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Force and Torque on a Small Magnetic Dipole

For a related topic, see [Permanent Magnet Motors](#).

On this page we give a simple derivation of the force and torque on a small magnetic dipole which is in a non-uniform magnetic field.

On this page, we will take a dipole of magnitude $|\mu|=IA$ to be a loop of current of magnitude I , with area of the loop equal to A .

The dipole moment, μ , is a vector which points perpendicular to the plane of the loop, and which points "up" when the dipole is oriented so that the current runs counterclockwise, looking down on the loop. For simplicity in the derivation, we're going to cheat and use a square loop ([figure 1](#)). This will save us a lot of niggling little sines and cosines in the integrals, and in fact will let us entirely avoid explicitly taking integrals. We'll just assume without proof that our result is the same as what we would get with a circular loop.

Throughout most of this page we'll also assume that the loop lies in the xy plane, with the sides of the square lying parallel to the x and y axes. The dipole vector points along the z axis. The magnetic ("B") field, on the other hand, has arbitrary orientation. (We can rotate the axes so that the position of the dipole is easy to describe, or so the B field lies along one axis ... but we can't do both at once.) The angle between the direction of the magnetic field and the dipole vector, which we're assuming lies on the z axis, is θ .

Torque on the Loop

With a circular loop, the z component of the torque, τ_z , would be identically zero. With our square loop the z component is zero to second order in the gradient of the B field, and vanishes as the loop size shrinks; we will assume that it actually is exactly zero with no further comment.

The x and y components of the torque, on the other hand, depend directly on the magnitudes of the y and x components of the B field. In [figure 2](#), we've shown the forces on the left and right sides of the loop as they would point if the y component of the B field were positive. Noting that the x axis points out of the page, we see that forces in the directions of the arrows would produce a negative torque along the x axis. The x component of the torque must then be

$$(1) \quad \tau_x = -\frac{dy}{2} (f_{zy1} + f_{zy2})$$

Recalling that the current, I , runs counter-clockwise, we can insert the expressions for the forces on the loop, obtaining

$$(2) \quad \tau_x = -\frac{dy}{2} (I \cdot B_y \cdot dx + I \cdot B_y \cdot dx)$$

or, replacing $dx \cdot dy$ with A ,

$$(3) \quad \tau_x = -IA \cdot B_y$$

Similar reasoning leads to the y component of the torque, which we find to be

Figure 1 -- Square Dipole:

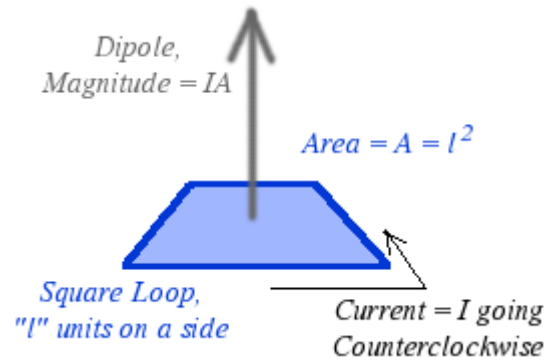
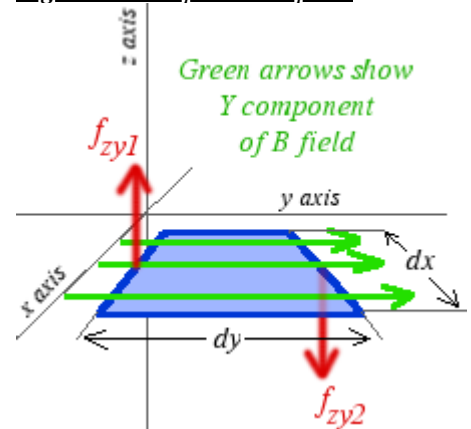


