mathrm{d}\tau + \int_{\mathrm{W}} \boldsymbol{H}_{\mathrm{a}}\cdot(\mu_{\mathrm{e}}\mu_0\boldsymbol{H}_{\mathrm{m}})\,\mathrm{d}\tau.$$

But
$$\int_{\mathrm{W}}(\mu_{\mathrm{e}}\mu_0\boldsymbol{H}_{\mathrm{a}})\cdot\boldsymbol{H}_{\mathrm{m}}\,\mathrm{d}\tau = -\int_{\mathrm{V}}(\mu_{\mathrm{e}}\mu_0\boldsymbol{H}_{\mathrm{a}})\cdot\boldsymbol{H}_{\mathrm{m}}\,\mathrm{d}\tau$$

as $\operatorname{curl}\boldsymbol{H}_{\mathrm{m}} = 0$ and $\operatorname{div}(\mu_{\mathrm{e}}\mu_0\boldsymbol{H}_{\mathrm{a}}) = 0$ everywhere (Abraham & Becker 1937, p. 39), so

$$U_{\mathrm{am}} = \int_{\mathrm{V}} \boldsymbol{H}_{\mathrm{a}}[(\mu_{\mathrm{i}}-\mu_{\mathrm{e}})\,\mu_0\boldsymbol{H}_{\mathrm{m}}+\mu_0\boldsymbol{M}_0]\,\mathrm{d}\tau.$$

If in fact μ_{e} and $\boldsymbol{H}_{\mathrm{a}}$ are constant in the region of interest it follows that the torque on the magnet is

$$\boldsymbol{T} = \int_{\mathrm{V}} [(\mu_{\mathrm{i}}-\mu_{\mathrm{e}})\,\mu_0\boldsymbol{H}_{\mathrm{m}}+\mu_0\boldsymbol{M}_0]\,\mathrm{d}\tau\times\boldsymbol{H}_{\mathrm{a}}.$$

For uniformly magnetized ellipsoide this gives the same result as §7.

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