Figure 2 -- Torque on a dipole:



$$(4) \quad \tau_y = IA \cdot B_x$$

Since τ_z is zero, the magnitude of the torque is therefore

$$(5) \quad |\tau| = IA \cdot \sqrt{B_x^2 + B_y^2}$$

Recalling that the angle between \mathbf{B} and the dipole vector is θ , we can see that this is

$$(6) \quad |\tau| = IA|B| \sin \theta$$

So the magnitude of the torque matches the magnitude of the cross product of the dipole and the \mathbf{B} field. The torque lies in the xy plane and so must be perpendicular to the dipole vector. From (3) and (4) we can compute the dot product of the torque with the \mathbf{B} field; that's zero, so the torque is also perpendicular to the \mathbf{B} field. So, the torque is +/- the cross product of μ and \mathbf{B} . Some fiddling with the right hand rule while looking at [figure 2](#) to determine the sign on the torque vector convinces us that the torque must be

$$(7) \quad \tau = \mu \times B$$

This expression is independent of rotations of the coordinates.

Force on the Loop

The net force on the loop is zero if the \mathbf{B} field is uniform, as we can see by looking at [figure 2](#): the current on opposite sides of the square is going in opposite directions, so the forces on each side due to the \mathbf{B} field are also opposite, and they cancel pairwise. There is, however, a net force which is first order in the *gradient* of \mathbf{B} .

Z component of the force

If the gradient of \mathbf{B} in the y direction is nonzero, then the forces shown in figure 2 will not cancel; rather, we can see that, to first order in the gradient of \mathbf{B} ,

$$(8) \quad \begin{aligned} f_{zy1} - f_{zy2} &= I \cdot dx \cdot (-\Delta B_y) \\ &= -I \cdot dx \cdot dy \cdot \partial_y B_y \\ &= -IA \cdot \partial_y B_y \end{aligned}$$

Similarly, the net force on the two sides of the loop which are parallel to the y axis will be

$$(9) \quad \begin{aligned} f_{zx1} - f_{zx2} &= I \cdot dy \cdot (-\Delta B_x) \\ &= -IA \cdot \partial_x B_x \end{aligned}$$

and, replacing IA with $|\mu|$ and summing (8) and (9), we see that the net force in the z direction is

$$(10) \quad f_z = -|\mu|(\partial_y B_y + \partial_x B_x)$$

Since we know that $\nabla \cdot \mathbf{B} = 0$, we must have

$$(11) \quad \partial_z B_z = -(\partial_x B_x + \partial_y B_y)$$

so we can rewrite (10) as

$$(12) \quad f_z = |\mu| \partial_z B_z$$

X and Y components of the force

In a uniform field the two segments of the loop parallel to the x axis feel equal and opposite forces (figure 3). But if the field is nonuniform, then we can see that, to first order in the gradient, the y component of the net force will be

$$(13) \quad \begin{aligned} f_y &= I dx B_{z_{y2}} - I dx B_{z_{y1}} \\ &= I dx (dy \cdot \partial_y B_z) \\ &= |\mu| \partial_y B_z \end{aligned}$$

Similarly, just by rotating the diagram we can see that

$$(14) \quad \begin{aligned} f_x &= I dx B_{z_{x2}} - I dx B_{z_{x1}} \\ &= |\mu| \partial_x B_z \end{aligned}$$

Putting together (12), (13), and (14) we see that the total force must be

$$(15) \quad f = |\mu| \nabla B_z$$

or, since θ is the angle between the \mathbf{B} field and the dipole vector, which we've assumed is aligned with the z axis, we have

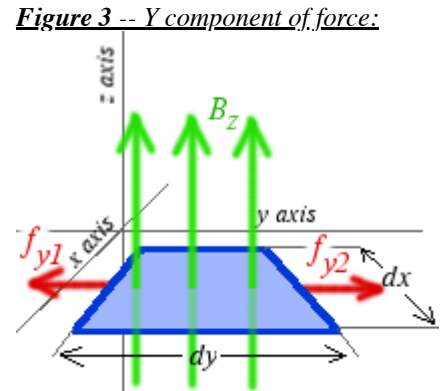
$$(16) \quad f = |\mu| \nabla(|B| \cos \theta)$$

Since the magnitude of the *dipole* is not a function of position, we can move $|\mu|$ under the gradient sign; we then recognize the formula for dot product, and rewrite (16) as:

$$(17) \quad f = \nabla(\mu \cdot B)$$

Is the Force Conservative?

In the case of an actual current loop, asking whether the force on the dipole is "conservative" is somewhat meaningless. In that case, the \mathbf{B} field's action on the current in the loop is via the Lorentz force law for charged particles, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and, since the force due to the \mathbf{B} field on each charge is always perpendicular to the charge's line of motion, all the energy gained by the loop must actually be provided by the current in the loop. However, in the case of a *permanent* dipole, such as that of an electron, or a permanent magnet which gains its field from large numbers of electrons with aligned spins, there is no apparent "internal" source of energy when the dipole is accelerated by the magnetic field. We could also say that, for permanent dipoles, nothing is "consumed" as the dipole moves through the field. In such a case, it makes sense to ask if the action of the magnetic field on the dipole is "conservative".



A conservative force can be described in terms of a potential function. If we can find a potential function for the force (*and* torque) on a dipole, then we'll have shown that it's conservative. And we can.

First, we need to choose coordinates to describe the position of a dipole in a magnetic field. For the location in space, we'll use ordinary Cartesian spacial coordinates, (x,y,z) . For the orientation of the dipole, we'll use angle between the dipole and the \mathbf{B} field, θ , along with the azimuth of the dipole around the \mathbf{B} field's direction, which we'll call σ . In all, then, we have five dimensions: (x,y,z,θ,σ) . Note that these coordinates are not the same as those used earlier in this page; neither the \mathbf{B} field nor the dipole is assumed to be aligned with the z axis.

Furthermore, the angle $\theta=0$ is not connected with the direction of any of the Cartesian coordinates -- it's pinned to the \mathbf{B} field.

We already know the forces acting on the dipole in these coordinates; they're given by equations (7) and (17).

Since there's no force acting in the σ direction, σ is a "cyclic" coordinate and won't enter into the potential energy in any meaningful way; we'll ignore it on the rest of this page.

We'll guess the potential, then prove that the guess is correct. From (17), a good candidate seems to be:

$$(18) \quad \phi = -\boldsymbol{\mu} \cdot \mathbf{B}$$

The negative of its gradient is certainly (17), the force on the dipole. However, we also need to confirm that the negative of its partial derivative with respect to θ is given by (7), the torque on the dipole. The potential can also be written as:

$$(19) \quad \phi = -|\boldsymbol{\mu}| |\mathbf{B}| \cos \theta$$

And the negative of its partial is then:

$$(20) \quad -\frac{\partial \phi}{\partial \theta} = -|\boldsymbol{\mu}| |\mathbf{B}| \sin \theta$$

That's certainly the correct magnitude. Is the direction correct? Certainly, looking at (7), the torque must lie along the line which is perpendicular to both $\boldsymbol{\mu}$ and \mathbf{B} , as this necessarily does. But what of the sign? If the angle between the dipole and \mathbf{B} is small, we can see from (7) that the torque will twist the dipole toward \mathbf{B} , which is the direction of decreasing θ . Looking at (20), we can see that for small θ , the direction of the "twist" will also be toward decreasing θ . So, the signs agree, and we conclude that the potential function correctly describes the torque as well as the force on the dipole.

Finally, we should say a few words about the coordinates. Deriving the potential was trivial in the coordinates I've given, but the kinetic energy of a physical dipole in these coordinates would be pretty messy: linear motion through space with fixed (θ,σ) coordinates would actually involve rotations of the dipole. In consequence, the Lagrangian for the dipole in these coordinates would be awkward. To solve a real problem one would more typically use Cartesian coordinates for the position along with polar coordinates which were referenced to the Cartesian z axis for the dipole orientation. Since (18) is independent of the coordinates, such a change in coordinates would not affect the potential function.

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